Discrete Comput Geom 25:1–22 (2001)
DOI: 10.1007/s004540010071

Discrete & Computational
Geometry

The Honeycomb Conjecture

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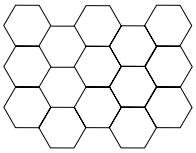
Abstract. This article gives a proof of the classical honeycomb conjecture: any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling.



1. Introduction

Around 36 B.C., Marcus Terentius Varro, in his book on agriculture, wrote about the hexagonal form of the bee's honeycomb [V]. There were two competing theories of the hexagonal structure. One theory held that the hexagons better accommodated the bee's six feet. The other theory, supported by the mathematicians of the day, was that the structure was explained by an isoperimetric property of the hexagonal honeycomb. Varro wrote, "Does not the chamber in the comb have six angles . . . The geometricians prove that this hexagon inscribed in a circular figure encloses the greatest amount of space."

The origin of this problem is somewhat obscure. Varro was aware of it long before Pappus of Alexandria, who mentions it in his fifth book [P1]. Much of Book V follows Zenodorus's much earlier work *Isometric Figures* (ca 180 B.C.). However, only fragments of Zenodorus's book remain, and it is not known whether the honeycomb is discussed there.

The argument in Pappus is incomplete. In fact it involves nothing more than a comparison of three suggestive cases. It was known to the Pythagoreans that only three regular

polygons tile the plane: the triangle, the square, and the hexagon. Pappus states that if the same quantity of material is used for the constructions of these figures, it is the hexagon that will be able to hold more honey. Pappus's reason for restricting to the three regular polygons that tile are not mathematical (bees avoid dissimilar figures). He also excludes gaps between the cells of the honeycomb without mathematical argument. If the cells are not contiguous "foreign matter could enter the interstices between them and so defile the purity of their produce" [H, p. 390].

In part because of the isoperimetric property of the honeycomb, there is is a vast literature through the centuries mentioning the bee as a geometer. Thompson devotes nearly 20 pages to the literature on the bee's cell in [T2]. Some background also appears in [K1, Intro.], [W3] and [W1]. During the 18th century, the mathematical architecture of the honeycomb was viewed as evidence of a great teleological tendency of the universe. Darwin explained the same structures by natural selection. "That motive power of the process of natural selection having been economy of wax; that individual swarm that wasted least honey in the secretion of wax, having succeeded best" [D, p. 235].

The honeycomb problem has never been solved, except under special hypotheses. An unsolved special case of the problem is attributed to Steinhaus in Problem C15 of [CFG]. Extending Pappus's results, in 1943, L. Fejes Tóth proved the honeycomb conjecture under the hypothesis that the cells are convex [FT1]. L. Fejes Tóth predicted that a proof of the honeycomb conjecture without the convexity hypothesis would "involve considerable difficulties" [FT2, p. 183]. Elsewhere, he writes about his proof for convex cells, "There is no doubt that the same is true for general cells. Nevertheless, this conjecture resisted all attempts at proving it" [FT3]. This paper gives a proof without the assumption of convexity.

Convexity is a highly restrictive hypothesis. This hypothesis immediately forces the boundaries of the cells to be polygons. By the isoperimetric inequality, we expect potential counterexamples to be regions bounded by circular arcs. One of the two regions bounded by a positively curved arc will not be convex. Thus, the assumption of convexity eliminates at once almost all the candidates that should be studied the most closely.

The geometrical properties of the three-dimensional honeycomb cells have also been studied extensively. The three-dimensional honeycomb cell is a hexagonal prism built on a base of three congruent rhombuses. The shape of the rhombic base of the three-dimensional cell suggested the rhombic dodecahedron to Kepler, the Voronoi cell of the face-centered cubic lattice. During the 18th century, many mathematicians studied the isoperimetric properties of the base of the cells. MacLaurin, in his analysis of the honeycomb, wrote, in 1743, "The sagacity of the bees in making their cells of an hexagonal form, has been admired of old." "The cells, by being hexagonal, are the most capacious, in proportion to their surface, of any regular figures that leave no interstices between them, and at the same time admit of the most perfect bases" [M1]. In a reversal of MacLaurin's conclusions and upsetting the prevailing opinion, L. Fejes Tóth discovered that the three-dimensional honeycomb cell is not the most economical (that is, it is not surface area minimizing) [FT3].

The honeycomb conjecture is the two-dimensional version of the three-dimensional Kelvin problem. The Kelvin problem asks for the surface minimizing partition of space into cells of equal volume. According to Lhuilier's memoir of 1781, the problem has been described as one of the most difficult in geometry [L1, p. 281]. The solution

proposed by Kelvin is a natural generalization of the hexagonal honeycomb in two dimensions. Take the Voronoi cells of the dual lattice of the lattice giving the densest sphere packing. In two dimensions this is the honeycomb arrangement. In three dimensions this gives truncated octahedra, the Voronoi cells of the body-centered cubic. A small deformation of the faces produces a minimal surface, which is Kelvin's proposed solution.

Phelan and Weaire produced a remarkable counterexample to the Kelvin conjecture. As a result, the honeycomb problem in two dimensions has come under increased scrutiny, and the need for a solution has become more acute. Morgan remarks, "In 1994, D. Weaire and R. Phelan improved on Lord Kelvin's candidate for the least-area way to partition space into regions of unit volume. Contrary to popular belief, even the planar question remains open" [M4].

It seems that the honeycomb is minimal with respect to various optimization problems. Even the classical problem can be expressed as a minimization of perimeter for fixed areas or as a maximization of areas for fixed perimeters. The first of the two presents greater difficulties and is treated here. (See Section 26 of [FT2]).

Morgan states several versions and points out that the versions are not known to be equivalent. Here the situation is similar to the sphere packing problem, which also has several competing versions (find the densest, the solid, or tight packings [CS], the finitely stable or the uniformly stable sphere packings [BBC]). (Also see [K2].) These notions have isoperimetric analogues. However, here the situation is even more perplexing because there is no upper bound on the diameter of the cells of a partition of the plane into equal areas. In this paper we follow the first approach outlined by Morgan (Section 2 of his paper). A topic for future research might be to determine to what extent the methods of this paper can be adapted to the other optimization problems.

Steiner's proofs of the isoperimetric problem were criticized by Weierstrass because they did not prove the existence of a solution. Today, general theorems assuring the existence and regularity of solutions to isoperimetric problems are available. (See [T1], [A], [M2], and compare [B].) This paper depends on these results, assuring the existence of a solution to our isoperimetric problems.

To solve the problem, we replace the planar cluster with a cluster on a flat torus. The torus has the advantages of compactness and a vanishing Euler characteristic. This part of the proof is reminiscent of [FT1], which transports the planar cluster to a sphere. The key inequality, called the hexagonal isoperimetric inequality, appears in Theorem 4. It asserts that a certain functional is uniquely minimized by a regular hexagon of area 1. The isoperimetric properties of the functional force the minimizing figure to be convex. A penalty term prevents the solution from becoming too "round." The optimality of the hexagonal honeycomb results.

2. Statement of the Theorem

We follow [M4] in the formulation of Theorem 1-A. Let $\pi_N = N\tan(\pi/N)$ be the isoperimetric constant for a regular N-gon. That is, $4\pi_N$ is the ratio of the circumference squared to the area of a regular N-gon. The particularly important case, the perimeter

$2\sqrt{\pi_6} = 2\sqrt[4]{12}$ of a regular hexagon with unit area, is used frequently. Let $B(0, r)$ be a disk of radius r at the origin.

Theorem 1-A (Honeycomb Conjecture). *Let Γ be a locally finite graph in $\mathbb{R}^2$, consisting of smooth curves, and such that $\mathbb{R}^2 \backslash \Gamma$ has infinitely many bounded connected components, all of unit area. Let C be the union of these bounded components. Then*

$$\limsup_{r \mapsto \infty} \frac{\text{perim}(C \cap B(0, r))}{\text{area}(C \cap B(0, r))} \geq \sqrt[4]{12}.$$

Equality is attained for the regular hexagonal tile.

The limit is insensitive to compact alterations. Therefore, there is no uniqueness statement for the theorem in this form. The uniqueness of the hexagonal tile appears in the compact version of Theorem 3 below.

Theorem 1-A has stronger hypotheses than necessary. It assumes that the curves are piecewise smooth. Each cell must be connected with unit area. There can be no *interstices* between the cells. Why are disks $B(0, r)$ used for the truncation? Why must the inequality involve lim sup?

We present a second version (1-B) of the theorem that has weaker hypotheses. Before stating the theorem, we discuss the form such a theorem might take. Let $T_1, \ldots, T_k, \ldots$ be a countable sequence of disjoint subsets of $\mathbb{R}^2$, representing the cells of a general cluster. It is natural to assume that for each i, the topological boundary of T_i has finite one-dimensional Hausdorff measure. By a result of Federer, this implies that T_i is measurable and that the current boundary ∂T_i is rectifiable (see Section 2.1 of [M2] and Sections 4.5.12 and 2.10.6 of [F]). In general, the one-dimensional Hausdorff measure of $\bigcup_i \partial T_i$ will be infinite. To get a finite perimeter, we truncate by fixing a compact set $K \subset \mathbb{R}^2$ (for example, a disk of radius ρ). Let $R_i \subset T_i \cap K$ be such that $\mathcal{H}^1(\partial R_i \backslash \partial T_i) = 0$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. For example, we could take R_i to be the union of connected components of T_i contained in K. We can measure the characteristics of the candidate $\{T_i\}$ through the Hausdorff measure of the sets $\bigcup \partial R_i$, and by taking a compact exhaustion of the plane with sets K. To state the honeycomb inequality, it is not necessary to refer to the original cells T_i; it can be formulated in terms of R_i and K, where we now allow the area of R_i to be less than 1. This provides motivation for the honeycomb problem in the following general form.

Theorem 1-B (Honeycomb Conjecture for Disconnected Regions). *Let K be a compact set in the plane containing disjoint measurable sets $R_1, R_2, \ldots$. Assume that each R_i has a rectifiable current boundary ∂R_i. Set $\alpha_i = \min(1, \text{area}(R_i))$. Set $\Gamma = \bigcup_i \partial R_i$. Assume $\alpha_i > 0$ for some i. Then*

$$\mathcal{H}^1(\Gamma) > \sqrt[4]{12} \sum \alpha_i.$$

Asymptotically, this inequality is sharp. For example, take T_i to be the regions of the honeycomb tile, and $K = B(0, \rho)$, a disk of radius ρ. Take $R_i = T_i$, if $T_i \subset B(0, \rho)$, and $R_i = \emptyset$, otherwise. Then $\alpha_i = 0$ or 1, and $\sum \alpha_i$ is the number (or area) of the hexagonal

tiles contained entirely in $B(0, \rho)$. Moreover, $\mathcal{H}^1(\Gamma)$ is asymptotic to $\sqrt[4]{12}\sum \alpha_i$, as $\rho \mapsto \infty$.

We combine regions R_i if the sum of their areas is less than 1. This does not change $\mathcal{H}^1(\Gamma)$ or $\sum \alpha_i$, so that the validity of the theorem is not affected by combining regions. Regions can be combined as long as there are at least two regions with area between 0 and $\frac{1}{2}$. This allows us to assume without generality, when we prove Theorem 1-B, that there are only finitely many $R_1, \ldots, R_n$.

To apply regularity results, we consider an optimization problem that will lead to a lower bound on $\mathcal{H}^1(\Gamma)$. We fix the constants $\alpha_i \in [0, 1]$, for $i = 1, \ldots, n$. Consider the optimization problem of varying $R_1, \ldots, R_n$ so as to minimize

$$\mathcal{H}^1\left(\bigcup \partial R_i\right)$$

subject to the constraint that area$(R_i) \geq \alpha_i$ for $i = 1, \ldots, n$. (We no longer require that $R_i \subset K$.)

By the existence and regularity results of [M2], there exists $\Gamma = \Gamma(\alpha_1, \ldots, \alpha_n)$ that minimizes the one-dimensional Hausdorff measure of the boundary $\bigcup \partial R_i$ of the corresponding regions R_i. The boundary Γ consists of finitely many arcs of circles (possibly reducing to straight lines) meeting at vertices of degree 3. (Morgan formulates the optimization problem with equality constraints area$(R_i) = \beta_i$, where $\beta_i \geq \alpha_i$ are fixed areas. To get the existence and regularity of $\Gamma(\alpha_1, \ldots)$ from this, we apply Morgan's optimization to a set of constants $\beta_i \geq \alpha_i$ giving the shortest perimeter.) The minimizing set Γ is connected [CHH$^+$]. Each connected component of each R_i is simply connected. Let R_0 be the union of the connected components of $\mathbb{R}^2 \backslash \Gamma$, other than the components of $R_1, \ldots, R_n$. R_0 is connected (otherwise remove edges between R_i and R_0 to shorten Γ and increase area(R_i)).

Theorem 2 (Honeycomb Conjecture, Finite Version). *Assume* $0 < \alpha_i \leq 1$ *for* $i = 1, \ldots, n$. *Let* $A = \sum_i \alpha_i$. *Then*

$$\text{perim}(\Gamma(\alpha_1, \ldots, \alpha_n)) > A\sqrt[4]{12}.$$

Remark 2.1. By an argument in [M4], the case $A = \sum_i \alpha_i \leq 398$ is elementary. Assume we have $R_1, \ldots, R_n$ with area$(R_i) \geq \alpha_i \in [0, 1]$. Applying the isoperimetric inequality to each R_i and then applying it once again to the union of the R_i, we find

$$\begin{aligned} 2\text{perim}(\Gamma(\alpha_1, \ldots, \alpha_n)) &\geq \sum 2\sqrt{\pi\alpha_i} + 2\sqrt{\pi A} \\ &\geq 2(\sqrt{\pi} + \sqrt{\pi}/\sqrt{A})A \\ &> 2\sqrt[4]{12}A. \end{aligned}$$

Lemma 2.2 (Morgan). *Theorem* 2 *implies Theorem* 1-A.

Proof. We consider the particular case $\alpha_1 = \cdots = \alpha_n = 1$. Let

$$\rho_n = \text{perim}(\Gamma(1, \ldots, 1))/n.$$

Let ρ_∞ be the infimum of the left-hand side of the inequality of Theorem 1, as C runs over all planar clusters satisfying the conditions of the theorem. Let C be the hexagonal honeycomb tiling to see that $\sqrt[4]{12} \geq \rho_\infty$. Theorem 2 gives

$$\liminf_n \rho_n \geq \sqrt[4]{12} \geq \rho_\infty.$$

By Section 2.1 of [M4], we have $\rho_\infty \geq \liminf_n \rho_n$. The result follows. □

Remark 2.3. In the third edition of [M3], Morgan extends his truncation lemma [M4, Section 2.1] to areas less than 1. This permits a generalization of Theorem 1-A to cells that are not connected.

Lemma 2.4. *Theorem* 2 *implies Theorem* 1-B.

Proof. If Γ is the current boundary in Theorem 1-B, then the optimization problem described above yields $\Gamma(\alpha_1, \ldots, \alpha_n)$. Its perimeter gives a lower bound on the one-dimensional Hausdorff measure of Γ. □

Remark 2.5. Assume that Γ is a finite connected collection of analytic arcs, and that Γ is the boundary of (finite unions of) simply connected bounded regions R_i. Let $\alpha_i = \min(1, \text{area}(R_i))$, and set $A = \sum_i \alpha_i$. Set

$$F(\Gamma, A) = \text{perim}(\Gamma) - A\sqrt[4]{12}.$$

Theorem 2 is false iff $F(\Gamma, A) \leq 0$ for some Γ, A. To check Theorem 2, we can therefore make a finite number of modifications that decrease the value of $F(\Gamma, A)$.

Remark 2.6. We claim that for the proof of Theorem 2, we can assume that the regions R_i are connected. Each R_i is a disjoint union of connected components R_{ij}. Let $\alpha_{ij} = \min(1, \text{area}(R_{ij}))$ and $A' = \sum_{ij} \alpha_{ij}$. We have $A \leq A'$. Apply Remark 2.5.

Remark 2.7. Let $a_0 = 2\pi\sqrt{3}/3 = 4\pi/\pi_6$. If R is one of the regions, let $M = M(R)$ be the number of sides that its boundary has (counted by the number of different regions neighboring R along analytic arcs). We claim that we may assume that each R satisfies

$$\text{area}(R) \geq a_0/M^2.$$

In fact, let R be a region of area a less than a_0/M^2. By the isoperimetric inequality, the perimeter of R is at least $2\sqrt{\pi a}$, so it shares at least the length $(2/M)\sqrt{\pi a}$ with some neighboring region. Let Γ' be the collection of analytic arcs obtained by deleting the edges shared with this neighboring region. By deleting these edges, we "pop" the bubble R and combine it with the neighboring region, so that Γ' bounds one fewer region than Γ. If A' is the sum of the α_i with respect to Γ', then $A' \geq A - a$. We claim that

$F(\Gamma, A) > F(\Gamma', A')$. In fact,

$$\begin{aligned} F(\Gamma, A) - F(\Gamma', A') &= \operatorname{perim}(\Gamma) - \operatorname{perim}(\Gamma') + (A' - A)\sqrt[4]{12} \\ &\geq (2/M)\sqrt{\pi a} - a\sqrt[4]{12} \\ &= \sqrt{a}\left((2/M)\sqrt{\pi} - \sqrt{a}\sqrt[4]{12}\right) \\ &> \sqrt{a}M^{-1}(2\sqrt{\pi} - \sqrt{a_0}\sqrt[4]{12}) \\ &= 0. \end{aligned}$$

Deleting edges in this way may lead to regions that are not simply connected. However, when an edge is deleted, arbitrarily short edges can be added to make the regions simply connected again. (Translate the boundary components of Γ so that they are arbitrarily close to one another, and then connect them with a short edge.) We can choose this short edge to be so short that this modification to (Γ', A') still gives a value of F less than the original $F(\Gamma, A)$.

3. Honeycombs on a Torus

Let $\mathbb{R}^2/\Lambda$ be a torus of area at least 1. Take a partition of the torus into a finite number of simply connected regions. Assume that the boundary consists of a finite number of simple rectifiable curves, meeting only at endpoints.

In view of Remark 2.7, we define $a(M) = \min(2\pi\sqrt{3}/(3M^2), 1)$. Assume the connected, simply connected regions are $R_1, \ldots, R_n$, and assume that $\operatorname{area}(R_i) \geq a(M_i)$.

Theorem 3 (Honeycomb Conjecture on a Torus).

$$\operatorname{perim}\left(\bigcup \partial R_i\right) \geq \sum_{i=1}^{n} \alpha_i \sqrt[4]{12}.$$

Equality is attained if and only if every R_i is a regular hexagon of area 1 *and each* $\alpha_i = 1$.

This follows as an immediate consequence of an isoperimetric inequality proved in the next section. The existence of a honeycomb tiling depends on the shape and size of the lattice: Λ must be a sublattice of a lattice formed by the tiling by unit area regular hexagonal tiles. In particular, the area of $\mathbb{R}^2/\Lambda$ must be an integer.

Lemma 3.1. *This theorem implies the honeycomb conjecture for a finite number of cells* (*Theorem* 2).

Proof. Consider $(\Gamma(\alpha_1, \ldots, \alpha_n), A = \sum \alpha_i)$ appearing in Theorem 2. By Remarks 2.6 and 2.7, we may modify the example so that the regions in the example are connected, simply connected, and such that each region R satisfies the inequality

$$\operatorname{area}(R) \geq 2\pi\sqrt{3}/(3M^2) \geq a(M).$$

Let (Γ', A') denote this modification. If we have $A' \leq 398$, then Theorem 2 follows from Remark 2.1. Assume that $A' \geq 398$. To complete the proof of the lemma, we move the cluster to the torus. This involves adding an additional edge of length $\sqrt{1/A'}$ and an additional region of area at least 1. Theorem 3 applied to this situation gives the inequality

$$\mathrm{perim}(\Gamma') + \sqrt{1/A'} \geq (1 + A')\sqrt[4]{12}.$$

When $A' \geq 398$, this yields the inequality of Theorem 2.

Before moving the cluster to a torus, we first move it to a cylinder. Pick a diameter to the cluster (a segment between maximally separated points p_1 and p_2 on Γ). Then move the cluster to the cylinder $\mathbb{R}^2/\mathbb{Z}v$, where v is the translation along the length of the diameter. The map of the cluster to the cylinder is injective, except at the points p_1 and p_2, which become identified. Since p_1 and p_2 are maximally separated, the cluster fits inside a square of edge length $|v|$ with a pair of sides parallel to v. Thus the area of the cluster is at most the area $|v|^2$ of the square. To simplify notation, we now drop the primes from (Γ', A'). This gives $|v| \geq \sqrt{A}$, with $A = \sum \alpha_i$, and $\alpha_i = \min(1, \mathrm{area}(R_i))$.

Let w be a unit vector perpendicular to v. Pick $\mu > 0$ to be the largest real number for which $\Gamma + \mu w$ touches Γ without overlap. Let

$$\Lambda = \mathbb{Z}v + \mathbb{Z}(\mu + \sqrt{1/A})w.$$

The cluster descends to $\mathbb{R}^2/\Lambda$, injective except at p_1 and p_2. We add a segment of length $\sqrt{1/A}$ to join the cluster with its translate. On the torus, the region "at infinity" becomes simply connected. Call it R_0. By adding this extra edge, we avoid the complications of a component with a loop representing a nontrivial homology class in $\mathbb{R}^2/\Lambda$.

Since $|v| \geq \sqrt{A}$, and the component R_0 has height at least $\sqrt{1/A}$ in the direction of w at every point, the area of R_0 is at least 1. Now we have a partition of the torus $\mathbb{R}^2/\Lambda$ into connected, simply connected regions of total area at least $1 + A$. □

3.1. *Torus Modifications*

The combinatorial structure is described by a finite torus graph in which each face is simply connected.

Loops (edges joined at both ends to the same vertex) can be eliminated from the graph as follows. If the vertex has degree greater than 3, then it can be considered a limit of multiple vertices of degree 3 and edges of length 0. This can be done in such a way that the vertex on the loop has multiplicity at least 2. If the vertex has degree 3, let e be the other edge that meets the loop at the vertex. Both sides of the edge e bound the same region E. Removing e leads to a nonsimply connected component. The loop Γ_1 can be moved arbitrarily close to another boundary component Γ_2. It can be joined to Γ_2 with two edges (one edge moving from a point vertex v_1 of Γ_1 down to w on Γ_2, and another edge moving back to v_2 in the opposite direction, as in Fig. 1). This decreases the area of E, which is not permissible. However, by scaling the entire torus by a homothety $(x + \Lambda \mapsto tx + t\Lambda)$, its area is restored. Since we can make the decrease in area arbitrarily small, we can make the increase in perimeter due to the homothety arbitrarily small. In particular, we can arrange that the total increase in perimeter by this

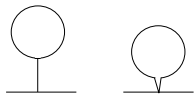


Fig. 1

process is offset by the length of the edge e that was removed. (Alternatively, we could let $v_1 = v_2$, and let the return path to v_1 be the same as the path from v_1 to w in the opposite direction. Then v_1 is considered to be a vertex of multiplicity 2, etc.)

If the torus graph has any vertices of degree greater than 3, we view them as vertices of degree 3 with degenerate edges of length 0. We can do this in a way that avoids creating any new loops. Each vertex has degree 3.

We are now in the situation where the boundary of every region P is a combinatorial N-gon, for some $N \geq 2$, where $N = N(P)$ is the number of directed edges bounding the region. (The boundary of P might traverse a segment twice in opposite directions.) We have the Euler characteristic of the torus graph

$$0 = V - E + F = \sum_P (1 - N(P)/6), \tag{3.1}$$

where P runs over the regions.

4. A Hexagonal Isoperimetric Inequality for Closed Plane Curves

Let Γ be a closed piecewise simple rectifiable curve in the plane. In our application we take Γ to be a lift from the torus to the plane of one of the ∂R_j from Section 3.

We use the parametrization of the curve to give it a direction, and use the direction to assign a signed area to the bounded components of the plane determined by the curve. For example, if Γ is a piecewise smooth curve, the signed area is given by Green's formula

$$\int_\Gamma x\, dy.$$

Generally, we view Γ as an integral current [M3, p. 44]. We let P be an integral current with boundary Γ. (In applications, $P = R_j$, for some j.) Expressed differently, we give a signed area by assigning a multiplicity $m(U) \in \mathbb{Z}$ to each bounded component U of $\mathbb{R}^2 \backslash \Gamma$. (An illustration appears in Fig. 2.) The area is $\sum m(U)\text{area}(U)$. P is represented by the formal sum $P = \sum m(U)U$.

Let $v_1, \ldots, v_t$, $t \geq 2$, be a finite list of points on Γ. We do not assume that the points are distinct. Index the points $v_1, v_2, \ldots, v_t$, in the order provided by the parametrization of Γ_i. Join v_i to v_{i+1} by a directed line segment f_i (take $v_{t+1} = v_1$). The chords f_i form a generalized polygon, and from the direction assigned to the edges, it has a signed area $A_P \in \mathbb{R}$.

Let e_i be segment of Γ between v_i to v_{i+1}. Let f^{op} be the chord f with the orientation reversed. Let $x(e_i) \in \mathbb{R}$ be the signed area of the integral current bounded by $(e_i,\ f_i^{\text{op}})$. Let $E(P) = \{e_i\}$ denote the set of edges of P. Accounting for multiplicities and orientations,

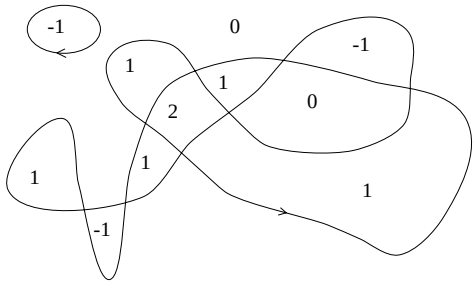


Fig. 2

we have

$$\text{area}(P) = A_P + \sum_{e \in E(P)} x(e).$$

Let $\alpha(P) = \min(1, \text{area}(P))$.

Define a truncation function $\tau\colon \mathbb{R} \to \mathbb{R}$ by

$$\tau(x) = \begin{cases} \frac{1}{2}, & x \geq \frac{1}{2}, \\ x, & |x| \leq \frac{1}{2}, \\ -\frac{1}{2}, & x \leq -\frac{1}{2} \end{cases}$$

Set $\tau_0 = \frac{1}{2}$. Set $T(P) = \sum_{E(P)} \tau(x(e))$. Recall that the perimeter of a regular hexagon of unit area is $2\sqrt[4]{12}$. Let $L(P)$ be the length of Γ. Let $N(P)$ be the number of points v_i on Γ, counted with multiplicities. Recall from Remark 2.7 that $a(N) = \min(2\pi\sqrt{3}/(3N^2), 1)$.

Theorem 4 (Hexagonal Isoperimetric Inequality). *Define P, $L(P)$, $N(P)$, and $a(N)$ as above. Assume that the signed area of P is at least $a(N(P))$. Then*

$$L(P) \geq -T(P)\sqrt[4]{12} - (N(P) - 6)0.0505 + 2\alpha(P)\sqrt[4]{12}.$$

Equality is attained if and only if P is a regular hexagon of area 1.

The theorem will be proved below.

Lemma 4.1. *The hexagonal isoperimetric inequality implies the honeycomb conjecture for a torus* (*Theorem* 3).

Proof. We apply this inequality to the n different regions $R_1, \ldots, R_n$ in the torus partition. We let the points v_i be the endpoints of the simple curves described in Section 3.

Let P be one of the regions R_i. The regions in Theorem 3 satisfy $\text{area}(P) \geq a(M(P))$, where M is the number of different regions bounding P. We have $N \geq M$, so $\text{area}(P) \geq a(M(P)) \geq a(N(P))$. Thus, P satisfies the area constraint of the hexagonal isoperimetric inequality.

Each edge of P occurs with opposite orientation e^{op} on a neighboring region P', and by construction $x(e) + x(e^{\mathrm{op}}) = 0$. Thus, summing over all directed edges of the partition, we get

$$\sum_e x(e) = 0.$$

Since τ is an odd function, we have $\tau(x(e)) + \tau(x(e^{\mathrm{op}})) = 0$, so also

$$\sum_e \tau(x(e)) = \sum_P T(P) = 0. \tag{4.1}$$

By Theorem 4,

$$\begin{aligned} 2\,\mathrm{perim}(\Gamma) &= \sum_P L(P) \\ &\geq -\sqrt[4]{12}\sum_P T(P) + 6(0.0505)\sum_P (1 - N(P)/6) + 2\sum_P \alpha(P)\sqrt[4]{12}. \end{aligned}$$

Thus, the lemma follows from Euler (3.1) and (4.1). □

Set

$$\begin{aligned} \Delta(P) &= L(P) + \varepsilon(N(P), \alpha(P), T(P)), \\ \varepsilon(N, \alpha, T) &= T\sqrt[4]{12} + (N-6)0.0505 - 2\alpha\sqrt[4]{12}, \\ X(P) &= \sum_{E(P)} x(e). \end{aligned}$$

Remark 4.2. The inequality is false without the truncation. For example, let P be constructed as a simple closed curve of area 1 bounded by three inverted circular arcs of the same curvature and the same length (Fig. 3). When the circular arcs are sufficiently long, $L(P) + \varepsilon(3, 1, X(P)) < 0$.

Remark 4.3. If the region P is a polygon of area 1 with v_i as vertices, then $\alpha(P) = 1$, $x(e) = 0$ for all e, and $T(P) = 0$. The inequality in this case is essentially the one used by Phelan and Weaire in [P2].

Remark 4.4. The sharp case of the inequality occurs for $N(P) = 6$, $\alpha(P) = 1$, and $x(e)$ near 0, so that $X(P) = T(P)$. $y = -\varepsilon(6, 1, X)$ is the tangent line to $y = L(P_X)$ at

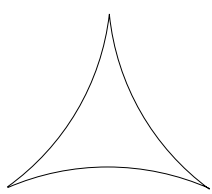


Fig. 3

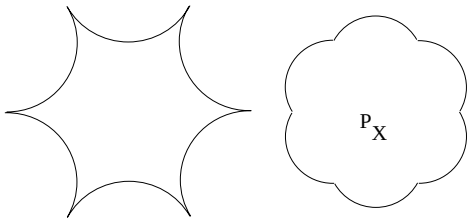


Fig. 4

$X = 0$ of the following region P_X (Figs. 4 and 5). Take a regular hexagon of area $1 - X$ and add six circular arcs of the same curvature to the six edges to make the total area 1 (inverting the arcs if $X < 0$).

5. Preliminaries for the Proof

Without loss of generality, we modify P to decrease the perimeter, maintaining the lower bound on the area, and holding ε fixed, or we may also modify P by fixing the perimeter, maintaining the lower bound on the area, and decreasing ε.

For each chord f, we may apply the isoperimetric inequality to the curve (e, f^{op}) to replace e with a circular arc with the same enclosed signed area. The isoperimetric inequality for integral currents appears in Section 4.5.14 of [F]. Uniqueness and regularity follow along the lines of [M2].

We may replace the polygon $(f_1, \ldots, f_N)$ by a convex polygon with the same edge lengths in the same order, that has at least the area as the original polygon. Thus, we may assume that the polygon has positive area.

If any $x(e) < -\tau_0$, we decrease the curvature of e until $x(e) = -\tau_0$. This leaves $T(P)$ unchanged, decreases the perimeter, and increases area(P). Thus, there is no loss in generality, if we assume $x(e) \geq -\tau_0$ for all e.

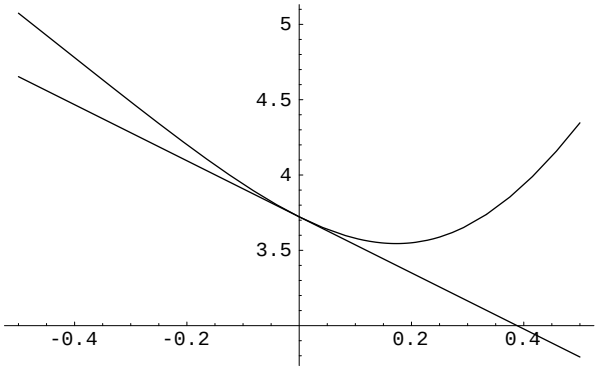


Fig. 5. The graph of $y = L(P_X)$ and its tangent line $y = -\varepsilon(6, 1, X)$.

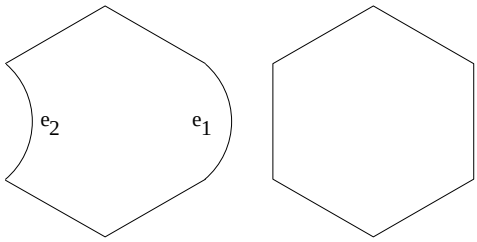


Fig. 6. A deformation reducing the perimeter at fixed area.

We leave edges e with $x(e) > \tau_0$ untouched.

If there are two edges e_1, e_2 of P with $0 < x(e_1) \leq \tau_0$ and $x(e_2) < 0$ (and so also $x(e_2) \geq -\tau_0$), we deform P by decreasing the curvature of the arcs e_1 and e_2 preserving $x(e_1) + x(e_2)$, thereby decreasing the perimeter $L(P)$ (see Fig. 6). $T(P)$ is unchanged. Continuing in this way, we may assume without loss of generality that all $x(e) \in [-\tau_0, \tau_0]$ have the same sign. We consider two cases depending on whether there is an edge e with $x(e) > \tau_0$.

Case I. For all e, $x(e) \geq -\tau_0$, and there exists e such that $x(e) > \tau_0$. All $x(e) \in [-\tau_0, \tau_0]$ have the same sign.

Case II. For all e, $|x(e)| \leq \tau_0$. All $x(e)$ have the same sign.

The theorem will be proved by a separate argument for the two cases.

6. Lower Bounds on $L(P)$

We were not able to find a single estimate of $L(P)$ that leads to the theorem in all cases. Instead, we rely on number of lower bounds on $L(P)$ (and $\Delta(P)$). Most are based on the isoperimetric inequality.

Lower Bound $L(N, \alpha, X)$, $N = 3, 4, \ldots$. The perimeter of a regular N-gon of area 1 is $2\sqrt{\pi_N}$, where $\pi_N = N \tan(\pi/N)$. The polygon $(f_1, \ldots, f_N)$ has area at least $\alpha - X$. By the isoperimetric inequality for polygons, it has perimeter at least

$$L(N, \alpha, X) := 2\sqrt{(\alpha - X)\pi_N}.$$

Each arc e_i has length at least that of f_i, so $L(P) \geq L(N, \alpha, X(P))$.

Lower Bound L_+. By the isoperimetric inequality,

$$L(P) \geq L_+(\alpha) := 2\sqrt{\pi\alpha},$$

the perimeter of a circle of area α.

Lower Bound L_-. Let J be the set of indices of the edges with $x(e_j) < 0$, for $j \in J$. Let $X_J = \sum_J x(e_j)$. By reflecting each edge e_j, $j \in J$, across the chord f_j, P is replaced

with a region of the same perimeter and area at least $\alpha - 2X_J$. By the isoperimetric inequality,

$$L(P) \geq L_-(\alpha, X_J) = 2\sqrt{(\alpha - 2X_J)\pi}.$$

Lower Bound L_D. Assume here that $|x(e)| \leq \tau_0$, for each e. Dido found the curve of minimum length with both endpoints on a given line, subject to the condition that the curve and the line bound a given area. The solution is a semicircle cut through the center by the line. Applying this to an arc e, we find that the length of e is at least $\sqrt{2\pi|x(e)|} \geq |x(e)|\sqrt{2\pi/\tau_0}$. Thus,

$$L(P) \geq L_D(X_D) := X_D\sqrt{2\pi/\tau_0},$$

where $X_D = \sum_e |x(e)|$. In Case II, $X_D = |T| = |X|$.

Lower Bound $L'(N, \alpha, X)$. Only this last bound is new. For $\ell, x \geq 0$, let $\mathrm{arc}(\ell, x)$ be the length of a circular arc chosen so that together with a chord of length ℓ joining its endpoints, the enclosed area is x. For example, $\mathrm{arc}(\ell, 0) = \ell$ and $\mathrm{arc}(0, x) = 2\sqrt{x\pi}$. Let $L(N, \alpha, x)$ be as above. Let

$$L'(N, \alpha, X) = L(N, \alpha, X)\,\mathrm{arc}(1, |X|/L(N, \alpha, X)).$$

Proposition 6.1. *If all the chords f_i of P have length at most* 1, *if* $|X| \leq 0.119$, $0.996 \leq \alpha \leq 1$, *and if* $N \leq 7$, *then*

$$L(P) \geq L'(N, \alpha, X),$$

where $X = X(P)$, $N = N(P)$, $\alpha = \alpha(P)$.

This is proved in the Appendix.

6.1. *Equal Curvature Condition*

Here is a simple observation about the lower bounds on perimeters that we refer to as the *equal curvature condition*. (A version for polygons was known to Zenodorus [H, p. 210].)

Suppose that we have two chords f_1 and f_2 of circular arcs e_1 and e_2. Minimize the sum of the lengths of e_1 and e_2, fixing f_1 and f_2, and constrained so the sum of the two enclosed areas is fixed. Two arcs of equal curvature give the minimum. If an arc is more than a semicircle, it occurs along the chord of greater length (or one of the two if the chords are equal in length).

To see this result, form a triangle with the two chords and a third edge of variable length t. Adjust t until the circumscribing circle gives arcs of the correct combined enclosing area on the two chords. Any shorter perimeter contradicts the isoperimetric inequality.

7. Case I of the Proof of the Hexagonal Isoperimetric Inequality

Digons. Before treating Case I, we treat the case of digons separately for both Cases I and II. Here $A_P = 0$ and $N(P) = 2$, so

$$\text{area}(P) = x(e_1) + x(e_2) \geq \alpha.$$

Also, $\alpha \geq a(N) = a(2) > \frac{1}{4}$. In Case I, $\sum x(e_i) = x_{\max} + x_{\min} \geq \tau_0 - \tau_0 \geq 0$. In Case II, $T(P) = \sum \tau(x(e_i)) = \sum x(e_i) \geq \alpha$.

If $T(P) > 0.21$, then $\Delta(P) > 0$, by the bound L_+: $L(P) \geq 2\sqrt{\pi\alpha} \geq 2\alpha\sqrt{\pi}$.

Assume $T(P) \leq 0.21$. We are now in Case I, so $x_{\max} > \tau_0 = \tau_{\max}$,

$$x_{\min} \leq \tau_{\min} \leq 0.21 - \tau_{\max} = -0.29.$$

By reflecting the arc e corresponding to $x_{\min}$ across the corresponding chord f, the area becomes at least $(\alpha - 2x_{\min})$, without changing the perimeter. We then have

$$L(P) \geq 2\sqrt{\pi(\alpha - 2x_{\min})} \geq 2\sqrt{\pi(\alpha + 2(0.29))}.$$

$\Delta(P) > 0$ follows.

For the rest of the proof, we assume $N(P) \geq 3$, so that, in particular, $\varepsilon(N, \alpha, T) \geq \varepsilon(3, \alpha, T)$.

Case I. We assume that for all e, $x(e) \geq -\tau_0$ and that for some e, $x(e) > \tau_0$. All $x(e)$ satisfying $|x(e)| \leq \tau_0$ have the same sign. In treating this case, we only need to assume that $\alpha \geq 0$, rather than $\alpha \geq a(N)$.

Assume $T(P) > 0.177$. We have the bounds L_+: $L(P) \geq 2\alpha\sqrt{\pi}$ and $\varepsilon(N, \alpha, T) \geq \varepsilon(3, \alpha, 0.177)$. It follows that $\Delta(P) > 0$.

Assume next that $T(P) < -0.36$. There exists $x(e) < 0$. Index so that $x(e_i) > \tau_0$ for $i \in I$ and $x(e_j) \leq 0$ for $j \in J$. Set $X_I = \sum x(e_i)$, $X_J = \sum x(e_j)$, so that $X(P) = X_I + X_J$. The area of P is $X_I + X_J + A_P \geq \alpha$. Let $k = |I|$. We have $T(P) = k\tau_0 + X_J$. By Dido,

$$L(P) \geq \sum_I \sqrt{2\pi|x_i|} + \sum_J \sqrt{2\pi|x_j|} \geq k\sqrt{2\pi\tau_0} - X_J\sqrt{2\pi/\tau_0}.$$

Then

$$\Delta(P) \geq k\sqrt{2\pi\tau_0} - X_J\sqrt{2\pi/\tau_0} + \varepsilon(3, 1, k\tau_0 + X_J).$$

Substituting the upper bound $X_J \leq -0.36 - k\tau_0$ for X_J, and then the lower bound $k \geq 1$ for k, we find that $\Delta(P) > 0$.

Assume finally that $T(P) \in [-0.36, 0.177]$. With the same notation, we have $T(P) = k\tau_0 + X_J \leq 0.177$, which gives $X_J \leq -0.323$. Reflecting the arcs corresponding to negative signed areas as above, we get

$$L(P) \geq 2\sqrt{\pi(\alpha + 2(0.323))}.$$

This gives $\Delta(P) > 0$.

8. Case II of the Proof of the Hexagonal Isoperimetric Inequality

Assume that $|x(e)| \leq \tau_0$ for all e, and that all $x(e)$ have the same sign. Then $X(P) = T(P)$ and $X_D = |X(P)|$.

First, we treat the case $\alpha \in [2\pi\sqrt{3}/(3N^2), \frac{1}{4}]$, and then we treat the case $\alpha \geq \frac{1}{4}$. If $\frac{1}{4} \geq \alpha \geq 2\pi\sqrt{3}/(3N^2)$, we have $N \geq 4$. If $T \geq 0$, we use $L(P) \geq L_+(\alpha)$ to get

$$\Delta(P) \geq 2\sqrt{\pi\alpha} + (N-6)0.0505 - 2\alpha\sqrt{\pi}.$$

The second derivative in α is negative, so it is enough to check that this is positive for $\alpha = 2\pi\sqrt{3}/(3N^2)$, and $\alpha = \frac{1}{4}$. This is elementary.

Next, if $T \in [-2.4, 0]$, we use $L(P) \geq L_-(P)$. We show that the following is positive:

$$2\sqrt{\pi(\alpha - 2T)} + T\sqrt[4]{12} + (N-6)0.0505 - 2\alpha\sqrt[4]{12}.$$

The second derivative in α is negative as well as that for T. Hence, it is enough to check positivity for $T = 0, -2.4$, and $\alpha = 2\pi\sqrt{3}/(3N^2), \frac{1}{4}$. Again, the verification is elementary.

Finally, if $T \leq -2.4$, we use $L(P) \geq L_D$. It is clear that

$$\Delta(P) \geq -T\sqrt{2\pi/\tau_0} + \varepsilon(N, 1, T) \geq 0.$$

This completes our discussion of the case $\alpha \in [2\pi\sqrt{3}/(3N^2), \frac{1}{4}]$.

The rest of Section 8 is devoted to the case $\alpha \geq \frac{1}{4}$. We pick a lower bound $\tilde{L}(N, \alpha, X)$ for $L(P)$ from the stock of lower bounds developed in Section 6, according to the schematic in the (N, X) plane shown in Fig. 7.

The boundary between $L_+(1)$ and $L(N, \cdot)$ is the curve $X_N^+ = 1 - \pi/\pi_N > 0$, determined by the condition $L_+(1) = L(N, 1, X_N^+)$. (For $N = 3$, we set $X_3^+ = 0.177$, instead of $1 - \pi/\pi_3$.) Also, set $X_N^- = (-\pi + \pi_N)/(-2\pi + \pi_N) < 0$, which satisfies $L_-(1, X_N^-) = L(N, 1, X_N^-)$. When $X < 0$, we have $X = X_J$, and $L_-(\alpha, X_J) =$

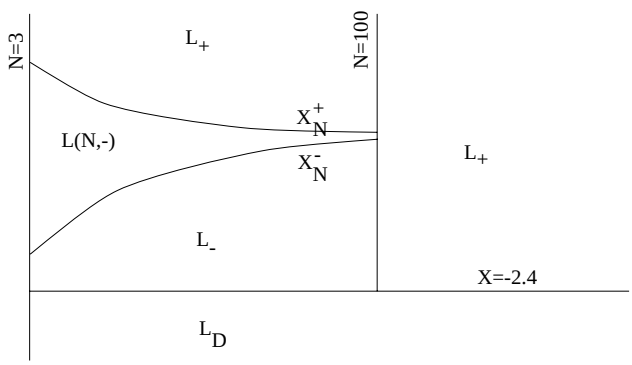


Fig. 7

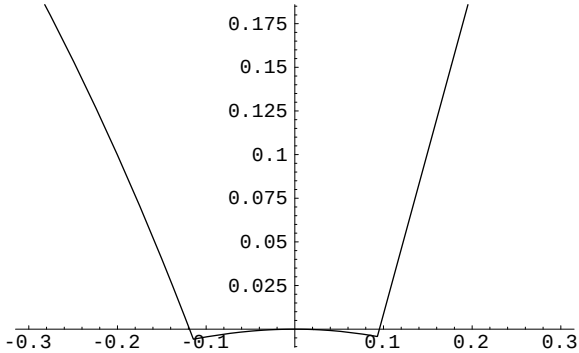


Fig. 8. The graph of $\tilde{L}(6, 1, X) + \varepsilon(6, 1, X)$.

$L_-(\alpha, X)$. We omit the details of a routine calculation that shows

$$\tilde{L}(N, \alpha, X) + \varepsilon(N, \alpha, X) \geq \tilde{L}(N, 1, X) + \varepsilon(N, 1, X),$$

for $N > 4$.

The function $\tilde{L}(N, 1, X) + \varepsilon(N, 1, X)$ is piecewise analytic and has a negative second derivative in X. This makes it trivial to check that this function is positive on a given analytic interval by checking the values at the endpoints. The function $\tilde{L}(N, \alpha, X) + \varepsilon(N, \alpha, X)$, for $N = 3, 4$, is also easily checked to be positive. We find that

$$\Delta(P) \geq \tilde{L}(N, \alpha, X) + \varepsilon(N, \alpha, X) > 0,$$

except in the following two situations that will be treated below:

(1) $N = 6, \alpha \in [0.996, 1]$, and $X \in [-0.119, 0.1]$,
(2) $N = 7, \alpha \in [0.996, 1]$, and $X \in [-0.082, 0.0684]$.

Figures 8 and 9 show $\tilde{L}(N, 1, X) + \varepsilon(N, 1, X)$, for $N = 6, 7$.

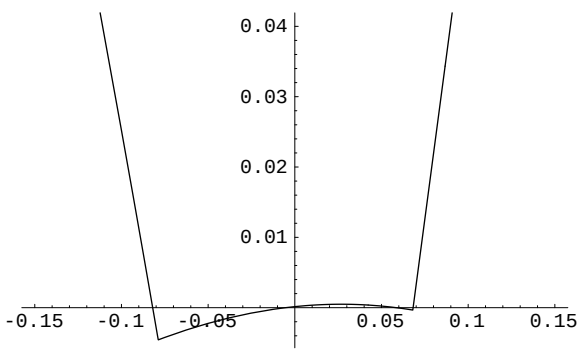


Fig. 9. The graph of $\tilde{L}(7, 1, X) + \varepsilon(7, 1, X)$.

For the rest of the argument, we assume we are in one of these two situations. If the lengths of the chords f_i are at most 1, we have

$$\Delta(P) \geq L'(N, \alpha, X) + \varepsilon(N, \alpha, X) \geq 0.$$

The inequality on the left is Proposition 6.1.

The inequality on the right was checked by taking a Taylor approximation at $X = 0$ (and by taking the worst-case values for α). We give a few details of the approximation in the special case $N = 6$, $|X| \leq 0.06$. The constant α satisfies $0 < \alpha \leq 1$. We also assume that $0 \leq \theta \leq 0.5$, where θ is a variable defined below. The function takes the form

$$L'(6, \alpha, X) + \varepsilon(6, \alpha, X) = \sqrt[4]{12}(2q(\theta)\sqrt{\alpha - X} - 2\alpha + X),$$

where $q(\theta) = \theta/\sin\theta$, and $\theta \geq 0$ is defined implicitly by the equation

$$p(\theta) := \frac{\theta - \sin\theta\cos\theta}{\sin^2\theta} = \frac{2|X|}{\sqrt{\alpha - X}\sqrt[4]{12}}.$$

(This formula for L' is presented in greater detail in the Appendix.) We show that

$$2q(\theta)\sqrt{\alpha - X} - 2\alpha + X \geq 0,$$

with equality exactly when $X = 0$. The following inequalities are easily verified, when the variables lie in the indicated intervals:

$$\sqrt{1 - X} \geq 1 - \frac{X}{2} - 0.22X^2;$$

$$\frac{2|X|}{\sqrt[4]{12}\sqrt{1 + 0.06}} \leq \frac{2|X|}{\sqrt[4]{12}\sqrt{\alpha - X}} = p(\theta) \leq 0.87\theta;$$

$$q(\theta) \geq 1 + \frac{\theta^2}{6} \geq 1 + \frac{4X^2}{6(0.87^2)(1.06)\sqrt{12}};$$

$$\begin{aligned} &2q(\theta)\sqrt{\alpha - X} - 2\alpha + X \\ &\qquad \geq 2\left(1 + \frac{4X^2}{6(0.87^2)(1.06)\sqrt{12}}\right)\left(1 - \frac{X}{2} - 0.22X^2\right) - 2 + X = X^2 f, \end{aligned}$$

where f is a quadratic polynomial in X taking positive values for $|X| \leq 0.06$. It follows that if the function vanishes, then $X = 0$. This implies $\theta = 0$, and $q(\theta) = 1$. We then have

$$L'(6, \alpha, 0) + \varepsilon(6, \alpha, 0) = \sqrt[4]{12}(2\sqrt{\alpha} - 2\alpha).$$

If this vanishes, then $\alpha = 1$.

This completes the hardest case. Similar calculations are left to the reader when $N = 7$, $|X| \geq 0.06$, or $\theta \geq 0.5$. The result is that $\Delta(P) = 0$, if and only if $X = 0$ and P is a regular hexagon with area 1. This is the tight case of the hexagonal isoperimetric

inequality. (The graph of $y = L'(6, 1, X)$ lies between the graphs of $y = L(P_X)$ and $y = -\varepsilon(6, 1, X)$ in Fig. 5.)

Now assume that some chord f_i has length at least 1. The area of the polygon $P_f = (f_1, \ldots, f_N)$ is at least $\alpha - X$. A lower bound on $L(P)$ is the perimeter $L(P_f)$ of this polygon.

Lemma 8.1. $L(P_f) + \varepsilon(N, \alpha, X) > 0$, *in situations* (1) *and* (2).

Proof. We develop an isoperimetric inequality for polygons of area at least $\alpha - X$, constrained so that one of the edges has length at least 1. We may assume that the area is $0.996 - X$. By well-known principles, the optimal polygon is inscribed in a circle with unconstrained edge lengths t and constrained edge length $\max(1, t)$. The area $A_N(r)$ and perimeter $\max(1, t) + (N - 1)t$ are monotonic increasing functions of the circle's radius r.

Now $X = X_N(r) = 0.996 - A_N(r)$, so

$$\Delta(P) \geq g_N(r) := \max(1, t) + (N - 1)t + \varepsilon(N, 1, X_N(r)).$$

The function $g_N(r)$ is easily estimated because of the monotonicity of t and A_N. For $a < b$, we write

$$g_N(a, b) = \max(1, t(a)) + (N - 1)t(a) + \varepsilon(N, 1, 0.996 - A_N(b)).$$

Then $\Delta(P) \geq g_N(r) \geq g_N(a, b)$, for $r \in [a, b]$. In situation (1),

$$X_6(0.671) < -0.119 \leq X \leq 0.1 < X_6(0.61).$$

Thus, it can be seen that $\Delta(P) > 0$, by computing the constants

$$g_6(0.61 + 0.001\,k, 0.611 + 0.001\,k) > 0,$$

for $i = 0, \ldots, 60$. (The constants are all at least 0.02.) Situation (2) is similar. □

Appendix. A Proof of Proposition 6.1

Let P have chords f_i of length ℓ_i. In this appendix we give a proof of the following result.

Proposition 6.1-A. *Choose constants s and ℓ so that $\ell_i \leq s < \ell \leq \sum \ell_i$, for all i. If $|X(P)| \leq \pi s^2/8$, then*

$$L(P) \geq \ell \operatorname{arc}(s, |X(P)|/\ell).$$

We obtain the version of the proposition that is stated in Section 6 by taking $s = 1$, $\ell = L(N, \alpha, X(P))$, $N(P) \leq 7$, and $\alpha \in [0.996, 1]$. Note that under these conditions, $0.119 < \pi s^2/8$ and $\ell \operatorname{arc}(s, |X(P)|/\ell) = L'(N, \alpha, X)$.

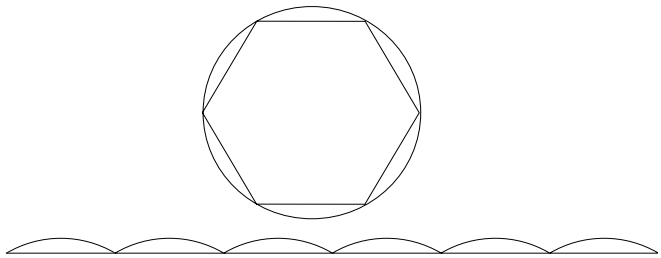


Fig. 10

Proof. We flatten out the perimeter of the polygon by arranging its edges f_i along a line as shown in Fig. 10. Without loss of generality, we may assume that all $x(e)$ have the same sign. (See Section 5, noting that the truncation τ_0 is not used in this appendix.)

Next we analyze two particular edges of lengths $(u, v) = (\ell_i, \ell_j)$. Let the lengths of the two circular arcs be $2\rho\theta_u$ and $2\rho\theta_v$, where $1/\rho$ is common curvature of the two arcs (see Section 6.1). That is, θ_u is the angle subtended by the arc and its chord (Fig. 11).

Set

$$\begin{aligned} p(t) &= \frac{t - \sin(t)\cos(t)}{\sin^2(t)}, \\ q(t) &= \frac{t}{\sin(t)}. \end{aligned}$$

We have

$$\ell_i + \ell_j = u + v, \tag{A.1}$$

$$4(|x(e_i)| + |x(e_j)|) = u^2 p(\theta_u) + v^2 p(\theta_v), \tag{A.2}$$

$$\mathrm{perim}(e_i) + \mathrm{perim}(e_j) = uq(\theta_u) + vq(\theta_v), \tag{A.3}$$

$$0 = u \sin\theta_v - v \sin\theta_u. \tag{A.4}$$

These four equations give the length of the chords, the enclosed area, the arc length of the circular arcs, and the equal curvature condition for the two edges. Let $\xi(u)$ be the area $|x(e_i)|+|x(e_j)|$, viewed as a function of u by fixing the combined length $\ell_i+\ell_j$, perimeter $\mathrm{perim}(e_i) + \mathrm{perim}(e_j)$, and the equal curvature condition. It is defined implicitly by (A.1)–(A.4).

Lemma A.1.

$$\xi'(u) = \rho(\cos\theta_v - \cos\theta_u).$$

In particular, $\xi'(u) \geq 0$, *if* $0 \leq \theta_v \leq \theta_u \leq \pi$.

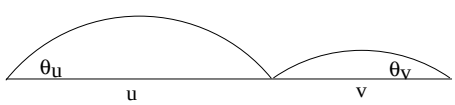


Fig. 11

Proof. We compute the derivative $\xi'(u)$ by implicit differentiation of (A.1)–(A.4). The differentials of these four equations give four homogeneous linear relations among du, dv, $d\theta_u$, $d\theta_v$, and $d\xi$. Solving the linear system for $d\xi/du$, we obtain the result. □

Remark A.2. Johnson and Morgan have observed that the inequality $\xi'(u) \geq \rho\cos\theta_v - \rho\cos\theta_u$ can be seen geometrically without a calculation. This inequality is all that is needed for the proof of Proposition 6.1-A. (In fact, $\xi' \geq 0$ is all that is needed.) The heights of the two bumps in Fig. 11 are $h(\theta_u)$, $h(\theta_v)$, where $h(\theta) = \rho - \rho\cos\theta$. We can increase the area by

$$(h(\theta_u) - h(\theta_v))\Delta u,$$

by cutting a vertical slice of area $h(\theta_v)\Delta u$ from the middle of the small bump, and adding a vertical slice of area $h(\theta_u)\Delta u$ to the middle of the large bump. To first order, this keeps the length of the perimeter constant. The optimal increase in area $\Delta\xi$ is at least the increase in area obtained by this strategy. Hence the inequality.

The lemma leads to a proof of the proposition. By the equal curvature condition (Section 6.1), if $u \geq v$, then $\theta_u \geq \theta_v$, so lengthening longer chords decreases the perimeter for fixed areas.

By continuity, it is enough to prove the proposition when $\ell = a/b$ is a rational number. We apply the lemma to pairs of chords, increasing the longer chord $u < s$ and decreasing the shorter chord $v > 0$, keeping the sum $u + v$ fixed, continuing until every segment has length 0 or s, except for one of length between 0 and s.

$\pi s^2/8$ is the area of a semicircle with diameter s. By the equal curvature condition (Section 6.1), any circular arc greater than a semicircle must lie along a chord of length s. The area under such an arc is greater than $\pi s^2/8$, contrary to hypothesis.

If $\sum \ell_i > \ell$, pick an edge of length s. The arc e_i along that edge is less than a semicircle. Decreasing the diameter ℓ_i while fixing $x(e_i)$ will decrease the length of e_i. (This is a standard argument: decrease the obtuse angle of the triangle joining the midpoint of e_i to the endpoints of f_i. This increases the area of the triangle, keeping the perimeter fixed.) Continuing in this manner, we can decrease $\sum \ell_i$ until $\ell = \sum \ell_i$. Again, we may assume that all lengths but one are s or 0.

We replicate b times these arcs and chords, enclosing a total area of at most $b\pi s^2/8$. We continue to apply the lemma to pairs of chords until all edges have length 0 or s. At no stage do the circular arcs along the chords of length s become semicircles. The perimeter of the replicated version is $a\,\mathrm{arc}(s, b|X|/a)$, so the perimeter of the unreplicated version is $\ell\,\mathrm{arc}(s, |X(P)|/\ell)$ as desired. □

Acknowledgments

I thank J. Sullivan for many helpful comments. I am particularly grateful to F. Morgan for many comments, suggestions, and corrections. I thank D. Weaire for his recommendation, "Given its celebrated history, it seems worth a try . . . ."

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Received August 30, 1999, *and in revised form April* 17, 2000. *Online publication September* 22, 2000.