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Lan Ma · Sophia K. Vassiliadou

L^p estimates for the Cauchy–Riemann operator on q-convex intersections in $\mathbb{C}^n$

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Abstract. We construct a new solution operator for $\overline{\partial}$ on certain piecewise smooth q-convex intersections. L^p estimates are obtained for the solution operators of $\overline{\partial}$-closed forms on such domains.

Introduction

Great progress has been made recently in understanding the $\overline{\partial}$-Neumann problem on piecewise smooth domains. Henkin–Iordan–Kohn [8], Michel-Shaw [15] obtained subelliptic $\frac{1}{2}$-estimates for the $\overline{\partial}$-Neumann operator on piecewise smooth intersections of strongly pseudoconvex domains. Henkin–Iordan, [7] showed compactness of the $\overline{\partial}$-Neumann operator on bounded pseudoconvex domains D with B-regular boundary (i.e. there exists a continuous function ρ in a neighborhood U of ∂D such that $D \cap U = \{\rho < 0\}, dd^c\rho \geq dd^c|z|^2$ where $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$). Straube [17] obtained subelliptic δ estimates ($\delta < \frac{1}{2}$) for piecewise smooth intersections of finite 1-D'Angelo type domains. The key ingredient in the proof of all of the results above is an exhaustion of the piecewise smooth domain by smooth (or uniformly Lipschitz) strongly pseudoconvex domains on which the $\overline{\partial}$-Neumann operators exist and satisfy uniform L^2 or subelliptic ϵ-estimates.

Much less is known when the domains are not pseudoconvex. Due to its intimate connection to the tangential Cauchy–Riemann operator operator it would be of great interest to understand the $\overline{\partial}$-Neumann operator on more general piecewise smooth domains that arise as piecewise smooth intersections of smooth q-convex domains.

Definition 1. *A bounded smooth domain D in $\mathbb{C}^n$ is called strongly q-convex (resp. weakly q-convex) if there exists a bounded neighborhood W of ∂D and a smooth defining function r of D such that for every $z \in \partial D$ the Levi form of $r|_{\partial D}$ at z has at least $n - q$ positive (resp. non-negative) eigenvalues.*

L. Ma: Mathematisches Institut der Universität Bonn, Beringstrasse 6, 53115 Bonn, Germany. e-mail: malan@math.uni-bonn.de
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S. K. Vassiliadou: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA. e-mail: sophia@math.lsa.umich.edu

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In particular, a strongly 1-convex domain is strongly pseudoconvex.

The category of q-convex domains when $q > 1$ is still rather unexplored. In contrast to the pseudoconvex case, there are major differences between 1-convex and q-convex domains when $q > 1$.

α) The intersection of any two q-convex domains ($q > 1$) is not necessarily q-convex (that will lead to problems in smoothing q-subharmonic functions).

β) There are no L^2-estimates for $\overline{\partial}$ with "good" constants on smooth strongly q-convex domains and there are no unweighted L^2 estimates for $\overline{\partial}$ on smooth weakly q-convex domains.

γ) There are no global holomorphic support functions for smooth strongly q-convex domains that can be used to construct global integral solution operators for $\overline{\partial}$ on such domains.

In this paper we shall study the $\overline{\partial}$ and $\overline{\partial}$-Neumann problem for the following type of piecewise smooth domains:

Definition 2. *A bounded domain Ω in $\mathbb{C}^n$ shall be called a C^3 q-convex intersection if there exists a bounded neighborhood W in $\mathbb{C}^n$ of $\overline{\Omega}$ and a finite number of real C^3 functions $\rho_1, \ldots, \rho_N$ where $n \geq N + 2$ defined on W such that $\Omega = \{z \in W | \rho_1(z) < 0, \ldots, \rho_N(z) < 0\}$ and the following are true:*

i) *For $1 \leq i_1 < i_2 < \cdots < i_\ell \leq N$ the 1-forms $d\rho_{i_1}, \ldots, d\rho_{i_\ell}$ are $\mathbb{R}$-linearly independent on $\bigcap_{j=1}^{j=\ell}\{\rho_{i_j} \leq 0\}$.*

ii) *For $1 \leq i_1 < \cdots < i_\ell \leq N$, for every $z \in \bigcap_{j=1}^{j=\ell}\{\rho_{i_j} \leq 0\}$, if we set $I = (i_1, \ldots, i_\ell)$, there exists a linear subspace T_z^I of $\mathbb{C}^n$ of complex dimension at least $n - q + 1$ such that for $i \in I$ the Levi forms $L\rho_i$ restricted on T_z^I are positive definite.*

Condition ii) was introduced by Grauert [5]. It implies that at every "corner" the Levi forms of the corresponding $\{\rho_i\}$ have their positive eigenvalues along the same directions.

Thiebaut–Leiterer [13] solved the $\overline{\partial}$-problem with Hölder estimates for piecewise smooth intersections of q-convex domains where instead of condition ii) they required that the Levi form of any nontrivial convex combination of $\{\rho_i\}_{i=1}^N$ has at least $n - q + 1$ positive eigenvalues. These type of domains were originally considered by Henkin [6]. However, their solution operators are not suitable for proving L^2(or more generally L^p, $1 \leq p \leq \infty$) estimates. In this paper we construct a different solution operator for $\overline{\partial}$ by means of Berndtsson–Andersson operators with multiple weights. The idea of multiple weights appeared in a paper of Berndtsson [3] on a division and interpolation problem in $\mathbb{C}^n$.

The main results are the following theorems:

Theorem 1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a $C^3 q$-convex intersection. Let $p, q, s \in \mathbb{N}, 1 \leq q \leq n, 1 \leq p \leq \infty, s \geq q$. Given $f \in L^p_{0,s}(\Omega)\, \overline{\partial} f = 0$ in Ω, there exists $u \in L^p_{0,s-1}(\Omega)$ such that $\overline{\partial} u = f$ in Ω. More precisely we have:*

$$\|u\|_{L^p(\Omega)} \leq c_1 \|f\|_{L^p(\Omega)}$$

where the constant c_1 is independent of f.

Under the same assumptions as Theorem 1 we obtain:

Corollary 1. $\overline{\partial} : L^2_{(0,s-1)}(\Omega) \to L^2_{(0,s)}(\Omega)$ *has closed range.*

The above will imply the L^2 existence of the $\overline{\partial}$-Neumann operator on q-convex intersections.

In fact, one can prove more:

Theorem 2. *Let Ω, f, u, p, s, q be as in Theorem 1. Then, there exists a $\nu \in \mathbb{N}^+$ (which depends on the maximal number of nonempty intersections of $\{\rho_i = 0\}_{i=1}^N$) such that*

$$\|u\|_{L^r_{(0,s-1)}(\Omega)} \leq C\|f\|_{L^p_{(0,s)}(\Omega)},$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{\lambda} - 1, \text{ where } 1 \leq \lambda < \frac{2n+2\nu}{2n-1+2\nu}.$$

More precisely,

i) *For any $1 < p < 2n+2\nu$, there exists $c_p(\Omega)$ positive constant such that*

$$\|u\|_{L^q_{(0,s-1)}(\Omega)} \leq c_p(\Omega)\|f\|_{L^p_{(0,s)}(\Omega)}$$

with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2n+2\nu}$.

ii) *For $p \geq 2n+2\nu$, we have $\|u\|_{L^\infty_{(0,s-1)}(\Omega)} \leq A_p(\Omega)\|f\|_{L^p_{(0,s)}(\Omega)}$ for some positive constant $A_p(\Omega)$.*

The paper is organized as follows: In Sect. 1, we present the generalized Berndtsson–Andersson [2] formula with multiple weights. In Sect. 2, we use the B-A formula to obtain homotopy formulas on special subdomains of our q-convex intersections for smooth $(0,s)$ forms with $s \geq q$. To show the L^p estimates – which form Sect. 3 – we have to overcome the difficulties caused by the existence of the characteristic points, at which $\partial\rho_{i_1}(z) \wedge \cdots \wedge \partial\rho_{i_l}(z)$ vanishes for some multi-indices $(i_1, \ldots, i_l)$. To do this we use certain affine transformations with respect to the variables ζ which depend on the point z. The crucial point is that the constants obtained in the estimates do not depend on z. We first solve the local $\overline{\partial}$-problem with good L^p-estimates, then we obtain the global solution to the $\overline{\partial}$-closed form which also satifies an L^p estimate. This is done in Sect. 4. Section 5 is devoted to proving Theorem 2.

1. Generalized Berndtsson–Andersson formula with multiple weights

We shall use the following notation: For $\xi, \eta \in \mathbb{C}^n$, $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n)$ we denote by

$$\langle \xi, \eta \rangle =: \sum_{j=1}^n \xi_j \eta_j, \quad |\xi|^2 =: \langle \xi, \xi \rangle .$$

For any C^1 $\mathbb{C}^n$-valued maps $\xi, \eta : X \to \mathbb{C}^n$ where X is a smooth manifold in $\mathbb{C}^n$ we define the differential forms

$$\omega'(\xi) =: \sum_{j=1}^{n} (-1)^{j-1} \xi_j \wedge \bigwedge_{j \neq i} d\xi_i,$$

$$\omega(\eta) =: d\eta_1 \wedge \cdots \wedge d\eta_n.$$

Let $D \subset \mathbb{R}^n$ be an open set and f be a differential form of degree s on D. Then for $x \in D$, we denote by $\|f(x)\|$ the euclidean length of the vector of the coefficients of $f(x)$ with respect to the canonical coordinates $x_1, \ldots, x_n$ in $\mathbb{R}^n$, that is if

$$f(x) = \sum_{1 \leq i_1 < \cdots < i_s \leq n} f_{i_1 \ldots i_s} dx_{i_1} \wedge \cdots \wedge dx_{i_s}, x \in D$$

then

$$\|f(x)\| =: \Big(\sum_{1 \leq i_1 < \cdots < i_s \leq n} |f_{i_1 \ldots i_s}|^2 \Big)^{\frac{1}{2}}, x \in D$$

$\|f(x)\|$ is called the norm of f at x.

Whenever we use the notation $A \preceq B$ we shall mean that there exists an absolute constant c (independent of z) such that $A \leq cB$.

Let $\Omega = \{z \in W; \rho_1 < 0, \ldots, \rho_N < 0\} \subset\subset \mathbb{C}^n$ be a piecewise smooth bounded domain in $\mathbb{C}^n$ such that the following conditions are satisfied:

α) There exists a C^1 $\mathbb{C}^n$-valued map

$$s =: (s_1, \ldots, s_n) : \overline{\Omega} \times \overline{\Omega} \to \mathbb{C}^n$$

such that for all $L \subset\subset \Omega$, there exist constants $C, c > 0$ such that for all $\zeta \in \overline{\Omega}$, $z \in L$ we have

$$\begin{aligned} &\text{i)} \quad |s_j(\zeta, z)| \leq C|\zeta - z|, \\ &\text{ii)} \quad |\langle s, \zeta - z \rangle| \geq c|\zeta - z|^2. \end{aligned} \tag{*}$$

β) For $i = 1, \ldots, N$ there exist C^1-maps

$$Q^i =: (Q^i_1, \ldots, Q^i_n) : \overline{\Omega} \times \overline{\Omega} \to \mathbb{C}^n.$$

Let $\{G_i\}_{i=1}^{i=N}$ be functions in one variable, holomorphic in a simply connected domain that contains the image of $\overline{\Omega} \times \overline{\Omega}$ under the map $(\zeta, z) \mapsto 1 + \langle Q^i(\zeta, z), z - \zeta \rangle$.

We shall use the same symbols s, Q^i to define the following $(1, 0)$ forms:

$$s = \sum_{j=1}^{n} s_j(\zeta, z) d(\zeta_j - z_j), \quad Q^i =: \sum_{j=1}^{n} Q^i_j(\zeta, z) d(\zeta_j - z_j).$$

We set:

$$K =: -\sum_{a_0+a_1+\cdots+a_N=n-1} \frac{(n-1)!}{a_1!\dots a_N!} G_1^{(a_1)}(1+\langle Q^1, z-\zeta\rangle)\dots G_N^{(a_N)}(1+\langle Q^N, z-\zeta\rangle) \wedge \wedge \frac{s\wedge (ds)^{a_0}\wedge (dQ^1)^{a_1}\wedge\cdots\wedge (dQ^N)^{a_N}}{< s, \zeta - z >^{a_0+1}}$$

$$P =: \sum_{a_1+\cdots+a_N=n} \frac{(n-1)!}{a_1!\dots a_N!} G_1^{(a_1)}(1+\langle Q^1, z-\zeta\rangle)\dots G_N^{(a_N)}(1+\langle Q^N, z-\zeta\rangle) \wedge \wedge (dQ^1)^{a_1}\wedge\cdots\wedge (dQ^N)^{a_N}$$

where $G^{(k)}$ denotes the kth derivative of G.

The assumption on s imply that K is integrable in $\zeta \in \overline{\Omega}$ uniformly for z in a compact set $L \subset \Omega$ and continuous off the diagonal.

By direct calculation we have the following:

Lemma 1.1. *Away from the diagonal Δ of $\mathbb{C}^n \times \mathbb{C}^n$ we have $d_{\zeta,z}K = P$.*

Proposition 1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a piecewise smooth domain that satisfies assumptions α), β) mentioned in the beginning of this section. Let $0 \le p \le n$, $1 \le r \le n$. Let $f \in C^1_{(p,r)}(\overline{\Omega})$. Then we have (in the sense of currents) for $z \in \Omega$:*

$$c_n f(z) = \int_{\zeta\in\partial\Omega} f(\zeta)\wedge K(\zeta,z) + (-1)^{p+r}\left\{\overline{\partial}_z \int_{\zeta\in\Omega} f(\zeta)\wedge K(\zeta,z) - \int_{\zeta\in\Omega} \overline{\partial} f(\zeta)\wedge K(\zeta,z)\right\} - \int_{\zeta\in\Omega} f(\zeta)\wedge P(\zeta,z)$$

where $c_n = (-1)^{\frac{n(n-1)}{2}}(2\pi i)^n$.

Proof. The proof is similar to that of Theorem 1 in Berndtsson–Andersson [2]. □

2. Homotopy formulas for local q-convex intersections

2.1. Preliminaries

Let $D \subset \mathbb{C}^n$ be a domain and ρ be a real C^3 function on D. We denote by $L\rho(\zeta)$ the Levi form of ρ at $\zeta \in \Omega$ and by $F_\rho(.,\zeta)$ the Levi polynomial of ρ at $\zeta \in D$, i.e.

$$L\rho(\zeta)t = \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial\zeta_j\partial\overline{\zeta}_k}(\zeta)t_j\overline{t}_k$$

for $z \in D, t \in \mathbb{C}^n$

$$F_\rho(\zeta, z) =: 2\sum_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial\zeta_j\partial\zeta_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k)$$

for $\zeta \in D, t \in \mathbb{C}^n$. By Taylor's expansion theorem we shall have for $\zeta, z \in \Omega$, ζ, z close to each other

$$\mathrm{Re} F_\rho(\zeta, z) = \rho(\zeta) - \rho(z) + L\rho(\zeta)(\zeta - z) + o(|\zeta - z|^2).$$

Let $\Omega \subset\subset \mathbb{C}^n$ be a q-convex intersection, described by $\Omega = \{z \in W; \rho_1 < 0, \ldots, \rho_N < 0\}$, $\Omega \subset\subset W$. For $1 \le i_1 < \cdots < i_\ell \le N$, $I = (i_1, \ldots, i_\ell)$ we set

$$\Omega_I = \{z \in W; \rho_i < 0 \text{ for } i \in I\} \quad S_I = \{z \in W; \rho_i = 0 \quad \text{for } i \in I\}$$

For $\xi \in S_I$ there exists D_* smoothly bounded strongly pseudoconvex domain described by $D_* = \{z \in W; \rho_* < 0\}$ such that ∂D_* intersects real transversally $\{z \in W; \rho_{i_1} = 0\}, \ldots, \{z \in W; \rho_{i_\ell} = 0\}$ and $\xi \in D_*$.

We set $\widetilde{I} =: (i_1, \ldots, i_\ell, *)$ and we define $\Omega_{\widetilde{I}} =: \{z \in W; \rho_j < 0 \text{ for } j \in \widetilde{I}\}$. $\Omega_{\widetilde{I}}$ will still be a q-convex intersection and shall be called a local q-convex intersection.

2.2. *Construction of the local kernels*

Since Ω_I is a q-convex intersection, for every $z \in \Omega_I$ there exists by definition an $(n - q + 1)$-linear subspace T_z^I of $\mathbb{C}^n$ such that the Levi forms $L\rho_i$ for $i \in I$ restricted on T_z^I are positive definite. Let $(T_z^I)^\perp$ denote its orthogonal complement in $\mathbb{C}^n$. Then $\dim_{\mathbb{C}}(T_z^I)^\perp = q - 1$. Let $\Theta_z^I : \mathbb{C}^n \to (T_z^I)^\perp$ be the orthogonal projection from $\mathbb{C}^n$ to $(T_z^I)^\perp$. Then Θ_z^I can be described by an $(n \times n)$ matrix $\Theta_z^I = ((\Theta_z^I)_{kj})_{n\times n}$.

Since Ω_I is a q-convex intersection if we consider the Levi polynomials of the defining functions ρ_i we shall have $\zeta, z \in \Omega$, ζ, z close to each other

$$\mathrm{Re}\, F_{\rho_i}(\zeta, z) \ge \rho_i(\zeta) - \rho_i(z) + c|\zeta - z|^2 - C\left|\Theta_z^I(\zeta - z)\right|^2.$$

For $k = 1, \ldots, \ell$, $j = 1, \ldots, n$ we define

$$w_j^{i_k}(\zeta, z) =: \frac{\partial\rho_{i_k}}{\partial\zeta_j}(\zeta) - \sum_{\mu=1}^{n} \frac{\partial^2\rho_{i_k}}{\partial\zeta_\mu\partial\zeta_j}(\zeta)(\zeta_\mu - z_\mu) + C\overline{\sum_{\mu=1}^{n} (\Theta_z^I)_{\mu j}(\zeta_\mu - z_\mu)}.$$

We also set

$$w_j^*(\zeta, z) =: \frac{\partial\rho_*}{\partial\zeta_j}(\zeta) H_j(\zeta, z) + O(|\zeta - z|) = \Pi_j(\zeta, z),$$

where $\{\Pi_j(\zeta, z)\}$ are holomorphic in z. The existence of H_j's follows from Heffer's decomposition theorem for smooth strongly pseudoconvex domains. H_j's are C^2 functions such that $0 < A_0 \le |H_j| \le A_1 < \infty$ for some positive constants A_0, A_1.

We set for $\nu = i_1, \dots, i_\ell, *$

$$\Phi_\nu(\zeta, z) = \langle w^\nu(\zeta, z), \zeta - z\rangle, \quad F_\nu(\zeta, z) =: \Phi_\nu(\zeta, z) - \rho_\nu(\zeta).$$

Let $\epsilon > 0$. We define

$$Q_\nu =: \frac{\sum_{j=1}^n w_j^\nu(\zeta, z) d(\zeta_j - z_j)}{F_\nu(\zeta, z)}, \qquad Q_\nu^\epsilon =: \frac{\sum_{j=1}^n w_j^\nu(\zeta, z) d(\zeta_j - z_j)}{F_\nu(\zeta, z) + \epsilon},$$

$$s(\zeta, z) =: \overline{\zeta} - \overline{z} = (\overline{\zeta_1 - z_1}, \dots, \overline{\zeta_n - z_n}), \quad s =: \sum_{j=1}^n s_j(\zeta, z) d(\zeta_j - z_j).$$

Then we have $1 + \langle Q_\nu^\epsilon(\zeta, z), z - \zeta\rangle = \dfrac{-\rho_\nu(\zeta) + \epsilon}{F_\nu(\zeta, z) + \epsilon}$.

Let $G_{i_1}(\alpha) = \cdots = G_{i_\ell}(\alpha) = G_*(\alpha) = \alpha^n$ where $n \in \mathbb{N}$.

It follows easily from the Taylor expansion of ρ_i's that we have the following estimates:

Lemma 2.1. *For $(\zeta, z) \in \overline{\Omega} \times \overline{\Omega}$ close to each other, $\nu = 1, \dots, \ell, *$ we have:*

$$2ReF_\nu(\zeta, z) \geq -\rho_\nu(\zeta) - \rho_\nu(z) + b|\zeta - z|^2$$

where b is a positive constant.

Remark. The singularities of $\{F_\nu\}_{\nu = i_1, \dots i_\ell, *}$ appear only when $\rho_\nu(\zeta) = 0 = \rho_\nu(z)$ and $\zeta = z$.

We are ready now to define the kernels K, P for the local q-convex intersection $\Omega_{\widetilde{I}}$.

Without loss of generality we shall assume $\widetilde{I} = (1, 2, \dots, \ell + 1)$. Then we define for $(\zeta, z) \in \overline{\Omega} \times \overline{\Omega}$ in an analogous way as in Sect. 1:

$$\begin{aligned} K_{\widetilde{I}}^\epsilon = - \sum_{a_0 + a_1 + \cdots + a_{\ell+1} = n-1} & \frac{(n-1)!}{a_1! \dots a_{\ell+1}!} G_1^{(a_1)}(1 + \langle Q_1^\epsilon, z - \zeta\rangle) \cdots \\ & \cdot G_{\ell+1}^{(a_{\ell+1})}(1 + \langle Q_{\ell+1}^\epsilon, z - \zeta\rangle) \\ & \wedge \frac{s \wedge (ds)^{a_0} \wedge (dQ_1^\epsilon)^{a_1} \wedge \cdots \wedge (dQ_{\ell+1}^\epsilon)^{a_{\ell+1}}}{(\epsilon + < s, \zeta - z >)^{a_0 + 1}}, \end{aligned}$$

$$\begin{aligned} P_{\widetilde{I}}^\epsilon =: \sum_{a_1 + \cdots + a_{\ell+1} = n} & \frac{(n-1)!}{a_1! \dots a_{\ell+1}!} G_1^{(a_1)}(1 + \langle Q_1^\epsilon, z - \zeta\rangle) \cdots \\ & \cdot G_{\ell+1}^{(a_{\ell+1})}(1 + \langle Q_{\ell+1}^\epsilon, z - \zeta\rangle) \\ & \wedge (dQ_1^\epsilon)^{a_1} \wedge \cdots \wedge (dQ_{\ell+1}^\epsilon)^{a_{\ell+1}}. \end{aligned}$$

From Proposition 1, of section 1, we have for every $f \in C^1_{(p,r)}(\overline{\Omega_{\widetilde{I}}})$ and $z \in \Omega_{\widetilde{I}}$ (in the sense of currents):

$$\begin{aligned} c_n f(z) = &\int_{\zeta \in \partial\Omega_{\widetilde{I}}} f(\zeta) \wedge K^{\epsilon}_{\widetilde{I}}(\zeta, z) + (-1)^{p+r}\overline{\partial}_z \int_{\zeta \in \Omega_{\widetilde{I}}} f(\zeta) \wedge K^{\epsilon}_{\widetilde{I}}(\zeta, z) \\ &- (-1)^{p+r} \int_{\zeta \in \Omega_{\widetilde{I}}} \overline{\partial} f(\zeta) \wedge K^{\epsilon}_{\widetilde{I}}(\zeta, z) \\ &- \int_{\zeta \in \Omega_{\widetilde{I}}} f(\zeta) \wedge P^{\epsilon}_{\widetilde{I}}(\zeta, z) \end{aligned}$$

where $c_n = (-1)^{\frac{n(n-1)}{2}} (2\pi i)^n$.

Remarks. i) The coefficients of $K^{\epsilon}_{\widetilde{I}}(\zeta, z) \big|_{\partial\Omega_{\widetilde{I}} \times \Omega_{\widetilde{I}}}$ are of type $O(\epsilon)$ and therefore the boundary integrals will vanish as $\epsilon \to 0$.

ii) Taking into account condition ii) in the definition of a q-convex intersection, we see that $P^{\epsilon}_{\widetilde{I}}$ will have at most $q - 1 d\overline{z}$'s. Hence if we examine the part of $P^{\epsilon}_{\widetilde{I}}$ of bidegree $(0, s)$ in z with $s \geq q$ we obtain for $f \in C^1_{(0,s)}(\overline{\Omega_{\widetilde{I}}})$, $z \in \Omega_{\widetilde{I}}$

$$\int_{\zeta \in \Omega_{\widetilde{I}}} f(\zeta) \wedge P^{\epsilon}_{\widetilde{I}}(\zeta, z) = 0.$$

Proposition 2. *Let $\Omega_{\widetilde{I}}$ be a local q-convex intersection defined as in the introduction of Sect. 2. Then for all $f \in C^1_{(0,s)}(\overline{\Omega_{\widetilde{I}}})$, $s \geq q$ we have for $z \in \Omega_{\widetilde{I}}$ (in the sense of currents):*

$$f(z) = c_{n,s}\left\{\overline{\partial}_z \int_{\zeta \in \Omega_{\widetilde{I}}} f(\zeta) \wedge K_{\widetilde{I}}(\zeta, z) - \int_{\zeta \in \Omega_{\widetilde{I}}} \overline{\partial} f(\zeta) \wedge K_{\widetilde{I}}(\zeta, z)\right\}$$

where $c_{n,s}$ is a positive constant.

Proof. It follows from Proposition 1, the remarks i), ii) and the fact that the kernels $\{K_{\Omega_{\widetilde{I}}}\}$ are actually absolutely integrable kernels in $\zeta \in \overline{\Omega_{\widetilde{I}}}$ uniformly for $z \in L \subset\subset \Omega_{\widetilde{I}}$. □

Corollary 2. *Let $\Omega_{\widetilde{I}}$ be as in Proposition 2. Given any $f \in C^1_{(0,s)}(\overline{\Omega_{\widetilde{I}}})$, $s \geq q$, such that $\overline{\partial} f = 0$ in $\Omega_{\widetilde{I}}$ there exists $u \in C^0_{(0,s-1)}(\overline{\Omega_{\widetilde{I}}})$ such that $\overline{\partial} u = f$ in $\Omega_{\widetilde{I}}$. More precisely we have for $z \in \Omega_{\widetilde{I}}$ $u(z) = \frac{1}{c_{n,r}} \int_{\zeta \in \Omega_{\widetilde{I}}} f(\zeta) \wedge K_{\widetilde{I}}(\zeta, z)$.*

3. L^p estimates

This section is devoted to the proof of the following Proposition:

Proposition 3. *Let $\Omega_{\widetilde{I}}$ be our local q-convex intersection, p, q, s as in the Theorem 1. Given $f \in L^p_{(0,s)}(\Omega_{\widetilde{I}})$ such that $\overline{\partial} f = 0$ in $\Omega_{\widetilde{I}}$ there exists $u \in L^p_{(0,s-1)}$ $(\Omega_{\widetilde{I}})$ $\overline{\partial} u = f$ in $\Omega_{\widetilde{I}}$. More precisely we have*

$$\|u\|_{L^p(\Omega_{\widetilde{I}})} \leq c\|f\|_{L^p(\Omega_{\widetilde{I}})}$$

where c is a constant independent of f and small C^3 perturbations of $\partial\Omega_{\widetilde{I}}$.

It will be sufficient to prove Proposition 3 for $f \in C^1_{(0,s)}(\overline{\Omega}_{\widetilde{I}})$, since the general case will follow by a standard regularization argument. To prove the L^p-estimates we shall use the following lemma:

Lemma 3.1. *Let $(X, d\mu)$, $(Y, d\nu)$ be two measured spaces, $H(x, y)$ be a kernel defined on $X \times Y$ such that*

$$\int_X \big|H(x, y)\big| d\mu(x) \leq C, \quad y \in Y$$
$$\int_Y \big|H(x, y)\big| d\nu(y) \leq C, \quad x \in X$$

for some positive constant C.

Let $Tf(y) =: \int_{x\in X} f(x)H(x, y)d\mu(x)$. Then for all p, $1 \leq p \leq \infty$, there exists $A_p > 0$ (independent of f) such that

$$\big\|Tf\big\|_{L^p(Y)} \leq A_p \big\|f\|_{L^p(X)}.$$

Proof. The reader may look at Appendix C in Range[16]. □

Remark. In what follows, by $\mathcal{E}^j(\zeta, z)$ we shall denote a double differential form in (ζ, z) such that its coefficients are of $O(|\zeta - z|^j)$ (i.e.. there exists an "absolute" constant $c > 0$ such that $\big|\mathcal{E}^j(\zeta, z)\big| \leq c|\zeta - z|^j$.

We wish to show that our kernel $K_{\widetilde{I}}(\zeta, z)$ satisfies the following two inequalities:

$$\int_{\zeta\in\Omega_{\widetilde{I}}} \big|K_{\widetilde{I}}(\zeta, z)\big| dV(\zeta) \leq C, \quad z \in \Omega_{\widetilde{I}},$$
$$\int_{z\in\Omega_{\widetilde{I}}} \big|K_{\widetilde{I}}(\zeta, z)\big| dV(z) \leq C, \quad \zeta \in \Omega_{\widetilde{I}}.$$

Without loss of generality we shall assume that $\widetilde{I} = (1, \dots, \ell+1)$. Then our kernel $K_{\widetilde{I}}$ can be written as finite sum of terms of the following form

$$\frac{\mathcal{E}^1}{|\zeta - z|^{2(n-a_1-\cdots-a_{\ell+1})}} \bigwedge_{j=1}^{\ell+1} \left(\frac{\mathcal{E}^0}{F_j^{a_j}} + a_j \frac{X_j}{F_j^{a_j+1}} \right)$$

where

$$\begin{cases} X_j^\zeta = \mathcal{E}^2 + \mathcal{E}^1 \partial_\zeta \rho_j(\zeta) - \mathcal{E}^1 \overline{\partial}_\zeta \rho_j(\zeta) + \partial_\zeta \rho_j(\zeta) \wedge \overline{\partial}_\zeta \rho_j(\zeta) \\ a_1 + \cdots + a_{\ell+1} \leq n - 1. \end{cases}$$

We shall denote by

$$X_j^z =: \mathcal{E}^2 + \mathcal{E}^1 \partial \rho_j(z) - \mathcal{E}^1 \overline{\partial} \rho_j(z) + \partial_\zeta \rho_j(z) \wedge \overline{\partial}_\zeta \rho_j(z). \tag{*}$$

Using the fact that $\partial_\zeta \rho_j(\zeta) - \partial_\zeta \rho_j(z) = \mathcal{E}^1$ for (ζ, z) in some convex neighborhood of $\overline{\Omega_{\widetilde{I}}} \times \overline{\Omega_{\widetilde{I}}}$ we can replace X_j^ζ in the above expression with X_j^z.

Thus using the multilinearity of the wedge product we shall obtain that $K_{\widetilde{I}}$ shall be a finite sum of terms of the form

$$\frac{\mathcal{E}^1}{|\zeta - z|^{2(n-a_1-\cdots-a_{\ell+1})}}\left\{\frac{\mathcal{E}^0}{F_1^{a_1}\ldots F_{a_{\ell+1}}^{\ell+1}} + \sum_{j=1}^{\ell+1} a_j \frac{X_j^z}{F_1^{a_1}\ldots F_j^{a_j+1}\ldots F_{\ell+1}^{a_{\ell+1}}}\right.$$
$$+ \sum_{2\le k\le \ell}\ \sum_{\substack{|J|=k\\ 1\le j_i<\cdots<j_k\le \ell+1}} a_{j_1}\ldots a_{j_k}\frac{X_{j_1}^z\wedge\cdots\wedge X_{j_k}^z}{\prod_{j\in J}F_j^{a_j+1}\prod_{j\notin J}F_j^{a_j}}$$
$$\left. + a_1\ldots a_{\ell+1}\frac{X_1^z\wedge\cdots\wedge X_{\ell+1}^z}{\prod_{j=1}^{\ell+1}F_j^{a_j+1}}\right\}$$

where $a_1+\cdots+a_{\ell+1}\le n-1$.

Let us consider an arbitrary term in the above sum, for example

$$\frac{\mathcal{E}^1 X_{j_1}^z\wedge\cdots\wedge X_{j_k}^z}{|\zeta - z|^{2(n-a_1-\cdots-a_{\ell+1})}\prod_{j\in J}F_j^{a_j+1}\prod_{j\notin J}F_j^{a_j}}$$

where $1\le k\le \ell+1$, $J=(j_1,\ldots,j_k)\subset\{1,\ldots,\ell+1\}$.

Replacing X_j^z by the right-hand side of (*) and using once again the multilinearity of the wedge product, we obtain that

$$X_{j_1}^z\wedge\cdots\wedge X_{j_k}^z = \sum_{i_1,\ldots,i_s)\subset\{j_1,\overline{j}_1,\ldots,j_k,\overline{j}_k\}}\mathcal{E}^{2k-s}\omega_{i_1}\wedge\cdots\wedge\omega_{i_s}$$

where

$$\text{if}\quad\begin{cases} i_\nu\in\{j_1,\ldots,j_k\}, & \omega_{i_\nu}=\partial_\zeta\rho_{i_\nu}(z)\\ i_\nu\in\{\overline{j}_1,\ldots,\overline{j}_k\}, & \omega_{i_\nu}=\overline{\partial}_\zeta\rho_{i_\nu}(z).\end{cases}$$

Therefore the arbitrary term of the kernel $K_{\widetilde{I}}$ can be written as a finite sum of terms of the following form

$$\frac{\mathcal{E}^1}{|\zeta - z|^{2(n-a_1-\cdots-a_{\ell+1})}}\frac{\mathcal{E}^{2k-s}\omega_{i_1}\wedge\cdots\wedge\omega_{i_s}}{\prod_{j\in\{j_1,\ldots,j_k\}}F_j^{a_j+1}\prod_{j\in\{1,\ldots,\ell+1\}\setminus J}F_j^{a_j}}$$

where $\{j_1,\ldots,j_k\}\subset\{1,\ldots,\ell+1\}$ any multiindex with strictly increasing components and $\{i_1,\ldots,i_s\}\subset\{j_1,\overline{j}_1,\ldots,j_k,\overline{j}_k\}$ and we use the same convention for ω_{i_ν} as before.

Then we wish to estimate for $z\in\Omega_{\widetilde{I}}$ and $\delta>0$ small, the following integral

$$\int_{\zeta\in\Omega_{\widetilde{I}}\cap B(z,\delta)}\frac{\mathcal{E}^1\mathcal{E}^{2k-s}\|\omega_{i_1}(z)\wedge\cdots\wedge\omega_{i_s}(z)\|\wedge V_{2n}(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{\ell+1}a_i)}\prod_{j=1}^{k}F_j^{a_j+1}\prod_{j\notin J}F_j^{a_j}}. \tag{3.1}$$

Without loss of generality we shall examine the case where $J = \{1, \ldots k\}$, $\{i_1, \ldots, i_s\} = \{1, \ldots, s\}, s \leq k$. The remaining cases will follow in a similar manner.

We shall use the following lemma:

Lemma 3.2. *If we set $F_j^*(\zeta, z) = F_j(z, \zeta)$ then there exist positive constants c_1, c_2 independent of (ζ, z) such that for $(\zeta, z) \in \Omega_{\tilde{I}} \times \Omega_{\tilde{I}}$ sufficiently close to each other we have*

$$c_1|F_j(\zeta, z)| \leq |F_j^*(\zeta, z)| \leq c_2|F_j(\zeta, z)|.$$

Proof. It is based on the following observations:

$$F_j^*(\zeta, z) + F_j(\zeta, z) = \mathcal{E}^2 - \rho_j(\zeta) - \rho(z),$$
$$Re F_j(\zeta, z) \geq (-\rho_j(\zeta) - \rho_j(z) + |\zeta - z|^2)$$

for $\zeta, z \in \Omega_{\tilde{I}}$ and close to each other. □

Remarks. 1. Lemma 3.2 allows us to replace $|F_j|$ in the above estimates by $|F_j^*|$ which is bounded from below in its turn by

$$|F_j^*(\zeta, z)| \geq (|\langle \partial\rho_j(z), \zeta - z\rangle| + |\rho_j(\zeta)| + |\rho_j(z)| + |\zeta - z|^2)$$

for $\zeta, z \in \Omega_{\tilde{I}}$ sufficiently close to each other.

2. There is a relation between the dimension of the ambient space (in our case n) and the number N of intersecting domains. More precisely we have $n \geq N + 2$.

Lemma 3.3. *Let $n, d \geq 1$, be integers. Then there exists a constant $C = C(n, d) < \infty$ such that the following statement is true: Let $p_1(x), \ldots p_s(x)$, $(1 \leq s \leq n)$ be real-valued polynomials of degree $\leq d$, defined on $\mathbb{R}^n$, let $\epsilon, \mu, \nu, \delta, \gamma \geq 0$ and let $D \subset \mathbb{R}^n$. Then,*

$$\int_{x \in D} \frac{\|dp_1(x) \wedge \cdots \wedge dp_s(x)\| dx_1 \ldots dx_{2n}}{[\epsilon + \sum_{j=1}^{s} |p_j(x)| + \gamma(|x| + \sum_{j=1}^{s} |p_j(x)|^{\mu})]^{\delta}[|x| + \sum_{j=1}^{s} |p_j(x)|]^{\nu}}$$
$$\leq C \int_{t \in T(D)} \frac{dt_1 \ldots dt_n}{(\epsilon + \sum_{j=1}^{s} |t_j| + \gamma|t|)^{\mu}|t|^{\nu}} \qquad (**)$$

where

$$T(D) =: \{(|p_1(x)|, \ldots, |p_s(x)|)\ x \in D\} \quad \text{if } s = n,$$
$$=: \bigcup_{1 \leq i_1 < \ldots i_{2n-s} \leq n} \{(|p_1(x)|, \ldots, |p_s(x)|, |x_{i_1}|, \ldots, |x_{i_{n-s}}|) : x \in D\}$$
$$\text{if } s < n.$$

Proof. Lemma 3.3 is Proposition 4 in Appendix 1, of [9]. The key idea of its proof shall be used repeatedly in our estimates. Therefore we shall present the argument in detail. From the definition of $\|dp_1 \wedge \cdots \wedge dp_s\|$ it follows that

$$\|dp_1(x) \wedge \wedge dp_s(x)\| \leq \max_{1 \leq j_1 < \cdots < j_s \leq n} |[\det(\frac{\partial p_\ell(x)}{\partial x_{jk}})_{k,\ell=1}^s]|.$$

If $1 \leq j_1 < \cdots < j_s \leq n$ and $1 \leq i_1 < \cdots < i_{n-s} \leq n$ such that $\{j_1, \ldots, j_s\} \cup \{i_1, \ldots, i_{n-s}\} = \{1, \ldots, n\}$ then

$$|dp_1(x) \wedge \ldots dp_s(x) \wedge dx_{i_1} \ldots dx_{i_{n-s}}| = |\det[(\frac{\partial p_\ell(x)}{\partial x_{jk}})_{k,\ell=1}^s] dx_1 \wedge \cdots \wedge dx_n|.$$

This implies that

$$\|dp_1(x) \wedge \cdots \wedge dp_s(x)\| dV(x) \leq \max_{1 \leq i_1 < \cdots < i_{n-s}} |dp_1(x) \wedge \ldots dp_s(x) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{n-s}}|.$$

Therefore the left-hand side of the above integral can be estimated by the maximum over all collections $1 \leq i_1 < \cdots < i_{n-s} \leq n$ of

$$\int_{x \in D} \frac{dp_1(x) \wedge \cdots \wedge dp_s(x) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{n-s}}|}{[\epsilon + \sum_{j=1}^s |p_j(x)| + \gamma(\sum_{j=1}^s |p_j(x)| + \sum_{k=1}^{n-s} |x_{i_k}|)^\mu]^\delta [\sum_{j=1}^s |p_j(x)| + \sum_{k=1}^{n-s} |x_{i_k}|]^\nu}.$$

For every collection $1 \leq i_1 < \cdots < i_{n-s} \leq n$, the integrand of the last integral is the pull-back of the integrand on the right-hand side of (**) with respect to the map:

$$a_{i_1 \ldots i_{n-s}}(x) = (|p_1(x)|, \ldots, |p_s(x)|, |x_{i_1}|, \ldots |x_{i_{n-s}}|).$$

It is a corollary of Bezout's theorem that for every $y \in \mathbb{R}^n$, there are no more than d^s points $x \in \mathbb{R}^n$ such that $(p_1(x), \ldots, p_s(x), x_{i_1}, \ldots, x_{i_{n-s}}) = y$. Consequently, for every $y \in \mathbb{R}^n$ there are no more than $2^n d^s$ points $x \in \mathbb{R}^n$ such that $a_{i_1 \ldots i_{n-s}}(x) = y$. $\square$

So we need to estimate for $z \in \Omega_{\tilde{I}}$, $\delta > 0$

$$\int_{\zeta \in \Omega_{\tilde{I}} \cap B(z,\delta)} \frac{|\zeta - z|^{1+2k-s} \|\partial_\zeta \rho_1(z) \wedge \cdots \wedge \partial_\zeta \rho_s(z)\| dV(\zeta)}{|\zeta - z|^{2(n - \sum_{i=1}^{\ell+1} a_i)} \prod_{j=1}^k |F_j^*|^{a_j+1} \prod_{j=k+1}^{\ell+1} |F_j^*|^{a_j}}. \tag{3.2}$$

We shall have

$$(3.2) \leq \int_{\zeta\in\Omega_{\tilde{I}}\cap B(z,\delta)} \frac{|\zeta - z|^{1+2k-s}\|\partial_\zeta\rho_1(z)\wedge\cdots\wedge\partial_\zeta\rho_s(z)\|dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{\ell+1}a_i)}\prod_{j=1}^{s}|F_j^*|^{a_j+1}|\zeta - z|^{2(\sum_{i=s+1}^{\ell+1}a_i+k-s)}}$$

$$\leq \int_{\zeta\in\Omega_{\tilde{I}}\cap B(z,\delta)} \frac{|\zeta - z|^{s+1}\|\partial_\zeta\rho_1(z)\wedge\cdots\wedge\partial_\zeta\rho_s(z)\|dV(\zeta)}{|\zeta - z|^{2(n-\sum_{j=1}^{s}a_i)}\prod_{j=1}^{s}|F_j^*|^{a_j+1}}$$

$$\leq \int_{\zeta\in\Omega_{\tilde{I}}\cap B(z,\delta)} \frac{\|\partial_\zeta\rho_1(z)\wedge\cdots\wedge\partial_\zeta\rho_s(z)\|dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{s}a_i)-1}\prod_{j=1}^{s}|F_j^*|^{a_j+\frac{1}{2}}}.$$

Using an argument similar to the proof of Lemma 3.3 we have

$$(3.2) \leq \int \frac{\|\partial_\zeta\rho_1(z)\wedge\cdots\wedge\partial_\zeta\rho_s(z)\|dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{s}a_i)-1}\prod_{j=1}^{s}(|<\partial_\zeta\rho_j(z),\zeta - z>| + |\zeta - z|^2)^{a_j+\frac{1}{2}}}.$$

We set $\zeta_j - z_j = x_{2j-1} + ix_{2j}$ for $j = 1,\ldots,n$.

We define for $k = 1,\ldots,\ell+1$

$$w_k(\zeta,z) =: \left\langle\partial_\zeta\rho_k(z),\zeta - z\right\rangle = \mathrm{Re}w_k(\zeta,z) + i\,\mathrm{Im}w_k(\zeta,z).$$

Then,

$$d_\zeta w_k(\zeta,z) = \partial_\zeta\rho_k(z) = d_\zeta\mathrm{Re}w_k(\zeta,z) + id_\zeta\mathrm{Im}w_k(\zeta,z).$$

Thus

$$\begin{aligned}
&\partial_\zeta\rho_1(z)\wedge\cdots\wedge\partial_\zeta\rho_s(z)\\
&= (d_\zeta\mathrm{Re}w_1(\zeta,z) + id_\zeta\mathrm{Im}w_1(\zeta,z))\wedge\cdots\wedge(d_\zeta\mathrm{Re}w_s(\zeta,z) + id_\zeta\mathrm{Im}w_s(\zeta,z))\\
&\qquad = \sum_{0\leq\tau\leq s} c_\tau d_\zeta\mathrm{Re}w_{j_1}(\zeta,z)\wedge\cdots\wedge d_\zeta\mathrm{Re}w_{j_\tau}(\zeta,z)\\
&\qquad\qquad \wedge d_\zeta\mathrm{Im}w_{j_{\tau+1}}(\zeta,z)\wedge\cdots\wedge d_\zeta\mathrm{Im}w_{j_s}(\zeta,z)
\end{aligned}$$

where $c_\tau\in\mathbb{C}$ are absolute constants, $\{j_1,\ldots,j_\tau\}\subset\{1,\ldots,s\}$, $\{j_{\tau+1},\ldots,j_s\}\subset\{1,\ldots,s\}\setminus\{j_1,\ldots,j_\tau\}$.

Hence the above integral can be split into finite sum of terms of the form (3.3)

$$\int\frac{\|d_\zeta\mathrm{Re}w_{j_1}(\zeta,z)\wedge\ldots\wedge d_\zeta\mathrm{Re}w_{j_\tau}(\zeta,z)\wedge d_\zeta\mathrm{Im}w_{j_{\tau+1}}(\zeta,z)\wedge\ldots\wedge d_\zeta w_{j_s}(\zeta,z)\|dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{s}a_i)-1}\prod_{j=1}^{s}(|\mathrm{Re}w_j(\zeta,z)| + |\mathrm{Im}w_j(\zeta,z)| + |\zeta - z|^2)^{a_j+\frac{1}{2}}}.$$

Using the real coordinates $\{x_j\}_{j=1}^{2n}$ we can write

$$\mathrm{Re}w_k(\zeta, z) = \frac{1}{2}\sum_{j=1}^{2n}\left(\frac{\partial\rho_k(0)}{\partial x_{2j-1}}x_{2j-1} + \frac{\partial\rho_k(0)}{\partial x_{2j}}x_{2j}\right) =: p_k(x, 0),$$

$$\mathrm{Im}w_k(\zeta, z) = \frac{1}{2}\sum_{j=1}^{2n}\left(\frac{\partial\rho_k(0)}{\partial x_{2j-1}}x_{2j} - \frac{\partial\rho_k(0)}{\partial x_{2j}}x_{2j-1}\right) =: q_k(x, 0)$$

where $p_k(x, 0), q_k(x, 0)$ are real-valued polynomials in x of degree $d = 1$.

Without loss of generality we shall estimate the case where $\tau = s$, $\{j_1, \ldots, j_s\} = \{1, \ldots, s\}$. The other cases will follow similarly. Hence we need to estimate

$$\int_{|x|<\delta} \frac{\|d_x p_1(x, 0) \wedge \cdots \wedge d_x p_s(x, 0)\| dV_{2n}(x)}{|x|^{2(n-\sum_{i=1}^{s} a_i)-1} \prod_{j=1}^{s} (|p_j(x, 0)| + |x|^2)^{a_j+\frac{1}{2}}}. \tag{3.4}$$

Arguing in a similar manner as in Lemma 3.3 the last integral can be majorized by

$$C \int_{t\in T(B(0,\delta))} \frac{dt_1 \ldots dt_s dV_{2n-s}(t)}{|t|^{2(n-\sum_{i=1}^{s} a_i)-1} \prod_{j=1}^{s} (|t_j| + |t|^2)^{a_j+\frac{1}{2}}}, \tag{3.5}$$

where $T(B(0, \delta)) = \bigcup_{1\le i_1<\cdots<i_{2n-s}\le 2n}\{(|p_1(x, 0)|, \ldots, |p_s(x, 0)|, |x_{i_1}|, \ldots |x_{i_s}|) : |x| < \delta\}$ if $s < 2n$. In our case of course, $s \le k \le \ell + 1 \le N + 1 \le n$.

It is not hard to check that $T(B(0, \delta))$ is a relatively compact, open neighborhood of 0.

Hence, if we set $r^2 = t_{s+1}^2 + \cdots + t_{2n}^2$, (3.5) can be majorized by

$$\int \frac{dt_1 \ldots dt_s r^{2n-s-1} dr}{r^{2(n-\sum_{i=1}^{s} a_i)-1} \prod_{j=1}^{s} (|t_j| + r^2)^{a_j+\frac{1}{2}}} \le \int \frac{r^{2n-s-1} dr}{r^{2(n-\sum_{i=1}^{s} a_i)-1} r^{2(\sum_{i=1}^{s} a_i - \frac{s}{2})}}$$

$$\int \frac{r^{2n-s-1} dr}{r^{2n-s-1}} \le \int dr \le c_1\delta,$$

where c_1 is a positive constant that depends on $\max\{\|\rho_k\|_{C^3}\}$.

4. Globalization

This section is devoted to the proof of the main theorem.

In his thesis [11], N. Kerzman, developed a method to obtain global solvability and regularity results for the $\overline{\partial}$-operator, in the L^p and Hölder category for smoothly bounded strongly pseudoconvex domains, once he had resolved the local $\overline{\partial}$-problem with good estimates. The method usually involves 3 steps, which for our case can be summarized as follows:

Step I. We enlarge our domain Ω, slightly into Ω_1 and "extend" f into f_1 such that

i) f_1 is defined in Ω_1, is $\overline{\partial}$ closed there and $f_1 = f - \overline{\partial}\psi$ in Ω.
ii) $\|f_1\|_{L^p(\Omega_1)} \leq C\|f\|_{L^p(\Omega)}$.
iii) $\|\psi\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$.

Step II. We try to fit into Ω_1, a strongly q-convex C^3 domain $\hat{\Omega}$ such that

$$\Omega \subset\subset \hat{\Omega} \subset\subset \Omega_1.$$

Step III. We apply the global solvability and regularity results for the $\overline{\partial}$-problem on strongly q-convex domains with C^3 boundary in [14] in conjunction with Step I to obtain the global solution and the estimates mentioned in the main theorem.

Step I is based on Proposition 4. Let $\xi \in \partial\Omega$. Then there exists I_ξ a multiindex of maximal length such that $\xi \in S_{I_\xi}$. Let $\Omega_{\tilde{I}_\xi}$ denote the local q-convex intersection on which we can solve $\overline{\partial}$ with L^p estimates. We may assume $\Omega_{\tilde{I}_\xi} = \{\rho^\xi_{i_1}, \dots \rho^\xi_{i_\ell}, \rho^\xi_* < 0\}$. Then

$$\partial\Omega \subset\subset \bigcup_{\xi \in \partial\Omega} \{\rho^\xi_* < 0\}.$$

Since $\partial\Omega$ is compact there will exist finitely many $\{\xi_i\}_{i=1}^M$ such that $\partial\Omega \subset\subset \bigcup_{i=1}^M \{\rho^{\xi_i}_* < 0\}$. Let $\theta_i \in C_0^\infty(\{\rho^{\xi_i}_* + \epsilon^{\xi_i} < 0\}), \epsilon^{\xi_i} > 0$ sufficiently small $0 \leq \theta_i \leq 1, \sum_{i=1}^M \theta_i = 1$ in a neighborhood $V_{\partial\Omega}$. We choose $V'_{\partial\Omega} \subset\subset V_{\partial\Omega} \subset\subset W$. We enlarge successively our q-convex C^3 intersection in the following way: For $\delta > 0$ sufficiently small to be chosen appropriately later on, we define

$$\Omega_0^\delta =: \Omega$$

$$\Omega_i^\delta =: \{z \in \Omega \cup V'_{\partial\Omega};\, \rho_1 < \delta \sum_{k=1}^{i} \theta_k, \dots \rho_N < \delta \sum_{k=1}^{i} \theta_k\} \text{ for } i = 1, \dots M.$$

Claim. Given $f_i \in L^p_{(0,r)}(\Omega_i^\delta)$ $1 \leq p \leq \infty$ such that $\overline{\partial} f_i = 0$ in Ω_i^δ for $0 \leq i \leq M-1$ there exist f_{i+1}, ψ_i such that the following are true;

$\alpha)$ $f_{i+1} \in L^p_{(0,r)}(\Omega^\delta_{i+1}), \overline{\partial} f_{i+1} = 0$ in Ω^δ_{i+1}.
$\beta)$ $f_{i+1} - f_i = \overline{\partial}\psi_i$ in Ω_i^δ.
$\gamma)$
$$\|f_{i+1}\|_{L^p(\Omega^\delta_{i+1})} \leq c\|f_i\|_{L^p(\Omega^\delta_i)}$$
$$\|\psi_i\|_{L^p(\Omega^\delta_i)} \leq c\|f_i\|_{L^p(\Omega^\delta_i)}.$$

Proof. Similar to the proof of Lemma 2.2.1 in [11]. □

Remark. We shall choose $\delta = \delta_0 > 0$ sufficiently small such that we can apply Proposition 3 to small perturbations of our q-convex intersection).

Thus we can find $\Omega_M =: \Omega_1 \subset\subset \mathbb{C}^n, \Omega \subset\subset \Omega_1$ $f_1 \in L^p(\Omega_1), \overline{\partial} f_1 = 0$ in Ω_1 and $\psi = \sum_{i=0}^{M-1} \psi_i \in L^p(\Omega)$ such that $f_1 - f = \overline{\partial}\psi$ in Ω.

Step II, is validated by the following lemma.

Lemma 4.1. *Let $V''_{\partial\Omega} \subset\subset V'_{\partial\Omega}$. Let Ω be as in Theorem. Let $\tau > 0$ and define $\Omega_\tau =: \{z \in \Omega \cup V''_{\partial\Omega} : \rho_1 < \tau, \dots, \rho_N < \tau\}$. Then, there exists a strongly q-convex domain $\hat{\Omega} \subset\subset \mathbb{C}^n$ with C^3 defining function such that*

$$\Omega \subset\subset \hat{\Omega} \subset\subset \Omega_\tau.$$

Proof. For $\gamma > 0$ we define

$$\phi_1 =: \max_\gamma(\rho_1, \rho_2), \phi_2 = \max_\gamma(\phi_1, \rho_3), \dots, \phi_N = \max_\gamma(\phi_{N-1}, \rho_N),$$

where $\max_\gamma(.,.)$=: the regularized max-function introduced by Andreotti–Hill. □

Definition. *For $\gamma > 0$ let $\chi_\gamma \in C^\infty(\mathbb{R})$, convex such that*

i) $\chi_\gamma(t) = \chi_\gamma(-t)$,
ii) $|t| \le \chi_\gamma(t) \le |t| + \gamma$,
iii) $|\chi'_\gamma(t)| < 1$ *if* $|t| < \frac{\gamma}{2}$,
iv) $\chi_\gamma(t) = |t|$ *if* $|t| \ge \frac{\gamma}{2}$.

For $t_1, t_2 \in \mathbb{R}$ we set

$$\max_\gamma(t_1, t_2) = \frac{t_1 + t_2}{2} + \chi_\gamma\left(\frac{t_1 - t_2}{2}\right).$$

Then some of the key properties of the regularized max function can be summarized in the following lemma

Lemma 4.2. *For $\phi, \psi \in C^k (k \ge 2)$ functions defined in $\mathbb{C}^n$ we have*

i) $\max(\phi, \psi) \le \max_\gamma(\phi, \psi) \le \max(\phi, \psi) + \gamma$,
ii) $\max_\gamma(\phi, \psi) = \max(\phi, \psi)$ *if* $|\phi - \psi| > \gamma$,
iii) $d \max_\gamma(\phi, \psi)(z) = \lambda(z)d\phi(z) + (1 - \lambda(z))d\psi(z)$ *for* $\lambda(z) \in [0, 1], z \in \mathbb{C}^n$,
iv) *If ϕ and ψ are strictly q-convex functions in $\mathbb{C}^n$ such that they have at least $n - q + 1$ positive eigenvalues in the same directions then $\max_\gamma(\phi, \psi)$ is a strictly q-convex function (as smooth as ϕ, ψ).*

Proof. See Lemma 4.13, Corollary 4.14, p. 64 in [10] for the proof of i), ii), iii), iv) follows by direct calculation.

Given $\tau > 0$ we can choose $0 < \gamma =: \frac{\tau}{2(N+1)}$ sufficiently small and $0 < \alpha = (N + 1)\gamma = \frac{\tau}{2}$, $V'''_{\partial\Omega} \subset\subset V''_{\partial\Omega}$ such that

$$\Omega \subset\subset \hat{\Omega} =: \{z \in \Omega \cup V'''_{\partial\Omega}; \phi_N - \alpha < 0\} \subset\subset \Omega_\tau.$$

Then $\hat{\Omega}$ is a strongly q-convex domain with C^3 defining function and by choosing τ sufficiently small we can even guarantee that $\Omega \subset\subset \Omega_\tau \subset\subset \Omega_1$.

Lemma 4.3. *Let $G \subset\subset \mathbb{C}^n$ be a strongly q-convex domain with C^3 defining function. Then for every $f \in L^p_{(0,r)}(G)$, $\overline{\partial} f = 0, r \ge q, p \in \mathbb{N}, 1 \le p \le \infty$, there exists $u \in L^p_{(0,r-1)}(G)$ such that*

α) $\overline{\partial} u = f$ in G (in the distribution sense)

β) $\|u\|_{L^p(G)} \leq K \|f\|_{L^p(G)}$

where K is a positive constant independent of f.

Proof. Satz 1.7.1 in [14].

Let $\hat{f} = f_1|_{\hat{\Omega}}$. Then $\hat{f}$ is $\overline{\partial}$ closed in $\hat{\Omega}$. By Lemma 4.3 there exists υ such that $\overline{\partial}\upsilon = \hat{f}$. But then we shall have $f = \overline{\partial}(\psi + \upsilon)$in the distribution sense in Ω. Hence $u =: \psi + \upsilon$ is a global solution that satisfies the estimates in Theorem 1. □

5. Improved L^p-estimates

The proof of Theorem 2 shall be based on the following Lemmas:

Lemma 5.1 (Generalized Young inequality). *Let $(X, d\mu)$, $(Y, d\nu)$ be two measured spaces, $H(x, y)$ be a measurable function defined on $X \times Y$ such that*

$$\int_X \left|H(x,y)\right|^\tau d\mu(x) \leq M^\tau \quad \textit{for almost all}, \quad y \in Y$$
$$\int_Y \left|H(x,y)\right|^\tau d\nu(y) \leq M^\tau \quad \textit{for almost all} \quad x \in X$$

for some positive constant $M < \infty$, $\tau \geq 1$. Then the linear operator defined ν-a.e. by

$$Tf(y) = \int_X f(x)K(x,y)d\mu(x)$$

is bounded from $L^p(X)$ to $L^q(Y)$ with norm $\leq M$ for all $1 \leq p, q \leq \infty$ with

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{\tau} - 1$$

with the usual conventions in the case q, or p are ∞.

Proof. See Appendix B in Range [16].

Lemma 5.2. *The local solution operators $T_{\tilde{I}} f(z) =: \int_{\zeta \in \Omega_{\tilde{I}}} f(\zeta) \wedge K_{\tilde{I}}(\zeta, z)$ satisfy the following estimates: For $1 \leq p \leq \infty$, $f \in L^p_{(0,s)}(\Omega_{\tilde{I}})$ we have*

$$\|T_{\tilde{I}} f\|_{L^r_{(0,s-1)}(\Omega_{\tilde{I}})} \leq c_p(\Omega) \|f\|_{L^p_{(0,s)}(\Omega_{\tilde{I}})}$$

with

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{\lambda} - 1,$$

where $1 \leq \lambda < \frac{2n+2\mu}{2n-1+2\mu}$, $\mu =$ the maximal number of nonempty intersections of $\{\rho_i = 0\}_{i=1}^{i=\ell+1}$ and $c_p(\Omega)$ a positive constant that depends on $\max\{\|\rho_i\|_{C^3}\ i = 1, \ldots \ell + 1\}$ and Ω, p.

Proof. We shall apply Lemma 5.1 for $H = K_{\widetilde{I}}$, $\tau \geq 1$, to be determined later on.

An arbitrary term of the kernel $K_{\widetilde{I}}$ can be written as a finite sum of terms of the following form

$$\frac{\mathcal{E}^1}{|\zeta - z|^{2(n-\sum_{i=1}^{\ell+1} a_i)}} \frac{\mathcal{E}^{2k-s}\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}}{\prod\limits_{j\in\{j_1,\ldots,j_k\}} F_j^{a_j+1} \prod\limits_{j\in\{1,\ldots,\ell+1\}\setminus J} F_j^{a_j}},$$

where $\{j_1, \ldots, j_k\} \subset \{1, \ldots, \ell+1\}$ any multiindex with strictly increasing components and $\{i_1, \ldots, i_s\} \subset \{j_1, \overline{j}_1, \ldots, j_k, \overline{j}_k\}$ and we use the same convention for ω_{i_ν} as in Sect. 3. □

Then we wish to estimate for $z \in \Omega_{\widetilde{I}}$ and $\delta > 0$ small, the following integral

$$\int_{\zeta\in\Omega_{\widetilde{I}}\cap B(z,\delta)} \frac{\mathcal{E}^{(2k-s+1)\tau} \|\omega_{i_1}(z) \wedge \cdots \wedge \omega_{i_s}(z)\| \wedge dV_{2n}(\zeta)}{|\zeta - z|^{2(n-\sum a_i)\tau} \prod\limits_{j=1}^{k} F_j^{(a_j+1)\tau} \prod\limits_{j\notin J} F_j^{a_j\tau}}. \tag{5.1}$$

Without loss of generality we shall examine the case where $J = \{1, \ldots k\}$, $\{i_1, \ldots, i_s\} = \{1, \ldots, s\}$, $s \leq k$. The remaining cases will follow in a similar manner.

So we need to estimate for $z \in \Omega_{\widetilde{I}}$, $\delta > 0$

$$\int_{\zeta\in\Omega_{\widetilde{I}}\cap B(z,\delta)} \frac{|\zeta - z|^{(1+2k-s)\tau} \|\partial_\zeta \rho_1(z) \wedge \cdots \wedge \partial_\zeta \rho_s(z)\| dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{\ell+1} a_i)\tau} \prod\limits_{j=1}^{k} |F_j^*|^{(a_j+1)\tau} \prod\limits_{j=k+1}^{\ell+1} |F_j^*|^{a_j\tau}}. \tag{5.2}$$

We shall have

$$\begin{aligned}
(5.2) &\leq \int_{\zeta\in\Omega_{\widetilde{I}}\cap B(z,\delta)} \frac{|\zeta - z|^{(1+2k-s)\tau} \|\partial_\zeta \rho_1(z) \wedge \cdots \wedge \partial_\zeta \rho_s(z)\| dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{\ell+1} a_i)\tau} \prod\limits_{j=1}^{s} |F_j^*|^{(a_j+1)\tau} |\zeta - z|^{2(\sum_{i=s+1}^{\ell+1} a_i + k - s)\tau}} \\
&\leq \int_{\zeta\in\Omega_{\widetilde{I}}\cap B(z,\delta)} \frac{|\zeta - z|^{(s+1)\tau} \|\partial_\zeta \rho_1(z) \wedge \cdots \wedge \partial_\zeta \rho_s(z)\| dV(\zeta)}{|\zeta - z|^{2(n-\sum_{j=1}^{s} a_i)\tau} \prod\limits_{j=1}^{s} |F_j^*|^{(a_j+1)\tau}} \\
&\leq \int_{\zeta\in\Omega_{\widetilde{I}}\cap B(z,\delta)} \frac{\|\partial_\zeta \rho_1(z) \wedge \cdots \wedge \partial_\zeta \rho_s(z)\| dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{s} a_i)\tau - \tau} \prod\limits_{j=1}^{s} |F_j^*|^{(a_j+\frac{1}{2})\tau}}.
\end{aligned}$$

Using an argument similar to the proof of Lemma 3.3 we have

$$(5.2) \leq \int_{\zeta\in\Omega_{\widetilde{I}}\cap B(z,\delta)} \frac{\|\partial_\zeta \rho_1(z) \wedge \cdots \wedge \partial_\zeta \rho_s(z)\| dV(\zeta)}{|\zeta - z|^{2(n-\sum_{i=1}^{s} a_i)\tau - \tau} \prod\limits_{j=1}^{s} (| < \partial_\zeta \rho_j(z), \zeta - z > | + |\zeta - z|^2)^{(a_j+\frac{1}{2})\tau}}.$$

We set $\zeta_j - z_j = x_{2j-1} + ix_{2j}$ for $j = 1, \dots, n$.
We define for $k = 1, \dots, \ell + 1$

$$w_k(\zeta, z) =:< \partial_\zeta \rho_k(z), \zeta - z >= \mathrm{Re} w_k(\zeta, z) + i \mathrm{Im} w_k(\zeta, z).$$

Then,

$$d_\zeta w_k(\zeta, z) = \partial_\zeta \rho_k(z) = d_\zeta \mathrm{Re} w_k(\zeta, z) + i d_\zeta \mathrm{Im} w_k(\zeta, z).$$

Thus

$$\begin{aligned}
&\partial_\zeta \rho_1(z) \wedge \cdots \wedge \partial_\zeta \rho_s(z) \\
&= (d_\zeta \mathrm{Re} w_1(\zeta, z) + i d_\zeta \mathrm{Im} w_1(\zeta, z)) \wedge \cdots \wedge (d_\zeta \mathrm{Re} w_s(\zeta, z) + i d_\zeta \mathrm{Im} w_s(\zeta, z)) \\
&\qquad = \sum_{0 \le \sigma \le s} c_\sigma d_\zeta \mathrm{Re} w_{j_1}(\zeta, z) \wedge \cdots \wedge d_\zeta \mathrm{Re} w_{j_\sigma}(\zeta, z) \wedge d_\zeta \\
&\qquad\qquad \cdot \mathrm{Im} w_{j_{\sigma+1}}(\zeta, z) \wedge \cdots \wedge d_\zeta \mathrm{Im} w_{j_s}(\zeta, z),
\end{aligned}$$

where $c_\sigma \in \mathbb{C}$ are absolute constants, $\{j_1, \dots, j_\sigma\} \subset \{1, \dots, s\}$, $\{j_{\sigma+1}, \dots, j_s\} \subset \{1, \dots, s\} \setminus \{j_1, \dots, j_\sigma\}$.

Hence the above integral can be split into finite sum of terms of the form (5.3)

$$\int \frac{\| d_\zeta \mathrm{Re} w_{j_1}(\zeta, z) \wedge \ldots \wedge d_\zeta \mathrm{Re} w_{j_\sigma}(\zeta, z) \wedge d_\zeta \mathrm{Im} w_{j_{\sigma+1}}(\zeta, z) \wedge \ldots \wedge d_\zeta \mathrm{Im} w_{j_s}(\zeta, z) \| dV(\zeta)}{|\zeta - z|^{2(n - \sum_{i=1}^s a_i)\tau - \tau} \prod_{j=1}^s (|\mathrm{Re} w_j(\zeta, z)| + |\mathrm{Im} w_j(\zeta, z)| + |\zeta - z|^2)^{(a_j + \frac{1}{2})\tau}}.$$

Using the real coordinates $\{x_j\}_{j=1}^{2n}$ we can write

$$\mathrm{Re} w_k(\zeta, z) = \frac{1}{2} \sum_{j=1}^{2n} (\frac{\partial \rho_k(0)}{\partial x_{2j-1}} x_{2j-1} + \frac{\partial \rho_k(0)}{\partial x_{2j}} x_{2j}) =: p_k(x, 0),$$

$$\mathrm{Im} w_k(\zeta, z) = \frac{1}{2} \sum_{j=1}^{2n} (\frac{\partial \rho_k(0)}{\partial x_{2j-1}} x_{2j} - \frac{\partial \rho_k(0)}{\partial x_{2j}} x_{2j-1}) =: q_k(x, 0),$$

where $p_k(x, 0)$, $q_k(x, 0)$ are real-valued polynomials in x of degree $d = 1$.

Without loss of generality we shall estimate the case where $\sigma = s$, $\{j_1, \dots, j_s\} = \{1, \dots, s\}$. The other cases will follow similarly. Hence we need to estimate

$$\int_{|x| < \delta} \frac{\| d_x p_1(x, 0) \wedge \cdots \wedge d_x p_s(x, 0) \| dV_{2n}(x)}{|x|^{2(n - \sum_{i=1}^s a_i)\tau - \tau} \prod_{j=1}^s (|p_j(x, 0)| + |x|^2)^{(a_j + \frac{1}{2})\tau}}. \tag{5.4}$$

Arguing in a similar manner as in Lemma 3.3 (5.4) can be majorized by

$$C \int_{t \in T(B(0, \delta))} \frac{dt_1 \ldots dt_s dV_{2n-s}(t)}{|t|^{2(n - \sum_{i=1}^s a_i)\tau - \tau} \prod_{j=1}^s (|t_j| + |t|^2)^{(a_j + \frac{1}{2})\tau}}, \tag{5.5}$$

where

$$T(B(0,\delta)) = \bigcup_{1\le i_1<\cdots<i_{2n-s}\le 2n} \{(|p_1(x,0)|,\dots,p_s(x,0)|,|x_{i_1}|,\dots|x_{i_s}|) : |x|<\delta\}$$

if $s < 2n$. In our case of course, $s \le k \le \ell + 1 \le N + 1 \le n - 1$.

It is not hard to check that $T(B(0,\delta))$ is a relatively compact, open neighborhood of 0.

Hence, if we set $r^2 = t_{s+1}^2 + \cdots + t_{2n}^2$, (3.5) can be majorized by

$$\int \frac{dt_1 \dots dt_s r^{2n-s-1} dr}{r^{2(n-\sum_{i=1}^s a_i)\tau-\tau} \prod_{j=1}^{s} (|t_j| + r^2)^{(a_j+\frac{1}{2})\tau}}$$
$$\le \int \frac{r^{2n-s-1}dr}{r^{2(n-\sum_{i=1}^s a_i)\tau-\tau} r^{2(\sum_{i=1}^s a_i\tau-\frac{s}{2}\tau)-2s}}$$
$$\le \int \frac{r^{2n-s-1}dr}{r^{2n\tau-2s-\tau s-\tau}} \le \int r^{2n-2n\tau+\tau-\tau s+s-1} dr < \infty$$

if and only if $2n - 2n\tau + \tau - \tau s + s > 0$ i.e. if $\tau < \frac{2n+s}{2n-1+s}$.

Taking into account that $0 \le s \le 2k \le 2\mu$ and choosing $\tau = \frac{2n+2\mu}{2n-1+2\mu} < \frac{2n+s}{2n-1+s}$ we can obtain the finiteness of the above integral. □

Remark. In the case where we have a smooth strongly pseudoconvex domain similar estimates were obtained by Krantz [12].

We need also a stronger version of Lemma 4.3 of the previous section.

Lemma 5.3. *Let G be a bounded smooth strongly $\underline{q}$-convex domain in $\mathbb{C}^n$ with C^3 defining function. Let $1 \le p \le \infty$, $f \in L^p_{(0,s)}(G)$, $\overline{\partial} f = 0$ in G, $s \ge q$. Then there exists $u \in L^p_{(0,s-1}(G)$, $\overline{\partial} u = f$ in G such that*

i) *If $1 < p < 2n+2$, $u \in L^r_{(0,s-1)}(G)$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{2n+2}$.*
ii) *If $p = 1$, $u \in L^{\frac{2n+2}{2n+1}-\eta}_{(0,s-1)}(G)$ for any $\eta > 0$.*
iii) *If $p = 2n+2$, $u \in L^r_{(0,s-1)}(G)$ where $p < r < \infty$.*
iv) *If $2n+2 < p \le \infty$, $u \in C^{0,\epsilon}_{(0,s-1)}(\overline{G})$ with $\epsilon = \frac{1}{2} - \frac{n+1}{p}$.*

Proof. The proof is based on Theorem 1 in Bonneau–Diederich [4] (which gives local estimates) and standard arguments based on ideas in Lemmas 2.3.1–2.3.5 of Henkin–Leiterer [9].

Using Lemmas 5.2, 5.3 and arguing along the same lines as in Sect. 4 we can show Theorem 2.

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