STABILITY IN THE FULL TWO-BODY PROBLEM

D. J. SCHEERES
Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, U.S.A., e-mail: scheeres@umich.edu

Abstract. Stability conditions are established in the problem of two gravitationally interacting rigid bodies, designated here as the full two-body problem. The stability conditions are derived using basic principles from the N-body problem which can be carried over to the full two-body problem. Sufficient conditions for Hill stability and instability, and for stability against impact are derived. The analysis is applicable to binary small-body systems such as have been found recently for asteroids and Kuiper belt objects.

Key words: binary asteroids, stability, N-body problem

1. Introduction

In this paper, some classical results from the N-body problem are applied to the problem of two interacting rigid bodies, each with an arbitrary gravity field. Such a problem can serve as a model for the dynamics of a binary small-body system, such as an asteroid or a Kuiper belt object, especially during the early stages of its evolution following a disruptive impact or planetary flyby. The specific interest expressed in this paper concerns the long term stability of the binary against disruption (escape) or impact. Stability against disruption for this problem is essentially Hill stability, and we find sufficient conditions for this stability and sufficient conditions for violation of this stability. Necessary conditions are more difficult, and this is explained. In the N-body problem, stability against impact is often related to Lagrange stability which restricts both the positions and velocities of the bodies to be bounded. For interacting point masses, such a restriction guarantees that impact will not occur. However, for rigid bodies with distributed mass, impacts can occur with finite velocity; thus we introduce a definition called stability against impact (SAI). We find sufficient conditions for SAI in the full two-body problem.

2. Basics of the N-body Problem

Assume N mutually gravitating bodies with mass, positions, and velocities $[m_i, \mathbf{r}_i, \mathbf{v}_i; i = 1, 2, \ldots, N]$, specified relative to a barycentric reference frame:

$$m_i \ddot{\mathbf{r}}_i = \frac{\partial U}{\partial \mathbf{r}_i}; \qquad i = 1, 2, \ldots, N, \tag{1}$$

$$U = \sum_{1 \leqslant j < k \leqslant N} \frac{\mathcal{G} m_j m_k}{|\mathbf{r}_j - \mathbf{r}_k|}, \tag{2}$$

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$$\sum_{i=1}^{N} m_i \mathbf{r}_i = 0, \tag{3}$$

$$\sum_{i=1}^{N} m_i \mathbf{v}_i = 0, \tag{4}$$

where U is the force potential of the system.

There are four non-trivial integrals of motion for these equations, the total energy of the system and the total angular momentum vector of the system, expressed as

$$E = T - U, \tag{5}$$

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \mathbf{v}_i, \tag{6}$$

$$\mathbf{K} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times \mathbf{v}_i, \tag{7}$$

where E is the total energy, T is the kinetic energy, and $\mathbf{K}$ is the total angular momentum vector. Additionally, define the polar moment of inertia of the system:

$$I_p = \sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \mathbf{r}_i. \tag{8}$$

One of the basic results from the N-body problem is the Lagrange–Jacobi identity, found by taking the second time derivative of the polar moment of inertia:

$$\ddot{I}_p = 4T + 2 \sum_{i=1}^{N} \mathbf{r}_i \cdot \frac{\partial U}{\partial \mathbf{r}_i}, \tag{9}$$

$$= 4E + 4U + 2 \sum_{i=1}^{N} \mathbf{r}_i \cdot \frac{\partial U}{\partial \mathbf{r}_i}, \tag{10}$$

$$= 4E + 2U, \tag{11}$$

where the final step occurs since U is homogeneous of degree -1 for the standard N-body problem.

Another basic result is found by applying the Cauchy inequality to the angular momentum (Pollard, 1976). From this we find:

$$K^2 \leqslant 2 I_p T, \tag{12}$$

where $K^2 = \mathbf{K} \cdot \mathbf{K}$. This is usually combined with the Lagrange–Jacobi identity to derive the Sundmann inequality, which places constraints on the growth of an N-body system (Pollard, 1976).

3. Extension to the Full Two-body Problem

Starting from the N-body problem we will carry our analysis over to the 'full Two-Body problem', defined as the problem of two gravitationally interacting rigid bodies. This problem has been studied extensively in the literature, and is a well-posed problem of celestial mechanics. We refer the reader to Maciejewski (1995) for a complete development of this system. For our purposes, we wish to re-evaluate the basic quantities of force potential, kinetic energy, total energy, angular momentum, and polar moment of inertia for such a system. To carry out the development, we note that the summations in the N-body problem naturally generalize to integrals over a continuum of mass distribution, and that a series of holonomic constraints can be applied to the N-body problem without changing the basic definitions and values of the quantities defined above for the N-body problem.

3.1. DEFINITION OF THE TWO BODIES

In the following we assume that the N particles are divided into two groups, denoted as β_1 and β_2, the particles in each group rigidly connected amongst themselves. These groups can be defined using two collections of integers, denoted as I_1 and I_2 for body 1 and body 2, with the number of points in each set being N_1 and N_2, respectively. Of course, we have that $N = N_1 + N_2$. Then the bodies are defined as $\beta_I = \{\mathbf{r}_i; i \in I_I\}$, $I = 1, 2$. Each body will have its own center of mass, computed as:

$$\mathbf{R}_I = \frac{\sum_{i\in I_I} m_i \mathbf{r}_i}{\sum_{i\in I_I} m_i}; \qquad I = 1, 2. \tag{13}$$

Then the set of relative vectors $\vec{\rho}_i$, are defined:

$$\vec{\rho}_i = \begin{cases} \mathbf{r}_i - \mathbf{R}_1, & i \in I_1, \\ \mathbf{r}_i - \mathbf{R}_2, & i \in I_2, \end{cases} \tag{14}$$

as well as the difference of the center of mass vectors $\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1$, and the difference of the relative vectors $\vec{\rho}_{ij} = \vec{\rho}_j - \vec{\rho}_i$. The latter are defined without ambiguity, given the sets I_1 and I_2. With this notation, the vector difference $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ can equal one of four possibilities, depending on which set the i and j indices lie in:

$$\mathbf{r}_{ij} = \begin{cases} \mathbf{r} + \vec{\rho}_{ij}, & i \in I_1, j \in I_2, \\ -\mathbf{r} + \vec{\rho}_{ij}, & i \in I_2, j \in I_1, \\ \vec{\rho}_{ij}, & i, j \in I_1 \text{ or } i, j \in I_2. \end{cases} \tag{15}$$

3.2. REDUCTION OF THE FORCE POTENTIAL

Using these definitions, the potential can be rewritten as:

$$U = \frac{\mathcal{G}}{2}\sum_{i,j=1}^{N}\frac{m_i m_j}{|\mathbf{r}_{ij}|} \tag{16}$$

$$= \frac{\mathcal{G}}{2}\left[\sum_{i\in I_1, j\in I_2}\frac{m_i m_j}{|\mathbf{r}+\vec{\rho}_{ij}|} + \sum_{i\in I_2, j\in I_1}\frac{m_i m_j}{|-\mathbf{r}+\vec{\rho}_{ij}|} + \right.$$

$$\left. + \sum_{i\in I_1, j\in I_1}\frac{m_i m_j}{|\vec{\rho}_{ij}|} + \sum_{i\in I_2, j\in I_2}\frac{m_i m_j}{|\vec{\rho}_{ij}|}\right], \tag{17}$$

where we assume that double summations over i and j exclude $i = j$. The first two summations can be reduced to the same general form, yielding:

$$U = \mathcal{G}\left[\sum_{i\in I_1, j\in I_2}\frac{m_i m_j}{|\mathbf{r}+\vec{\rho}_{ij}|} + \right.$$

$$\left. +\frac{1}{2}\sum_{i\in I_1, j\in I_1}\frac{m_i m_j}{|\vec{\rho}_{ij}|} + \frac{1}{2}\sum_{i\in I_2, j\in I_2}\frac{m_i m_j}{|\vec{\rho}_{ij}|}\right]. \tag{18}$$

Now note that each mass m_i is uniquely identified with a particular body, β_1 or β_2. Thus each summation can be generalized to an integral over one of the bodies, defining the mass elements as a function of position: $m_i = dm_I(\vec{\rho}_I)$, $\vec{\rho}_I \in \beta_I$, where $I = 1, 2$ depending on which set i belongs to. The vectors $\vec{\rho}$ are integration variables over the mass distributions, leading to

$$U = \mathcal{G}\int_{\beta_1}\int_{\beta_2}\frac{\mathrm{d}m_1(\vec{\rho}_1)\mathrm{d}m_2(\vec{\rho}_2)}{|\mathbf{r}+\vec{\rho}_2-\vec{\rho}_1|} +$$

$$+\frac{\mathcal{G}}{2}\left[\int_{\beta_1}\int_{\beta_1}\frac{\mathrm{d}m_1(\vec{\rho})\mathrm{d}m_1(\vec{\rho})}{|\vec{\rho}|} + \int_{\beta_2}\int_{\beta_2}\frac{\mathrm{d}m_2(\vec{\rho})\mathrm{d}m_2(\vec{\rho})}{|\vec{\rho}|}\right]. \tag{19}$$

For representational purposes, we must introduce the relative orientation between the two rigid bodies. First define the transformation dyad of each body, $\mathbf{A}_I$, which takes a vector from an inertial frame into a body-fixed frame, and its inverse, $\mathbf{A}_I^T$, which takes a vector from the body-fixed frame into an inertial frame. These transformation dyads can be used to define the inertial orientation of each body. In the force potential we perform the integrations in the frames fixed to the bodies β_1 and β_2 and require each relative vector to be specified in the inertial frame, and thus can replace each $\vec{\rho}_I$ with $\mathbf{A}_I^T \cdot \vec{\rho}_I$ without loss of generality. Then the mutual potential between the two bodies (defined to be U_{12}) is:

$$U_{12} = \mathcal{G}\int_{\beta_1}\int_{\beta_2}\frac{\mathrm{d}m_1(\vec{\rho}_1)\mathrm{d}m_2(\vec{\rho}_2)}{|\mathbf{r}+\mathbf{A}_2^T\cdot\vec{\rho}_2-\mathbf{A}_1^T\cdot\vec{\rho}_1|}. \tag{20}$$

With these definitions in hand we can rewrite the potential as:

$$U = U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) + U_{11} + U_{22}. \tag{21}$$

Since we have a set of rigid holonomic constraints fixing the relative distance and orientation of each body internal to itself, we note that the self potentials U_{11} and U_{22} are each constant. We will sometimes suppress the dependance of U_{12} on $\mathbf{A}_1$ and $\mathbf{A}_2$ in the following.

Before proceeding, we discuss three results involving the mutual force potential U_{12}: upper and lower bounds on U_{12} as a function of $r = |\mathbf{r}|$, monotonicity of U_{12} as a function of r, and a limitation on the magnitude of U_{12} as compared to its gradient. For all the discussions we note that each of the vectors $\vec{\rho}_1$ and $\vec{\rho}_2$ in the integral for U_{12} will be bounded when evaluated over their respective bodies, yielding:

$$|\vec{\rho}_1| \leqslant \rho_{M_1}, \tag{22}$$

$$|\vec{\rho}_2| \leqslant \rho_{M_2}, \tag{23}$$

$$|\Delta\vec{\rho}| \leqslant \rho_M, \tag{24}$$

where ρ_{M_I} is the largest radius of body I as measured from its center of mass, $\rho_M = \rho_{M_1} + \rho_{M_2}$, and $\Delta\vec{\rho} = \vec{\rho}_2 - \vec{\rho}_1$ (see Figure 1).

3.2.1. *Upper and lower bounds on* U_{12}

First we note that the force potential can be bounded as a simple function of radius:

$$U_M(r) = \max_{\mathbf{A}_1,\mathbf{A}_2} U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) \geqslant U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2), \tag{25}$$

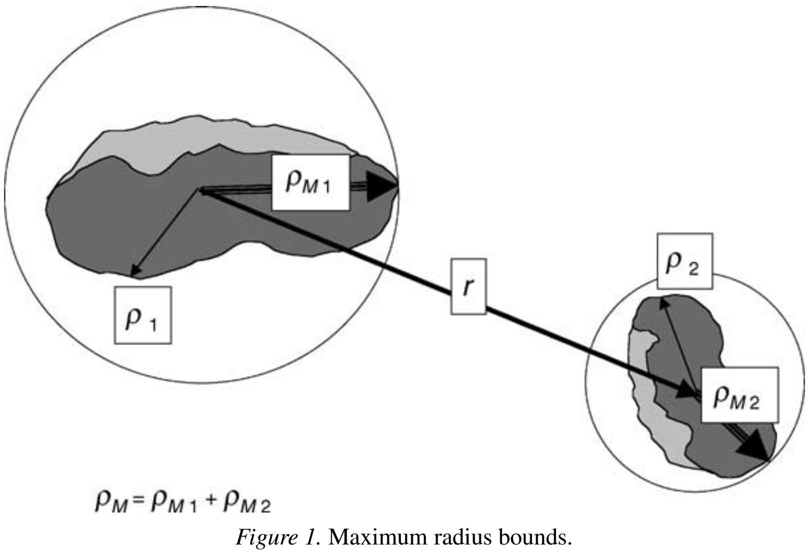

Figure 1. Maximum radius bounds.

where the maximization of U_{12} occurs over all possible orientations between the bodies for a fixed vector $\mathbf{r}$. In a similar fashion we can define a lower bound $U_m(r) = \min_{\mathbf{A}_1,\mathbf{A}_2} U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) \leqslant U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2)$. Note that U_m and U_M are only a function of the distance between the two centers of mass.

If $r > \rho_M$ we can derive an explicit upper bound on U_{12}:

$$U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) \leqslant \frac{\mathcal{G}M_1M_2}{r - \rho_M}. \tag{26}$$

To establish this we must show:

$$\int_{\beta_1}\int_{\beta_2} \frac{\mathrm{d}M_1\mathrm{d}M_2}{|\mathbf{r} + \Delta\vec{\rho}|} \leqslant \int_{\beta_1}\int_{\beta_2} \frac{\mathrm{d}M_1\mathrm{d}M_2}{r - \rho_M}. \tag{27}$$

A sufficient condition for this is

$$r - \rho_M \leqslant |\mathbf{r} + \Delta\vec{\rho}|\,, \tag{28}$$

which can easily be shown to be true given $r > \rho_M$ and Equation (24).

3.2.2. *Monotonicity of* U_{12}

Next we develop conditions under which the potential U_{12} is monotonically decreasing for an increasing radius, r. Specifically, given two vectors $\mathbf{r}$ and $\mathbf{d}$ which are parallel, that is, $\mathbf{r} \cdot \mathbf{d} = rd$, we will show that a sufficient condition for $U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) \leqslant U_{12}(\mathbf{d}, \mathbf{A}_1, \mathbf{A}_2)$ if $d < r$ is that $r + d \geqslant 2\rho_M$.

A sufficient condition for $U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) \leqslant U_{12}(\mathbf{d}, \mathbf{A}_1, \mathbf{A}_2)$ is

$$|\mathbf{r} + \Delta\vec{\rho}| \geqslant |\mathbf{d} + \Delta\vec{\rho}| \tag{29}$$

over all admissible values of $\Delta\vec{\rho}$. Squaring both sides and reducing yields

$$r^2 - d^2 + 2(r - d)\hat{\mathbf{r}} \cdot \Delta\vec{\rho} \geqslant 0 \tag{30}$$

since $\mathbf{r}$ and $\mathbf{d}$ are parallel. Now divide by $(r - d) > 0$ to find:

$$r + d \geqslant -2\hat{\mathbf{r}} \cdot \Delta\vec{\rho}. \tag{31}$$

But $-\hat{\mathbf{r}} \cdot \Delta\vec{\rho} \leqslant \rho_M$ in general, producing a sufficient condition for monotonicity to hold: $r + d \geqslant 2\rho_M$.

3.2.3. *Relation between magnitude and gradient of* U_{12}

In the classical N-body problem, use is made of Euler's theorem of homogenous functions, where we find that $U + \sum_{i=1}^{N} (\partial U/\partial \mathbf{r}_i) \cdot \mathbf{r}_i = 0$. This is used to simplify the Lagrange–Jacobi identity and establish other important results. For the full two-body problem, as stated here, this result no longer holds due to the holonomic constraints placed on the mass distributions. We can establish a variant of this result, however, that is useful in computing a sufficient condition for a binary system to escape.

The condition we find is that $2U_{12} + \frac{\partial U_{12}}{\partial \mathbf{r}} \cdot \mathbf{r} > 0$ if $r > 2\rho_M$.

To establish this result we must show that:

$$\int_{\beta_1}\int_{\beta_2} \mathrm{d}m_1 \mathrm{d}m_2 \left[\frac{2}{|\mathbf{r} + \Delta\vec{\rho}|} - \frac{(\mathbf{r} + \Delta\vec{\rho}) \cdot \mathbf{r}}{|\mathbf{r} + \Delta\vec{\rho}|^3}\right] \geqslant 0. \tag{32}$$

A sufficient condition is that the integrand be positive, or:

$$2(\mathbf{r} + \Delta\vec{\rho}) \cdot (\mathbf{r} + \Delta\vec{\rho}) - (\mathbf{r} + \Delta\vec{\rho}) \cdot \mathbf{r} \geqslant 0. \tag{33}$$

Factoring out $(\mathbf{r} + \Delta\vec{\rho})$ results in:

$$(\mathbf{r} + \Delta\vec{\rho}) \cdot (\mathbf{r} + 2\Delta\vec{\rho}) \geqslant 0. \tag{34}$$

We see that this will be satisfied if $r \geqslant 2\rho_M$, giving a sufficient condition.

3.3. TOTAL ENERGY

Next we consider the form of the total energy of our general two-body system. To develop this we first note the following results:

$$\sum_{i \in I} m_i \vec{\rho}_i = 0, \tag{35}$$

$$M_I = \sum_{i \in I} m_i, \tag{36}$$

$$\mathbf{I}_I = -\sum_{i \in I} m_i \tilde{\rho}_i \cdot \tilde{\rho}_i, \tag{37}$$

where $\mathbf{I}_I$ is the inertia dyad of body I and $\tilde{\rho}$ is the cross product dyad defined such that $\vec{\omega} \times \vec{\rho} = \vec{\omega} \cdot \tilde{\rho} = \tilde{\omega} \cdot \vec{\rho}$. Additionally, we assume that $M_1\mathbf{R}_1 + M_2\mathbf{R}_2 = 0$, which yields explicit formulae for the location of the positions and velocities of the mass centers in terms of their relative distance:

$$\mathbf{R}_1 = \frac{-M_2}{M_1 + M_2}\mathbf{r}, \tag{38}$$

$$\mathbf{R}_2 = \frac{M_1}{M_1 + M_2}\mathbf{r}, \tag{39}$$

and their immediate generalization to velocity by replacing $\mathbf{R}_I$ with $\mathbf{V}_I$ and $\mathbf{r}$ with $\mathbf{v}$. Finally, we note that the velocity of a particle in the Ith body is equal to $\mathbf{v}_i = \mathbf{V}_I + \vec{\omega}_I \times \vec{\rho}_i$, where $\vec{\omega}_I$ is the rotational velocity of body I. Applying these definitions to the N-body kinetic energy in Equation (6) results in

$$T = \frac{1}{2}\frac{M_1 M_2}{M_1 + M_2}\mathbf{v} \cdot \mathbf{v} + \frac{1}{2}\vec{\omega}_1 \cdot \mathbf{I}_1 \cdot \vec{\omega}_1 + \frac{1}{2}\vec{\omega}_2 \cdot \mathbf{I}_2 \cdot \vec{\omega}_2. \tag{40}$$

The total energy of the N-body problem is $E = T - U$, thus for our system the total energy becomes:

$$E = T - U_{12} - U_{11} - U_{22}. \tag{41}$$

Since the self potentials are constant and arise nowhere else in our set of N-body conditions, we redefine the total energy as E plus the self-potentials:

$$\begin{aligned} E_{12} &= \frac{1}{2}\frac{M_1 M_2}{M_1 + M_2}\mathbf{v}\cdot\mathbf{v} + \\ &\quad + \frac{1}{2}\vec{\omega}_1\cdot\mathbf{I}_1\cdot\vec{\omega}_1 + \frac{1}{2}\vec{\omega}_2\cdot\mathbf{I}_2\cdot\vec{\omega}_2 - U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2). \end{aligned} \tag{42}$$

This quantity is conserved under evolution of the system.

3.4. TOTAL ANGULAR MOMENTUM

Now consider the total angular momentum of the system. Equation (7) for $\mathbf{K}$ can be expressed as:

$$\mathbf{K} = \sum_{i\in I_1} m_i(\mathbf{R}_1 + \vec{\rho}_i) \times (\mathbf{V}_1 + \vec{\omega}_1 \times \vec{\rho}_i) + \tag{43}$$

$$+ \sum_{i\in I_2} m_i(\mathbf{R}_2 + \vec{\rho}_i) \times (\mathbf{V}_2 + \vec{\omega}_2 \times \vec{\rho}_i). \tag{44}$$

Using the center of mass definitions and inertia dyad definitions this reduces to:

$$\mathbf{K} = M_1\mathbf{R}_1 \times \mathbf{V}_1 + M_2\mathbf{R}_2 \times \mathbf{V}_2 + \mathbf{I}_1\cdot\vec{\omega}_1 + \mathbf{I}_2\cdot\vec{\omega}_2. \tag{45}$$

Using Equations (38) and (39) leads to

$$M_1\mathbf{R}_1 \times \mathbf{V}_1 + M_2\mathbf{R}_2 \times \mathbf{V}_2 = \frac{M_1 M_2}{M_1 + M_2}\mathbf{r}\times\mathbf{v} \tag{46}$$

reducing the angular momentum to its simplest form:

$$\mathbf{K} = \frac{M_1 M_2}{M_1 + M_2}\mathbf{r}\times\mathbf{v} + \mathbf{I}_1\cdot\vec{\omega}_1 + \mathbf{I}_2\cdot\vec{\omega}_2. \tag{47}$$

This is also conserved under evolution of the system.

3.5. POLAR MOMENT OF INERTIA

The final reduction we make is for the polar moment of inertia. The general expression becomes

$$I_p = \sum_{i\in I_1} m_i(\mathbf{R}_1 + \vec{\rho}_i)\cdot(\mathbf{R}_1 + \vec{\rho}_i) + \sum_{i\in I_2} m_i(\mathbf{R}_2 + \vec{\rho}_i)\cdot(\mathbf{R}_2 + \vec{\rho}_i), \tag{48}$$

which reduces to:

$$I_p = M_1 \mathbf{R}_1 \cdot \mathbf{R}_1 + M_2 \mathbf{R}_2 \cdot \mathbf{R}_2 + \sum_{i \in I_1} m_i \vec{\rho}_i \cdot \vec{\rho}_i + \sum_{i \in I_2} m_i \vec{\rho}_i \cdot \vec{\rho}_i. \tag{49}$$

Now, the final two summations can be re-expressed in terms of the inertia dyads of the bodies using the well-known identity (MacMillan, 1960): $\sum_i m_i \vec{\rho}_i \cdot \vec{\rho}_i = \frac{1}{2}\mathrm{Tr}[\mathbf{I}]$ where $\mathrm{Tr}[\mathbf{I}]$ denotes the trace of the inertia dyad $\mathbf{I}$, leading to:

$$\sum_{i \in I_1} m_i \vec{\rho}_i \cdot \vec{\rho}_i + \sum_{i \in I_2} m_i \vec{\rho}_i \cdot \vec{\rho}_i = \tfrac{1}{2}\mathrm{Tr}[\mathbf{I}_1 + \mathbf{I}_2]. \tag{50}$$

Restating Equation (49) in terms of $\mathbf{r}$ yields the final form of I_p. It is important to note that I_p is only a function of the radius magnitude.

$$I_p(r) = \frac{1}{2}\mathrm{Tr}[\mathbf{I}_1 + \mathbf{I}_2] + \frac{M_1 M_2}{M_1 + M_2} \mathbf{r} \cdot \mathbf{r}. \tag{51}$$

Even though the trace of the inertia dyads is constant, we do not redefine the polar moment of inertia so that we can still apply it to Cauchy's inequality.

3.6. THE CAUCHY INEQUALITY

The Cauchy inequality still applies to our system since it concerns $\mathbf{K}$, I_p, and T, each of which is directly related to the N-body results with no change. Thus, in the full two-body problem the Cauchy inequality can be explicitly stated as:

$$K^2 \leqslant 2 I_p(r) T, \tag{52}$$

$$T = E_{12} + U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2). \tag{53}$$

Later we will use this inequality to establish sufficient conditions for stability against impact. Before doing this, we use the inequality to define the minimum energy of the system.

For a given value of angular momentum, there exists a minimum system energy. From Cauchy's inequality we have:

$$E_{12} \geqslant \frac{K^2}{2 I_p(r)} - U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2). \tag{54}$$

For a given value of r we can define the maximum force potential, $U_M(r)$, allowing us to write

$$\frac{K^2}{2 I_p(r)} - U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) \geqslant E_m(r), \tag{55}$$

$$E_m(r) = \frac{K^2}{2 I_p(r)} - U_M(r), \tag{56}$$

and the right-hand side of Equation (55) is only a function of the distance between the centers of mass. Thus this can be minimized over all values of radius $r \geqslant \rho_M$ to find:

$$E_m^* = \min_{r \geqslant \rho_M} E_m(r). \tag{57}$$

Note that we cannot use our explicit bound on U_{12} given in Equation (26), since it has an unnatural singularity at $r = \rho_M$. The existence of this minimum energy for a given value of K^2 is important and will be used later.

3.7. EQUATIONS OF MOTION

For completeness, we also give the equations of motion for the coupled system. The resultant equations of motion for such a system are well-defined and involve the combined translational and rotational motion of the two bodies (Maciejewski, 1995).

$$\frac{M_1 M_2}{M_1 + M_2}\ddot{\mathbf{r}} = \frac{\partial U_{12}}{\partial \mathbf{r}}, \tag{58}$$

$$\dot{\mathbf{H}}_I = \mathbf{H}_I \times \vec{\omega}_I + \mathbf{M}_I, \tag{59}$$

$$\vec{\omega}_I = \mathbf{I}_I^{-1} \cdot \mathbf{H}_I, \tag{60}$$

$$\dot{\mathbf{A}}_I = \mathbf{A}_I \times \vec{\omega}_I, \tag{61}$$

$$I = 1, 2,$$

where $\mathbf{H}_I$ is the angular momentum, $\mathbf{I}_I$ is the inertia dyad, $\vec{\omega}_I$ is the rotational velocity vector in the body-fixed frame, and $\mathbf{A}_I$ is the transformation matrix for the bodies $I = 1, 2$, respectively. The torques $\mathbf{M}_I$ are derivable from the mutual force potential U_{12}.

4. Hill Stability

We say that our two-body system is Hill stable if $r(t) < C$ for all time, where C is a positive, finite, constant value. It is Hill unstable if $\lim_{t\to\infty} r(t) \to \infty$. We can develop two sufficient conditions, one for Hill stability and one for Hill instability.

4.1. SUFFICIENT CONDITION FOR HILL STABILITY

A sufficient condition for Hill stability is $E_{12} < 0$.

The proof of this is simple. If $E_{12} < 0$ and the system is Hill unstable, then $r \to \infty$. But we can directly show from Equation (26) that:

$$\lim_{r\to\infty} U_{12} = 0. \tag{62}$$

This leads to $E_{12} = T < 0$, which cannot be true since T is the sum of positive definite quadratic forms.

The necessary conditions for Hill stability are more difficult, and must involve estimates of the strength of interaction between translational and rotational motion. As a case in point, the 'full' problem of two interacting spheres can be given an arbitrary value of energy by increasing the sphere's rotation rate. In this instance the total energy is independent of the system stability, as there will be no exchange of energy and angular momentum between rotational and translational motion. If, however, one of the bodies is given a non-spherical mass distribution, it is possible for coupling between the translation and rotation to occur (Scheeres, 2001) and the stability of the system becomes a relevant question.

4.2. SUFFICIENT CONDITION FOR HILL INSTABILITY

A sufficient condition can be developed for Hill instability. If:

$$E_{12} - \frac{1}{2}\sum_{i=1}^{2} \vec{\omega}_i \cdot \mathbf{I}_i \cdot \vec{\omega}_i > \epsilon > 0, \tag{63}$$

$$\mathbf{r} \cdot \mathbf{v} > 0, \tag{64}$$

$$r > 2\rho_M, \tag{65}$$

then $r \to \infty$. We will discuss the necessary magnitude of ε in a moment.

To prove this we first develop an analogue of the Lagrange–Jacobi identity. The first and second time derivatives of I_p are:

$$\dot{I}_p = 2\frac{M_1 M_2}{M_1 + M_2}\mathbf{r} \cdot \mathbf{v}, \tag{66}$$

$$\ddot{I}_p = 2\frac{M_1 M_2}{M_1 + M_2}\mathbf{v} \cdot \mathbf{v} + 2\frac{M_1 M_2}{M_1 + M_2}\mathbf{r} \cdot \ddot{\mathbf{r}}, \tag{67}$$

Equation (67) can be rewritten using Equations (42) and (58) as:

$$\ddot{I}_p = 4E_{12} - 2\sum_{i=1}^{2} \vec{\omega}_i \cdot \mathbf{I}_i \cdot \vec{\omega}_i + 4U_{12} + 2\frac{\partial U_{12}}{\partial \mathbf{r}} \cdot \mathbf{r}. \tag{68}$$

As discussed previously, the potential U_{12} is no longer homogeneous, meaning that we cannot apply the standard reduction to the final two terms of the expression. However, in Section 3.2.3 we established that $2U_{12} + \frac{\partial U_{12}}{\partial \mathbf{r}} \cdot \mathbf{r} > 0$ if $r > 2\rho_M$.

Note that $I_p > 0$ by definition. Next, if inequality (64) is satisfied then $\dot{I}_p > 0$. Finally, if inequalities (63) and (65) are satisfied then $\ddot{I}_p > 0$. If ε is large enough, this will always be true, leading to $I_p \to \infty$ which immediately implies $r \to \infty$.

The number ε must be chosen to bound the amount of energy the rotating body can extract from the translational motion of the system. To establish such a bound requires that more details of the system be specified. We will only consider a very simple system, consisting of the interaction between a sphere and a body with a C_{22} gravity coefficient. Assume that the bodies are initially travelling away from each other along a rectilinear orbit with sufficient energy to escape (barring any additional interaction). This scenario provides a reasonable pre-condition for our case of Hill instability discussed above. In (Scheeres et al., 2000) this situation was considered and an equation for the change in rotational angular momentum of the body with the gravity coefficient was developed (assuming motion in the equatorial plane):

$$\dot{H}_z = \frac{6\mathcal{G} M_c M_s C_{22}}{r^3} \sin[2(\lambda - \theta)], \tag{69}$$

where M_c is the mass of the body with the gravity coefficient, M_s is the mass of the sphere, C_{22} is the gravity coefficient (with dimensions of km^2), r is the radius between the centers of mass, λ is the longitude of the sphere, and θ is the rotational phase of the body with gravity coefficient. We assume that all motion is in the plane. Then applying the first step of Picard's method of successive approximation yields:

$$\Delta H_z = 6\mathcal{G} M_c M_s C_{22} \int_{t=0}^{\infty} \frac{\sin[2(\lambda - \theta)]}{r^3} \mathrm{d}t, \tag{70}$$

where we assume that $\theta = \Omega_z t$, where Ω_z is the initial rotation rate of the body. The largest value of ΔH_z occurs if the rectilinear motion is parabolic:

$$r^{3/2} = r_o^{3/2} + \tfrac{3}{2} t \sqrt{2\mu}. \tag{71}$$

Substituting for θ and r as functions of time, the resulting integral can be expressed as:

$$\begin{aligned} \Delta H_z \; = \; & 2\sqrt{2} \sqrt{\frac{\mathcal{G}(M_c + M_s)}{r_o^3}} \frac{M_c M_s}{M_c + M_s} C_{22} \Bigg\{ \sin(2\lambda) + \\ & + 2\alpha\Omega_z \Bigg[\left(S_i(2\alpha\Omega_z) - \frac{\pi}{2} \right) \sin 2(\lambda + \alpha\Omega_z) + \\ & + C_i(2\alpha\Omega_z) \cos 2(\lambda + \alpha\Omega_z) \Bigg] \Bigg\}, \end{aligned} \tag{72}$$

$$\alpha = \sqrt{\frac{2r_o^3}{9\mu}}, \tag{73}$$

$$S_i(x) = \int_0^x \frac{\sin(x)}{x} \mathrm{d}x, \tag{74}$$

$$C_i(x) = -\int_x^{\infty} \frac{\cos(x)}{x} \mathrm{d}x, \tag{75}$$

where $\mu = \mathcal{G}(M_c + M_s)$ and S_i and C_i are the sine and cosine integrals, respectively. All of the above results can be generalized to fully three-dimensional, non-rectilinear orbits (Scheeres, 2001). For the sake of argument we will make one final simplification and assume that the rotation rate Ω_z is slow, reducing the expression to:

$$\Delta H_z \sim 2\sqrt{2}\sqrt{\frac{\mathcal{G}(M_c + M_s)}{r_o^3}}\frac{M_c M_s}{M_c + M_s}C_{22}\sin(2\lambda), \tag{76}$$

where we note that $\Delta H_z = I_z \Delta\Omega_z$.

Now ε must be chosen to be large enough so that the change in rotational energy over the entire escape trajectory is bounded:

$$E_{12} - \tfrac{1}{2}I_z(\Omega_z + \Delta\Omega_z)^2 > 0, \tag{77}$$

where Ω_z is the initial rotation rate of our system when our sufficient condition is applied and $\Delta\Omega_z$ denotes the maximum possible increase in rotation rate due to interaction between the bodies, a result we derived above. Ignoring higher orders of $\Delta\Omega_z$, we find that

$$\varepsilon \sim 2\sqrt{2}\sqrt{\frac{\mathcal{G}(M_c + M_s)}{r_o^3}}\frac{M_c M_s}{M_c + M_s}C_{22}\Omega_z \tag{78}$$

should ensure escape, leading to the more specific sufficiency condition:

$$E_{12} - \frac{1}{2}\sum_{i=1}^{2}\vec{\omega}_i \cdot \mathbf{I}_i \cdot \vec{\omega}_i > 2\sqrt{2}\sqrt{\frac{\mathcal{G}(M_c + M_s)}{r_o^3}}\frac{M_c M_s}{M_c + M_s}C_{22}\Omega_z. \tag{79}$$

We note that this is a conservative result for the system of an interacting sphere and a body with C_{22} gravity coefficient as the maximum addition of energy to a rotating body would be found by such an analysis (i.e. slowly rotating body, rectilinear parabolic orbit, in the equatorial plane).

5. Stability Against Impact

A binary will be stable against impact (SAI) if $|\mathbf{r}(t)| > \rho_M$ for all time. A sufficient condition for SAI can be derived. Assume a system with a current separation r, energy E, and angular momentum K. A sufficient condition for the system to be SAI is that a number $d \in (\rho_M, r)$ can be found such that:

$$I_p(d) \leqslant \frac{K^2}{2(E + U_M(d))}. \tag{80}$$

To prove this, rewrite Cauchy's inequality as:

$$\frac{K^2}{2(E + U_{12})} \leqslant I_p. \tag{81}$$

Now if two vectors **r** and **d** are parallel, then $U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) < U_{12}(\mathbf{d}, \mathbf{A}_1, \mathbf{A}_2)$ if $r > d$ and $r + d > 2\rho_M$, as established in Section 3.2.2. Then, by definition, $U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2) < U_{12}(\mathbf{d}, \mathbf{A}_1, \mathbf{A}_2) \leqslant U_M(d)$ and for $r > d > \rho_M$ we have:

$$\frac{K^2}{2(E + U_M(d))} < \frac{K^2}{2(E + U_{12}(\mathbf{r}, \mathbf{A}_1, \mathbf{A}_2))}. \tag{82}$$

If the proper value of d can be found to satisfy Equation (80), we immediately know that the system cannot evolve to the mutual distance d without violating the Cauchy inequality, thus:

$$I_p(d) < I_p(r), \tag{83}$$

which reduces to:

$$d < |\mathbf{r}(t)| \tag{84}$$

guaranteeing SAI.

This proof does not tell us what value of d will supply SAI, but given a set of constants, initial conditions, and potential for the system, it is an easy process to search for this value. If such a value of d cannot be found, mutual impact cannot be ruled out.

We can develop a condition for when such a number d will exist for a given value of angular momentum, however. If

$$E_m^* < \frac{K^2}{2I_p(\rho_M)} - U_M(\rho_M) = E_m(\rho_M), \tag{85}$$

then there exists a d such that the SAI sufficiency condition is satisfied if $r > \rho_M$ and the system energy $E \in [E_m^*, E_m(\rho_M)]$. If these conditions are met, then there exists a $d \in (\rho_M, r)$ such that

$$E = \frac{K^2}{2I_p(d)} - U_M(d), \tag{86}$$

which corresponds to the sufficient condition in Equation (80).

Observing Equation (80), we note that small values of angular momentum or large values of energy may cause the inequality to be violated. While we have a lower bound on energy for a given angular momentum, there is no upper bound on energy for a given angular momentum, thus a decrease in angular momentum will never violate a given energy value. From these realizations we can derive a limit on K^2 and E which, if violated, means that the sufficient SAI condition cannot occur. To derive this we consider Equation (80) again and note that for the numbers $\rho_M < d < r$,

$$I_p(\rho_M) < I_p(d), \tag{87}$$

$$\frac{K^2}{2(E + U_M(d))} < \frac{K^2}{2(E + U_m(r))}. \tag{88}$$

Thus, should the inequality:

$$I_p(\rho_M) \geqslant \frac{K^2}{2(E + U_m(r))} \tag{89}$$

ever be satisfied, we know immediately that a d that satisfies the sufficient SAI condition cannot be found. This inequality is immediately computable for a given set of initial conditions and models.

6. Conclusions

This paper derives some basic stability results in the problem of two gravitationally interacting rigid bodies. To find these stability results we use classical results from the N-body problem and, by the application of holonomic constraints on the masses, reduce them to the problem of two interacting rigid bodies. This results in sufficiency conditions for Hill stability, Hill instability, and stability against impact. The conditions found here are directly applicable to the problem of asteroid binaries, and can be used to better understand the final outcomes of motion in such systems.

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