On Bounding the Betti Numbers and Computing the Euler Characteristic of Semi-Algebraic Sets[*]

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Abstract. In this paper we prove new bounds on the sum of the Betti numbers of closed semi-algebraic sets and also give the first single exponential time algorithm for computing the Euler characteristic of arbitrary closed semi-algebraic sets.

Given a closed semi-algebraic set $S \subset R^k$ defined as the intersection of a real variety, $Q = 0, \deg(Q) \leq d$, whose real dimension is k', with a set defined by a quantifier-free Boolean formula with no negations with atoms of the form $P_i = 0, P_i \geq 0, P_i \leq 0, \deg(P_i) \leq d, 1 \leq i \leq s$, we prove that the sum of the Betti numbers of S is bounded by $s^{k'}(O(d))^k$. This result generalizes the Oleinik–Petrovsky–Thom–Milnor bound in two directions. Firstly, our bound applies to arbitrary unions of basic closed semi-algebraic sets, not just for basic semi-algebraic sets. Secondly, the combinatorial part (the part depending on s) in our bound, depends on the dimension of the variety rather than that of the ambient space. It also generalizes the result in [4] where a similar bound is proven for the number of connected components. We also prove that the sum of the Betti numbers of S is bounded by $s^{k'} 2^{O(k^2 m^4)}$ in case the total number of monomials occurring in the polynomials in $\mathcal{P} \cup \{Q\}$ is m. Using the tools developed for the above results, as well as some additional techniques, we give the first single exponential time algorithm for computing the Euler characteristic of arbitrary closed semi-algebraic sets.

1. Introduction

Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset R[X_1, \ldots, X_k]$ be a family of polynomials whose degrees are bounded by d and S is a closed semi-algebraic set defined by a quantifier-free Boolean

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formula without negations, whose atoms are of the form $P_i \sigma 0$ where $\sigma \in \{\geq, \leq, =\}$ for $1 \leq i \leq s$.

Thus S is defined by a disjunction of weak sign conditions on the family $\mathcal{P}$, where a weak sign condition is a conjunction of the form

$$P_1 \sigma_1 0, \ldots, P_s \sigma_s 0 \qquad \text{where} \quad \sigma_i \in \{\geq, \leq, =\} \quad \text{for} \quad 1 \leq i \leq s.$$

The Betti numbers $\beta_i(S)$, the ranks of the singular homology groups of S, are a measure of the topological complexity of S and can be bounded in terms of s, d, and k as follows. Collins' algorithm [9] for cylindrical algebraic decomposition gives a cellular decomposition of S into $(sd)^{2^{O(k)}}$ cells and thus the same bound applies to the $\beta_i(S)$. In case S is a basic closed semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with $\deg(P_i) \leq d$, the tighter bound of $(O(sd))^k$ on the sum of the Betti numbers of S was proved in separate papers by Oleinik and Petrovsky [25], Thom [29], and Milnor [23]. (Note that when S is the set of real zeros of a set of polynomials this bound can be reduced to $(O(d))^k$.) Similar bounds on the number of connected components of basic semi-algebraic sets were also proved by Warren [30]. These bounds play an important role in algorithmic real algebraic geometry [17], in discrete geometry [12], [13], and have been used recently in proving lower bounds in the algebraic computation tree model (see [31] and [24]).

A difficulty in extending this bound to an arbitrary semi-algebraic set defined by the polynomials $\mathcal{P}$, lies in the fact that the Betti numbers of the union of two disjoint sets can be arbitrarily greater than the sum of the individual Betti numbers of the two sets (see [31] for an easy example).

We extend this bound to arbitrary unions of basic closed semi-algebraic sets by proving the following theorem:

Theorem 1. *Let $S \subset R^k$ be the intersection of a closed semi-algebraic set defined by a quantifier-free Boolean formula without negations involving a family $\mathcal{P} = \{P_1, \ldots, P_s\}$, of s polynomials whose atoms are of the form $P_i \sigma 0$, $\sigma \in \{\geq, \leq, =\}$ with $Z(Q)$, the zero set of a polynomial Q. If the geometric dimension of $Z(Q)$ is k', and the degrees of the polynomials in $\mathcal{P} \cup \{Q\}$ are all bounded by d. Then the sum of the Betti numbers of S is bounded by $s^{k'}(O(d))^k$.*

If S can be more simply described as $Q = 0$, $P_1 > 0, \ldots, P_s > 0$, then we have the slightly better bound of $\binom{s}{k'}(O(d))^k$ on the sum of the Betti numbers of S.

If Q is the zero polynomial, then $Z(Q) = R^k$, $k' = k$, and we obtain the Oleinik–Petrovsky–Thom–Milnor bound extended to arbitrary closed semi-algebraic sets as a special case. Note that a bound of $\binom{s}{k'}(O(d))^k$ on the zeroth Betti number for arbitrary semi-algebraic sets (which is just the number of connected components) was known before (see [4]) and our bound is a generalization of this result to the sum of the Betti numbers in the case of closed semi-algebraic sets.

Note also that a lower bound of $(sd/k')^{k'}$ on the zeroth Betti number is easily obtained by considering the set of nonzeros of s polynomials, each of them a product of d linear polynomials, restricted to a k' dimensional linear subspace.

The dependence of the combinatorial part (the part depending on s) of the bound in Theorem 1 on k' instead of k is important when we consider low-dimensional semi-algebraic sets embedded in a higher-dimensional space, and this is sometimes important

in applications. For example, the bound on the number of connected components in [4] plays a crucial role in the proof of the main result in [13], where the variety is the real Grassmannian $G_{m,n}$ (the space of m-dimensional subspaces of R^n), embedded as an $m(n-m)$-dimensional variety in R^{n^2}.

Remark 1. We note that a bound of $(O(sd))^{2k}$ on the sum of the Betti numbers of compact semi-algebraic sets can be deduced from Theorem 1 in [31] or from Theorem 1.10 in [24]. Also note that recently Barvinok [1] has proved a new bound on the sum of Betti numbers of semi-algebraic sets defined by *quadratic* inequalities which is polynomial in the number of variables when the number of inequalities is constant.

Remark 2. Even though we state our results for closed semi-algebraic sets over the reals the same bounds hold over any real closed field through the application of well-known transfer principles. We refer the reader to the article by Knebusch [21] and to [6] for the intricacies of semi-algebraic topology over general real closed fields.

In order to achieve the bound in Theorem 1 we prove that an arbitrary closed semi-algebraic set has the same homology groups as a compact semi-algebraic set defined by polynomials in general position. This result generalizes a similar result in [7] where it was proved for semi-algebraic sets defined by a single sign condition, and this intermediate result might be of independent interest.

Next we consider the problem of computing the Euler characteristic, $\chi(S)$, of a closed semi-algebraic set S. The Euler characteristic, which is the alternating sum of the Betti numbers of S, is an important topological invariant and thus can be used as a test to rule out topological equivalence. Also, computing the Euler characteristic of semi-algebraic sets occurs as an important subproblem in some recent work due to Gabrielov [11] on computing multiplicities of the zeros of polynomial functions along the trajectories of a polynomial vector field.

We prove the following theorem.

Theorem 2. *Let $S \subset R^k$ be a real closed semi-algebraic set defined by a quantifier-free Boolean formula without negation, with atoms of the form $P_i \sigma 0$ where $\sigma \in \{\geq, \leq, =\}$ for $1 \leq i \leq s$, where $\mathcal{P} = \{P_1, \ldots, P_s\} \subset Z[X_1, \ldots, X_k]$, is a family of s polynomials whose degrees are bounded by d, and the bit lengths of the coefficients of P_i are bounded by L. Then there exists an algorithm for computing $\chi(S)$ which performs at most $(ksd)^{O(k)} L^{O(1)}$ bit operations.*

We remark that computing the homology groups of semi-algebraic sets in single exponential time is a central open problem of computational real algebraic geometry. Single exponential algorithms for determining certain other (weaker) topological properties of semi-algebraic sets are known. For example, it is possible to compute the number of connected components [15], [7], [14], [18], the semi-algebraic description of the connected components [8], [19], as well as to decide whether two points are in the same connected component of a semi-algebraic set [7], [5], in single exponential time.

Collins' algorithm for computing a cylindrical algebraic decomposition [9] gives sufficient topological information for computing the Euler characteristic, and in fact the homology groups of a given semi-algebraic set [27]. However, this algorithm has double exponential complexity $(sd)^{2^{O(k)}} L^{O(1)}$ where L is a bound on the number of bits used in specifying the coefficients of the input polynomials. Previously, this was the best algorithm for computing the Euler characteristic of general semi-algebraic sets. A single exponential algorithm for computing the Euler characteristic of a smooth algebraic hypersurface is mentioned in [26].

The rest of the paper is organized as follows. In Section 2 we show how to perturb the polynomials to bring them into general position without changing the homology groups of the given semi-algebraic set. In Section 3 we prove our bound on the Betti numbers of closed semi-algebraic sets. In Section 4 we prove a bound on the sum of the Betti numbers of a closed semi-algebraic set, in which the algebraic part of the complexity depends only on the number of monomials appearing in the polynomials and is independent of the degree. In Section 5 we describe our algorithm for computing the Euler characteristic. In Section 5.2 we give an algorithm for computing the Euler characteristic of a basic semi-algebraic set, and in Section 5.3 we give an algorithm for computing the Euler characteristic of an arbitrary union of such sets.

2.   Going to General Position

In the proofs of our theorems as well as in the algorithms to be described later we often make use of perturbations by *infinitesimals*. For the purposes of this paper it is enough to consider an infinitesimal as a sufficiently small positive real number. We also use the notation $\varepsilon_1 \gg \varepsilon_2 \gg \varepsilon_3 \cdots$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are infinitesimals and ε_1 is positive and sufficiently small, ε_2 is positive and sufficiently small with respect to ε_1, and so on.

2.1.   *General Position*

We say that a family of polynomials $\mathcal{P}$ in k variables is in *general position* if no $k + 1$ of them have a common real zero. In this section we show that given a semi-algebraic set, S, defined by a family, $\mathcal{P}$, of s polynomials with degrees bounded by d, we can define a new compact semi-algebraic set S', which has the same homology groups as S, but which is defined by a family, $\mathcal{P}'$, of polynomials in general position. Moreover, $|\mathcal{P}'| \le 4s + 1$, and the degrees of the polynomials in $\mathcal{P}'$ are bounded by d', where d' is the least even number greater than d. Similar results appear in [7], where they are proved for semi-algebraic sets defined by one single sign condition. Our proof techniques are similar to those used in [7], but our results apply to arbitrary closed semi-algebraic sets.

We first recall a few facts from algebraic topology. Given a semi-algebraic set, $S \subset R^k$, we denote by $H_*(S)$ the graded singular homology group of S.

The following theorem appears in [28] (Theorem 4.1.7).

Theorem 3.   *The singular homology functor commutes with direct limits*.

We also need the following fundamental property of the $\check{C}ech$ homology groups of a compact space (see p. 257 of [10]).

Theorem 4. *If X is compact and the inverse limit of compact sets X_α, then the $\check{C}ech$ homology group $H_*(X)$ is the inverse limit of the $\check{C}ech$ homology groups $H_*(X_\alpha)$.*

Note that, since compact semi-algebraic sets are triangulable, their singular and $\check{C}ech$ homologies coincide.

We first ensure that a given semi-algebraic set has the same homology groups as a bounded one.

Lemma 1. *Let S be any closed semi-algebraic set. Let $S_\Omega = S \cap (X_1^2 + \cdots + X_k^2 \leq \Omega)$. Then, for sufficiently large Ω, $H_*(S) \cong H_*(S_\Omega)$.*

Proof. Let $S' \subset R^{k+1}$ be the set defined by the same formula as S plus the new inequality $T(X_1^2 + \cdots + X_k^2) \leq 1$. Let π and π_x denote the projections onto the T and the X coordinates, respectively. Then, by Hardt's triviality theorem [16], for all sufficiently large Ω, and for all $t \in (0, 1/\Omega)$, $S' \cap \pi^{-1}(t)$ is homeomorphic to $S' \cap \pi^{-1}(1/\Omega)$.

Moreover, each $S' \cap \pi^{-1}(t)$ is compact and $S = \bigcup_{t>0} \pi_x(S' \cap \pi^{-1}(t))$. It is also clear that the singular chain complex of S is the direct limit of the singular chain complexes of $\pi_x(S' \cap \pi^{-1}(t))$. Now, by Theorem 3, $H_*(S)$ is the direct limit of the groups $H_*((S' \cap \pi^{-1}(t)))$.

Hence, $H_*(S) \cong H_*(S' \cap \pi^{-1}(1/\Omega)) \cong H_*(S_\Omega)$. $\square$

Let S be a compact semi-algebraic set defined by the formula

$$\left(\bigvee_{1 \leq j \leq L} \sigma_j(\mathcal{P}) \right) \wedge (X_1^2 + \cdots + X_k^2 \leq \Omega),$$

where each σ_j is a nonempty weak sign condition for the family $\mathcal{P}$, and Ω is positive and sufficiently large.

For any weak sign condition σ and a new variable ε we denote by $\sigma(\mathcal{P}, \varepsilon)$ the following formula: For every weak inequality in σ we replace the corresponding conjunct $P_j \geq 0$ (resp. $P_j \leq 0$) by $P_j > -\varepsilon$ (resp. $P_j < \varepsilon$). For every equality in σ we replace the corresponding conjunct $P_j = 0$ by the conjunct $(P_j > -\varepsilon) \wedge (P_j < \varepsilon)$.

Let S_ε denote the semi-algebraic set defined by the formula $\bigvee_{1 \leq j \leq L} \sigma_j(\mathcal{P}, \varepsilon)$.

We have the following lemma.

Lemma 2. *Let S be compact and let S_ε be as above. Then, for sufficiently small $\varepsilon > 0$, $H_*(S) \cong H_*(S_\varepsilon)$.*

Proof. Let S'_ε denote the semi-algebraic set defined as S_ε with the difference that every inequality is replaced by the corresponding weaker version and the constraint $X_1^2 + \cdots + X_k^2 \leq R$ is added, where R is the radius of a ball containing S. It follows that

S'_ε is compact. Again by Hardt's triviality theorem, for sufficiently small $\varepsilon > 0$, and all $0 < \delta < \varepsilon$, S'_δ is homeomorphic to S'_ε.

Now, $S \subset S_\varepsilon \subset S'_\varepsilon$. Also, $S = \bigcap_{\varepsilon > 0} S'_\varepsilon$. Hence, by Theorem 4, $H_*(S)$ is the inverse limit of the groups $H_*(S'_\varepsilon)$, and thus by the previous observation $H_*(S) \cong H_*(S'_\varepsilon)$. Finally, note that $S_\varepsilon = \bigcup_{0 < t < \varepsilon} S'_t$, and the singular chain complex of S_ε is the direct limit of the singular chain complexes of S'_t.

Thus, by Theorem 3, $H_*(S_\varepsilon)$ is the direct limit of the groups $H_*(S'_t)$. However, $H_*(S'_t) \cong H_*(S'_\varepsilon)$ for $0 < t < \varepsilon$. Thus, we have that $H_*(S_\varepsilon) \cong H_*(S'_\varepsilon)$ which implies that $H_*(S) \cong H_*(S_\varepsilon)$.   □

Note that the set S_ε is defined by a disjunction of conjunctions of strict inequalities involving $2s$ polynomials,

$$\bigcup_{P \in \mathcal{P}} \{P + \varepsilon, P - \varepsilon\},$$

and that there are no equalities in the formula.

We next prove that given a semi-algebraic set defined by a disjunct of conjuncts of the form $\bigwedge_j Q_j s_j 0$ where $s_j \in \{<, >\}$, it is then possible to define a new semi-algebraic set defined by polynomials in general position having the same homology group.

Let S be a semi-algebraic set defined by the formula $\bigvee_{1 \le j \le L} \sigma_j(\mathcal{Q}_j)$, where each σ_j is a sign-condition on a subset $\mathcal{Q}_j$ of a family of polynomial $\mathcal{Q}$, and σ_j does not contain equality. Let $\mathcal{Q} = \{Q_1, \ldots, Q_s\}$ and $\deg(Q_i) < d, \ 1 \le i \le s$.

For $1 \le i \le 2s$, let $H_i = 1 + \sum_{1 \le j \le k} i^j X_j^{d'}$, where d' is the least even number larger than d. For $Q_i \in \mathcal{Q}$, let $Q_i^+ = (1 - \delta)Q_i + \delta H_{2i-1}$ and $Q_i^- = (1 - \delta)Q_i - \delta H_{2i}$, where δ is a new variable. Let $\mathcal{Q}'$ be the family $\bigcup_i \{Q_i^+, Q_i^-\}$.

Lemma 3. *For sufficiently small $\delta > 0$, the family of polynomials $\mathcal{Q}'$ is in general position.*

Proof.   See [3].   □

For a sign condition σ on $\mathcal{Q}$, without any equalities, define $\sigma(\mathcal{Q}, \delta)$ to be the formula obtained as follows:

Every conjunct $Q_i > 0$ is replaced by $Q_i^- \ge 0$, while every conjunct $Q_i < 0$ is replaced by $Q_i^+ \le 0$.

Let S_δ^- be the set defined by $\bigvee_{1 \le j \le L} \sigma_j(\mathcal{Q}_j, \delta)$.

Lemma 4. *Let S and S_δ^- be as above. Then, for sufficiently small $\delta > 0$, $H_*(S_\delta^-) \cong H_*(S)$.*

Proof.   Again by Hardt's triviality theorem, $H_*(S_t^-)$ are isomorphic for all $t \in (0, \delta)$ for sufficiently small δ. Also, $S = \bigcup_{0 < t} S_t^-$, and the singular chain complex of S is the direct limit of the singular chain complexes of S_t^-. Again applying Theorem 3 we have that $H_*(S) \cong H_*(S_\delta^-)$, for all sufficiently small $\delta > 0$.   □

Combining the previous four lemmas we obtain the following proposition.

Proposition 1. *Let S be any semi-algebraic set defined by a family of s polynomials with degrees bounded by d. Then there exists a set $S(\Omega, \varepsilon, \delta)$ defined by a family of at most $4s + 1$ polynomials, in $X_1, \ldots, X_k, \varepsilon, \delta, \Omega$ with degrees bounded by 2d such that, for sufficiently small $1/\Omega \gg \varepsilon \gg \delta > 0$, this new family of polynomials is in general position and $H_*(S(\Omega, \varepsilon, \delta)) \cong H_*(S)$. Moreover, the set $S(\Omega, \varepsilon, \delta)$ is compact, and defined as a disjunction of conjunctions of weak inequalities.*

In our proofs we need a slightly stronger notion of general position.

Let Q be a polynomial such that $Z(Q)$ has geometric dimension k'. We say that a family of polynomials $\mathcal{P}$ is in *general position with respect to* Q, if no $k' + 1$ of the polynomials in $\mathcal{P}$ have a real zero in common with Q.

We need the following proposition whose proof appears in [4].

Proposition 2. *Given a family $\{P_1, \ldots, P_s\}$ of polynomials in $R[X_1, \ldots, X_k]$ and a variety $Z(Q)$ of real dimension k' and infinitesimals $\delta_1 \gg \cdots \gg \delta_s \gg \delta$, the perturbed family $\mathcal{P}^* = \bigcup_{1 \leq i \leq s} \{P_i - \delta_i, P_i + \delta_i, P_i - \delta\delta_i, P_i + \delta\delta_i\}$ is in general position with respect to the variety $Z(Q)$.*

Using the above proposition we have the following more general version of Proposition 1.

Proposition 3. *Let S be any semi-algebraic set defined by a family of s polynomials with degrees bounded by d and contained in a real variety $Z(Q)$ of real dimension k'. Then there exists a set $S(\Omega, \varepsilon, \delta_1, \ldots, \delta_s, \delta)$ defined by a family of at most $4s + 1$ polynomials, in $X_1, \ldots, X_k, \varepsilon, \delta, \delta_1, \ldots, \delta_s, \Omega$ with degrees bounded by 2d such that, for sufficiently small $1/\Omega \gg \varepsilon \gg \delta_1 \gg \cdots \gg \delta_s \gg \delta > 0$, this new family of polynomials is in general position with respect to Q, and $H_*(S(\Omega, \varepsilon, \delta_1, \ldots, \delta_s, \delta)) \cong H_*(S)$. Moreover, the set $S(\Omega, \varepsilon, \delta_1, \ldots, \delta_s, \delta)$ is compact, and defined as a disjunction of conjunctions of weak inequalities.*

Proof. The proof is the same as that of Proposition 1, always carrying along the extra condition $Q = 0$, and replacing the polynomials H_i by the infinitesimals δ_i. □

We also remark that if a family of polynomials $\mathcal{P}$ is in general position with respect to a polynomial Q, then for an infinitesimal ε the family $\bigcup_{P \in \mathcal{P}} \{P, P + \varepsilon, P - \varepsilon\}$ is also in general position with respect to Q as long as we consider zeros that are bounded over R.

3.   Bounding the Betti Numbers

We make use of the following facts from algebraic topology which follows easily from the Mayer–Vietoris sequence (see [28]). Given a semi-algebraic set S we define $r(S) = \sum_i \beta_i(S)$. Let S_1 and S_2 be two compact semi-algebraic sets. Then, from the Mayer–

Vietoris sequence

$$\cdots \to H_{i+1}(S_1 \cup S_2) \to H_i(S_1 \cap S_2) \to H_i(S_1) \oplus H_i(S_2)$$
$$\to H_i(S_1 \cup S_2) \to H_{i-1}(S_1 \cap S_2) \to \cdots,$$

it easily follows that

$$r(S_1) + r(S_2) \le r(S_1 \cup S_2) + r(S_1 \cap S_2) \tag{1}$$

and

$$r(S_1 \cup S_2) \le r(S_1) + r(S_2) + r(S_1 \cap S_2). \tag{2}$$

Lemma 5. *Let S be a semi-algebraic set defined by a conjunct $(Q = 0) \wedge (\sigma_1(\mathcal{P}) \vee \cdots \vee \sigma_L(\mathcal{P}))$, where Q is a polynomial, and σ_j, $1 \le j \le L$, are distinct sign conditions on a family of polynomials $\mathcal{P}$, such that none of the σ_j contain an equality. Then $\beta_i(S) = \sum_{1 \le j \le L} \beta_i(S_j)$ and $\chi(S) = \sum_{1 \le j \le L} \chi(S_j)$, where S_j is the set defined by $(Q = 0) \wedge \sigma_j(\mathcal{P})$.*

Proof. This follows immediately from the definition of singular homology groups. The singular chain of S being the direct sum of the singular chains of the S_j's. $\square$

3.1. *Proof of Theorem* 1

Proof. In view of Proposition 3 and the remarks following it we can assume without loss of generality that S is a compact set defined by s polynomials in general position with respect to Q.

We next prove two lemmas that will imply the theorem.

Given a polynomial Q and a family of polynomials $\mathcal{P} = \{P_1, \ldots, P_s\}$, we define the *combinatorial level* of the system $(Q, \mathcal{P})$ to be the least integer m such that no $m + 1$ of the polynomials in $\mathcal{P}$ have a common real zero with Q.

For example, the combinatorial level of $(Q, \mathcal{P})$ is bounded by k' if the dimension of $Z(Q)$ is k' and the polynomials in $\mathcal{P}$ are in general position with respect to Q.

Lemma 6. *Let S be a semi-algebraic set, defined by $Q = 0$, $P_1 > 0, \ldots, P_s > 0$, where $Z(Q)$ is bounded and Q is nonnegative everywhere. Let $\mathcal{P} = \{P_1, \ldots, P_s\}$, and let the combinatorial level of the system $(Q, \mathcal{P})$ be bounded by $m \le k$ and let the degrees of the polynomials Q and P_i be bounded by $2d$ and d, respectively. Then $r(S) = \binom{s}{m}(O(d))^k$.*

Proof. Consider the set S_ε defined by

$$Q = 0, P_1 \ge \varepsilon, \ldots, P_s \ge \varepsilon,$$

where $\varepsilon > 0$ is a positive infinitesimal. Then it is easy to prove that, for sufficiently small $\varepsilon > 0$, $H_*(S_\varepsilon) \cong H_*(S)$. Let T denote the set defined by $Q = 0$, $P_2 \ge \varepsilon, \ldots, P_s \ge \varepsilon$.

Consider the sets $U_1 = T \cap ((P_1 \geq \varepsilon) \cup (P_1 \leq \varepsilon))$, $V_1 = T \cap (-\varepsilon \leq P_1 \leq \varepsilon)$ and $W_1 = T \cap ((P_1 = \varepsilon) \cup (P_1 = -\varepsilon))$.

From inequality (1) we see that $r(U_1) + r(V_1) \leq r(T) + r(W_1)$ and from Lemma 5 that $r(S) \leq r(U_1)$. It immediately follows that $r(S) \leq r(T) + r(W_1)$.

Moreover, let $r(s, d, m, k)$ be the maximum possible value of $r(S)$ for any set defined by a system with these parameters. Then we have the recurrence,

$$r(s, d, m, k) \leq r(s - 1, d, m, k) + 2r(s - 1, d, m - 1, k), \qquad m \leq k.$$

Using the Oleinik–Petrovsky–Thom–Milnor bound for algebraic sets, we have

$$r(0, d, m, k) = (O(d))^k, \qquad r(s, d, 0, k) = (O(d))^k.$$

It follows easily that $r(s, d, m, k) = \binom{s}{m}(O(d))^k$. $\qquad\qquad\square$

Lemma 7. *Let S be a* compact *semi-algebraic set contained in the zero set of a polynomial Q and defined by a family of polynomials $\mathcal{P} = \{P_1, \ldots, P_s\}$. Suppose that Q is nonnegative everywhere, the combinatorial level of the system $(Q, \mathcal{P})$ is bounded by $m \leq k$, and the degrees of the polynomials Q and P_i are bounded by $2d$ and d, respectively. Then $r(S) = s^m (O(d))^k$.*

Proof. Let S be defined by the conjunct $(Q = 0) \wedge (\sigma_1 \vee \cdots \wedge \sigma_L)$ where the σ_i are sign conditions on the family $\mathcal{P}$. Let $\varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_s > 0$ be infinitesimals. Consider the sets T_1, U_1, V_1, W_1', W_1 defined as follows:

$$
\begin{aligned}
T_1 &= S \cap ((P_1 \geq \varepsilon_1) \cup (P_1 \leq -\varepsilon_1)), \\
U_1 &= S \cap (P_1 = \varepsilon_1), \\
V_1 &= S \cap (P_1 = -\varepsilon_1), \\
W_1' &= S \cap (-\varepsilon_1 \leq P_1 \leq \varepsilon_1),
\end{aligned}
$$

and

$$W_1 = S \cap (P_1 = 0).$$

Now $S = T_1 \cup W_1'$, and it is clear that $T_1 \cap W_1' = U_1 \cup V_1$ and $U_1 \cap V_1 = \emptyset$.

Using inequality (2) twice, along with the fact that $r(\emptyset) = 0$, we have

$$r(S) \leq r(T_1) + r(W_1') + r(U_1) + r(V_1).$$

Moreover, using the same arguments as in Lemmas 2 and 4, it can be shown that $H_*(W_1') \cong H_*(W_1)$. We omit the proof as the arguments are completely analogous.

Thus, we have

$$r(S) \leq r(T_1) + r(W_1) + r(U_1) + r(V_1).$$

Note that U_1, V_1, W_1 are defined by the systems $(Q + (P_1 - \varepsilon_1)^2, \mathcal{P})$, $(Q + (P_1 + \varepsilon_1)^2, \mathcal{P})$, $(Q + P_1^2, \mathcal{P} \setminus \{P_1\})$ (note that Q is nonnegative everywhere), respectively. Moreover, each of the above system has a combinatorial level at most $m - 1$.

We next consider T_1 which is defined by a set of sign conditions without the atom $P_1 = 0$, and eliminate the atom $P_2 = 0$. We do this by replacing $P_2 < 0$, $P_2 > 0$ by $P_2 \leq -\varepsilon_2$, $P_2 \geq \varepsilon_2$, respectively. In this way we obtain the inequality

$$r(T_1) \leq r(T_2) + r(W_2) + r(U_2) + r(V_2),$$

where U_2, V_2, W_2 are sets defined by systems with a combinatorial level at most $m - 1$.

The remaining set, T_2, has the same homology as the union of the sets defined by those sign conditions appearing in the definition of S, which contain neither $P_1 = 0$ nor $P_2 = 0$.

We continue this process till we have eliminated $P_s = 0$, and we get the inequality

$$r(S) \leq r(T_s) + \sum_{1 \leq i \leq s} (r(U_i) + r(V_i) + r(W_i)).$$

The sets U_i, V_i, W_i are defined by systems of at most s polynomials having a combinatorial level at most $m - 1$. Moreover, the remaining term $r(T_s)$ is the bound on the Betti numbers of a semi-algebraic set defined by a union of sign conditions of the form $Q = 0$, $P_i s_i 0$ for $1 \leq i \leq s$, with $s_i \in \{<, >\}$.

Again, by Lemma 5, the Betti numbers of this set are the sum of the the Betti numbers of the nonempty sets defined by each individual sign condition. Now consider the set T defined by $Q = 0$, $P_1^2 > 0, \ldots, P_s^2 > 0$. From the above remark it is clear that $r(T_s) \leq r(T)$. Moreover, applying the bound proved in Lemma 6 we have $r(T) = \binom{s}{m}(O(d))^k$.

Let $r(s, d, k, m)$ denote the sum of the Betti numbers of a semi-algebraic set defined by a system $(Q, \mathcal{P})$ with $|\mathcal{P}| = s$, a combinatorial level of the system bounded by m, and the degrees of the polynomial Q and those in $\mathcal{P}$ bounded by d. Then we have the recurrence

$$r(s, d, k, m) = \binom{s}{m}(O(d))^k + 3sr(s, d, k, m - 1), \qquad m \leq k,$$

$$r(s, d, k, 0) = (O(d))^k.$$

This recurrence solves to $r(s, d, k, m) = s^m(O(d))^k, m \leq k$, which proves the lemma. $\qquad\square$

The theorem now follows since the combinatorial level of a system $(Q, \mathcal{P})$ with the family $\mathcal{P}$ in general position with respect to Q is bounded by k'. $\qquad\square$

4.   A Fewnomial Bound

In this section we prove a bound on the sum of the Betti numbers of a closed semi-algebraic set, in which the algebraic part of the complexity depends only on the number of monomials appearing in the polynomials and is independent of the degree. This result is a consequence of the fewnomial bounds due to Khovansky [20] and the technique used to prove Theorem 1.

The following proposition is crucial for the proving the bound and is used as well in our algorithms to compute the Euler characteristic. Let S be a compact, basic, semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. The next proposition proves that S has the same homology groups as a certain set which is bounded by a smooth hypersurface.

Proposition 4. *Let $Q = \prod_{1 \leq i \leq s}(\zeta + (1 - \zeta)P_i) - \zeta^{s+1}(X_1^{2d'} + \cdots + X_k^{2d'} + 1)$, where ζ is a new variable, and $2d'$ is any even number greater than $\sum_{1 \leq i \leq s} \deg(P_i)$. Let S_ζ be the set defined by $(Q \geq 0) \wedge_{1 \leq i \leq s} ((1 - \zeta)P_i + \zeta \geq 0)$. Then, for all sufficiently small $\zeta > 0$, $H_*(S) = H_*(S_\zeta)$. Moreover, S_ζ is bounded by connected components of a smooth hypersurface $Z(Q)$, which has a finite number of critical points for the projection map onto the X_1 coordinate and these critical points are nondegenerate and have distinct X_1 coordinates.*

Proof. Let $S_\zeta = (Q \geq 0) \wedge_{1 \leq i \leq s} ((1 - \zeta)P_i + \zeta \geq 0)$. Since S is compact there exists a constant R such that $x \in S \Rightarrow x_1^{2d'} + \cdots + x_k^{2d'} + 1 < R$. For $0 < \zeta < 1/R$, any point $x = (x_1, \ldots, x_k)$ satisfying $P_1(x) \geq 0, \ldots, P_s(x) \geq 0$ will also satisfy $Q \geq 0$. This follows directly from the definition of Q and the fact that $x_1^{2d'} + \cdots + x_k^{2d'} + 1 < R$.

Thus, for every connected component C of S there exists a connected component C' of $Q \geq 0$ such that $C \subset C'$. Moreover, the signs of the polynomials $\zeta + (1 - \zeta)P_i$, $1 \leq i \leq s$, cannot change over C' because if one of them became zero Q would be negative at that point. However, since C' contains C, and ζ is sufficiently small, it is clear that $\zeta + (1 - \zeta)P_i > 0$, $1 \leq i \leq s$, over C'. Thus, for ζ small enough, $S \subset S_\zeta$.

Moreover, $S = \bigcap_{t>0} S_t$, and S and S_t are compact. Thus using Hardt's triviality and Theorem 4, we have that, for small enough $\zeta > 0$, $H_*(S) \cong H_*(S_\zeta)$.

We next show that the set S_ζ is bounded by connected components of the smooth hypersurface defined by $Q = 0$.

First observe that the set $Q \geq 0$ is bounded. This follows from the fact that $2d' > sd$ and thus the second term in Q dominates the first as $|x|$ becomes large.

Secondly, the polynomials $\zeta + (1 - \zeta)P_i$ are all strictly positive over S_ζ. Hence, S_ζ must be bounded by the hypersurface $Z(Q)$.

It remains to show that the hypersurface $Z(Q)$ is smooth and has a finite number of critical points for the projection map onto the X_1 coordinate, and that these critical points are nondegenerate with distinct X_1 coordinate.

Let $Q_t = \prod_{1 \leq i \leq s}(t + (1 - t)P_i) - t^{s+1}(X_1^{2d'} + \cdots + X_k^{2d'} + 1)$.

Now, $Z(Q_t)$ is smooth if the following system of equation has no solutions over complex projective space:

$$\bar{Q}_t = \frac{\partial \bar{Q}_t}{\partial X_0} = \cdots = \frac{\partial \bar{Q}_t}{\partial X_k} = 0,$$

where $\bar{Q}_t$ is the homogenization of Q_t.

The set T of complex t's for which this system has no solutions in complex projective space is Zariski constructible, open, and contains $t = 1$. Hence, it contains ζ for all sufficiently small real $\zeta > 0$, and thus $Z(Q)$ is smooth.

Also note that, for all $t \neq 0$, the ideal generated by

$$\left\{ Q_t, \frac{\partial Q_t}{\partial X_2}, \ldots, \frac{\partial Q_t}{\partial X_k} \right\}$$

is zero-dimensional. This follows from a simple Grobner basis argument which can be found in [3]. Moreover, for $t = 1$, all distinct projective solutions over the algebraic closure of the zero-dimensional homogenized system

$$\bar{Q}_t = \frac{\partial \bar{Q}_t}{\partial X_2} = \cdots = \frac{\partial \bar{Q}_t}{\partial X_k} = 0$$

have distinct X_1 coordinates. Moreover, all these critical points are nondegenerate (the Hessian matrix is nonzero at these points).

Again, the set of t's for which the above conditions are met is Zariski constructible, open, and contains $t = 1$, and hence contains ζ for all sufficiently small $\zeta > 0$. $\qquad\square$

We also need the following proposition.

Proposition 5. *Let Q be the same polynomial as in the previous proposition and let $Q_1 = X_{k+1}^{2d'} - Q$, where X_{k+1} is a new variable.*

Let $S_1 \subset R^{k+1}$ be the set defined by $(Q_1 = 0) \wedge_{1 \leq i \leq s} ((1 - \zeta)P_i + \zeta \geq 0)$, let $S_2 \subset R^k$ be the set defined by $(Q = 0) \wedge_{1 \leq i \leq s} ((1 - \zeta)P_i + \zeta \geq 0)$, and let $S_3 \subset R^k$ be the set defined by $(Q \geq 0) \wedge_{1 \leq i \leq s} ((1 - \zeta)P_i + \zeta \geq 0)$. Then, for all sufficiently small $\zeta > 0$, $\chi(S_1) = 2\chi(S_3) - \chi(S_2)$.

Moreover, the connected components of S_1 are the connected components of a smooth hypersurface $Z(Q_1)$, which has a finite number of critical points for the projection map onto the X_1 coordinate and these critical points are nondegenerate and have distinct X_1 coordinates.

Proof. Firstly, note that $Z(Q_1)$ consists topologically of two copies of the set defined by $Q \geq 0$ glued along $Z(Q)$. Moreover, S_1, S_2, S_3 are all compact and unions of connected components of the sets defined by $Q_1 = 0, Q = 0, Q \geq 0$, respectively. A straightforward application of the Mayer–Vietoris sequence gives $\chi(S_1) = 2\chi(S_3) - \chi(S_2)$.

The proof of the second part of the proposition is entirely similar to the proof of the corresponding properties for $Z(Q)$ given earlier. $\qquad\square$

We first state the theorem for an algebraic set. This theorem appears in [20], but without a precise bound.

Theorem 5. *Let $Z(Q) \subset R^k$ be a real algebraic set, and let m be the number of distinct monomials that appear in Q. Then the sum of the Betti numbers of $Z(Q)$ is bounded by $2^{O(k^2 m^4)}$.*

Proof. Replace $Z(Q)$ by the set V defined by $Q_1 = (\zeta + (1-\zeta)Q^2) - \zeta^2(X_1^{2d} + \cdots + X_k^{2d} + 1) \geq 0$, $(\zeta + (1-\zeta)Q^2) \geq 0$ where $2d > \deg(Q)$, and $\zeta > 0$ and sufficiently

small. Then as in the proof of Proposition 4 we can show that V is bounded by a smooth hypersurface and has a finite number of nondegenerate critical points for the projection map onto the X_1 coordinates. Moreover, V has the same homology groups as $Z(Q)$. The number of critical points bounds the sum of the Betti numbers of V and hence of $Z(Q)$. Moreover, the number of critical points is bounded by the number of real solutions of the system $Q_1 = \partial Q_1 / \partial X_2 = \cdots = \partial Q_1 / \partial X_k = 0$. The total number of nonzero monomials appearing in this system is bounded by $km^2 + 2k = O(km^2)$.

We can now apply Khovansky's bound on the number of real solutions of a system of k polynomials in k variables [20] to obtain the bound stated in the theorem. $\square$

We can now state and prove the theorem in the general case.

Theorem 6. *Let $S \subset R^k$ be the intersection of a closed semi-algebraic set defined by a quantifier-free Boolean formula without negations involving a family, $\mathcal{P} = \{P_1, \ldots, P_s\}$, of s polynomials, whose atoms are of the form $P_i \sigma 0$, $\sigma \in \{\geq, \leq, =\}$, with the zero set $Z(Q)$, of a polynomial Q. Let the geometric dimension of $Z(Q)$ be k', and let the total number of monomials occurring in the polynomials in $\mathcal{P} \cup \{Q\}$ be bounded by m. Then the sum of the Betti numbers of S is bounded by $s^{k'} 2^{O(k^2 m^4)}$.*

Secondly, if S is defined by $Q = 0$, $P_1 > 0, \ldots, P_s > 0$, we have a slightly tighter bound of $\binom{s}{k'} 2^{O(k^2 m^4)}$.

Proof. The proof is the same as that of Theorem 1 except that in the case of algebraic sets we use the bound in Theorem 5 rather than the Thom-Milnor bound of $O(d)^k$. $\square$

Note that this bound is again separated into a combinatorial part of $s^{k'}$, and an algebraic part which depends only on m and k. Also note that the Thom–Milnor method does not give this bound even for the basic semi-algebraic sets. The reason is that in their method one replaces a basic semi-algebraic set by a set defined by a single polynomial which is a perturbed version of their product. However, the number of monomials in the product of s polynomials is exponentially large in s, and thus the separation of the combinatorial and the algebraic parts is no longer possible.

The above techniques have been extended recently to prove similar bounds on the Betti numbers of semi-Pffafian sets, with separation of combinatorial and algebraic parts [32].

5. Computing the Euler Characteristic

By Proposition 3 we can assume, without loss of generality, that the given semi-algebraic set is compact. If the given set is not compact, then we make the perturbations described in Section 2 and compute the Euler characteristic of the perturbed set. The Euler characteristic of this new set is equal to the Euler characteristic of the original set. The new system will have at most $4s + 1$ polynomials with degrees at most $2d$. Moreover, we now have to compute in a larger ring $Z[\delta, \varepsilon, \Omega]$. However, since we have introduced only three infinitesimals, the asymptotic complexity of the algorithm is not affected.

5.1. *Algorithmic Preliminaries*

In our algorithm we utilize several other algorithms from real algebraic geometry as subroutines. In this section we recall some of the algorithms that we use as subroutines, with appropriate pointers to the literature.

We use a subroutine from [3] that constructs univariate representations of the zeros of a zero-dimensional variety. This subroutine takes as input a Gröbner basis of a zero-dimensional ideal, I, of polynomials in k variables and outputs a set consisting of $(k+2)$-tuples of univariate polynomials $(f, g_0, \ldots, g_k)$ such that the complex zeros of I are among the points obtained by evaluating the rational functions $(g_1/g_0, \ldots, g_k/g_0)$ at the roots of the univariate polynomial f, for all the tuples $(f, g_0, \ldots, g_k)$ in the output. We say that the real points corresponding to the tuple $(f, g_0, \ldots, g_k)$ are *associated* to the tuple, and the tuple itself is a *univariate representation* of these points. Moreover, if the degrees of the polynomials in the input are bounded by d, the degrees of the polynomials in the output as well as the complexity of this subroutine is bounded by $d^{O(k)}$.

We also make use of an algorithm, called the sample points subroutine [3], that computes a finite set of points which intersects every connected component of every nonempty sign condition (referred to as *cells* henceforth) of a family of polynomials $\mathcal{P}$, of size s and degrees bounded by d. The subroutine also outputs the sign vector of the polynomials of $\mathcal{P}$ at each output point. The complexity of this subroutine is $\binom{O(s)}{k}sd^{O(k)} = (s/k)^k sd^{O(k)}$.

Lastly, we make use of a multivariate sign determination subroutine [3]. The input is a system T of polynomial equations in k variables, with a finite number of zeros, along with a Gröbner basis for the ideal generated by the polynomials in T, and a set of s polynomials $\mathcal{P} = \{P_1, \ldots, P_s\}$. The output is the list of nonempty sign conditions $\sigma_1, \ldots, \sigma_M$ of $\mathcal{P}$ at the real zeros of the system T and the numbers $c_1, \ldots, c_M$, where c_i is the number of real zeros of T at which the sign vector of $\mathcal{P}$ is σ_i. Moreover, if the polynomials in the input have degrees bounded by d, the complexity of this subroutine is $d^{O(k)}$.

We also use this subroutine in the special case of computing the index of a symmetric square matrix of size $k \times k$, with polynomial entries, at the real zeros of a zero-dimensional system. Again, if the degrees of the polynomials in the input are bounded by d, the complexity of the subroutine is bounded by $(kd)^{O(k)}$ (see [26]).

5.2. *The Algorithm for a Semi-Algebraic Set Defined by One Sign Condition*

In this section we describe an algorithm for computing the Euler characteristic of a semi-algebraic set defined by one single sign condition on a family of polynomials. Using Lemma 4 we can assume, without loss of generality, that the semi-algebraic set S is defined by $P_1 \geq 0, \ldots, P_s \geq 0$.

From Propositions 5 and 4 it follows that $\chi(S) = (\chi(S_1) + \chi(S_2))/2$. We actually compute $\chi(S_1)$ and $\chi(S_2)$ by computing the indices of the critical points of the projections map to the X_1 coordinate.

The details are as follows:

5.2.1. *Description of the Algorithm.*   Given a basic semi-algebraic set S defined by $P_1 \geq 0, \ldots, P_s \geq 0$, where $\deg(P_i) < d$ for $1 \leq i \leq s$, the algorithm computes $\chi(S)$. We first introduce a positive infinitesimal ζ and construct the following two polynomials:

$$Q = \prod_{1 \leq i \leq s} (\zeta + (1 - \zeta)P_i) - \zeta^{s+1}(X_1^{2d'} + \cdots + X_k^{2d'} + 1),$$

$$Q_1 = X_{k+1}^{2d'} - Q,$$

where X_{k+1} is a new variable.

For the following two zero-dimensional systems, $Q = \partial Q/\partial X_2 = \cdots = \partial Q/\partial X_k = 0$ and $Q_1 = \partial Q_1/\partial X_2 = \cdots = \partial Q_1/\partial X_{k+1} = 0$, we first check, using the univariate representation subroutine and for every real zero p of these systems, whether the polynomials $\zeta + (1 - \zeta)P_1, \ldots, \zeta + (1 - \zeta)P_s$ are all nonnegative at the point p. We retain only those real solutions for which this is satisfied.

For each real zero (critical point) p retained so far for the first system, we compute the index of the critical point, which is the index of the Hessian matrix $(\partial^2 Q/\partial X_i \, \partial X_j)_{ij}$, $2 \leq i, j \leq k$, evaluated at p. For $0 \leq i \leq k$ let c_i denote the number of critical points of index i.

Similarly, for each real zero (critical point) p retained so far for the second system, we compute the index of the critical point, which is the index of the Hessian matrix $(\partial^2 Q_1/\partial X_i \, \partial X_j)_{ij}$, $2 \leq i, j \leq k + 1$, evaluated at p. For $0 \leq i \leq k + 1$ let c_i denote the number of critical points of index i.

We output $\chi(S) = (\sum_{0 \leq i \leq k}(-1)^i c_i + \sum_{0 \leq i \leq k+1}(-1)^i d_i)/2$.

5.2.2. *Proof of Correctness.*   The proof correctness follows immediately from Propositions 4 and 5 and the standard facts of Morse theory [22].

5.2.3. *Complexity of the Algorithm.*   The polynomials Q and Q_1 have degrees bounded by $O(sd)$. The cost of computing the indices of the Hessian matrices in the last step dominates the cost. We use the sign determination subroutine in the special case of computing the index of a symmetric square matrix of size $k \times k$, with polynomial entries, at the real zeros of a zero-dimensional system. If the degrees of the polynomials in the input are bounded by D, the complexity is bounded by $(kD)^{O(k)}$ (see [26]). Thus the total complexity of this algorithm is bounded by $(ksd)^{O(k)}$.

5.3.   *The Case of a General Semi-Algebraic Set*

5.3.1. *Description of the Algorithm.*   Using the algorithm in [3] list all nonempty sign conditions, $\sigma_1, \ldots, \sigma_m$ of the family $\mathcal{P}$, such that the set S_j defined by σ_j is contained in S for $1 \leq j \leq m$.

For each j, $1 \leq j \leq m$, do the following. Without loss of generality assume that σ_j is of the form $P_1 = \cdots = P_l = 0, P_{l+1} > 0, \ldots, P_s > 0$. Let U_j be the set defined by $P_1 = \cdots = P_l = 0, P_{l+1} \geq 0, \ldots, P_s \geq 0$, and let $V_j = U_j \cap Z(\prod_{l < i \leq s} P_i)$.

Using the algorithm described in Section 5.2 for computing the Euler characteristic of basic semi-algebraic sets, compute $\chi(U_j)$ and $\chi(V_j)$.

Output $\chi(S) = \sum_{1 \leq j \leq m} (\chi(U_j) - \chi(V_j))$.

5.3.2. *Proof of Correctness.*   First note that the set S is compact.

Let Σ be the set of nonempty sign conditions on the family of polynomials $\mathcal{P} \cup \{Q\}$, whose realizations are contained in S. Also, let S_σ be the set of points satisfying the sign condition σ. Thus,

$$S = \bigcup_{\sigma \in \Sigma} S_\sigma.$$

For any set A, following [31], we define $\beta_i'(A)$ to be the rank of the ith homology group $H_i(\bar{A}, \partial A)$, where $\bar{A}$ is the closure of A in the topology of R^k and $\partial A = \bar{A} - A$. Similarly we define $\chi'(A) = \sum_{0 \leq i \leq k} (-1)^i \beta_i'(A)$. If A is compact, then $\beta_i'(A) = \beta_i(A)$ and $\chi'(A) = \chi(A)$.

We first prove a preliminary lemma. Recall that a semi-algebraic set is *semi-closed* if it is the difference of two closed semi-algebraic sets.

Lemma 8.   *Let X be a semi-algebraic set in R^k which is bounded and semiclosed. For some polynomial*

$$f \in R[X_1, \ldots, X_k],$$

let $A = X \cap (f \geq 0)$ and $B = X - A$. Then $\chi'(X) = \chi'(A) + \chi'(B)$.

Proof.   The proof is the same as the proof of Lemma 5 in [31] and is reproduced here.

From the exact sequence

$$\cdots \to H_i(\bar{A} \cup \partial X, \partial X) \to H_i(\bar{X}, \partial X) \to H_i(\bar{X}, \bar{A} \cup \partial X) \to \cdots,$$

we have $\chi(\bar{X}, \partial X) = \chi(\bar{X}, \bar{A} \cup \partial X) + \chi(\bar{A} \cup \partial X, \partial X)$. Moreover, $\beta_i(\bar{X}, \bar{A} \cup \partial X) = \beta_i'(B)$ and $\beta_i(\bar{A} \cup \partial X, \partial X) = \beta_i'(A)$ (see [31] for a proof). It follows easily that $\chi'(X) = \chi'(A) + \chi'(B)$.   □

Next, we prove

Lemma 9. $\chi(S) = \sum_{\sigma \in \Sigma} \chi'(S_\sigma)$.

Proof.   Note that since S is compact, $\chi'(S) = \chi(S)$. Also note that for all i, and for all sign conditions $P_1 \sigma_1 0, \ldots, P_i \sigma_i 0$, on the family of polynomials $\{P_1, \ldots, P_i\}$, the set $S \cap (P_1 \sigma_1 0) \cap \cdots \cap (P_i \sigma_i 0)$ is semiclosed. This follows from the fact that S is closed. Now consider the sets $A = S \cap (P_1 > 0)$, $B = S \cap (P_1 = 0)$, $C = S \cap (P_1 < 0)$. Now $S = A \cup B \cup C$. Applying Lemma 8 twice we get that $\chi'(X) = \chi'(A) + \chi'(B) + \chi'(C)$. Also, A, B, C are themselves semiclosed. We can continue the argument with the polynomial P_2 and the sets $X \cap (P_2 > 0)$, $X \cap (P_2 = 0)$, $X \cap (P_2 < 0)$ for $X = A, B, C$, and so on. A simple induction completes the argument.   □

It is easy to show, following the notation introduced in the algorithm, that $\beta_i'(S_j) = \beta_i(U_j, V_j)$, and thus $\chi'(S_j) = \chi(U_j, V_j)$. The proof of this appears on p. 621 of [31] and is omitted here.

It follows from the exact sequence

$$\cdots \rightarrow H_i(V_j) \rightarrow H_i(U_j) \rightarrow H_i(U_j, V_j) \rightarrow H_{i-1}(V_j) \rightarrow \cdots,$$

that $\chi(U_j, V_j) = \chi(U_j) - \chi(V_j)$. This in conjunction with Lemma 9 shows that $\chi(S) = \sum_{1 \le j \le m}(\chi(U_j) - \chi(V_j))$, and this proves the correctness of the algorithm.

5.3.3. *Complexity Analysis.*   The cost of computing all the nonempty sign conditions of the family $\mathcal{P}$ is $s^{k+1}d^{O(k)}$. Moreover, there can be only $\binom{s}{k}(O(d))^k$ such nonempty sign conditions. For each such sign condition included in S, we call the algorithm for computing the Euler characters of basic semi-algebraic sets twice. The sum of the degrees of the polynomials involved in each such call is $O(sd)$. Thus each call costs $(ksd)^{O(k)}$. Hence, the total complexity of the algorithm is bounded by $(ksd)^{O(k)}$.

The bound in the bit model follows easily once we note that bit sizes of the intermediate values are bounded by $L(skd)^{O(k)}$.

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