Invent. math. 150, 1–44 (2002)
DOI: 10.1007/s00222-002-0227-x

Inventiones mathematicae

Real zeros of quadratic Dirichlet L-functions*

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Oblatum 31-X-2001 & 21-II-2002
Published online: 29 April 2002 –

1. Introduction

A small part of the Generalized Riemann Hypothesis asserts that L-functions do not have zeros on the line segment $(\frac{1}{2}, 1]$. The question of vanishing at $s = \frac{1}{2}$ often has deep arithmetical significance, and has been investigated extensively. A persuasive view is that L-functions vanish at $\frac{1}{2}$ either for trivial reasons (the sign of the functional equation being negative), or for deep arithmetical reasons (such as the L-function of an elliptic curve of positive rank) and that the latter case happens very rarely. N. Katz and P. Sarnak [7] have formulated precise conjectures on the low lying zeros in families of L-functions which support this view.

In the case of Dirichlet L-functions it is expected that $L(\frac{1}{2}, \chi)$ is never zero, and so $L(\sigma, \chi) \neq 0$ for all $\frac{1}{2} \leq \sigma \leq 1$. This conjecture appears to have been first enunciated by S.D. Chowla [2] in the special case of quadratic characters χ. Progress towards these non-vanishing questions has been in two directions: zero-density type results which establish that very few L-functions have a zero in $(\frac{1}{2} + \epsilon, 1]$ (see for example A. Selberg [10], M. Jutila [6] and D.R. Heath-Brown [4]), and a growing body of work on non-vanishing at $\frac{1}{2}$ (see for example R. Balasubramanian and V.K. Murty [1], H. Iwaniec and Sarnak [5], and K. Soundararajan [11]). Further much numerical evidence for the GRH has been accumulated, and these calculations support Chowla's conjecture (see [8] and [9]). However the state of knowledge could not exclude the possibility that every Dirichlet L-function of sufficiently large conductor has a non-trivial real zero. In this

* Research of both authors is supported by the American Institute of Mathematics (AIM), and in part through grants from the NSF, including the FRG grant DMS0074028.

paper we eliminate this possibility, and prove that a positive proportion of quadratic Dirichlet L-functions do not vanish on $[\frac{1}{2}, 1]$.

For an integer $d \equiv 0$, or $1 \pmod 4$ we put $\chi_d(n) = \left(\frac{d}{n}\right)$, so that χ_d is a real character with conductor at most $|d|$. If d is an odd, positive, square-free integer then χ_{-8d} is a real, primitive character with conductor $8d$, and with $\chi_{-8d}(-1) = -1$.

Theorem 1. *For at least* 20% *of the odd square-free integers* $d \geq 0$ *we have* $L(\sigma, \chi_{-8d}) > 0$ *for* $0 \leq \sigma \leq 1$. *More precisely, for all large x the number of odd positive square-free integers $d \leq x$ such that* $L(\sigma, \chi_{-8d}) > 0$ *for all* $0 \leq \sigma \leq 1$ *exceeds* $\frac{1}{5}(\frac{4x}{\pi^2})$.

While in this paper we have restricted our attention to fundamental discriminants of the form $-8d$, our methods would apply to fundamental discriminants in any arithmetic progression. Also our proof yields that there are many L-functions having no non-trivial zeros in a thin rectangle containing the real axis. Precisely, there is a constant $c > 0$ such that for at least 20% of the fundamental discriminants $-8d$ with $0 < d \leq x$, the rectangle $\{\sigma + it : \ \sigma \in [0, 1], \ \ |t| \leq c/\log x\}$ is free of zeros of $L(s, \chi_{-8d})$. As another consequence of our work we find that the number of fundamental discriminants $-8d$ with $0 < d \leq x$ such that $L(s, \chi_{-8d})$ has a zero in the interval $[\sigma, 1]$ is $\ll x^{1-(1-\epsilon)(\sigma-\frac{1}{2})}$ for any fixed $\epsilon > 0$.

2. Outline of the proof

We begin with the following version of the argument principle, due to Selberg [10], whose proof we reproduce for completeness.

Lemma 2.1. *Let f be a holomorphic function, which is non-zero in some half-plane $Re(z) \geq W$. Let $\mathcal{B}$ be the rectangular box with vertices $W_0 \pm iH$, $W_1 \pm iH$ where $H > 0$ and $W_0 < W < W_1$. Then*

$$4H \sum_{\substack{\beta+i\gamma \in \mathcal{B} \\ f(\beta+i\gamma)=0}} \cos\left(\frac{\pi\gamma}{2H}\right) \sinh\left(\frac{\pi(\beta - W_0)}{2H}\right)$$

$$= \int_{-H}^{H} \cos\left(\frac{\pi t}{2H}\right) \log|f(W_0 + it)| dt$$

$$+ \int_{W_0}^{W_1} \sinh\left(\frac{\pi(\alpha - W_0)}{2H}\right) \log|f(\alpha + iH) f(\alpha - iH)| d\alpha$$

$$- Re \int_{-H}^{H} \cos\left(\pi \frac{W_1 - W_0 + it}{2iH}\right) \log f(W_1 + it) dt.$$

Proof. From the box $\mathcal{B}$ we exclude the line segments $x + i\gamma$ with $W_0 \leq x \leq \beta$ for every zero $\beta + i\gamma$ of f lying in $\mathcal{B}$. Denoting by Γ the boundary of the resulting domain we see that

$$\int_\Gamma \cos\Big(\pi\frac{s-W_0}{2iH}\Big)\log f(s)ds = 0.$$

Since the value of $\log f(s)$ differs by $2\pi i$ on the upper and lower sides of the "cuts" from $\beta + i\gamma$ to $W_0 + i\gamma$, we conclude from the above that

$$2\pi i \sum_{\substack{\beta+i\gamma\in\mathcal{B}\\ f(\beta+i\gamma)=0}} \int_{W_0+i\gamma}^{\beta+i\gamma} \cos\Big(\pi\frac{s-W_0}{2iH}\Big)ds$$
$$= \Big(\int_{W_0-iH}^{W_0+iH} + \int_{W_0+iH}^{W_1+iH} - \int_{W_1-iH}^{W_1+iH} - \int_{W_0-iH}^{W_1-iH}\Big)\cos\Big(\pi\frac{s-W_0}{2iH}\Big)\log f(s)ds.$$

The imaginary part of the LHS above equals the LHS of the equality of the lemma. The imaginary part of the first integral on the RHS above equals the first term on the RHS of the lemma. The second and fourth integrals on the RHS above have combined imaginary part equal to the second term on the RHS of the lemma. Lastly the imaginary part of the third term on the RHS above equals the third term on the RHS of the lemma. Thus Lemma 2.1 is proved.

Let X be large, and let d be any odd square-free number in $[X, 2X]$. We shall apply Lemma 2.1 to a mollified version of $L(s, \chi_{-8d})$. Precisely, for a parameter $X^\epsilon \le M \le X$ to be fixed later[1], let

$$M(s,d) = \sum_{n\le M}\frac{\lambda(n)}{n^s}\chi_{-8d}(n),$$

where the $\lambda(n)$ are real numbers $\ll n^\epsilon$ to be specified later. We apply Lemma 2.1 with $f(s,d) := L(s,\chi_{-8d})M(s,d)$ and $W_0 = \frac{1}{2} - \frac{R}{\log X}$, $H = \frac{S}{\log X}$, and $W_1 = \sigma_0$ where R and S are fixed positive parameters in the interval $(\epsilon, 1/\epsilon)$ to be chosen later, and $\sigma_0 > 1$ will be chosen later such that $f(s,d)$ has no zeros in $\mathrm{Re}\, s > \sigma_0$. Since the LHS of Lemma 2.1 consists of positive terms we glean that

$$4S \sum_{\substack{\beta\ge \frac{1}{2}-\frac{R}{\log X}\\ L(\beta,\chi_{-8d})=0}} \sinh\Big(\frac{\pi(R+\log X(\beta-1/2))}{2S}\Big) \le I_1(d) + I_2(d) + I_3(d), \tag{2.1}$$

[1] Here and throughout, ϵ denotes a small positive real number. The reader should be warned that it might be a different ϵ from line to line.

where (after obvious changes of variables)

$$\text{(2.2a)}\qquad I_1(d) = \int_{-S}^{S} \cos\left(\frac{\pi t}{2S}\right) \log\left| f\Big(\frac{1}{2} - \frac{R}{\log X} + i\frac{t}{\log X}, d\Big)\right| dt,$$

$$\text{(2.2b)}\qquad I_2(d) = \int_{-R}^{(\sigma_0-\frac{1}{2})\log X} \sinh\left(\frac{\pi(x+R)}{2S}\right) \log\left| f\left(\frac{1}{2} + \frac{x}{\log X} + i\frac{S}{\log X}, d\right)\right|^2 dx,$$

and

$$\text{(2.2c)}\qquad I_3(d) = -\operatorname{Re}\int_{-S}^{S} \cos\Big(\pi\frac{(\sigma_0-1/2)\log X - R + it}{2iS}\Big) \log f\Big(\sigma_0 + i\frac{t}{\log X}, d\Big) dt.$$

Suppose that $L(\beta, \chi_{-8d}) = 0$ for some $\beta \in [\frac{1}{2}, 1]$. We claim that the LHS of (2.1) exceeds $8S\sinh\left(\frac{\pi R}{2S}\right)$. To see this, suppose first that $L(s, \chi_{-8d})$ has a zero $\beta > \frac{1}{2} + \frac{R}{\log X}$. Then the contribution of this zero alone would be $\geq 4S\sinh(\frac{\pi}{R}S) \geq 8S\sinh\left(\frac{\pi R}{2S}\right)$ since $\sinh(2x) \geq 2\sinh x$ for $x \geq 0$. On the other hand, if $L(s, \chi_{-8d})$ has a zero at $\frac{1}{2} + \frac{\xi}{\log X}$ for some $0 \leq \xi \leq R$ then by the functional equation it also has a zero at $\frac{1}{2} - \frac{\xi}{\log X}$. In case $\xi = 0$ note that there is at least a double zero at $\frac{1}{2}$. Both these zeros are included in the LHS of (2.1), and together they contribute $4S\left(\sinh\left(\frac{\pi(R-\xi)}{2S}\right) + \sinh\left(\frac{\pi(R+\xi)}{2S}\right)\right) \geq 8S\sinh\left(\frac{\pi R}{2S}\right)$, since the minimum value of $\sinh(x-y) + \sinh(x+y)$ for $0 \leq y \leq x$ is attained at $y = 0$. We document this below:

$$\text{(2.3)}\qquad I_1(d) + I_2(d) + I_3(d) \geq 8S\sinh\left(\frac{\pi R}{2S}\right) \quad \text{if } L(s, \chi_{-8d}) \text{ has a non-trivial real zero.}$$

The plan now is to obtain upper bounds for $I_1(d) + I_2(d) + I_3(d)$ on average over d, and thereby conclude that the inequality (2.3) cannot hold too often. To elaborate on this, we first fix some notation. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence of complex numbers, and let F denote a smooth function supported in the interval $[1, 2]$. Throughout this paper we adopt the notation

$$\mathcal{S}(a_d; F) = \mathcal{S}(a_d; F, X) = \frac{1}{X}\sum_{d \text{ odd}} \mu^2(d) a_d F\left(\frac{d}{X}\right).$$

Let Φ be a smooth non-negative function supported in $[1, 2]$. For a complex number w we define

$$\text{(2.4a)}\qquad \check{\Phi}(w) = \int_0^{\infty} \Phi(y) y^w dy.$$

For integers $\nu \geq 0$ we define

$$\Phi_{(\nu)} = \max_{0\leq j\leq \nu} \int_1^2 |\Phi^{(j)}(t)|dt. \tag{2.4b}$$

Integrating by parts ν times we get that

$$\check{\Phi}(w) = \frac{1}{(w+1)\cdots(w+\nu)} \int_0^\infty \Phi^{(\nu)}(y) y^{w+\nu} dy,$$

so that for Re $w > -1$ we have

$$|\check{\Phi}(w)| \ll_\nu \frac{2^{\mathrm{Re}\, w}}{|w+1|^\nu} \Phi_{(\nu)}. \tag{2.4c}$$

Let $\mathcal{N}(X, \Phi)$ count, with weight $\Phi(d/X)$, the odd, square-free integers $d \in [X, 2X]$ such that $L(s, \chi_{-8d})$ has a non-trivial real zero. In view of (2.1) and (2.3) we see that

$$\mathcal{N}(X, \Phi) \leq X\Big(8S \sinh\Big(\frac{\pi R}{2S}\Big)\Big)^{-1} \mathcal{S}(I_1(d) + I_2(d) + I_3(d); \Phi).$$

For a complex number δ_1 we define

$$\mathcal{W}(\delta_1, \Phi) = \frac{\mathcal{S}(|L(\frac{1}{2}+\delta_1, \chi_{-8d}) M(\frac{1}{2}+\delta_1, \chi_{-8d})|^2; \Phi)}{\mathcal{S}(1; \Phi)}. \tag{2.5}$$

Since the arithmetic mean exceeds the geometric mean we have that

$$\mathcal{S}(\log |f(\tfrac{1}{2}+\delta_1, d)|^2; \Phi) \leq \mathcal{S}(1; \Phi) \log \mathcal{W}(\delta_1; \Phi).$$

Using this in (2.1), and recalling the definitions (2.2a,b), we conclude that

$$\begin{aligned} \mathcal{N}(X, \Phi) \leq & \frac{X\mathcal{S}(1; \Phi)}{8S \sinh\left(\frac{\pi R}{2S}\right)} \big(J_1(X; \Phi) + J_2(X; \Phi)\big) \\ & + \frac{X}{8S \sinh\left(\frac{\pi R}{2S}\right)} \mathcal{S}(I_3(d); \Phi), \end{aligned} \tag{2.6}$$

where

$$J_1(X; \Phi) = \int_0^S \cos\Big(\frac{\pi t}{2S}\Big) \log \mathcal{W}\Big(-\frac{R}{\log X} + i\frac{t}{\log X}; \Phi\Big) dt, \tag{2.7a}$$

(2.7b)

$$J_2(X; \Phi) = \int_{-R}^{(\sigma_0-\frac{1}{2})\log X} \sinh\Big(\frac{\pi(x+R)}{2S}\Big) \log \mathcal{W}\Big(\frac{x}{\log X} + i\frac{S}{\log X}; \Phi\Big) dx.$$

At this juncture we specify more carefully the choice of our mollifier coefficients. To counter the rapid growth of the $\sinh(\pi(x+R)/(2S))$ term in (2.7b), we would like $\mathcal{W}((x+iS)/\log X;\Phi)$ to tend rapidly to 1. One way to ensure this is to choose $\lambda(n)=0$ if n is even, or if $n>M$, and for odd integers $n\le M$ define

(2.8)
$$\lambda(n):=\mu(n)Q\Big(\frac{\log(M/n)}{\log M}\Big):=\begin{cases}\mu(n) & \text{if } n\le M^{1-b}\\ \mu(n)P\left(\frac{\log(M/n)}{\log M}\right) & \text{if } M^{1-b}\le n\le M.\end{cases}$$

Here b is a parameter in $[\epsilon,1-\epsilon]$, and $P(x)$ is a polynomial such that $P(0)=P'(0)=0$, and $P(b)=1$, $P'(b)=0$.

Proposition 2.1. *Suppose Φ is a non-negative smooth function supported on $[1,2]$ such that $\Phi(t)\ll 1$, and with $\int_1^2\Phi(t)dt\gg 1$. If $M\le\sqrt{X}$ and δ_1 is a complex number with $\frac{3}{4}\ge Re\ \delta_1>\epsilon$ then*

$$\mathcal{W}(\delta_1,\Phi)=1+O\big(\Phi_{(2)}X^{\epsilon}\big(M^{-2Re\,\delta_1(1-b)}+M^{(\frac12-Re\,\delta_1)(1-b)}X^{-\frac12}\big)\big).$$

Further $f(s,d)$ has no zeros in $Re\ s>1+3\log\log M/\log M$, and taking $\sigma_0=1+3\log\log M/\log M$, we have

$$\mathcal{S}(I_3(d);\Phi)\ll\exp\Big(\pi\frac{(1/2+\epsilon)\log X}{2S}\Big)M^{-(1-b)}X^{\epsilon}.$$

The implied constants above, and elsewhere, may depend upon ϵ, and the polynomial P. Proposition 2.1 allows us to mollify a little away from $\frac{1}{2}$, and we now turn to the more delicate question of mollifying near $\frac{1}{2}$.

From now on we let ϑ denote a fixed positive real number below $\frac{1}{100}$, which we shall choose later. Let δ_1 and δ_2 be two complex numbers with $|\delta_1|$ and $|\delta_2|\le\vartheta$, and define $\tau=\frac{\delta_1+\delta_2}{2}$, and $\delta=\frac{\delta_1-\delta_2}{2}$. Note that both τ and δ are $\le\vartheta$ in magnitude. All our subsequent work may be carried out under the less stringent assumption that $\max(|\text{Re }\delta|,|\text{Re }\tau|)\le\frac14-\epsilon$, and some of the error terms that feature below may also be strengthened. However the more restrictive condition imposed here allows for a somewhat simpler exposition, and is quite adequate for our application.

Let

$$\xi(s,\chi_{-8d})=\Big(\frac{8d}{\pi}\Big)^{\frac{s}{2}-\frac14}\Gamma(\tfrac{s}{2}+\tfrac12)L(s,\chi_{-8d}),$$

denote the completed L-function which satisfies the functional equation $\xi(s,\chi_{-8d})=\xi(1-s,\chi_{-8d})$. We shall show how to evaluate

(2.9) $\mathcal{S}(\xi(\frac12+\delta_1,\chi_{-8d})\xi(\frac12+\delta_2,\chi_{-8d})M(\frac12+\delta_1,d)M(\frac12+\delta_2,d);\Psi)$,

where Ψ is a smooth function supported on $[1,2]$. By taking $\delta_2=\overline{\delta_1}$ and $\Psi(t)=\Phi(t)t^{-\tau}$ we obtain $(8X/\pi)^{\tau}\Gamma_{\delta}(\tau)\mathcal{S}(1;\Phi)\mathcal{W}(\delta_1,\Phi)$ where

$$\Gamma_{\delta}(s)=\Gamma\Big(\frac34+\frac{s}{2}+\frac{\delta}{2}\Big)\Gamma\Big(\frac34+\frac{s}{2}-\frac{\delta}{2}\Big).$$

To evaluate the expression (2.9), we first need an "approximate functional equation" for $\xi(\frac{1}{2}+\delta_1, \chi_{-8d})\xi(\frac{1}{2}+\delta_2, \chi_{-8d})$. For $\xi > 0$ we define

$$W_{\delta,\tau}(\xi) = \frac{1}{2\pi i}\int_{(c)} \Gamma_\delta(s)\xi^{-s}\frac{2s}{s^2-\tau^2}ds, \tag{2.10}$$

where $c > |\text{Re}\,\tau|$ is a real number. Here, and throughout, we abbreviate $\int_{c-i\infty}^{c+i\infty}$ to $\int_{(c)}$. In Lemma 3.1 we shall see that $W_{\delta,\tau}(\xi)$ is a smooth function on $(0,\infty)$, and that it decays exponentially as $\xi \to \infty$. For all integers $n \geq 1$, and complex numbers s we put

$$r_s(n) = \sum_{ab=n}\left(\frac{a}{b}\right)^s,$$

which is plainly an even function of s. Finally, for all integers $d > 0$ we define

$$A_{\delta,\tau}(d) := \sum_{n=1}^{\infty}\frac{r_\delta(n)}{\sqrt{n}}\left(\frac{-8d}{n}\right)W_{\delta,\tau}\left(\frac{n\pi}{8d}\right). \tag{2.11}$$

We establish in Lemma 3.2 that for fundamental discriminants $-8d(<0)$ we have

$$\xi\left(\frac{1}{2}+\delta_1, \chi_{-8d}\right)\xi\left(\frac{1}{2}+\delta_2, \chi_{-8d}\right) = A_{\delta,\tau}(d).$$

Thus our expression in (2.9) becomes $\mathcal{S}(A_{\delta,\tau}(d)M(\frac{1}{2}+\delta_1,d)M(\frac{1}{2}+\delta_2,d);\Psi)$.

Let $\sqrt{2X} \geq Y > 1$ be a real parameter to be chosen later and write $\mu^2(d) = M_Y(d) + R_Y(d)$ where

$$M_Y(d) = \sum_{\substack{l^2|d\\ l\leq Y}}\mu(l), \qquad \text{and} \qquad R_Y(d) = \sum_{\substack{l^2|d\\ l>Y}}\mu(l).$$

Given a sequence $\{a_n\}_{n=1}^{\infty}$, and a smooth function F supported on $[1,2]$, we define

$$\mathcal{S}_M(a_d;F) = \mathcal{S}_{M,X,Y}(a_d;F) = \frac{1}{X}\sum_{d\text{ odd}}M_Y(d)a_dF\left(\frac{d}{X}\right),$$

and

$$\mathcal{S}_R(a_d;F) = \mathcal{S}_{R,X,Y}(f_d;F) = \frac{1}{X}\sum_{d\text{ odd}}\left|R_Y(d)a_dF\left(\frac{d}{X}\right)\right|,$$

so that $\mathcal{S}(a_d;F) = \mathcal{S}_M(f_d;F) + O(\mathcal{S}_R(a_d;F))$.

Proposition 2.2. *Let Ψ be a smooth function supported on $[1,2]$, with $\Psi(t) \ll 1$. With notations as above, and supposing that $M \le \sqrt{X}$, we have*

$$\mathcal{S}_R(A_{\delta,\tau}(d)M(\tfrac12+\delta_1,d)M(\tfrac12+\delta_2,d);\Psi) \ll X^{\vartheta+\epsilon}\Big(\frac{1}{Y}+\frac{M^{-Re\,\delta_1}+M^{-Re\,\delta_2}+M^{-2Re\,\tau}}{Y^{\frac12}}+\frac{M^{1-2Re\,\tau}}{X^{\frac12}}\Big).$$

It remains lastly to evaluate $\mathcal{S}_M(A_{\delta,\tau}(d)M(\frac12+\delta_1,d)M(\frac12+\delta_2,d);\Psi)$. We evaluate more generally $\mathcal{S}_M(A_{\delta,\tau}(d)\left(\frac{-8d}{l};\Psi\right)$ for any odd integer l. To state our result, we need a few more definitions. For any two complex numbers s and w we define

$$Z(s;w)=\zeta(s-2w)\zeta(s)\zeta(s+2w). \tag{2.12}$$

We write the odd integer l as $l=l_1l_2^2$, where l_1 and l_2 are odd, and l_1 is square-free. For a complex number w with $|\text{Re}\,w|\le\frac14$, and a complex number s with $\text{Re}\,s>\frac12$ we define $\eta_w(s;l)=\prod_p\eta_{p;w}(s;l)$ where $\eta_{2;w}(s;l)=(1-2^{-s-2w})(1-2^{-s})(1-2^{-s+2w})$ and for primes $p\ge 3$ we have

$$\eta_{p;w}(s;l)= \begin{cases} \left(\frac{p}{p+1}\right)\left(1-\frac{1}{p^s}\right)\left(1+\frac1p+\frac{1}{p^s}-\frac{p^{2w}+p^{-2w}}{p^{s+1}}+\frac{1}{p^{2s+1}}\right) & \text{if } p\nmid l\\ \left(\frac{p}{p+1}\right)\left(1-\frac{1}{p^s}\right) & \text{if } p|l_1\\ \left(\frac{p}{p+1}\right)\left(1-\frac{1}{p^{2s}}\right) & \text{otherwise.}\end{cases} \tag{2.13}$$

Note that $\eta_w(s;l)$ is absolutely convergent in the range of our definition.

Proposition 2.3. *With notations as above, we may write*

$$\begin{aligned}&\mathcal{S}_M\left(\left(\frac{-8d}{l}\right)A_{\delta,\tau}(d);\Psi\right)\\ &=\frac{2}{3\zeta(2)\sqrt{l_1}}\sum_{\mu=\pm}\Big(r_\delta(l_1)\Gamma_\delta(\mu\tau)\left(\frac{8X}{l_1\pi}\right)^{\mu\tau}\check{\Psi}(\mu\tau)Z(1+2\mu\tau;\delta)\eta_\delta(1+2\mu\tau;l)\\ &\quad+r_\tau(l_1)\Gamma_\tau(\mu\delta)\left(\frac{8X}{l_1\pi}\right)^{\mu\delta}\check{\Psi}(\mu\delta)Z(1+2\mu\delta;\tau)\eta_\tau(1+2\mu\delta;l)\Big)\\ &\quad+\mathcal{R}(l)+O\Big(l_1^{\vartheta-\frac14+\epsilon}X^{-\frac14+\epsilon}+\frac{X^{\vartheta+\epsilon}l_1^{2\vartheta-\frac12}}{Y^{1-4\vartheta}}\Big).\end{aligned}$$

Here $\mathcal{R}(l)$ is a remainder term bounded for each individual l by

$$|\mathcal{R}(l)|\ll\frac{l^{\frac12+\epsilon}Y^{1+\epsilon}}{X^{\frac12-|Re\,\delta|-\epsilon}}\Psi_{(2)}\Psi_{(3)}^{\epsilon},$$

and bounded on average by

$$\sum_{l=L}^{2L-1} |\mathcal{R}(l)| \ll \Big(\frac{L^{1+\epsilon}Y^{1+\epsilon}}{X^{\frac{1}{2}-|Re\,\delta|-\epsilon}} + \frac{L^{1+\vartheta+\epsilon}Y^{2\vartheta+\epsilon}}{X^{\frac{1}{2}-|Re\,\delta|-\epsilon}}\Big)\Psi_{(2)}\Psi_{(3)}^{\epsilon}.$$

We shall prove Proposition 2.3 in Sect. 5. Observe that although each of the four main terms in Proposition 2.3 has singularities (for example the first term has poles when $\tau = 0$, or when $\tau = \pm\delta$), their sum is regular.

Plainly Propositions 2.2 and 2.3 can be used to evaluate the quantity in (2.9). However, carrying this out is complicated, and in an effort to keep the exposition simple we shall restrict our values of δ_1 and δ_2 to those necessary in evaluating $J_1(X;\Phi)$ and $J_2(X;\Phi)$.

Proposition 2.4. *Let Φ be a non-negative smooth function on $[1,2]$ with $\Phi(t) \ll 1$, and with $\int_1^2 \Phi(t)dt \gg 1$. Let δ_1 be a complex number such that $Re\,\delta_1 \geq -\frac{1}{\epsilon \log X}$, and with $\vartheta \geq |\delta_1| \geq \frac{\epsilon}{\log X}$. We take $\delta_2 = \overline{\delta_1}$ so that $\tau = Re\,\delta_1$, and $\delta = iIm\,\delta_1$. Then with the mollifier coefficients as in (2.8), and with $M = X^{\frac{1}{2}-5\vartheta}$ we have that $\mathcal{W}(\delta_1,\Phi)$ equals*

$$1 + \Big(\frac{1-(8X/\pi)^{-2\tau}}{2\tau\log M} - \Big(\frac{8X}{\pi}\Big)^{-\tau}\frac{(8X/\pi)^{\delta}-(8X/\pi)^{-\delta}}{2\delta\log M}\Big) \times \int_0^b M^{-2\tau(1-x)}\Big|Q'(x) + \frac{Q''(x)}{2\delta_1\log M}\Big|^2 dx$$

with an error $O(X^{-\vartheta+\epsilon}\Phi_{(2)}\Phi_{(3)}^{\epsilon} + M^{-2\tau(1-b)}|\delta_1|^6\log^5 X)$.

We emphasize that the conditions on $|\delta_1|$ and Re δ_1 were assumed only to ease our exposition. In fact, the stated result holds without these restraints. Armed with these results, we complete the proof of Theorem 1.

Proof of Theorem 1. We take Φ to be a smooth function supported in $(1,2)$ such that $\Phi(t) \in [0,1]$ for all t, $\Phi(t) = 1$ for $t \in (1+\epsilon, 2-\epsilon)$, and $|\Phi^{(\nu)}(t)| \ll_{\nu,\epsilon} 1$. Our mollifier is chosen as in (2.8), with $M = X^{\frac{1}{2}-5\vartheta}$, and $b = 0.64$. Further we take σ_0 as in Proposition 2.1, and $S = \pi/(2(1-b)(1-20\vartheta))$. Using Proposition 2.1 in (2.6) we get that

$$\mathcal{N}(X,\Phi) \leq \frac{X\mathcal{S}(1;\Phi)}{8S\sinh\left(\frac{\pi R}{2S}\right)}(J_1(X;\Phi) + J_2(X;\Phi)) + o(X),$$

where J_1 and J_2 are given in (2.7a,b).

Applying Proposition 2.4 we get that for real numbers u and v with $\vartheta\log X \geq |u+iv| \geq \epsilon$ and $u \geq -1/\epsilon$

$$\mathcal{W}\Big(\frac{u+iv}{\log X},\Phi\Big) = \mathcal{V}(u,v) + O\Big(M^{-2u(1-b)/\log X}\frac{(1+|u|+|v|)^6}{\log X}\Big),$$

where

$$\mathcal{V}(u,v) := 1 + \frac{e^{-u}\log X}{\log M}\Big(\frac{\sinh u}{u} - \frac{\sin v}{v}\Big) \times \int_0^b M^{-2u(1-x)/\log X}\Big|Q'(x) + \frac{Q''(x)\log X}{2(u+iv)\log M}\Big|^2 dx.$$

Plainly $\mathcal{V}(u,v) \geq 1$ always, and so we deduce that

$$J_1(X;\Phi) = \int_0^S \cos\left(\frac{\pi t}{2S}\right)\log \mathcal{V}(-R,t)dt + O\Big(\frac{1}{\log X}\Big).$$

Further, using the above together with Proposition 2.1, and keeping in mind our choice for S, we obtain that

$$J_2(X;\Phi) = \int_0^\infty \sinh\left(\frac{\pi u}{2S}\right)\log \mathcal{V}(u-R,S)du + o(1).$$

We conclude that

$$\mathcal{N}(X,\Phi) \leq \frac{X\mathcal{S}(1;\Phi)}{8S\sinh\left(\frac{\pi R}{2S}\right)}\Big(\int_0^S \cos\left(\frac{\pi t}{2S}\right)\log\mathcal{V}(-R,t)dt + \int_0^\infty \sinh\left(\frac{\pi u}{2S}\right)\log\mathcal{V}(u-R,S)du\Big) + o(X).$$

We now take $R = 6.8$, $P(x) = 3(x/b)^2 - 2(x/b)^3$, and $\vartheta = 10^{-10}$. Then a computer calculation showed that $\mathcal{N}(X,\Phi) \leq 0.79X\mathcal{S}(1;\Phi) + o(X)$. Taking $X = x/2, x/4, \ldots$, we obtain Theorem 1.

We end this section by reflecting on some features of the method used to prove Theorem 1. Our overall strategy was to estimate on average the number of zeros (weighted suitably) of the mollified L-function in a small box $\mathcal{B}$ as in Lemma 2.1. If we use the usual argument principle to estimate the zeros in $\mathcal{B}$, then we face the problem of trying to understand the argument of $f(s,d)$ on the horizontal sides of $\mathcal{B}$. This appears to be difficult because the argument of $L(s,\chi_{-8d})$ is intimately related to the location of its zeros. Selberg's argument principle (Lemma 2.1) allows us to circumvent this by introducing the kernel $\sin(\pi(s-W_0+iH)/(2iH))$ which is real on the left vertical edge of $\mathcal{B}$, and purely imaginary on the horizontal edges of $\mathcal{B}$. This enables us to deal only with $\log|f(s,d)|$ (a quantity well suited for estimating from above) on these three sides of $\mathcal{B}$, while on the left vertical edge of $\mathcal{B}$ we are in the region of absolute convergence of $L(s,\chi_{-8d})$ so that $\log f(s,d)$ is relatively easy to understand on this line.

The chief drawback with Selberg's lemma is the exponential growth of the the kernel $\sin(\pi(s-W_0+iH)/(2iH))$ on the horizontal sides of $\mathcal{B}$. To offset this it is necessary that $\log|f(s,d)|$ be very small on the horizontal sides of $\mathcal{B}$ (at least on average over d). This motivates our choice (see (2.8)) of the mollifier coefficients $\lambda(n)$: this choice guarantees that the Dirichlet series coefficients of $f(s, d)$ vanish for $2 \le n \le M^{1-b}$ so that we would expect $f(s, d)$ to be close to 1 on average (as confirmed by Proposition 2.1). Since the growth of Selberg's kernel is determined by the height H of the box $\mathcal{B}$, and the decay of $\log |f(s, d)|$ is controlled by how long a mollifier we can take, we see that there is a natural limitation on how small a box we can take in terms of how long a mollifier we can allow.

In this way we reduce the problem of estimating the weighted average of zeros in $\mathcal{B}$ to evaluating certain mollified mean values, and that is accomplished by extending the ideas in [11]. There are two features of this approach which are a little dissatisfying. Firstly the choice of mollifier coefficients is made in an *ad hoc* way through some numerical experimentation. This is in contrast with the classical situation of mollifying at a point where the optimal mollifier coefficients emerge as minimizers of a certain quadratic form while keeping a linear form fixed. The situation here is less clear because the final answer depends on a complicated integral over the sides of $\mathcal{B}$ of the mollified moments, and also because the initial mollifier coefficients are no longer free, as explained above. We have not understood this optimization problem fully, and it is quite possible that a better choice of mollifier exists.

Secondly, the proof of Theorem 1 relied crucially upon knowing that our weighted average of zeros is less than 1. Since this emerged only after an involved calculation we now indicate why it is reasonable to expect this average to be small. More precisely note that in the proof of Theorem 1 we bounded

(2.14)
$$\frac{1}{2\sinh\left(\frac{\pi R}{2S}\right)} \sum_{\substack{\beta+i\gamma\in\mathcal{B} \\ f(\beta+i\gamma,d)=0}} \cos\left(\frac{\pi\gamma\log X}{2S}\right) \sinh\left(\frac{\pi(R+\log X(\beta-1/2))}{2S}\right).$$

We showed that on average over d this quantity is bounded by 0.79, while if $L(s, \chi_{-8d})$ had a real zero this quantity exceeds 1; thus producing many $L(s, \chi_{-8d})$ having no real zeros. We now restrict our attention to the zeros in (2.14) arising from the $L(s, \chi_{-8d})$ term, and calculate (conjecturally) their contribution. We suspect that the contribution from zeros of the mollifier to (2.14) is negligible on average; at any rate (2.14) is at least as large as the contribution from zeros of $L(s, \chi_{-8d})$, and so it is necessary that this be small. If we assume the Generalized Riemann Hypothesis then the zeros of $L(s, \chi_{-8d})$ in $\mathcal{B}$ contribute to (2.14) the amount

$$\frac{1}{2} \sum_{\substack{|\gamma\log X|\le S \\ L\left(\frac{1}{2}+i\gamma,\chi_{-8d}\right)=0}} \cos\left(\frac{\pi\gamma\log X}{2S}\right). \tag{2.15}$$

The distribution of low lying zeros in families of L-functions has been studied extensively by Katz and Sarnak [7], and the conjectures they formulate there enable one to calculate sums like (2.15) on average. Our family of

L-functions is expected to have a symplectic symmetry, whose 1-level density function is conjecturally $1 - \sin(2\pi x)/(2\pi x)$ (see pp. 405–409 of [7]). Note that this density vanishes to order 2 at 0, indicating that the zeros of $L(s, \chi_{-8d})$ tend to repel the point 1/2. This philosophy predicts that the average value of (2.15) is

$$\int_0^{S/(2\pi)} \cos\left(\frac{\pi^2 x}{S}\right)\left(1 - \frac{\sin(2\pi x)}{2\pi x}\right) dx.$$

For the choice of S in Theorem 1 (namely $S \approx \pi/(0.72)$) the above evaluates to $0.1827\ldots$. Thus conjecturally there are very few zeros in our box, and this suggests an explanation for why the method works.

We may ask if results similar to Theorem 1 hold for other families of L-functions. Our remarks above indicate that perhaps the method would succeed in other families with a repulsion phenomenon at 1/2. One example of these is the family of modular forms (say, of large weight) with odd sign of the functional equation, where there is always a zero at 1/2 but the next zero is repelled. We hope to return to these questions later.

3. Preliminaries

3.1. The approximate functional equation

Lemma 3.1. *For $\xi \in (0, \infty)$, $W_{\delta,\tau}(\xi)$ is a smooth complex-valued function. For ξ near 0 we have the asymptotic*

$$W_{\delta,\tau}(\xi) = \Gamma_\delta(\tau)\xi^{-\tau} + \Gamma_\delta(-\tau)\xi^{\tau} + O(\xi^{1-\epsilon}).$$

For large ξ and any integer ν we have the estimate

$$W_{\delta,\tau}^{(\nu)}(\xi) \ll_\nu \xi^{2\nu+6} e^{-2\xi} \ll_\nu e^{-\xi}.$$

Proof. By moving the line of integration in (2.10) to $\mathrm{Re}\, s = -1+\epsilon$ we see immediately the asymptotic claimed for small ξ. Plainly the ν-th derivative of $W_{\delta,\tau}$ is given by the convergent integral

$$\frac{(-1)^\nu}{2\pi i}\int_{(c)} \Gamma_\delta(s)s(s+1)\cdots(s+\nu-1)\xi^{-s}\frac{2s}{s^2-\tau^2}ds$$

for any $c > |\mathrm{Re}\, \tau|$. Thus $W_{\delta,\tau}(\xi)$ is smooth. To prove the last estimate of the lemma we may suppose that $\xi > \nu+4$. Since $|\Gamma(x+iy)| \le \Gamma(x)$ for $x \ge 1$, and $s\Gamma(s) = \Gamma(s+1)$, we obtain that the integral above giving $W_{\delta,\tau}^{(\nu)}(\xi)$ is bounded by

$$\ll_\nu |\Gamma(c/2+\nu+3)|^2\xi^{-c}\int_{(c)}\frac{|ds|}{|s^2-\tau^2|} \ll_\nu \Gamma(c/2+\nu+3)^2\frac{\xi^{-c}}{c-|\mathrm{Re}\, \tau|}.$$

By Stirling's formula this is

$$\ll_\nu \Big(\frac{c+2\nu+6}{2e}\Big)^{c+2\nu+6} \frac{\xi^{-c}}{c-|\mathrm{Re}\ \tau|},$$

and taking $c = 2\xi - 2\nu - 6 (\geq 2)$ we get the lemma.

Lemma 3.2. *Let δ_1 and δ_2 be complex numbers less than $\frac{1}{5}$ in magnitude. For fundamental discriminants $-8d (< 0)$ we have*

$$\xi(\tfrac{1}{2}+\delta_1, \chi_{-8d})\xi(\tfrac{1}{2}+\delta_2, \chi_{-8d}) = A_{\delta,\tau}(d).$$

Proof. Consider for some $3/2 - |\mathrm{Re}\ \delta| > c > 1/2 + |\mathrm{Re}\ \delta|$

$$\frac{1}{2\pi i}\int_{(c)} \xi(\tfrac{1}{2}+\delta+s, \chi_{-8d})\xi(\tfrac{1}{2}-\delta+s, \chi_{-8d})\frac{2s}{s^2-\tau^2}ds.$$

Expanding $L(\frac{1}{2}+\delta+s, \chi_{-8d})L(\frac{1}{2}-\delta+s, \chi_{-8d})$ into its Dirichlet series $\sum_{n=1}^{\infty} \frac{r_\delta(n)}{n^{\frac{1}{2}+s}}\left(\frac{-8d}{n}\right)$, and integrating term by term, we get that this equals $A_{\delta,\tau}(d)$. Now move the path of integration to the line $\mathrm{Re}(s) = -c$. We encounter poles at $s = \tau, -\tau$, and the residues here give $\xi(\frac{1}{2}+\delta+\tau, \chi_{-8d})\xi(\frac{1}{2}-\delta+\tau, \chi_{-8d}) + \xi(\frac{1}{2}+\delta-\tau, \chi_{-8d})\xi(\frac{1}{2}-\delta-\tau, \chi_{-8d}) = 2\xi(\frac{1}{2}+\delta_1, \chi_{-8d})\xi(\frac{1}{2}+\delta_2, \chi_{-8d})$, upon using the functional equation. In the remaining integral on the $-c$ line, we let $s \to -s$ and use the functional equation. Then it evaluates to $-A_{\delta,\tau}(d)$, which completes our proof.

3.2. On Gauss-type sums

Let n be an odd integer. We define for all integers k

$$G_k(n) = \left(\frac{1-i}{2} + \left(\frac{-1}{n}\right)\frac{1+i}{2}\right)\sum_{a\,(\mathrm{mod}\ n)}\left(\frac{an}{e}\right)\left(\frac{ak}{n}\right),$$

and put

$$\tau_k(n) = \sum_{a\,(\mathrm{mod}\ n)}\left(\frac{a}{n}\right)e\left(\frac{ak}{n}\right) = \left(\frac{1+i}{2} + \left(\frac{-1}{n}\right)\frac{1-i}{2}\right)G_k(n).$$

If n is square-free then $\left(\frac{\cdot}{n}\right)$ is a primitive character with conductor n. Here it is easy to see that $G_k(n) = \left(\frac{k}{n}\right)\sqrt{n}$. For our later work, we require knowledge of $G_k(n)$ for all odd n. This is contained in the next lemma which is Lemma 2.3 of [11].

Lemma 3.3.
(i) (Multiplicativity) Suppose m and n are coprime odd integers. Then $G_k(mn) = G_k(m)G_k(n)$.

(ii) Suppose p^α is the largest power of p dividing k. (If $k = 0$ then set $\alpha = \infty$.) Then for $\beta \geq 1$

$$G_k(p^\beta) = \begin{cases} 0 & \text{if } \beta \leq \alpha \text{ is odd,} \\ \varphi(p^\beta) & \text{if } \beta \leq \alpha \text{ is even,} \\ -p^\alpha & \text{if } \beta = \alpha + 1 \text{ is even,} \\ \left(\frac{kp^{-\alpha}}{p}\right) p^\alpha \sqrt{p} & \text{if } \beta = \alpha + 1 \text{ is odd,} \\ 0 & \text{if } \beta \geq \alpha + 2. \end{cases}$$

3.3. Lemmas for estimating character sums

We collect here two lemmas that will be very useful in bounding the character sums that arise below. These are consequences of a recent large sieve result for real characters due to D.R. Heath-Brown [4] (see Lemmas 2.4 and 2.5 of [11]).

Lemma 3.4. *Let N and Q be positive integers and let $a_1, \ldots, a_N$ be arbitrary complex numbers. Let $S(Q)$ denote the set of real, primitive characters χ with conductor $\leq Q$. Then*

$$\sum_{\chi \in S(Q)} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \ll_\epsilon (QN)^\epsilon (Q + N) \sum_{n_1 n_2 = \square} |a_{n_1} a_{n_2}|,$$

for any $\epsilon > 0$. Let M be any positive integer, and for each $|m| \leq M$ write $4m = m_1 m_2^2$ where m_1 is a fundamental discriminant, and m_2 is positive. Suppose the sequence a_n satisfies $|a_n| \ll n^\epsilon$. Then

$$\sum_{|m| \leq M} \frac{1}{m_2} \left| \sum_{n \leq N} a_n \left(\frac{m}{n}\right) \right|^2 \ll (MN)^\epsilon N(M + N).$$

Lemma 3.5. *Let $S(Q)$ be as in Lemma 3.4, and suppose $\sigma + it$ is a complex number with $\sigma \geq \frac{1}{2}$. Then*

$$\sum_{\chi \in S(Q)} |L(\sigma + it, \chi)|^4 \ll Q^{1+\epsilon}(1 + |t|)^{1+\epsilon}, \quad \text{and}$$

$$\sum_{\chi \in S(Q)} |L(\sigma + it, \chi)|^2 \ll Q^{1+\epsilon}(1 + |t|)^{\frac{1}{2}+\epsilon}.$$

3.4. Poisson summation

For a Schwarz class function F we define

$$\tilde{F}(\xi) = \frac{1+i}{2}\hat{F}(\xi) + \frac{1-i}{2}\hat{F}(-\xi) = \int_{-\infty}^{\infty} (\cos(2\pi\xi x) + \sin(2\pi\xi x))F(x)dx.$$

We quote the following version of Poisson summation (see Lemma 2.6 of [11]):

Lemma 3.6. *Let F be a smooth function supported in $(1,2)$. For any odd integer n,*

$$\mathcal{S}_M\left(\left(\frac{d}{n}\right); F\right) = \frac{1}{2n}\left(\frac{2}{n}\right) \sum_{\substack{\alpha\le Y\\ (\alpha,2n)=1}} \frac{\mu(\alpha)}{\alpha^2} \sum_k (-1)^k G_k(n) \tilde{F}\left(\frac{kX}{2\alpha^2 n}\right).$$

4. Proofs of Propositions 2.1 and 2.2

We first record two applications of Lemma 3.4 which will be useful in the proofs of these Propositions. Write $\lambda_2(n) = \sum_{ab=n,a,b\le M} \lambda(a)\lambda(b)$. Note that $|\lambda_2(n)| \ll n^{\epsilon}$ and that $M(s,d)^2 = \sum_{n\le M^2} \lambda_2(n) n^{-s} \left(\frac{-8d}{n}\right)$. By Lemma 3.4 we see that for $N \le M^2 (\ll X^2)$ we have

$$\begin{aligned}
\sum_{X\le d\le 2X} \mu^2(2d) \Big| \sum_{N\le n\le 2N} \frac{\lambda_2(n)}{n^s}\left(\frac{-8d}{n}\right)\Big|^2 \\
&\ll X^{\epsilon}(X+N) \sum_{\substack{N\le n_1,n_2\le 2N\\ n_1 n_2=\square}} \frac{|\lambda_2(n_1)\lambda_2(n_2)|}{(n_1n_2)^{\mathrm{Re}\, s}} \\
&\ll X^{\epsilon}(X+N)N^{1-2\mathrm{Re}\, s} \sum_{\substack{N\le n_1,n_2\le 2N\\ n_1 n_2=\square}} \frac{1}{\sqrt{n_1 n_2}} \\
&\ll X^{\epsilon}(X+N)N^{1-2\mathrm{Re}\, s} \sum_{a\le M^2} \frac{d(a^2)}{a} \\
&\ll X^{\epsilon}(X+N)N^{1-2\mathrm{Re}\, s}.
\end{aligned}$$

From this we conclude that

$$\sum_{X\le d\le 2X} \mu^2(2d)|M(s,d)|^4 \ll X^{\epsilon}(X + XM^{2(1-2\mathrm{Re}\, s)} + M^{4(1-\mathrm{Re}\, s)}). \tag{4.1}$$

In a similar manner we see that if l is any odd integer $\le \sqrt{2X}$ then

(4.2)
$$\sum_{X/l^2\le m\le 2X/l^2} \mu^2(2m)|M(s,l^2m)|^4 \ll X^{\epsilon}\Big(\frac{X}{l^2} + \frac{X}{l^2}M^{2(1-2\mathrm{Re}\, s)} + M^{4(1-\mathrm{Re}\, s)}\Big).$$

4.1. Proof of Proposition 2.1

Since Φ is a non-negative smooth function supported on $[1,2]$ such that $\Phi(t) \ll 1$, and $\int_1^2 \Phi(t)dt \gg 1$ we see that $\mathcal{S}(1;\Phi) \gg X^{-1}\sum_{X\le d\le 2X} \mu^2(2d)$

$\gg 1$. We write $B(s,d) = L(s,\chi_{-8d})M(s,d) - 1$ so that

(4.3) $\mathcal{W}(\delta_1,\Phi) = 1 + O(\mathcal{S}(B(\frac{1}{2}+\delta_1,d);\Phi) + \mathcal{S}(|B(\frac{1}{2}+\delta_1,d)|^2;\Phi)).$

To estimate the unknown terms above, we consider

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s)B(\tfrac{1}{2}+\delta_1+s,d)X^s\,ds,$$

for any real number $c > \max(0, \frac{1}{2} - \text{Re }\delta_1)$. We move the line of integration to the line $\text{Re } s = -\text{Re }\delta_1$. The pole at $s=0$ contributes $B(\frac{1}{2}+\delta_1;d)$ and so we conclude that $B(\frac{1}{2}+\delta_1,d)$ equals

(4.4)
$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s)B(\tfrac{1}{2}+\delta_1+s,d)X^s\,ds - \frac{1}{2\pi i}\int_{(-\text{Re }\delta_1)} \Gamma(s)B(\tfrac{1}{2}+\delta_1+s,d)X^s\,ds.$$

Write the expression in (4.4) as $T_1(\frac{1}{2}+\delta_1,d) - T_2(\frac{1}{2}+\delta_1,d)$, say.

We first consider the contributions of the $T_2(d)$ terms to the unknown quantities in (4.3). We shall prove that

(4.5)
$$\mathcal{S}(|T_2(\tfrac{1}{2}+\delta_1,d)|^2;\Phi) \ll X^{-2\text{Re }\delta_1+\epsilon}, \quad \text{and}$$
$$\mathcal{S}(|T_2(\tfrac{1}{2}+\delta_1,d)|;\Phi) \ll X^{-\text{Re }\delta_1+\epsilon}.$$

Plainly the second estimate above follows from the first and Cauchy's inequality. To see the first estimate observe that by Cauchy's inequality

$$|T_2(\tfrac{1}{2}+\delta_1,d)|^2 \ll X^{-2\text{Re }\delta_1}\Big(\int_{(-\text{Re }\delta_1)} |\Gamma(s)B(\tfrac{1}{2}+\delta_1+s,d)^2 ds|\Big)\Big(\int_{(-\text{Re }\delta_1)} |\Gamma(s)ds|\Big),$$

and in view of the rapid decay of $|\Gamma(s)|$ as $|\text{Im s}| \to \infty$, we deduce that

$$|T_2(\tfrac{1}{2}+\delta_1,d)|^2 \ll X^{-2\text{Re }\delta_1} \times \Big(1 + \int_{(-\text{Re }\delta_1)} |\Gamma(s)||L(\tfrac{1}{2}+\delta_1+s,\chi_{-8d})M(\tfrac{1}{2}+\delta_1+s,d)|^2|ds|\Big).$$

Averaging this over the appropriate d, with another application of Cauchy's inequality we obtain that $\mathcal{S}(|T_2(\frac{1}{2}+\delta_1,d)|^2;\Phi)$ is bounded by

$$X^{-2\text{Re }\delta_1}\Big(1 + \int_{(-\text{Re }\delta_1)} |\Gamma(s)|\mathcal{S}(|L(\tfrac{1}{2}+\delta_1+s,\chi_{-8d})|^4;\Phi)^{\frac{1}{2}} \times \mathcal{S}(|M(\tfrac{1}{2}+\delta_1+s,d)|^4;\Phi)^{\frac{1}{2}}|ds|\Big),$$

and (4.5) follows upon using Lemma 3.5 and (4.1) above (keeping in mind that $M \le \sqrt{X}$).

It remains now to consider the T_1 contribution. In the region $\mathrm{Re}\, s > 1$ we may write

$$B(s,d) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}\left(\frac{-8d}{n}\right).$$

From the shape of our mollifier we see that $b(n) = 0$ for all $n \le M^{1-b}$, $b(n) = 0$ for all square values $n \le M^{2(1-b)}$ (because $b(m^2) = \sum_{d|m^2} \lambda(d) = \sum_{d|m} \lambda(d) = b(m)$, since λ is supported on square-free numbers), and lastly $|b(n)| \ll d(n) \ll n^{\epsilon}$ for all n. We write

$$T_1(\tfrac{1}{2}+\delta_1, d) = \frac{1}{2\pi i}\int_{(c)} \Gamma(s) \sum_{M^{1-b} \le n \le X \log^2 X} \frac{b(n)}{n^{\frac{1}{2}+\delta_1+s}}\left(\frac{-8d}{n}\right) X^s ds$$
$$+ \sum_{n > X\log^2 X} \frac{b(n)}{n^{\frac{1}{2}+\delta_1}}\left(\frac{-8d}{n}\right)\left(\frac{1}{2\pi i}\int_{(c)} \Gamma(s)\left(\frac{X}{n}\right)^s ds\right).$$

Since $\frac{1}{2\pi i}\int_{(c)} \Gamma(s)\left(\frac{X}{n}\right)^s ds = e^{-n/X}$, the second term above contributes $\ll X^{-5}$ say. In the first term above we move the line of integration to $\mathrm{Re}\, s = \frac{1}{\log X}$. Thus

(4.6) $$T_1(\tfrac{1}{2}+\delta_1, d) = \frac{1}{2\pi i}\int_{\left(\frac{1}{\log X}\right)} \Gamma(s) \sum_{M^{1-b} \le n \le X \log^2 X} \frac{b(n)}{n^{\frac{1}{2}+\delta_1+s}}\left(\frac{-8d}{n}\right) X^s ds + O(X^{-5}).$$

By Cauchy's inequality we get that

$$|T_1(\tfrac{1}{2}+\delta_1, d)|^2 \ll X^{-10} + \left(\int_{\left(\frac{1}{\log X}\right)} |\Gamma(s)| \Big| \sum_{M^{1-b} \le n \le X \log^2 X} \frac{b(n)}{n^{\frac{1}{2}+\delta_1+s}}\left(\frac{-8d}{n}\right)\Big|^2 |ds|\right) \times \left(\int_{\left(\frac{1}{\log X}\right)} |\Gamma(s) ds|\right)$$
$$\ll X^{-10} + X^{\epsilon}\int_{\left(\frac{1}{\log X}\right)} |\Gamma(s)| \Big| \sum_{M^{1-b} \le n \le X \log^2 X} \frac{b(n)}{n^{\frac{1}{2}+\delta_1+s}}\left(\frac{-8d}{n}\right)\Big|^2 |ds|.$$

Splitting the sum over n into dyadic blocks and using Lemma 3.4 we conclude that

$$\mathcal{S}(|T_1(\tfrac{1}{2}+\delta_1, d)|^2; \Phi) \ll M^{-2\mathrm{Re}\,\delta_1(1-b)} X^{\epsilon},$$

which when combined with (4.5) gives that

(4.7) $$\mathcal{S}(|B(\tfrac{1}{2}+\delta_1, d)|^2; \Phi) \ll M^{-2\mathrm{Re}\,\delta_1(1-b)} X^{\epsilon}.$$

We now show how to bound $\mathcal{S}(T_1(\frac{1}{2}+\delta_1, d); \Phi)$. By (4.6) we see that

$$\text{(4.8)} \quad \mathcal{S}(T_1(\tfrac{1}{2}+\delta_1, d); \Phi) \ll X^{-5} + X^{\epsilon} \sum_{M^{1-b} \le n \le X \log^2 X} \frac{|b(n)|}{n^{\frac{1}{2}+\text{Re }\delta_1}} \Big| \mathcal{S}\Big(\Big(\frac{-8d}{n}\Big); \Phi\Big) \Big|.$$

For each odd integer n let ψ_n denote the character $\psi_n(m) = \left(\frac{m}{n}\right)$ whose conductor is at most n. Note that ψ_n is non-trivial unless n is a square. Observe that for any sequence of numbers $a_n \ll n^{\epsilon}$, and any smooth function g with $g(0) = 0$ and $g(x)$ decaying rapidly as $x \to \infty$, we have the Mellin transform identity

$$\text{(4.9)} \quad \sum_{n=1}^{\infty} a_n g(n) = \frac{1}{2\pi i} \int_{(c)} \sum_{n=1}^{\infty} \frac{a_n}{n^w} \Big(\int_0^{\infty} g(t) t^{w-1} dt \Big) dw,$$

where $c > 1$. Hence we obtain that for any odd integer n

$$\begin{aligned} \mathcal{S}(\psi_n(-8d); \Phi) &= \frac{\psi_n(-8)}{2\pi i} \int_{(c)} \sum_{d=1}^{\infty} \frac{\mu^2(2d)\psi_n(d)}{d^w} X^{w-1} \check{\Phi}(w-1) dw \\ &= \frac{\psi_n(-8)}{2\pi i} \int_{(c)} \frac{L(w, \psi_n)}{L(2w, \psi_n)} (1 + \psi_n(2)/2^w)^{-1} X^{w-1} \check{\Phi}(w-1) dw, \end{aligned}$$

where $L(w, \psi_n) = \sum_{d=1}^{\infty} \psi_n(d)/d^w$ is the usual Dirichlet L-function. We move the line of integration above to the line Re $w = \frac{1}{2} + \frac{1}{\log X}$. We encounter a pole at $w = 1$ if and only if n is a square (in which case $L(w, \psi_n)$ is essentially $\zeta(w)$) and the residue of this pole is $\ll 1$. Thus we conclude that

$$|\mathcal{S}(\psi_n(-8d); \Phi)| \ll \delta(n = \square) + X^{-\frac{1}{2}+\epsilon} \int_{(\frac{1}{2}+\frac{1}{\log X})} |L(w, \psi_n)| |\check{\Phi}(w-1)| |dw|,$$

where $\delta(n = \square)$ is 1 if n is a square, and 0 otherwise. Since $b(n) = 0$ for all squares $\le M^{2(1-b)}$ we find that

$$\begin{aligned} \sum_{M^{1-b} \le n \le X \log^2 X} \frac{|b(n)|}{n^{\frac{1}{2}+\text{Re }\delta_1}} \Big| \mathcal{S}\Big(\Big(\frac{-8d}{n}\Big); \Phi\Big) \Big| &\ll X^{\epsilon} M^{-2\text{Re }\delta_1(1-b)} \\ &+ X^{-\frac{1}{2}+\epsilon} \int_{(\frac{1}{2}+\frac{1}{\log X})} \sum_{M^{1-b} \le n \le X \log^2 X} \frac{1}{n^{\frac{1}{2}+\text{Re }\delta_1}} |L(w, \psi_n)| |\check{\Phi}(w-1)| |dw|. \end{aligned}$$

An easy application of Lemma 3.5 gives that

$$\sum_{N\le n\le 2N} |L(w,\psi_n)| \ll N^{1+\epsilon}(1+|w|)^{\frac14+\epsilon},$$

and using this above, together with (2.4c) (taking $\nu = 2$ there), we get that

$$\sum_{M^{1-b}\le n\le X\log^2 X} \frac{|b(n)|}{n^{\frac12+\mathrm{Re}\,\delta_1}}\left|\mathcal{S}\left(\left(\frac{-8d}{n}\right);\Phi\right)\right| \ll X^{\epsilon}\Phi_{(2)}\big(M^{-2\mathrm{Re}\,\delta_1(1-b)} + X^{-\mathrm{Re}\,\delta_1} + M^{(\frac12-\mathrm{Re}\,\delta_1)(1-b)}X^{-\frac12}\big).$$

Using this in (4.8), and combining with (4.5) we deduce that (since $M \le \sqrt{X}$)

(4.10)
$$\mathcal{S}\left(B\left(\frac12+\delta_1,d\right);\Phi\right) \ll X^{\epsilon}\Phi_{(2)}\big(M^{-2\mathrm{Re}\,\delta_1(1-b)} + M^{(\frac12-\mathrm{Re}\,\delta_1)(1-b)}X^{-\frac12}\big).$$

Using (4.10) and (4.7) in (4.3), we deduce the first statement of the proposition.

To see the second assertion, note that

$$f(s,d) = 1 + B(s,d) = 1 + O\Big(\sum_{n\ge M^{1-b}} \frac{d(n)}{n^{\mathrm{Re}\,s}}\Big),$$

from which it follows easily that $f(s,d)$ has no zeros to the right of $1 + 3\log\log M/\log M$. Further for s in this region $\log f(s,d) = B(s,d) + O(|B(s,d)|^2)$, and so, with σ_0 as in the proposition, we have

$$\mathcal{S}(I_3(d);\Phi) \ll \exp\left(\pi\frac{(\frac12+\epsilon)\log X}{2S}\right) \times \Big(|\mathcal{S}(B(s,d);\Phi)| + \mathcal{S}(|B(s,d)|^2;\Phi)\Big).$$

Thus the second assertion also follows from (4.7) and (4.10).

4.2. *Proof of Proposition 2.2*

Observe that $R_Y(d) = 0$ unless $d = l^2 m$ where m is squarefree and $l > Y$. Further, note that $|R_Y(d)| \le \sum_{k|d} 1 \ll d^{\epsilon}$. Hence

(4.11)
$$\mathcal{S}_R(A_{\delta,\tau}(d)M(\tfrac12+\delta_1,d)M(\tfrac12+\delta_2,d);\Psi) \ll X^{-1+\epsilon}\sum_{\substack{Y<l\\(l,2)=1}}\ \sum_{X/l^2\le m\le 2X/l^2} |A_{\delta,\tau}(l^2m)M(\tfrac12+\delta_1,l^2m)M(\tfrac12+\delta_2,l^2m)|,$$

where the $\flat$ on the sum over m indicates that m is odd and squarefree. By two applications of Cauchy's inequality the sum over m above is

$$\ll \Big(\sum\nolimits_m |M(\tfrac{1}{2}+\delta_1, l^2m)|^4\Big)^{\frac14}\Big(\sum\nolimits_m |M(\tfrac{1}{2}+\delta_2, l^2m)|^4\Big)^{\frac14} \times \Big(\sum\nolimits_m |A_{\delta,\tau}(l^2m)|^2\Big)^{\frac12}. \tag{4.12}$$

Now observe that for any $c > \frac{1}{2} + |\mathrm{Re}\,\delta|$

$$A_{\delta,\tau}(l^2m) = \frac{1}{2\pi i}\int_{(c)} \Gamma_\delta(s)\left(\frac{8l^2m}{\pi}\right)^s \frac{2s}{s^2-\tau^2}\sum_{n=1}^{\infty}\frac{r_\delta(n)}{n^{s+\frac12}}\left(\frac{-8l^2m}{n}\right) ds. \tag{4.13}$$

Plainly

$$\sum_{n=1}^{\infty}\frac{r_\delta(n)}{n^{s+\frac12}}\left(\frac{-8l^2m}{n}\right) = L(\tfrac12+s+\delta,\chi_{-8m})L(\tfrac12+s-\delta,\chi_{-8m})\mathcal{E}(s,l) \tag{4.14}$$

where

$$\mathcal{E}(s,l) = \prod_{p|l}\left(1-\frac{1}{p^{s+\frac12+\delta}}\left(\frac{-8m}{p}\right)\right)\left(1-\frac{1}{p^{s+\frac12-\delta}}\left(\frac{-8m}{p}\right)\right).$$

Since χ_{-8m} is non-principal, it follows that the left side of (4.14) is analytic for all s.

Hence we may move the line of integration in (4.13) to the line from $\vartheta + 1/\log X - i\infty$ to $\vartheta + 1/\log X + i\infty$. We encounter no poles, and so $A_{\delta,\tau}(l^2m)$ is given by the integral on this new line. Since $|\mathcal{E}(s,l)| \le \prod_{p|l}(1+1/\sqrt{p})^2 \ll l^\epsilon \ll X^\epsilon$, $2s/(s^2-\tau^2) \ll X^\epsilon$, and $|\Gamma_\delta(s)|$ decays exponentially for large $|\mathrm{Im}\, s|$, we obtain by Cauchy's inequality that

$$|A_{\delta,\tau}(l^2m)|^2 \ll X^{2\vartheta+\epsilon}\int_{(\vartheta+\frac{1}{\log X})} |\Gamma_\delta(s)||L(\tfrac12+s+\delta,\chi_{-8m})L(\tfrac12+s-\delta,\chi_{-8m})|^2|ds|.$$

Summing this over m and using Lemma 3.5, we obtain that

$$\sum_{X/l^2\le m\le 2X/l^2} |A_{\delta,\tau}(l^2m)|^2 \ll \frac{X^{1+2\vartheta+\epsilon}}{l^2}\int_{(\vartheta+\frac{1}{\log X})} |\Gamma_\delta(s)|(1+|s|)^{1+\epsilon}|ds| \ll \frac{X^{1+2\vartheta+\epsilon}}{l^2}.$$

Using this together with (4.2) we conclude that the quantity in (4.12) is bounded by

$$\ll \frac{X^{\frac{1}{2}+\vartheta+\epsilon}}{l}\Big(\frac{X^{\frac{1}{4}}}{l^{\frac{1}{2}}}+\frac{X^{\frac{1}{4}}}{l^{\frac{1}{2}}}M^{-\mathrm{Re}\,\delta_1}+M^{\frac{1}{2}-\mathrm{Re}\,\delta_1}\Big) \times\Big(\frac{X^{\frac{1}{4}}}{l^{\frac{1}{2}}}+\frac{X^{\frac{1}{4}}}{l^{\frac{1}{2}}}M^{-\mathrm{Re}\,\delta_2}+M^{\frac{1}{2}-\mathrm{Re}\,\delta_2}\Big),$$

which when inserted in (4.11), and recalling that $M \le \sqrt{X}$, yields the proposition.

5. Proof of Proposition 2.3

Observe that

$$\mathcal{S}_M\left(\left(\frac{-8d}{l}\right)A_{\delta,\tau}(d);\Psi\right)=\sum_{n=1}^{\infty}\frac{r_\delta(n)}{\sqrt{n}}\mathcal{S}_M\left(\left(\frac{-8d}{ln}\right);F_n\right), \tag{5.1}$$

where

$$F_n(t)=F_n(\delta,\tau;t)=\Psi(t)W_{\delta,\tau}\left(\frac{n\pi}{8Xt}\right).$$

Using the Poisson summation formula, Lemma 3.6 above, we obtain for odd n (using $G_{-k}(ln)=\left(\frac{-1}{ln}\right)G_k(ln)$, and $\left(\frac{16}{ln}\right)=1$ when ln is odd)

(5.2)

$$\mathcal{S}_M\left(\left(\frac{-8d}{ln}\right);F_n\right)=\frac{1}{2ln}\sum_{\substack{\alpha\le Y\\(\alpha,2ln)=1}}\frac{\mu(\alpha)}{\alpha^2}\sum_{k=-\infty}^{\infty}(-1)^kG_{-k}(ln)\tilde{F}_n\left(\frac{kX}{2\alpha^2ln}\right).$$

Note that when n is even, the LHS above is zero.

Using this in (5.1), we deduce that

$$\mathcal{S}_M\left(\left(\frac{-8d}{l}\right)A_{\delta,\tau}(d);\Psi\right)=\mathcal{P}(l)+\mathcal{R}_0(l),$$

where $\mathcal{P}(l)$ is the main principal contribution (arising from the $k=0$ term in (5.2)), and $\mathcal{R}_0(l)$ includes all the non-zero terms k in (5.2). Thus

$$\mathcal{P}(l)=\frac{1}{2l}\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\frac{r_\delta(n)}{n^{\frac{3}{2}}}\sum_{\substack{\alpha\le Y\\(\alpha,2ln)=1}}\frac{\mu(\alpha)}{\alpha^2}G_0(ln)\tilde{F}_n(0),$$

and

(5.3)

$$\mathcal{R}_0(l)=\frac{1}{2l}\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\frac{r_\delta(n)}{n^{\frac{3}{2}}}\sum_{\substack{\alpha\le Y\\(\alpha,2ln)=1}}\frac{\mu(\alpha)}{\alpha^2}\sum_{\substack{k=-\infty\\k\ne0}}^{\infty}(-1)^kG_{-k}(ln)\tilde{F}_n\left(\frac{kX}{2\alpha^2ln}\right).$$

5.1. The principal $\mathcal{P}(l)$ contribution

Note that $\tilde{F}_n(0) = \hat{F}_n(0)$ and that $G_0(ln) = \varphi(ln)$ if $ln = \square$ and $G_0(ln) = 0$ otherwise. Using this together with

$$\sum_{\substack{\alpha \le Y \\ (\alpha, 2ln)=1}} \frac{\mu(\alpha)}{\alpha^2} = \frac{1}{\zeta(2)} \prod_{p|2ln} \Big(1 - \frac{1}{p^2}\Big)^{-1} \Big(1 + O\Big(\frac{1}{Y}\Big)\Big),$$

we deduce that

$$\mathcal{P}(l) = \frac{1 + O(Y^{-1})}{\zeta(2)} \sum_{\substack{n=1 \\ ln=\square \\ n \text{ odd}}}^{\infty} \frac{r_\delta(n)}{n^{\frac{1}{2}}} \prod_{p|2ln} \left(\frac{p}{p+1}\right) \hat{F}_n(0).$$

Recall that $l = l_1 l_2^2$ where l_1 and l_2 are odd, and l_1 is square-free. The condition that $ln = \square$ is thus equivalent to $n = l_1 m^2$ for some integer m. Hence

$$\mathcal{P}(l) = \frac{1 + O(Y^{-1})}{\zeta(2)\sqrt{l_1}} \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{r_\delta(l_1 m^2)}{m} \prod_{p|2lm} \left(\frac{p}{p+1}\right) \hat{F}_{l_1 m^2}(0).$$

For any $c > |\text{Re}\, \tau|$ we have

$$\begin{aligned}
\hat{F}_{l_1 m^2}(0) &= \int_0^\infty \Psi(t) W_{\delta,\tau}\left(\frac{l_1 m^2 \pi}{8Xt}\right) dt \\
&= \frac{1}{2\pi i} \int_{(c)} \Gamma_{\delta(s)} \left(\frac{8X}{l_1 m^2 \pi}\right)^s \Big(\int_0^\infty \Psi(t) t^s dt\Big) \frac{2s}{s^2 - \tau^2} ds \\
&= \frac{1}{2\pi i} \int_{(c)} \Gamma_{\delta(s)} \check{\Psi}(s) \left(\frac{8X}{l_1 m^2 \pi}\right)^s \frac{2s}{s^2 - \tau^2} ds.
\end{aligned}$$

Thus for any $c > \vartheta$

(5.4a) $$\mathcal{P}(l) = \frac{2}{3} \frac{1 + O(Y^{-1})}{\zeta(2)\sqrt{l_1}} I(l),$$

where

(5.4b)
$$I(l) = \frac{1}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left(\frac{8X}{l_1 \pi}\right)^s \check{\Psi}(s) \frac{2s}{s^2 - \tau^2} \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{r_\delta(l_1 m^2)}{m^{1+2s}} \prod_{p|lm} \left(\frac{p}{p+1}\right) ds.$$

Lemma 5.1. *Suppose $l = l_1 l_2^2$ is as above. Then for Re $s > 1 + 2|Re\ \delta|$*

$$\sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{r_\delta(l_1 m^2)}{m^s} \prod_{p|lm} \left(\frac{p}{p+1}\right) = r_\delta(l_1) Z(s;\delta)\eta_\delta(s;l)$$

where Z and η are as defined in (2.12) and (2.13).

Proof. This follows by comparing the Euler factors on both sides.

Using Lemma 5.1 in (5.4b), we deduce that

$$I(l) := \frac{r_\delta(l_1)}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left(\frac{8X}{l_1 \pi}\right)^s \check{\Psi}(s) \frac{2s}{s^2 - \tau^2} Z(1+2s;\delta)\eta_\delta(1+2s;l)ds.$$

Note first that taking $c = \vartheta + \epsilon$ here we deduce easily that $I(l) \ll |r_\delta(l_1)|(X/l_1)^{\vartheta+\epsilon}$. We now move the line of integration above to the line Re $s = -\frac{1}{4} + \epsilon$. We encounter simple poles at $s = \pm\tau, \pm\delta$. The remaining integral on the $-\frac{1}{4} + \epsilon$ line we bound as follows: From [3] we know that on this line $|Z(1+2s;\delta)| \ll (1+|s|)^3$, and plainly $|\eta_\delta(1+2s;l)| \ll \prod_{p|l_1} \left(1 + O\left(\frac{1}{\sqrt{p}}\right)\right) \prod_{p\nmid l_1} (1 + O(\frac{1}{p^{1+\epsilon}})) \ll l_1^\epsilon$. Hence the integral on the Re $s = -\frac{1}{4} + \epsilon$ line is

$$\ll \frac{|r_\delta(l_1)| l_1^{\frac{1}{4}+\epsilon}}{X^{\frac{1}{4}-\epsilon}} \int_{(-\frac{1}{4}+\epsilon)} |s|^2 |\check{\Psi}(s)| \Gamma_\delta(s)||ds| \ll \frac{|r_\delta(l_1)| l_1^{\frac{1}{4}+\epsilon}}{X^{\frac{1}{4}-\epsilon}}.$$

We deduce that

$$I(l) = r_\delta(l_1) \operatorname*{Res}_{s=\pm\delta,\pm\tau} \left\{ \Gamma_\delta(s) \left(\frac{8X}{l_1\pi}\right)^s \check{\Psi}(s) \frac{2s}{s^2-\tau^2} Z(1+2s;\delta)\eta_\delta(1+2s;l) \right\} + O\left(\frac{|r_\delta(l_1)| l_1^{\frac{1}{4}+\epsilon}}{X^{\frac{1}{4}-\epsilon}}\right).$$

Using this in (5.4a), we conclude that

$$(5.5)\quad \mathcal{P}(l) = \frac{2r_\delta(l_1)}{3\zeta(2)\sqrt{l_1}} \times \operatorname*{Res}_{\substack{s=\pm\delta \\ s=\pm\tau}} \left\{ \Gamma_\delta(s) \left(\frac{8X}{l_1\pi}\right)^s \check{\Phi}(s) \frac{2s}{s^2-\tau^2} Z(1+2s;\delta)\eta_\delta(1+2s;l) \right\} + O\left(\frac{|r_\delta(l_1)| X^{\vartheta+\epsilon}}{Y l_1^{\frac{1}{2}+\vartheta}} + \frac{|r_\delta(l_1)| X^\epsilon}{(Xl_1)^{\frac{1}{4}}}\right).$$

5.2. Extracting the secondary principal term from $\mathcal{R}_0(l)$

Define for all real numbers ξ, and all complex numbers w with Re $w > 0$,

$$f(\xi, w) = \int_0^\infty \tilde{F}_t\left(\frac{\xi}{t}\right) t^{w-1} dt. \tag{5.6}$$

Since $|\tilde{F}_t(\xi/t)| \le 2\int_{-\infty}^{\infty} |F_t(x)|dx \ll e^{-\frac{t}{20X}}$ by Lemma 3.1, the integral above is absolutely convergent for Re $w > 0$. We collect below some properties of $f(\xi, w)$ which are easily established by making minor modifications to the proof of Lemma 5.2 of [11].

Lemma 5.2. *For corresponding choices of sign define*

$$\mathbb{G}_\pm(u) = (2\pi)^{-u}\Gamma(u)\Big(\cos\left(\tfrac{\pi}{2}u\right) \pm \sin\left(\tfrac{\pi}{2}u\right)\Big).$$

If $\xi \neq 0$ then for any $1 + Re\ w > c > \max(|Re\ \tau|, Re\ w)$ we have

$$f(\xi, w) = |\xi|^w \check{\Phi}(w)\frac{1}{2\pi i}\int_{(c)} \Gamma_\delta(s)\left(\frac{8X}{\pi|\xi|}\right)^s \mathbb{G}_{sgn(\xi)}(s-w)\frac{2s}{s^2-\tau^2}ds. \tag{5.7}$$

For $\xi \neq 0$, $f(\xi, w)$ is a holomorphic function of w in $Re\ w > -1 + |Re\ \tau|$, and in the region $1 \ge Re\ w > -1 + |Re\ \tau|$ satisfies the bound

$$|f(\xi, w)| \ll (1+|w|)^{-Re\ w-\frac{1}{2}} \exp\Big(-\frac{1}{10}\frac{\sqrt{|\xi|}}{\sqrt{X(1+|w|)}}\Big)|\xi|^w|\check{\Phi}(w)|.$$

Using the Mellin transform identity (4.9), we may recast the expression for $\mathcal{R}_0(l)$ (see (5.3) above) as

$$\mathcal{R}_0(l) = \frac{1}{2l}\sum_{\substack{\alpha\le Y\\(\alpha,2l)=1}}\frac{\mu(\alpha)}{\alpha^2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{(-1)^k}{2\pi i}\int_{(c)}\sum_{\substack{n=1\\(n,2\alpha)=1}}^{\infty}\frac{r_\delta(n)}{n^{\frac{3}{2}+w}}G_{-4k}(ln)f\left(\frac{kX}{2\alpha^2 l}, w\right)dw, \tag{5.8}$$

for any $c > |\text{Re}\ \delta|$.

Lemma 5.3. *Write $-4k = k_1k_2^2$ where k_1 is a fundamental discriminant (possibly $k_1 = 1$, giving the trivial character), and k_2 is positive. In the region $Re\ s > 1 + |Re\ \delta|$*

$$\begin{aligned}\sum_{\substack{n=1\\(n,2a)=1}}^{\infty}\frac{r_\delta(n)}{n^s}\frac{G_{-4k}(ln)}{\sqrt{n}} &= L(s-\delta,\chi_{k_1})L(s+\delta,\chi_{k_1})\prod_p \mathcal{G}_{\delta;p}(s;-k,l,\alpha)\\ &=: L(s-\delta,\chi_{k_1})L(s+\delta,\chi_{k_1})\mathcal{G}_\delta(s;-k,l,\alpha),\end{aligned} \tag{5.9}$$

where $\mathcal{G}_{\delta;p}(s;-k,l,\alpha)$ *is defined as follows: If* $p|2\alpha$ *then*

$$\mathcal{G}_{\delta;p}(s;-k,l,\alpha)=\Big(1-\frac{1}{p^{s-\delta}}\Big(\frac{k_1}{p}\Big)\Big)\Big(1-\frac{1}{p^{s+\delta}}\Big(\frac{k_1}{p}\Big)\Big),$$

while, if $p\nmid 2\alpha$,

$$\mathcal{G}_{\delta;p}(s;-k,l,\alpha)$$
$$=\Big(1-\frac{1}{p^{s-\delta}}\Big(\frac{k_1}{p}\Big)\Big)\Big(1-\frac{1}{p^{s+\delta}}\Big(\frac{k_1}{p}\Big)\Big)\ \sum_{r=0}^{\infty}\frac{r_\delta(p^r)}{p^{rs}}\frac{G_{-4k}(p^{r+ord_p(l)})}{p^{\frac{r}{2}}}.$$

Then $\mathcal{G}_{\delta}(s;-k,l,\alpha)$ *is holomorphic in the region* $Re\ s>\frac{1}{2}+|Re\ \delta|$, *and for* $Re\ s\geq\frac{1}{2}+|Re\ \delta|+\epsilon$ *satisfies the bound*

$$|\mathcal{G}_{\delta}(s;-k,l,\alpha)|\ll\alpha^{\epsilon}|k|^{\epsilon}l^{\frac{1}{2}+\epsilon}\big(l,k_2^2\big)^{\frac{1}{2}}. \tag{5.10}$$

Proof. This follows by making minor changes to the proof of Lemma 5.3 of [11].

We use Lemma 5.3 in (5.8), and move the line of integration to the line Re $w=-\frac{1}{2}+|\text{Re }\delta|+\epsilon$. We encounter poles only when $-k=\square$ (so that $k_1=1$, and $L(s,\chi_{k_1})=\zeta(s)$): in this case, we have simple poles at $w=\pm\delta$, and the residues of these poles give rise to a second main term (see (5.12) below). Thus we may write $\mathcal{R}_0(l)=\mathcal{R}(l)+\mathcal{P}_2(l)$ where

$$\mathcal{R}(l)=\frac{1}{2l}\sum_{\substack{\alpha\leq Y\\(\alpha,2l)=1}}\frac{\mu(\alpha)}{\alpha^2}\times\sum_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{(-1)^k}{2\pi i}\int_{(-\frac{1}{2}+|\text{Re }\delta|+\epsilon)}L(1+w+\delta,\chi_{k_1})L(1+w-\delta,\chi_{k_1})\times\mathcal{G}_{\delta}(1+w;-k,l,\alpha)f\Big(\frac{kX}{2\alpha^2 l},w\Big)dw, \tag{5.11}$$

and (with an obvious change of notation, writing k^2 in place of $-k$)

$$\mathcal{P}_2(l)=\frac{1}{2l}\sum_{\substack{\alpha\leq Y\\(\alpha,2l)=1}}\frac{\mu(\alpha)}{\alpha^2}\sum_{\mu=\pm}\zeta(1+2\mu\delta)\times\sum_{k=1}^{\infty}(-1)^k\mathcal{G}_{\delta}(1+\mu\delta;k^2,l,\alpha)f\Big(-\frac{k^2X}{2\alpha^2 l},\mu\delta\Big). \tag{5.12}$$

5.3. *The secondary principal term* $\mathcal{P}_2(l)$

Let v be a complex number with $|\text{Re } v| \le \vartheta + \frac{1}{2\log X}$, and let u be a complex number with Re $u > \frac{1}{2}$. We define

$$\mathcal{H}(u, v; l, \alpha) = l^u \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2u}} \mathcal{G}_v(1+v; k^2, l, \alpha).$$

Note that the above series converges absolutely when Re $u > \frac{1}{2}$. Using Lemma 5.2, and noting that $\mathcal{G}_v(s; k^2, l, \alpha)$ is an even function of v, we see that

$$\sum_{\mu=\pm} \sum_{k=1}^{\infty} (-1)^k \mathcal{G}_\delta(1+\mu\delta; k^2, l, \alpha) f\Big(-\frac{k^2 X}{2\alpha^2 l}, \mu\delta\Big)$$

may be recast as

$$\text{(5.13)} \quad \frac{1}{2\pi i} \int_{(c)} \Gamma_\delta(s) \left(\frac{16\alpha^2}{\pi}\right)^s \times \sum_{\mu=\pm} \check{\Psi}(\mu\delta) \left(\frac{X}{2\alpha^2}\right)^{\mu\delta} \mathbb{G}_-(s-\mu\delta) \frac{2s}{s^2-\tau^2} \mathcal{H}(s-\mu\delta, \mu\delta; l, \alpha) ds,$$

where $c > \max(\frac{1}{2} + |\text{Re } \delta|, |\text{Re } \tau|) = \frac{1}{2} + |\text{Re } \delta|$.

From the definition of $\mathcal{G}_v$ we see that

$$\begin{aligned}\mathcal{H}(u, v; l, \alpha) &= -l^u(1-2^{1-2u}) \sum_{k=1}^{\infty} \frac{1}{k^{2u}} \mathcal{G}_v(1+v; k^2, l, \alpha) \\ &= -l^u(1-2^{1-2u}) \prod_p \sum_{b=0}^{\infty} \frac{\mathcal{G}_{v;p}(1+v; p^{2b}, l, \alpha)}{p^{2bu}}.\end{aligned}$$

Using the expression for $\mathcal{G}_{v;p}$ in Lemma 5.3 and then employing Lemma 3.3 to evaluate it, we see (after some straight-forward but tedious calculations) that we may write

$$\mathcal{H}(u, v; l, \alpha) = -l(1-2^{1-2u}) l_1^{u-\frac{1}{2}} \zeta(2u)\zeta(2u+1+4v)\mathcal{H}_1(u, v; l, \alpha)$$

where $\mathcal{H}_1 = \prod_p \mathcal{H}_{1;p}$ with

$$\mathcal{H}_{1;p} = \begin{cases} \Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{p^{1+2v}}\Big)\Big(1-\frac{1}{p^{2u+1+4v}}\Big) & \text{if } p|2\alpha \\ \Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{p^{1+2v}}\Big)\Big(1+\frac{1}{p}+\frac{1}{p^{1+2v}}-\frac{1}{p^{2u+2+4v}}\Big) & \text{if } p \nmid 2\alpha l \\ \Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{p^{1+2v}}\Big)\Big(1+\frac{1}{p^{2u+2v}}\Big) & \text{if } p|l_1 \\ \Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{p^{1+2v}}\Big)\Big(1+\frac{1}{p^{1+2v}}\Big) & \text{if } p|l,\ p \nmid l_1. \end{cases}$$

These expressions show that $\mathcal{H}(u, v; l, \alpha)$, viewed as a function of u, is meromorphic in the domain Re $u > -\frac{1}{2} - 2$Re v. Note that there is a simple pole at $u = -2v$. Thus we may move the line of integration in (5.13) to the line Re $s = \vartheta + \frac{1}{\log X}$, and since we encounter no poles, (5.13) is given by the resulting integral on this line. Using these observations in (5.12) we conclude that

$$\mathcal{P}_2(l) = \frac{1}{2\pi i} \int_{(\vartheta + \frac{1}{\log X})} \Gamma_\delta(s) \left(\frac{16}{\pi}\right)^s \frac{2s}{s^2 - \tau^2} \times \sum_{\mu=\pm} \check{\Psi}(\mu\delta) \left(\frac{X}{2}\right)^{\mu\delta} \zeta(1 + 2\mu\delta) \mathbb{G}_-(s - \mu\delta) \times \frac{1}{2l} \sum_{\substack{\alpha \le Y \\ (\alpha, 2l)=1}} \frac{\mu(\alpha)}{\alpha^{2-2s+2\mu\delta}} \mathcal{H}(s - \mu\delta, \mu\delta; l, \alpha) ds. \tag{5.14}$$

We now wish to show that the sum over α in (5.14) may be extended to infinity, at the cost of an acceptable error. Let $\mathcal{C}$ denote the circle, oriented counter-clockwise, with center at 0 and radius $\vartheta + \frac{1}{2\log X}$. Given s with Re $s = \vartheta + \frac{1}{\log X}$, the function $2z\check{\Psi}(z) \left(\frac{X}{2\alpha^2}\right)^z \zeta(1 + 2z)\mathbb{G}_-(s - z)\mathcal{H}(s - z, z; l, \alpha)$ is analytic for z inside $\mathcal{C}$. So by Cauchy's theorem

$$\sum_{\mu=\pm} \check{\Psi}(\mu\delta) \left(\frac{X}{2\alpha^2}\right)^{\mu\delta} \zeta(1 + 2\mu\delta)\mathbb{G}_-(s - \mu\delta)\mathcal{H}(s - \mu\delta, \mu\delta; l, \alpha) = \frac{1}{2\pi i} \int_{\mathcal{C}} \check{\Psi}(z) \left(\frac{X}{2\alpha^2}\right)^z \zeta(1 + 2z)\mathbb{G}_-(s - z)\mathcal{H}(s - z, z; l, \alpha) \frac{2z}{z^2 - \delta^2} dz. \tag{5.15}$$

For z on $\mathcal{C}$ we see that $2\vartheta + 3/(2\log X) \ge$ Re $(s - z) \ge 1/(2\log X)$. Further $|\frac{2z}{z^2-\delta^2}\check{\Psi}(z)\zeta(1 + 2z) \left(\frac{X}{2\alpha^2}\right)^z| \ll (\log^2 X)(X\alpha^2)^{\vartheta+1/\log X}$, and by Stirling's formula we see that $|\mathbb{G}_-(s - z)| \ll \log X(1 + |\text{Im } (s)|)^{2\vartheta-1/2+3/(2\log X)} \ll \log X$. Lastly from our expressions for $\mathcal{H}(u, w; l, \alpha)$ we deduce that $|\mathcal{H}(s - z, z; l, \alpha)| \ll l^{1+\epsilon} l_1^{2\vartheta + \frac{3}{2\log X} - \frac{1}{2}} \alpha^\epsilon (1 + |s|) \log X$. From these estimates we conclude that (5.15) is bounded by

$$l^{1+\epsilon} l_1^{2\vartheta - \frac{1}{2}} (X\alpha^2)^{\vartheta+\epsilon}(1 + |s|).$$

We deduce that

$$\frac{1}{2l} \sum_{\substack{\alpha > Y \\ (\alpha, 2l)=1}} \frac{\mu(\alpha)}{\alpha^{2-2s}} \times \sum_{\mu=\pm} \check{\Psi}(\mu\delta) \left(\frac{X}{2\alpha^2}\right)^{\mu\delta} \zeta(1 + 2\mu\delta)\mathbb{G}_-(s - \mu\delta)\mathcal{H}(s - \mu\delta, \mu\delta; l, \alpha)$$

is bounded by $\ll l^{\epsilon} l_1^{2\vartheta-\frac{1}{2}} X^{\vartheta+\epsilon}(1+|s|)Y^{-1+4\vartheta}$. Using this in (5.14) we conclude that the error incurred in extending the sum over α to infinity is

$$\ll l^{\epsilon} l_1^{2\vartheta-\frac{1}{2}} X^{\vartheta+\epsilon} Y^{-1+4\vartheta} \int_{(\vartheta+\frac{1}{\log X})} |\Gamma_\delta(s)|(1+|s|)\frac{|s|}{|s^2-\tau^2|}|ds|$$
$$\ll l^{\epsilon} l_1^{2\vartheta-\frac{1}{2}} X^{\vartheta+\epsilon} Y^{-1+4\vartheta}.$$

Thus, up to an error $O(l^{\epsilon} l_1^{2\vartheta-\frac{1}{2}} X^{\vartheta+\epsilon} Y^{-1+4\vartheta})$, $\mathcal{P}_2(l)$ is given by

$$(5.16) \quad \sum_{\mu=\pm} \check{\Psi}(\mu\delta)\left(\frac{X}{2}\right)^{\mu\delta}\zeta(1+2\mu\delta)\frac{1}{2\pi i}$$
$$\times \int_{(\vartheta+\frac{1}{\log X})} \Gamma_\delta(s)\left(\frac{16}{\pi}\right)^s \mathbb{G}_-(s-\mu\delta)\frac{2s}{s^2-\tau^2}\mathcal{K}(s,\mu\delta;l)ds,$$

where

$$\mathcal{K}(s,v;l) = \frac{1}{2l}\sum_{\substack{\alpha=1\\(\alpha,2l)=1}}^{\infty} \frac{\mu(\alpha)}{\alpha^{2-2s+2v}}\mathcal{H}(s-v,v;l,\alpha).$$

Using our expression for $\mathcal{H}$ a calculation gives

$$\mathcal{K}(s,v;l) = -\frac{1}{4l_1^{\frac{1}{2}+v}}\frac{\varphi(l)}{l}\prod_{p|2l}\left(1-\frac{1}{p^{1+2v}}\right)\prod_{\substack{p|l\\p\nmid l_1}}\left(1+\frac{1}{p^{1+2v}}\right)r_s(l_1)$$
$$\times\left(\frac{4^s+4^{-s}-2^{-1-2v}-2^{1+2v}}{4^s}\right)\zeta(2s-2v)\zeta(2s+1+2v)$$
$$\times\prod_{p\nmid 2l}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{1+2v}}\right)\left(1+\frac{1}{p}+\frac{1}{p^{1+2v}}+\frac{1}{p^{3+4v}}-\frac{(p^{2s}+p^{-2s})}{p^{2+2v}}\right).$$

Using this together with the functional equation for $\zeta(s)$ and the relations $\Gamma(z)\Gamma(1-z)=\pi\operatorname{cosec}(\pi z)$ and $\Gamma(z)\Gamma(z+\frac{1}{2})=\pi^{\frac{1}{2}}2^{1-2z}\Gamma(2z)$ we see that

$$\mathcal{J}(s,v;l) := \Gamma_v(s)\mathbb{G}_-(s-v)\left(\frac{16}{\pi}\right)^s\mathcal{K}(s,v;l)$$

satisfies the functional equation $\mathcal{J}(s,v;l)=\mathcal{J}(-s,v;l)$. In fact, we obtain the useful identity

(5.17)
$$\zeta(1+2v)\mathcal{J}(s,v;l) = \frac{2r_s(l_1)}{3\zeta(2)\sqrt{l_1}}\left(\frac{16}{\pi l_1}\right)^v\Gamma_s(v)Z(1+2v;s)\eta_s(1+2v;l);$$

it is plain that the left side above is invariant under $s\to -s$.

Consider now for $\mu = \pm$ the integral in (5.16): that is,

$$\frac{1}{2\pi i}\int_{(\vartheta+\frac{1}{\log X})} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2}ds. \tag{5.18}$$

We move the line of integration to the line $\mathrm{Re}(s) = -\vartheta - \frac{1}{\log X}$. We encounter simple poles at $s = \delta, -\delta, \tau$, and $-\tau$. Thus (5.18) equals

$$\operatorname*{Res}_{s=\pm\delta,\pm\tau} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2} + \frac{1}{2\pi i}\int_{(-\vartheta-\frac{1}{\log X})} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2}ds.$$

Changing s to $-s$ and using the relation $\mathcal{J}(s,\mu\delta;l) = \mathcal{J}(-s,\mu\delta;l)$ we see that the above is

$$= \operatorname*{Res}_{s=\pm\delta,\pm\tau} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2} - \frac{1}{2\pi i}\int_{(\vartheta+\frac{1}{\log X})} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2}ds.$$

Hence (5.18) equals

$$\frac{1}{2}\operatorname*{Res}_{s=\pm\delta,\pm\tau} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2} = \operatorname*{Res}_{s=\mu\delta} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2} + \mathcal{J}(\tau,\mu\delta;l),$$

using once again that $\mathcal{J}(s,\mu\delta;l) = \mathcal{J}(-s,\mu\delta;l)$.

We conclude that

$$\mathcal{P}_2(l) = \sum_{\mu=+,-} \check{\Psi}(\mu\delta)\left(\frac{X}{2}\right)^{\mu\delta}\zeta(1+2\mu\delta) \tag{5.19}$$
$$\times\left(\operatorname*{Res}_{s=\mu\delta} \mathcal{J}(s,\mu\delta;l)\frac{2s}{s^2-\tau^2} + \mathcal{J}(\tau,\mu\delta;l)\right) + O\left(\frac{l^{\epsilon}X^{\vartheta+\epsilon}l_1^{2\vartheta-\frac{1}{2}}}{Y^{1-4\vartheta}}\right).$$

5.4. The contribution of the remainder terms $\mathcal{R}(l)$

The contribution of the remainder terms $\mathcal{R}(l)$ is bounded in much the same manner as the analogous quantity in [11] (see Sect. 5.4 there). For the sake of completeness we give a detailed sketch of the main ideas of the proof.

First we bound $|\mathcal{R}(l)|$ for individual l. Using the bounds of Lemmas 5.2 and 5.3 in (5.11) we get that $|\mathcal{R}(l)|$ is

$$\ll \frac{l^{-|\mathrm{Re}\,\delta|+\epsilon}}{X^{\frac{1}{2}-|\mathrm{Re}\,\delta|-\epsilon}}\sum_{\alpha\le Y}\frac{1}{\alpha^{1+2|\mathrm{Re}\,\delta|-\epsilon}}\int_{(-\frac{1}{2}+|\mathrm{Re}\,\delta|+\epsilon)} |\check{\Psi}(w)|(1+|w|)^{-|\mathrm{Re}\,\delta|}$$
$$\times\sum_{\substack{k=-\infty\\k\ne 0}}^{\infty}\frac{|L(1+w+\delta,\chi_{k_1})L(1+w-\delta,\chi_{k_1})|}{|k_1|^{\frac{1}{2}-|\mathrm{Re}\,\delta|}}k_2^{2|\mathrm{Re}\,\delta|}$$
$$\times\exp\Big(-\frac{1}{10}\frac{\sqrt{|k|}}{\alpha\sqrt{l(1+|w|)}}\Big)|dw|.$$

Performing the sum over k_2 we see that this is bounded by

$$\frac{l^{\frac{1}{2}+\epsilon}}{X^{\frac{1}{2}-|\text{Re }\delta|-\epsilon}} \sum_{\alpha \le Y} \alpha^{\epsilon} \int_{(-\frac{1}{2}+|\text{Re }\delta|+\epsilon)} |\check{\Psi}(w)|(1+|w|)^{\frac{1}{2}} \times \sum_{k_1} \frac{|L(1+w+\delta,\chi_{k_1})L(1+w-\delta,\chi_{k_1})|}{|k_1|} \times \exp\Big(-\frac{1}{10}\frac{\sqrt{|k_1|}}{\alpha\sqrt{l(1+|w|)}}\Big)|dw|.$$

We split the k_1 into dyadic blocks and use Cauchy's inequality with Lemma 3.5 to estimate these contributions. We deduce that

$$|\mathcal{R}(l)| \ll \frac{l^{\frac{1}{2}+\epsilon}Y^{1+\epsilon}}{X^{\frac{1}{2}-|\text{Re }\delta|-\epsilon}} \int_{(-\frac{1}{2}+|\text{Re }\delta|+\epsilon)} |\check{\Psi}(w)|(1+|w|)|dw| \ll \frac{l^{\frac{1}{2}+\epsilon}Y^{1+\epsilon}}{X^{\frac{1}{2}-|\text{Re }\delta|-\epsilon}} \Psi_{(2)}\Psi_{(3)}^{\epsilon}.$$

We now sketch how a better bound for $\mathcal{R}(l)$ may be obtained on average. Let $\beta_l = \frac{\overline{\mathcal{R}(l)}}{|\mathcal{R}(l)|}$ if $\mathcal{R}(l) \neq 0$, and $\beta_l = 1$ otherwise. Then, from (5.11), $\sum_{l=L}^{2L-1} |\mathcal{R}(l)| = \sum_{l=L}^{2L-1} \beta_l \mathcal{R}(l)$ is

$$\ll \sum_{\substack{\alpha \le Y \\ (\alpha,2)=1}} \frac{1}{\alpha^2} \int_{(-\frac{1}{2}+|\text{Re }\delta|+\epsilon)} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |L(1+w+\delta,\chi_{k_1})L(1+w-\delta,\chi_{k_1})| \times \Big|\sum_{\substack{l=L \\ (l,\alpha)=1}}^{2L-1} \frac{\beta_l}{l}\mathcal{G}_\delta(1+w;-k,l,\alpha) f\Big(\frac{kX}{2\alpha^2 l},w\Big)\Big||dw|. \tag{5.20}$$

We now split the sum over k into dyadic blocks $K \le |k| \le 2K-1$. By Cauchy's inequality the sum over k in this range is bounded by the product of two terms. The first of these terms is

$$\Big(\sum_{|k|=K}^{2K-1} k_2 |L(1+w+\delta,\chi_{k_1})L(1+w-\delta,\chi_{k_1})|^2\Big)^{\frac{1}{2}} \ll (K(1+|w|))^{\frac{1}{2}+\epsilon},$$

upon using Cauchy's inequality again with Lemma 3.5. The second term in question is

$$\Big(\sum_{|k|=K}^{2K-1} \frac{1}{k_2}\Big|\sum_{\substack{l=L \\ (l,2\alpha)=1}}^{2L-1} \frac{\beta_l}{l}\mathcal{G}_\delta(1+w;-k,l,\alpha) f\Big(\frac{kX}{2\alpha^2 l},w\Big)\Big|^2\Big)^{\frac{1}{2}}. \tag{5.21}$$

Lemma 5.4. *Let $\alpha \le Y$, K and L be positive integers, and suppose w is a complex number with Re $w = -\frac{1}{2} + |\text{Re }\delta| + \epsilon$. Then for any choice of complex numbers γ_l with $|\gamma_l| \le 1$ we have*

$$\sum_{|k|=K}^{2K-1} \frac{1}{k_2} \left| \sum_{\substack{l=L \\ (l,2\alpha)=1}}^{2L-1} \frac{\gamma_l}{l} \mathcal{G}(1+w; -k, l, \alpha) f\left(\frac{kX}{2\alpha^2 l}, w\right) \right|^2$$

is bounded by

$$(1+|w|)^{-2|Re\,\delta|+\epsilon} |\check{\Psi}(w)|^2 \frac{\alpha^{2-4|Re\,\delta|+\epsilon} L^{2-2|Re\,\delta|+\epsilon} K^{2|Re\,\delta|+\epsilon}}{X^{1-2|Re\,\delta|-\epsilon}} \times \exp\left(-\frac{1}{20} \frac{\sqrt{K}}{\alpha\sqrt{L(1+|w|)}}\right),$$

and also by

$$((1+|w|)\alpha K L X)^{\epsilon} |\check{\Psi}(w)|^2 \left(\frac{\alpha^2 L(1+|w|)}{K}\right)^{2|Re\,\tau|-2|Re\,\delta|} \times \frac{\alpha^2 L}{K X^{1-2|Re\,\delta|}} (K+L).$$

We bound (5.21) using the first bound of the Lemma for $K \geq \alpha^2 L(1+|w|)\log^2 X$, and the second bound for smaller K. Inserting this bound in (5.20) gives (with a little calculation)

$$\sum_{l=L}^{2L-1} |\mathcal{R}(l)| \ll \sum_{\alpha \leq Y} \int_{(-\frac{1}{2}+|\mathrm{Re}\,\delta|+\epsilon)} |\check{\Psi}(w)|(1+|w|)^{1+\epsilon} \times \left(\frac{\alpha^{2+\epsilon} L^{1+\epsilon}}{X^{\frac{1}{2}-|\mathrm{Re}\,\delta|-\epsilon}} + \frac{\alpha^{1+2\vartheta+\epsilon} L^{1+\vartheta+\epsilon}}{X^{\frac{1}{2}-|\mathrm{Re}\,\delta|-\epsilon}}\right) |dw|$$
$$\ll \left(\frac{L^{1+\epsilon} Y^{1+\epsilon}}{X^{\frac{1}{2}-|\mathrm{Re}\,\delta|-\epsilon}} + \frac{Y^{2\vartheta+\epsilon} L^{1+\vartheta+\epsilon}}{X^{\frac{1}{2}-|\mathrm{Re}\,\delta|-\epsilon}}\right) \Psi_{(2)} \Psi_{(3)}^{\epsilon},$$

as desired.

Proof of Lemma 5.4. We follow closely the proof of Lemma 5.4 in [11]. Using the bound for $\mathcal{G}_\delta$ in Lemma 5.3, and the bound for $|f(\xi, w)|$ in Lemma 5.2 we easily obtain the first bound claimed.

Write the integral in (5.7) as $\frac{1}{2\pi i}\int_{(c)} g(s, w; \mathrm{sgn}(\xi))\left(\frac{8X}{|\xi|\pi}\right)^s ds$. Taking $c = |\mathrm{Re}\,\tau| + \epsilon$, we see that (for $K \leq |k| \leq 2K-1$)

$$\left| \sum_{\substack{l=L \\ (l,2\alpha)=1}}^{2L-1} \frac{\gamma_l}{l} \mathcal{G}_\delta(1+w; -k, l, \alpha) f\left(\frac{kX}{2\alpha^2 l}, w\right) \right|$$
$$\ll |\check{\Psi}(w)| \frac{\alpha^{1+2|\mathrm{Re}\,\tau|-2|\mathrm{Re}\,\delta|+\epsilon}}{K^{\frac{1}{2}+|\mathrm{Re}\,\tau|-|\mathrm{Re}\,\delta|-\epsilon} X^{\frac{1}{2}-|\mathrm{Re}\,\delta|-\epsilon}} \times \int_{(c)} \left| g(s, w; \mathrm{sgn}(k)) \sum_{\substack{l=L \\ (l,2\alpha)=1}}^{2L-1} \frac{\gamma_l}{l^{1+w-s}} \mathcal{G}_\delta(1+w; -k, l, \alpha) ds \right|.$$

Since $|g(s,w;\mathrm{sgn}(k))| \ll (1+|w|)^{c-\frac{1}{2}-\mathrm{Re}\,w+\epsilon}\exp(-\frac{\pi}{2}|\mathrm{Im}(s)|)$ by Stirling's formula, we get by Cauchy's inequality that the above is

$$\ll (1+|w|)^{|\mathrm{Re}\,\tau|-|\mathrm{Re}\,\delta|+\epsilon}|\check{\Psi}(w)|\frac{\alpha^{1+2|\mathrm{Re}\,\tau|-2|\mathrm{Re}\,\delta|+\epsilon}}{K^{\frac{1}{2}-|\mathrm{Re}\,\delta|+|\mathrm{Re}\,\tau|-\epsilon}X^{\frac{1}{2}-|\mathrm{Re}\,\delta|-\epsilon}}$$
$$\times\left(\int_{(c)}\exp\left(-\tfrac{\pi}{2}|\mathrm{Im}(s)|\right)\left|\sum_{\substack{l=L\\(l,2\alpha)=1}}^{2L-1}\frac{\gamma_l}{l^{1+w-s}}\mathcal{G}_\delta(1+w;-k,l,\alpha)\right|^2|ds|\right)^{\frac{1}{2}}.$$

The second bound of the lemma follows by combining this with Lemma 5.5 below.

Lemma 5.5. *Let $|\delta_l| \ll l^\epsilon$ be any sequence of complex numbers, with $\delta_l = 0$ if $(l,2\alpha)\neq 1$. Let w be any complex number with $Re(w) = -\frac{1}{2}+|Re\ \delta|+\epsilon$. Then*

$$\sum_{|k|=K}^{2K-1}\frac{1}{k_2}\left|\sum_{l=L}^{2L-1}\frac{\delta_l}{\sqrt{l}}\mathcal{G}_\delta(1+w;-k,l,\alpha)\right|^2 \ll (KL\alpha)^\epsilon(K+L)L.$$

Proof. For any integer $k=\pm\prod_{i,\ a_i\geq 1}p_i^{a_i}$ we define $a(k)=\prod_i p_i^{a_i+1}$, and put $b(k)=\prod_{i,\ a_i=1}p_i\ \prod_{i,\ a_i\geq 2}p_i^{a_i-1}$. Note that $\mathcal{G}_\delta(1+w;-k,l,\alpha)=0$ unless l can be written as dm where $d|a(k)$ and $(m,k)=1$ with m square-free. From the definition of $\mathcal{G}$ in Lemma 5.3, and using Lemma 3.3, we get

$$\mathcal{G}_\delta(1+w;k,l,\alpha)=\sqrt{m}\left(\frac{-k}{m}\right)\prod_{p|m}\left(1+\frac{r_\delta(p)}{p^{1+w}}\left(\frac{-k}{p}\right)\right)^{-1}$$
$$\times\,\mathcal{G}_\delta(1+w;-k,d,\alpha).$$

Using Lemma 5.3 to bound $|\mathcal{G}_\delta(1+w;-k,d,\alpha)|$ we see that our desired sum is

$$\ll (KL\alpha)^\epsilon\sum_{|k|=K}^{2K-1}\frac{1}{k_2}$$
$$\times\sum_{d|a(k)}d\left|\sum_{m=L/d}^{2L/d}\delta_{dm}\mu(m)^2\left(\frac{-k}{m}\right)\prod_{p|m}\left(1+\frac{r_\delta(p)}{p^{1+w}}\left(\frac{-k}{p}\right)\right)^{-1}\right|^2.$$

We interchange the sums over d and k. Note that $d|a(k)$ implies that that $b(d)|k$, so that $k=b(d)f$ for some integer f with $K/b(d)\leq|f|\leq 2K/b(d)$. Write $-4f=f_1f_2^2$ where f_1 is a fundamental discriminant, and

f_2 is positive. Notice that $k_2 \geq f_2$. Thus our desired sum is bounded by

$$(KL\alpha)^\epsilon \sum_{d \leq 2L} d \sum_{f=K/b(d)}^{2K/b(d)} \frac{1}{f_2} \times \left| \sum_{m=L/d}^{2L/d} \delta_{dm} \mu(m)^2 \left(\frac{-fb(d)}{m}\right) \prod_{p|m} \left(1 + \frac{r_\delta(p)}{p^{1+w}} \left(\frac{-fb(d)}{p}\right)\right)^{-1} \right|^2. \tag{5.22}$$

Observe that

$$\begin{aligned} \left(\frac{-fb(d)}{m}\right) \prod_{p|m} \left(1 + \frac{r_\delta(p)}{p^{1+w}} \left(\frac{-fb(d)}{p}\right)\right)^{-1} &= a_m \left(\frac{-fb(d)}{m}\right) \prod_{p|m} \left(1 - \frac{r_\delta(p)}{p^{1+w}} \left(\frac{-fb(d)}{p}\right)\right) \\ &= a_m \left(\frac{-fb(d)}{m}\right) \sum_{n|m} \frac{\mu(n) r_\delta(n)}{n^{1+w}} \left(\frac{-fb(d)}{n}\right), \end{aligned}$$

where $a_m = \prod_{p|m}(1 - \frac{r_\delta(p)^2}{p^{2+2w}})^{-1} \ll m^\epsilon$. Hence for appropriate $b_m(n,d) \ll (mnd)^\epsilon$ we have that the $|\sum_m|^2$ term in (5.22) is bounded by

$$\begin{aligned} &\left(\sum_{n \leq L/d} \frac{|r_\delta(n)|}{n^{\frac{1}{2}+|\mathrm{Re}\,\delta|+\epsilon}} \left| \sum_{m=L/(nd)}^{2L/(nd)} b_m(n,d) \mu^2(m) \left(\frac{-f}{m}\right) \right| \right)^2 \\ &\qquad \ll \left(\frac{L}{d}\right)^\epsilon \sum_{n \leq L/d} \left| \sum_{m=L/(nd)}^{2L/(nd)} b_m(n,d) \mu^2(m) \left(\frac{-f}{m}\right) \right|^2, \end{aligned}$$

using Cauchy's inequality. Using this in (5.22) and invoking Lemma 3.4 we conclude that our desired sum is

$$\begin{aligned} &\ll (KL\alpha)^\epsilon \sum_{d \leq L} d \sum_{n \leq L/d} \frac{L}{nd} \left(\frac{K}{b(d)} + \frac{L}{nd}\right) \\ &\ll (KL\alpha)^\epsilon (KL + L^2) \sum_{d \leq L} \frac{1}{b(d)} \ll (KL\alpha)^\epsilon (KL + L^2), \end{aligned}$$

as stated.

5.5. *Completion of the proof*

From our work above the remainder terms are under control; and we need only simplify the main term $\mathcal{P}(l) + \mathcal{P}_2(l)$ arising from (5.5) and (5.19).

Using (5.17) it is easy to see that the contribution to (5.19) from the poles at $\mu\delta$ cancel precisely the contribution to (5.5) from the poles at $\mu\delta$. Thus our main term includes only the contribution from the poles at $\pm\tau$ in both these expressions. Employing (5.17) we deduce that the main term is

$$\frac{2}{3\zeta(2)\sqrt{l_1}}\sum_{\mu=\pm}\bigg(r_\delta(l_1)\Gamma_\delta(\mu\tau)\left(\frac{8X}{l_1\pi}\right)^{\mu\tau}\check{\Psi}(\mu\tau)Z(1+2\mu\tau;\delta)\eta_\delta(1+2\mu\tau;l)$$
$$+r_\tau(l_1)\Gamma_\tau(\mu\delta)\left(\frac{8X}{l_1\pi}\right)^{\mu\delta}\check{\Psi}(\mu\delta)Z(1+2\mu\delta;\tau)\eta_\tau(1+2\mu\delta;l)\bigg).$$

This proves Proposition 2.3.

6. Mollification near $s=\frac{1}{2}$: proof of Proposition 2.4

Throughout this section we recall that $\delta_2=\overline{\delta_1}$ so that τ is purely real, and δ is purely imaginary. Further we recall that $\tau\geq-\frac{1}{\epsilon\log X}$, that $\vartheta\geq|\delta_1|\geq\frac{\epsilon}{\log X}$ and that $M=X^{\frac{1}{2}-5\vartheta}$. From Lemma 3.2, and the definition of $\xi(s,\chi_{-8d})$ we see that

$$\mathcal{S}(1;\Phi)\mathcal{W}(\delta_1,\Phi)=\frac{(8X/\pi)^{-\tau}}{\Gamma_\delta(\tau)}\mathcal{S}(A_{\delta,\tau}(d)|M(\tfrac{1}{2}+\delta_1,d)|^2;\Phi_{-\tau})$$

where $\Phi_{-\tau}(t)=t^{-\tau}\Phi(t)$. We choose $Y=X^{4\vartheta}$, and decompose the above as

$$\frac{(8X/\pi)^{-\tau}}{\Gamma_\delta(\tau)}\Big\{\mathcal{S}_M(A_{\delta,\tau}(d)|M(\tfrac{1}{2}+\delta_1,d)|^2;\Phi_{-\tau})$$
$$+O(\mathcal{S}_R(A_{\delta,\tau}(d)|M(\tfrac{1}{2}+\delta_1,d)|^2;\Phi_{-\tau}))\Big\}.$$

Applying Proposition 2.2 we conclude that

$$\mathcal{S}(1;\Phi)\mathcal{W}(\delta_1,\Phi)=\frac{(8X/\pi)^{-\tau}}{\Gamma_\delta(\tau)}\mathcal{S}_M(A_{\delta,\tau}(d)|M(\tfrac{1}{2}+\delta_1,d)|^2;\Phi_{-\tau})+O(X^{-\vartheta+\epsilon}). \tag{6.1}$$

Now

$$\mathcal{S}_M(A_{\delta,\tau}(d)|M(\tfrac{1}{2}+\delta_1,d)|^2;\Phi_{-\tau})$$
$$=\sum_l\Big(\sum_{rs=l}\frac{\lambda(r)\lambda(s)}{r^{\frac{1}{2}+\delta_1}s^{\frac{1}{2}+\delta_2}}\Big)\mathcal{S}_M\Big(A_{\delta,\tau}(d)\left(\frac{-8d}{l}\right);\Phi_{-\tau}\Big),$$

and we use Proposition 2.3 to evaluate these terms. First we note that the various remainder terms in Proposition 2.3 contribute (using $|\lambda(n)| \ll n^\epsilon$, $r_\delta(n) \ll n^\epsilon$, $d(n) \ll n^\epsilon$, and $\tau \geq -1/(\epsilon \log X)$)

$$\ll \sum_{l \leq M^2} \frac{l^\epsilon}{l^{\frac{1}{2}+\tau}} \Big(l_1^{\vartheta - \frac{1}{4}} X^{-\frac{1}{4}+\epsilon} + \frac{X^{\vartheta+\epsilon} l_1^{2\vartheta - \frac{1}{2}}}{Y^{1-4\vartheta}} + |\mathcal{R}(l)| \Big) \ll X^{-\vartheta+\epsilon} \Phi_{(2)} \Phi_{(3)}^\epsilon.$$

Thus we get that

$$\begin{aligned} \text{(6.2a)}\quad & \mathcal{S}(1;\Phi)\mathcal{W}(\delta_1,\Phi) \\ & = \frac{(8X/\pi)^{-\tau}}{\Gamma_\delta(\tau)} \sum_l \Big(\sum_{rs=l} \frac{\lambda(r)\lambda(s)}{r^{\frac{1}{2}+\delta_1} s^{\frac{1}{2}+\delta_2}} \Big) \mathcal{M}(l) + O(X^{-\vartheta+\epsilon} \Phi_{(2)} \Phi_{(3)}^\epsilon), \end{aligned}$$

where $\mathcal{M}(l) = \mathcal{M}_1(l) + \mathcal{M}_2(l)$ with

$$\begin{aligned} \text{(6.2b)}\quad \mathcal{M}_1(l) = & \frac{2}{3\zeta(2)\sqrt{l_1}} \sum_{\mu=\pm} r_\delta(l_1) \Gamma_\delta(\mu\tau) \left(\frac{8X}{l_1 \pi} \right)^{\mu\tau} \\ & \times \check{\Phi}(\mu\tau - \tau) Z(1+2\mu\tau;\delta) \eta_\delta(1+2\mu\tau; l), \end{aligned}$$

and

$$\begin{aligned} \text{(6.2c)}\quad \mathcal{M}_2(l) = & \frac{2}{3\zeta(2)\sqrt{l_1}} \sum_{\mu=\pm} r_\tau(l_1) \Gamma_\tau(\mu\delta) \left(\frac{8X}{l_1 \pi} \right)^{\mu\delta} \\ & \times \check{\Phi}(\mu\delta - \tau) Z(1+2\mu\delta;\tau) \eta_\tau(1+2\mu\delta; l). \end{aligned}$$

From our assumptions on δ_1 we know that $\tau^2 - \delta^2 = |\delta_1|^2 \geq \epsilon^2/(\log X)^2$. This enables us to evaluate the $\mathcal{M}_1(l)$ and $\mathcal{M}_2(l)$ contributions to (6.2a) separately. Let $\mathcal{C}$ denote a closed contour (oriented counter-clockwise) which contains the points $\pm\tau$ and such that for $w \in \mathcal{C}$ we have $|\text{Re } w| \leq |\tau| + C/\log X$, and $|\text{Im } w| \leq C/\log X$ for some absolute constant C, and such that $|w^2 - \tau^2| \geq \epsilon^2/(3\log^2 X)$, and $|w^2 - \delta^2| \geq \epsilon^2/(3\log^2 X)$, and finally such that the perimeter length of $\mathcal{C}$ is $\ll |\delta_1|$. Then the contribution of $\mathcal{M}_1(l)$ to (6.2a) is

$$\begin{aligned} \text{(6.3)}\quad & \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{2\check{\Phi}(w-\tau)}{3\zeta(2)\Gamma_\delta(\tau)} \left(\frac{8X}{\pi} \right)^{w-\tau} Z(1+2w;\delta) \Gamma_\delta(w) \frac{2w}{w^2-\tau^2} \\ & \times \left\{ \sum_l \frac{r_\delta(l_1)}{l_1^{\frac{1}{2}+w}} \Big(\sum_{rs=l} \frac{\lambda(r)\lambda(s)}{r^{\frac{1}{2}+\delta_1} s^{\frac{1}{2}+\delta_2}} \Big) \eta_\delta(1+2w\tau; l) \right\} dw. \end{aligned}$$

We focus first on simplifying the term in parenthesis above. Since λ is supported on square-free integers, we may write $r = \alpha a$, $s = \alpha b$ where

α, a, and b are square-free with $(a,b)=1$. Thus $l=\alpha^2 a\,b$, $l_1=ab$, and $l_2=\alpha$. With this notation the sum over l in (6.3) becomes

$$\sum_{\alpha}\sum_{\substack{a,b\\(a,b)=1}}\frac{r_\delta(ab)}{(ab)^{\frac12+w}}\frac{\lambda(\alpha a)\lambda(\alpha b)}{\alpha^{1+\delta_1+\delta_2}a^{\frac12+\delta_1}b^{\frac12+\delta_2}}\eta_\delta(1+2w;\alpha^2 ab) \tag{6.4}$$

$$=\sum_{\alpha}\frac{1}{\alpha^{1+2\tau}}\sum_{\substack{a,b\\(a,b)=1}}\frac{r_\delta(a)r_\delta(b)}{a^{1+\delta_1+w}b^{1+\delta_2+w}}\lambda(\alpha a)\lambda(\alpha b)\eta_\delta(1+2w;\alpha^2 ab).$$

Define, for odd primes p,

$$h_w(p)=\Big(1+\frac{1}{p}+\frac{1}{p^{1+2w}}-\frac{p^{-2\delta}+p^{2\delta}}{p^{2+2w}}+\frac{1}{p^{3+4w}}\Big)$$

and extend this multiplicatively to a function on odd, square-free integers. From the definition of η we see that

$$\eta_\delta(1+2w;\alpha^2 ab)=\frac{\eta_\delta(1+2w;1)}{h_w(\alpha)h_w(a)h_w(b)}\prod_{p|\alpha}\Big(1+\frac{1}{p^{1+2w}}\Big).$$

Hence our expression (6.4) may be recast as

$$\eta_\delta(1+2w;1)\sum_{\alpha}\frac{1}{\alpha^{1+2\tau}h_w(\alpha)}\prod_{p|\alpha}\Big(1+\frac{1}{p^{1+2w}}\Big)$$
$$\times\sum_{\substack{a,b\\(a,b)=1}}\frac{r_\delta(a)\lambda(\alpha a)}{a^{1+\delta_1+w}h_w(a)}\frac{r_\delta(b)\lambda(\alpha b)}{b^{1+\delta_2+w}h_w(b)}.$$

Using $\sum_{\beta|(a,b)}\mu(\beta)=1$ if $(a,b)=1$ and 0 otherwise, the above becomes

$$\eta_\delta(1+2w;1)\sum_{\alpha}\frac{\prod_{p|\alpha}(1+1/p^{1+2w})}{\alpha^{1+2\tau}h_w(\alpha)}\sum_{\beta}\frac{r_\delta(\beta)^2\mu(\beta)}{\beta^{2+2\tau+2w}h_w(\beta)^2} \tag{6.5}$$
$$\times\sum_{a,b}\frac{r_\delta(a)\lambda(a\alpha\beta)}{a^{1+\delta_1+w}h_w(a)}\frac{r_\delta(b)\lambda(b\alpha\beta)}{b^{1+\delta_2+w}h_w(b)}.$$

Define for odd primes p

$$H_w(p)=1+\frac{1}{p^{1+2w}}-\frac{r_\delta(p)^2}{p^{1+2w}h_w(p)},$$

and extend this multiplicatively to all odd, square-free integers. Then grouping terms according to $\gamma=\alpha\beta$, we see that (6.5) equals

(6.6)
$$\eta_\delta(1+2w;1)\sum_{\gamma}\frac{H_w(\gamma)}{\gamma^{1+2\tau}h_w(\gamma)}\Big(\sum_{a}\frac{r_\delta(a)\lambda(a\gamma)}{a^{1+\delta_1+w}h_w(a)}\Big)\Big(\sum_{b}\frac{r_\delta(b)\lambda(b\gamma)}{b^{1+\delta_2+w}h_w(b)}\Big).$$

Lemma 6.1. *Let R be a polynomial with $R(0) = R'(0) = 0$. Let g be a multiplicative function with $g(p) = 1 + O(p^{-\nu})$ for some fixed $\nu > 0$. Let y be a large real number, and suppose that u and v are bounded complex numbers such that $Re(u + v)$ and $Re(u - v)$ are $\geq -D/\log y$ where D is an absolute positive constant. When $Re\, s > 1 + D/\log y$ we have*

$$\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{r_v(n)\mu(nc)}{n^{s+u}} g(n) = \frac{\mu(c)G(s, c; u, v)}{\zeta(s + u + v)\zeta(s + u - v)},$$

where $G(s, c; u, v) = \prod_p G_p(s, c; u, v)$ with

$$G_p(s, c; u, v) := \begin{cases} \left(1 - \frac{1}{p^{s+u+v}}\right)^{-1}\left(1 - \frac{1}{p^{s+u-v}}\right)^{-1} & \text{if } p|2c \\ \left(1 - \frac{1}{p^{s+u+v}}\right)^{-1}\left(1 - \frac{1}{p^{s+u-v}}\right)^{-1}\left(1 - \frac{g(p)r_v(p)}{p^{s+u}}\right) & \text{otherwise}, \end{cases}$$

so that $G(s, c; u, v)$ is holomorphic in $Re\, s > \max(\frac{1}{2}, 1 - \nu) + D/\log y$. For any odd integer $c \leq y$ we have

$$\sum_{\substack{n \leq y/c\\ n \text{ odd}}} \frac{r_v(n)\mu(nc)}{n^{1+u}} g(n) R\left(\frac{\log(y/(cn))}{\log y}\right)$$
$$= O\left(\frac{E(c)}{\log^2 y}\left(\frac{y}{c}\right)^{-Re\, u + |Re\, v|} \exp(-A_0\sqrt{\log(y/c)})\right)$$
$$+ \operatorname*{Res}_{s=0} \frac{\mu(c)G(s + 1, c; u, v)}{s\zeta(1 + s + u + v)\zeta(1 + s + u - v)} \sum_{k=0}^{\infty} \frac{1}{(s \log y)^k} R^{(k)}\left(\frac{\log(y/c)}{\log y}\right)$$

for some absolute constant $A_0 > 0$, and where $E(c) = \prod_{p|c}(1 + 1/\sqrt{p})$.

Proof. Our assertion about the generating function $\sum r_v(n)\mu(nc)g(n)/n^{s+u}$ follows readily upon comparing Euler products. In proving the other statements we may plainly suppose that $c \leq y/2$. Using the Taylor expansion $R(x) = \sum_{j=0}^{\infty} \frac{R^{(j)}(0)}{j!} x^j = \sum_{j=2}^{\infty} \frac{R^{(j)}(0)}{j!} x^j$, we see that our sum is

$$\sum_{j=2}^{\infty} \frac{R^{(j)}(0)}{(\log y)^j} \frac{1}{j!} \sum_{\substack{n \leq y/c\\ n \text{ odd}}} \frac{r_\beta(n)\mu(nc)}{n^{1+\alpha}} g(n) \log^j\left(\frac{y}{cn}\right)$$
$$= \sum_{j=2}^{\infty} \frac{R^{(j)}(0)}{(\log y)^j} \frac{1}{2\pi i} \int_{\left(\frac{D+1}{\log(y/c)}\right)} \frac{\mu(c)G(s + 1, c; u, v)}{\zeta(1 + s + u + v)\zeta(1 + s + u - v)} \left(\frac{y}{c}\right)^s \frac{ds}{s^{j+1}}.$$

The integral above is evaluated by a standard procedure: First one truncates the above integral to the line segment $\frac{D+1}{\log(y/c)} - iT$ to $\frac{D+1}{\log(y/c)} + iT$ where $T := \exp(\sqrt{\log(y/c)})$. The error involved in doing so is $\ll E(c)(\log y/c)^2/T^2$. Next we shift the integral on this line segment to the left onto the line segment $-\text{Re}\, u + |\text{Re}\, v| - A_1/\log T$ where A_1 is a positive constant such that $\zeta(1+s+u+v)\zeta(1+s+u-v)$ has no zeros in the region traversed. We encounter a (multiple) pole at $s = 0$ whose residue we shall calculate presently. The integrals on the three other sides are bounded using standard estimates for $1/\zeta(s)$ in the zero-free region, and contribute an amount $\ll E(c)(\log(Ty/c))^2(T^{-2} + (y/c)^{-\text{Re}\, u+|\text{Re}\, v|-A_1/\log T})$. We conclude that for an appropriate positive constant A_0 the above is

$$= \operatorname*{Res}_{s=0} \frac{\mu(c)G(s+1, c; u, v)}{s\zeta(1+s+u+v)\zeta(1+s+u-v)} \sum_{j=2}^{\infty} \frac{R^{(j)}(0)(y/c)^s}{s^j \log^j y} + O\left(\frac{E(c)}{\log^2 y}\left(\frac{y}{c}\right)^{-\text{Re}\, u+|\text{Re}\, v|} \exp(-A_0\sqrt{\log(y/c)})\right).$$

For the purpose of the residue calculation we may replace $\sum_{j=2}^{\infty} \frac{R^{(j)}(0)(y/c)^s}{s^j \log^j y}$ with

$$\sum_{j=2}^{\infty} \frac{R^{(j)}(0)}{s^j \log^j y}\Big(\sum_{l\le j} \frac{s^l}{l!}(\log(y/c))^l\Big) = \sum_{k=0}^{\infty} \frac{s^{-k}}{\log^k y} \sum_{l=0}^{\infty} \frac{R^{(k+l)}(0)}{l!}\left(\frac{\log(y/c)}{\log y}\right)^l,$$

upon grouping terms according to $k = j - l$, and bearing in mind that $R(0) = R'(0) = 0$. This clearly equals

$$\sum_{k=0}^{\infty} \frac{s^{-k}}{\log^k y} R^{(k)}\left(\frac{\log(y/c)}{\log y}\right),$$

completing our proof of the lemma.

We now return to the evaluation of the expression (6.6). We first deal with the contribution arising from the terms $\gamma \le M^{1-b}$. We shall apply Lemma 6.1 twice. In both cases we take $u = \delta_1 + w$, $v = \delta$, and $g(n) = 1/h_w(n)$, and we shall denote the corresponding $G(s,\gamma; u, v)$ by $G_w(s,\gamma; u, v)$. In the first application we take $y = M$, and $R(x) = P(x)$; and in the second application we take $y = M^{1-b}$ and $R(x) = (1 - P(b + x(1-b)))$. Adding these two applications we deduce that

$$\sum_{\substack{a\le M/\gamma \\ a \text{ odd}}} \frac{r_\delta(a)\mu(a\gamma)}{a^{1+\delta_1+w}h_w(a)} Q\left(\frac{\log(M/a\gamma)}{\log M}\right) = \frac{\mu(\gamma)G_w(1,\gamma;\delta_1+w,\delta)}{\zeta(1+\delta_1+w+\delta)\zeta(1+\delta_1+w-\delta)} + O\Big(\frac{E(\gamma)}{\log^2 M}\left(\frac{M^{1-b}}{\gamma}\right)^{-\tau-\text{Re}\, w} \exp(-A_0\sqrt{\log M^{1-b}/\gamma})\Big).$$

Note that the main term above came from the $k = 0$ contribution in the applications of Lemma 6.1, and that the contributions from $k \ge 1$ in the two applications cancel each other. Indeed the appropriate term for $k \geq 1$ in the first application is

$$\frac{1}{(\log M)^k} P^{(k)}\left(\frac{\log(M/\gamma)}{\log M}\right),$$

and the corresponding term in the second application is

$$\frac{1}{(\log M^{1-b})^k}(1 - P(b + x(1-b)))^{(k)}\Big|_{x=\log(M^{1-b}/\gamma)/\log M},$$

and these clearly cancel. This justifies our preceding expression, and now observe that, because of our choice of the contour $\mathcal{C}$, the main term there is $\ll |\delta_1|^2$. An analogous expression holds for the sum over b in (6.6), with the only change being that δ_1 above gets replaced by δ_2. We deduce that the contribution of the $\gamma \leq M^{1-b}$ terms to (6.6) equals

$$\eta_\delta(1+2w;1) \times \sum_{\substack{\gamma \leq M^{1-b} \\ \gamma \text{ odd}}} \frac{H_w(\gamma)}{\gamma^{1+2\tau} h_w(\gamma)} \Bigg(\frac{\mu^2(\gamma) G_w(1,\gamma;\delta_1+w,\delta) G_w(1,\gamma;\delta_2+w,\delta)}{\prod_{\mu=\pm} \zeta(1+\delta_1+w+\mu\delta)\zeta(1+\delta_2+w+\mu\delta)} + O\Big(E(\gamma)\frac{|\delta_1|^2}{\log^2 M}\left(\frac{M^{1-b}}{\gamma}\right)^{-\tau-\text{Re } w} \exp(-A_0\sqrt{\log(M^{1-b}/\gamma)})\Big)\Bigg).$$

This is readily seen to be

$$\eta_\delta(1+2w;1) \times \sum_{\substack{\gamma \leq M^{1-b} \\ \gamma \text{ odd}}} \frac{H_w(\gamma)}{\gamma^{1+2\tau} h_w(\gamma)} \frac{\mu^2(\gamma) G_w(1,\gamma;\delta_1+w,\delta) G_w(1,\gamma;\delta_2+w,\delta)}{\prod_{\mu=\pm} \zeta(1+\delta_1+w+\mu\delta)\zeta(1+\delta_2+w+\mu\delta)} + O\Big(\frac{|\delta_1|^2}{\log^2 M} M^{(1-b)(-\tau-\text{Re } w)}\Big).$$

We use this expression in (6.3) to evaluate the contribution of the $\gamma \leq M^{1-b}$ terms to the integral there. From our choice of $\mathcal{C}$, and since $M = X^{\frac{1}{2}-5\vartheta}$, we see that the error term arising from the above is

$$O\big(\log^2 X |\delta_1|^3 M^{-2\tau(1-b)}\big). \tag{6.7}$$

The main term arising there is

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{2\check{\Phi}(w-\tau)}{3\zeta(2)\Gamma_\delta(\tau)}\left(\frac{8X}{\pi}\right)^{w-\tau} Z(1+2w;\delta)\Gamma_\delta(w)\frac{2w}{w^2-\tau^2}\eta_\delta(1+2w;1) \times \sum_{\substack{\gamma \leq M^{1-b} \\ \gamma \text{ odd}}} \frac{H_w(\gamma)}{\gamma^{1+2\tau} h_w(\gamma)} \frac{\mu^2(\gamma) G_w(1,\gamma;\delta_1+w,\delta) G_w(1,\gamma;\delta_2+w,\delta)}{\prod_{\mu=\pm} \zeta(1+\delta_1+w+\mu\delta)\zeta(1+\delta_2+w+\mu\delta)} dw.$$

A priori the integrand has two poles (at $\pm\tau$) inside $\mathcal{C}$, but since $\prod_{\mu=\pm}\zeta(1+\delta_1+w+\mu\delta)^{-1}\zeta(1+\delta_2+w+\mu\tau)^{-1}$ vanishes (indeed to order 2) at $w=-\tau$, in fact we have only the one simple pole at $w=\tau$. Thus by Cauchy's theorem the main term above equals

$$\frac{2\check{\Phi}(0)}{3\zeta(2)}\frac{\eta_\delta(1+2\tau;1)}{\zeta(1+2\tau)} \times \sum_{\substack{\gamma\le M^{1-b}\\ \gamma \text{odd}}} \frac{\mu^2(\gamma)H_\tau(\gamma)}{\gamma^{1+2\tau}h_\tau(\gamma)}G_\tau(1,\gamma;\delta_1+\tau,\delta)G_\tau(1,\gamma;\delta_2+\tau,\delta). \tag{6.8}$$

Lemma 6.2. *With w on the contour $\mathcal{C}$, and other notations as above we have for $x\ge 2$*

$$\eta_\delta(1+2w;1)\sum_{\substack{\gamma\le x\\ \gamma\text{ odd}}}\frac{\mu^2(\gamma)H_w(\gamma)}{\gamma^{1+2\tau}h_w(\gamma)}G_w(1,\gamma;\delta_1+w,\delta)G_w(1,\gamma;\delta_2+w,\delta)$$
$$=\zeta(1+2\tau)(1-x^{-2\tau})(1+O(|w-\tau|))+O(x^{-2\tau}).$$

Further if $1\le y\le x$ then for any smooth function R on $[0,1]$

$$\eta_\delta(1+2w;1)\sum_{\substack{y\le\gamma\le x\\ \gamma\text{ odd}}}\frac{\mu^2(\gamma)H_w(\gamma)}{\gamma^{1+2\tau}h_w(\gamma)}G_w(1,\gamma;\delta_1+w,\delta)$$
$$\times G_w(1,\gamma;\delta_2+w,\delta)R\Big(\frac{\log\gamma}{\log x}\Big)$$
$$=(1+O(|\delta_1|))\int_y^x R\left(\frac{\log t}{\log x}\right)\frac{dt}{t^{1+2\tau}}.$$

Proof. Upon recalling the definition of $G_w(s,\gamma;u,v)$ from Lemma 6.1 we see that our desired expression equals

$$\eta_\delta(1+2w;1)G_w(1,1;\delta_1+w,\delta)G_w(1,1;\delta_2+w,\delta)\sum_{\substack{\gamma\le x\\ \gamma\text{odd}}}\frac{f_w(\gamma)}{\gamma^{1+2\tau}},$$

say, where $f_w(\gamma)$ is the multiplicative function given by

$$f_w(\gamma)=\mu^2(\gamma)\frac{H_w(\gamma)}{h_w(\gamma)}\prod_{p|\gamma}\Big(1-\frac{r_\delta(p)}{p^{1+\delta_1+w}h_w(p)}\Big)^{-1}\Big(1-\frac{r_\delta(p)}{p^{1+\delta_2+w}h_w(p)}\Big)^{-1}.$$

Plainly $f_w(p)=1+O(1/\sqrt{p})$, say, and so the calculation of the sum over γ becomes a standard exercise. Writing the generating function $\sum_{\gamma\text{ odd}}f_w(\gamma)/\gamma^s=\zeta(s)F_w(s)$, (note that F is holomorphic in Re $s>\frac12$), using Perron's formula, and shifting contours appropriately we deduce easily that

$$\sum_{\substack{\gamma\le x\\ \gamma \text{ odd}}} \frac{f_w(\gamma)}{\gamma^{1+2\tau}} = \zeta(1+2\tau)F_w(1+2\tau) - F_w(1)\frac{x^{-2\tau}}{2\tau} + O(x^{-(\frac{1}{3}+2\tau)})$$
$$= \zeta(1+2\tau)(1-x^{-2\tau})F_w(1+2\tau) + O(x^{-2\tau}),$$

upon using $\zeta(1+2\tau) = 1/(2\tau)+O(1)$, and that $F_w(1+2\tau) = F_w(1)+O(\tau)$. Denote $\eta_\delta(1+2w;1)G_w(1,1;\delta_1+w,\delta)G_w(1,1;\delta_2+w,\delta)F_w(1+2\tau)$ by $H(w)$ say. Then with a little calculation we may check that $H(w) = H(\tau) + O(|w-\tau|)$, and that $H(\tau) = 1$. This proves the first assertion of the Lemma. Our second claim follows upon using partial summation, and arguing along similar lines.

Using Lemma 6.2 in (6.8) above, and combining with the error term estimate (6.7), we conclude that the part of the $\mathcal{M}_1(l)$ contribution arising from the $\gamma \le M^{1-b}$ terms equals

$$\text{(6.9)} \qquad \frac{2\check{\Phi}(0)}{3\zeta(2)}(1-M^{-2\tau(1-b)}) + O(\log^2 X|\delta_1|^3 M^{-2\tau(1-b)}).$$

We now turn to the corresponding contribution from the terms $\gamma > M^{1-b}$ in (6.6). Applying Lemma 6.1 with u, v, $g(n)$, and $G_w(s,\gamma;u,v)$ as above, and with $R(x) = P(x)$, and $y = M$. We get that for any odd $M^{1-b} \le \gamma < M$, (since $Q\left(\frac{\log(M/a\gamma)}{\log M}\right) = P\left(\frac{\log(M/a\gamma)}{\log M}\right)$ for γ in this range)

$$\sum_{\substack{a\le M/\gamma\\ a \text{ odd}}} \frac{r_\delta(a)\mu(a\gamma)}{a^{1+\delta_1+w}h_w(a)} Q\left(\frac{\log(M/a\gamma)}{\log M}\right)$$
$$= O\Big(\frac{E(\gamma)}{\log^2 M}\left(\frac{M}{\gamma}\right)^{-\tau-\text{Re } w} \exp(-A_0\sqrt{\log(M/\gamma)})\Big)$$
$$+ \operatorname*{Res}_{s=0} \frac{\mu(\gamma)G_w(1+s,\gamma;\delta_1+w,\delta)}{s\zeta(1+s+\delta_1+w+\delta)\zeta(1+s+\delta_1+w-\delta)}$$
$$\times \sum_{k=0}^{\infty} \frac{1}{(s\log M)^k} Q^{(k)}\left(\frac{\log(M/\gamma)}{\log M}\right).$$

Write the Taylor expansion of $G_w(1+s,\gamma;u,v)/(\zeta(1+s+\delta_1+w+\delta)\zeta(1+s+\delta_1+w-\delta))$ as $a_0+a_1s+a_2s^2+\ldots$. Then we see that $a_0 = (\delta_1+w+\delta)(\delta_1+w-\delta)G_w(1,\gamma;\delta_1+w,\delta) + O((|\delta_1|+|w|)^3)$, $a_1 = 2(\delta_1+w)G_w(1,\gamma;\delta_1+w,\delta) + O((|\delta_1|+|w|)^2)$, $a_3 = G_w(1,\gamma;\delta_1+w,\delta)+O(|\delta_1|+|w|)$, and that $a_n \ll_n 1$ for $n \ge 4$. From this it follows that the residue term above equals

$$\mu(\gamma)G_w(1,\gamma;\delta_1+w,\delta)\Bigg((\delta_1+w+\delta)(\delta_1+w-\delta)Q\left(\frac{\log(M/\gamma)}{\log M}\right) + 2\frac{\delta_1+w}{\log M}Q'\left(\frac{\log(M/\gamma)}{\log M}\right)+\frac{1}{\log^2 M}Q''\left(\frac{\log(M/\gamma)}{\log M}\right)\Bigg)+O(|\delta_1|^3).$$

An analogous espression holds for the sum over b in (6.6), replacing δ_1 above with δ_2. We use these expressions to evaluate the contribution to (6.6) from the terms $\gamma > M^{1-b}$. Firstly, we see that the remainder terms that accrue are (bearing in mind that w is on the contour $\mathcal{C}$)

$$\ll \sum_{M^{1-b}\le\gamma\le M}\frac{1}{\gamma^{1+2\tau}}\Bigg(\frac{E(\gamma)}{\log^2 M}|\delta_1|^2\left(\frac{M}{\gamma}\right)^{-\tau-\mathrm{Re}\,w} \times \exp(-A_0\sqrt{\log(M/\gamma)})+|\delta_1|^5\Bigg)$$
$$\ll \frac{|\delta_1|^2}{\log^2 M}M^{-\tau-\mathrm{Re}\,w}+M^{-2\tau(1-b)}|\delta_1|^5\log M.$$

Secondly, we get that the main term here is (denoting for brevity $Q^{(j)}(\log(M/\gamma)/\log M)$ by $Q_\gamma^{(j)}$)

$$\eta_\delta(1+2w;1)\sum_{\substack{M^{1-b}<\gamma\le M\\ \gamma\ \text{odd}}}\frac{\mu^2(\gamma)H_w(\gamma)}{\gamma^{1+2\tau}h_w(\gamma)}G_w(1,\gamma;\delta_1+w,\delta)G_w(1,\gamma;\delta_2+w,\gamma)$$
$$\times\Big((\delta_1+w+\delta)(\delta_1+w-\delta)Q_\gamma+2\frac{\delta_1+w}{\log M}Q'_\gamma+\frac{1}{\log^2 M}Q''_\gamma\Big)$$
$$\times\Big((\delta_2+w+\delta)(\delta_2+w-\delta)Q_\gamma+2\frac{\delta_2+w}{\log M}Q'_\gamma+\frac{1}{\log^2 M}Q''_\gamma\Big).$$

Applying Lemma 6.2 (and a suitable change of variables) we conclude that this equals

$$\log M\int_0^b M^{-2\tau(1-x)} \times\Big((\delta_1+w+\delta)(\delta_1+w-\delta)Q(x)+2\frac{\delta_1+w}{\log M}Q'(x)+\frac{Q''(x)}{\log^2 M}\Big) \times\Big((\delta_2+w+\delta)(\delta_2+w-\delta)Q(x)+2\frac{\delta_2+w}{\log M}Q'(x)+\frac{Q''(x)}{\log^2 M}\Big)dx + O(M^{-2\tau(1-b)}|\delta_1|^5\log M). \tag{6.10}$$

We use these expressions in (6.3) to evaluate the contribution of the $\gamma > M^{1-b}$ terms to the integral there. From our choices of M and $\mathcal{C}$ we get that the error term arising from the above is

$$O\big(|\delta_1|^3 M^{-2\tau}\log^2 X + |\delta_1|^6 M^{-2\tau(1-b)}\log^5 X\big). \tag{6.11}$$

Call the main term in (6.10) $N(w)$. Inserting this into the integral in (6.3), we seek to evaluate

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{2\check{\Phi}(w-\tau)}{3\zeta(2)\Gamma_\delta(\tau)}\left(\frac{8X}{\pi}\right)^{w-\tau} Z(1+2w;\delta)\Gamma_\delta(w)N(w)\frac{2w}{w^2-\tau^2}dw.$$

Now $\check{\Phi}(w-\tau)\Gamma_\delta(w)/\Gamma_\delta(\tau) = \check{\Phi}(0) + O(|\delta_1|)$, and $2wZ(1+2w;\delta) = \frac{1}{4(w^2-\delta^2)} + O(|\delta_1|\log^2 X)$, and $N(w) \ll M^{-2\tau(1-b)}|\delta_1|^4\log M$, whence we deduce that the above integral is

$$\begin{aligned} &\frac{2\check{\Phi}(0)}{3\zeta(2)}\frac{1}{2\pi i}\int_{\mathcal{C}}\left(\frac{8X}{\pi}\right)^{w-\tau} N(w)\frac{1}{4(w^2-\delta^2)}\frac{1}{w^2-\tau^2}dw \\ &\qquad + O(|\delta_1|^6 M^{-2\tau(1-b)}\log^5 X) \\ &\quad = \frac{2\check{\Phi}(0)}{3\zeta(2)}\frac{1}{8\delta_1\delta_2\tau}\left(N(\tau) - \left(\frac{8X}{\pi}\right)^{-2\tau} N(-\tau)\right) \\ &\qquad + O(|\delta_1|^6 M^{-2\tau(1-b)}\log^5 X). \end{aligned} \tag{6.12}$$

Using integration by parts together with $Q(0) = Q'(0) = 0$, and $Q(b) = 1$, $Q'(b) = 0$ we may simplify the expression for $N(\tau)$ to

$$8\delta_1\delta_2\tau M^{-2\tau(1-b)} + \frac{4\delta_1\delta_2}{\log M}\int_0^b M^{-2\tau(1-x)}\left|Q'(x) + \frac{Q''(x)}{2\delta_1\log M}\right|^2 dx.$$

Similarly we find that

$$N(-\tau) = \frac{4\delta_1\delta_2}{\log M}\int_0^b M^{-2\tau(1-x)}\left|Q'(x) + \frac{Q''(x)}{2\delta_1\log M}\right|^2 dx.$$

Using these identities we conclude that our expression in (6.12) equals

$$\begin{aligned} \frac{2\check{\Phi}(0)}{3\zeta(2)}\Bigg(M^{-2\tau(1-b)} &+ \frac{1-(8X/\pi)^{-2\tau}}{2\tau\log M}\int_0^b M^{-2\tau(1-x)}\Bigg|Q'(x) \\ &+ \frac{Q''(x)}{2\delta_1\log M}\Bigg|^2 dx\Bigg) + O(M^{-2\tau(1-b)}|\delta_1|^6\log^5 X). \end{aligned}$$

Combining this with (6.11) we conclude that the part of the $\mathcal{M}_1(l)$ contribution arising from the $M^{1-b} \le \gamma \le M$ terms equals

$$\begin{aligned} &\frac{2\check{\Phi}(0)}{3\zeta(2)}\Bigg(M^{-2\tau(1-b)} + \frac{1-(8X/\pi)^{-2\tau}}{2\tau\log M}\int_0^b M^{-2\tau(1-x)}\Bigg|Q'(x) \\ &\quad + \frac{Q''(x)}{2\delta_1\log M}\Bigg|^2 dx\Bigg) + O(|\delta_1|^3 M^{-2\tau}\log^2 X + M^{-2\tau(1-b)}|\delta_1|^6\log^5 X). \end{aligned}$$

Taking this together with (6.9) we have determined the $\mathcal{M}_1(l)$ contribution to be

$$\frac{2\check{\Phi}(0)}{3\zeta(2)}\Big(1+\frac{1-(8X/\pi)^{-2\tau}}{2\tau\log M}\int_0^b M^{-2\tau(1-x)}\Big|Q'(x)+\frac{Q''(x)}{2\delta_1\log M}\Big|^2dx\Big) + O(M^{-2\tau(1-b)}|\delta_1|^6\log^5 X). \tag{6.13}$$

The calculation of the $\mathcal{M}_2(l)$ contribution to (6.2a) is entirely similar. We obtain that this contribution equals

$$-\frac{2\check{\Phi}(0)}{3\zeta(2)}\left(\frac{8X}{\pi}\right)^{-\tau}\frac{(8X/\pi)^{\delta}-(8X/\pi)^{-\delta}}{2\delta\log M} \times\int_0^b M^{-2\tau(1-x)}\Big|Q'(x)+\frac{Q''(x)}{2\delta_1\log M}\Big|^2dx + O(X^{-\tau}M^{-2\tau(1-b)}|\delta_1|^6\log^5 X).$$

Inputing this and (6.13) into (6.2a), we obtain Proposition 2.4.

Acknowledgements. We thank the referee for a thorough reading of the manuscript.

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