# BOUNDS ON ROTATION PERIODS OF DISRUPTED BINARIES IN THE FULL 2-BODY PROBLEM

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**Abstract.** Constraints on the final rotation rates of a Hill unstable binary in the full 2-body problem are derived and analyzed. Application of these constraints are made to the problem of a constant density body spun to disruption. This analysis has relevance to the evolution of asteroid spin rates.

**Key words:** asteroids, full 2-body problem, gravitational potentials, rotation

## 1. Introduction

A proposed source of slowly rotating asteroids is from the disruption of binary bodies formed during flybys of the Earth and Venus [3, 6]. As a result of such an interaction, it is possible for the asteroid to be spun at a rate fast enough for the centrifugal accelerations to overcome the gravitational attraction. Thus, due to a planetary flyby it is possible for an asteroid to be spun to disruption, in particular to be spun into two or more distinct bodies. If these bodies do not reimpact at their first periapsis passage, a binary (or more) asteroid has been formed. Depending on the total energy of this binary system, it is possible (indeed it is likely) for the resulting binary system to be Hill unstable, and thus to eventually have a mutual escape of the two (or more) bodies [7]. When this occurs, the rotational kinetic energy of both bodies must decrease from their initial, post disruption rate due to conservation of energy. It is possible that this reduction in spin rate could be significant and observable in the asteroid rotation rate population [3]. In this paper we carry out an analysis of such a system and find constraints on the final spin rates of a disrupted binary formed by spinning a single body beyond its disruption limit.

This paper first derives the total energy of a rotating body, for both a general and ellipsoidal body. Next, conditions for such a body to 'disassemble' into a binary are stated, along with the resulting energy constraints. Then, the problem of specifying the mass distributions of the newly created binaries is considered, and a reasonable approach is defined. Assuming a system with positive free energy that is Hill unstable, constraints on the final rotation rates of the binary are found after it has disrupted into two single bodies. These results are also specialized to the

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case of ellipsoids. Finally, a mathematical analysis of these constraints is carried out and a few example computations are given.

This paper presents two results of special significance. First is the realization that the 'fission' of a rigid body into two or more components liberates potential energy, enabling it to interact with the kinetic energy of the system. Second is that strict bounds on the final rotation rate of at least one component of a Hill unstable binary asteroid can be found.

## 2. Total Energy of a Rotating Body

### 2.1. GENERAL CASE

Consider a body of mass $M$ rotating about its largest moment of inertia. It has a kinetic energy of:

$$\mathcal{T} = \frac{1}{2} I_z \omega^2, \tag{1}$$

where $I_z$ is the maximum moment of inertia. A natural body will always tend to this rotation state [1]. The total angular momentum of the body is simply computed as:

$$K = I_z \omega \tag{2}$$

and is related to the kinetic energy by

$$\mathcal{T} = \frac{1}{2} K \omega. \tag{3}$$

To compute the total energy of the body we must also consider its self gravitational potential. For an arbitrary mass distribution this is defined as:

$$\mathcal{U} = -\frac{\mathcal{G}}{2} \int_{\mathcal{B}} \int_{\mathcal{B}} \frac{\mathrm{d}m_1 \, \mathrm{d}m_2}{|\rho_{12}|}, \tag{4}$$

where $\mathcal{G}$ is the gravitational constant, $\mathcal{B}$ is the mass distribution of the body, $\mathrm{d}m_i$ is the elemental mass of the body at a position vector $\rho_i$, and $\rho_{12} = \rho_2 - \rho_1$ is the relative distance between two mass elements.

A lower bound on $\mathcal{U}$ can be inferred from physical principles, as the spherical self-potential of the same mass must minimize all possible mass distributions. Thus, we can assert:

$$\mathcal{U} \geqslant -\frac{3\mathcal{G}M^2}{5r}, \tag{5}$$

where the lower bound is the self gravitational potential of a sphere, $M$ is the total mass of the body, and $r$ is its mean radius, computed from:

$$r = \left( \frac{3M}{4\pi\sigma} \right)^{1/3}, \tag{6}$$

where $\sigma$ is the body's mean density. For a constant density tri-axial ellipsoid with semi-major axes $\alpha$, $\beta$, and $\gamma$, the mean radius is defined as the geometric mean of the semi-major axes, $r = (\alpha\beta\gamma)^{1/3}$.

Thus, the total energy of a rotating body can be stated as:

$$E = \tfrac{1}{2} I_z \omega^2 + \mathcal{U}. \tag{7}$$

### 2.2. ELLIPSOIDAL CASE

If the body is a tri-axial ellipsoid with semi-major axes $\alpha \geqslant \beta \geqslant \gamma$, we can provide greater detail on the above equation. First, the maximum moment of inertia of the body is now:

$$I_z = \tfrac{1}{5} M(\alpha^2 + \beta^2). \tag{8}$$

If the body is a constant density ellipsoid, we can carry out the integration for the self-potential in closed form. First we start from the gravitational potential for an interior or surface point [2]:

$$V(\mathbf{r}) = -\mathcal{G}\pi\sigma\alpha\beta\gamma \int_0^\infty \phi(\mathbf{r}, u) \frac{\mathrm{d}u}{\Delta(u)}, \tag{9}$$

$$\phi(\mathbf{r}, u) = 1 - \frac{x^2}{\alpha^2 + u} - \frac{y^2}{\beta^2 + u} - \frac{z^2}{\gamma^2 + u}, \tag{10}$$

$$\Delta(u) = \sqrt{(\alpha^2 + u)(\beta^2 + u)(\gamma^2 + u)} \tag{11}$$

where $\mathbf{r}$ is the position vector with coordinates $x$, $y$, and $z$ oriented along the maximum, intermediate and minimum semi-major axes of the ellipsoid, respectively.

The self-potential is then computed as:

$$\mathcal{U} = \tfrac{1}{2} \int_{\mathbf{r}\in\mathcal{B}} V(\mathbf{r})\sigma \, \mathrm{d}\mathbf{r}, \tag{12}$$

where $\mathcal{B} = \{\mathbf{r} | 0 \leqslant \phi(\mathbf{r}, 0) \leqslant 1\}$. Direct integration yields:

$$\mathcal{U} = -\frac{3}{8}\mathcal{G}M^2 \int_0^\infty \frac{\mathrm{d}u}{\Delta(u)} \left[1 - \frac{1}{5}\left\{\frac{\alpha^2}{\alpha^2 + u} + \frac{\beta^2}{\beta^2 + u} + \frac{\gamma^2}{\gamma^2 + u}\right\}\right], \tag{13}$$

where $M = 4\pi\sigma\alpha\beta\gamma/3$. This expression can be simplified by a certain identity [2]:

$$\int_0^\infty \left[1 - \left\{\frac{\alpha^2}{\alpha^2 + u} + \frac{\beta^2}{\beta^2 + u} + \frac{\gamma^2}{\gamma^2 + u}\right\}\right] \frac{\mathrm{d}u}{\Delta(u)} = 0. \tag{14}$$

This allows the result to be simplified to:

$$\mathcal{U} = -\tfrac{3}{10}\mathcal{G}M^2\mathcal{I}, \tag{15}$$

where

$$\mathcal{I} = \int_0^\infty \frac{\mathrm{d}u}{\Delta(u)}. \tag{16}$$

The integral $\mathcal{I}$ has some interesting properties. First, due to the identity asserted above, it is homogenous of degree $-1$ in its arguments, that is, $\alpha\mathcal{I}_\alpha + \beta\mathcal{I}_\beta + \gamma\mathcal{I}_\gamma = -\mathcal{I}$, where $\mathcal{I}_\alpha = \partial\mathcal{I}/\partial\alpha$. Second, a sharp bound on its value can be delimited, $\mathcal{I} \leqslant 2/(\alpha\beta\gamma)^{1/3}$, which corresponds again to the fact that a spherical body minimizes the self gravitational potential.

Thus, if the body is a constant density, rotating ellipsoid the total energy is

$$E = \tfrac{1}{10}M(\alpha^2 + \beta^2)\omega^2 - \tfrac{3}{10}\mathcal{G}M^2\mathcal{I}. \tag{17}$$

It is interesting to note that minimizing $E$ over the variables $\alpha$, $\beta$, and $\gamma$, while maintaining $\alpha\beta\gamma$ and $I_z\omega$ constant, directly yields the conditions for Jacobi and Maclaurin ellipsoids. This should be expected, however, as these special shapes are known to be minimum energy configurations for rotating ellipsoids of inviscid fluid [2].

## 3. Disassembly of the Body

One sufficient condition for the physical disruption of a body with no tensile strength is that the body spin fast enough for material on its surface to experience greater centrifugal than gravitational accelerations. In the following we will outline these necessary conditions and introduce an idealized model to track the total energy and angular momentum before and after the body has 'disassembled'. Again, we first compute the general conditions and then give details for an ellipsoid.

To define the condition for disassembly for the general case, first define the set:

$$\phi = \left\{ \mathbf{r} \in \mathcal{B} \middle| \omega^2 |\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \mathbf{r})| \geqslant \left| \frac{\partial V(\mathbf{r})}{\partial \mathbf{r}} \right| \right\}, \tag{18}$$

where $V(\mathbf{r}) = -\mathcal{G}\int_{\mathcal{B}} \mathrm{d}m/|\mathbf{r} + \rho|$ is the gravitational potential at a location $\mathbf{r}$ measured from the body center of mass and $\rho$ is the position vector to the mass element $\mathrm{d}m$. For disassembly to occur we need the set $\phi$ to not be empty. More sophisticated statements of instability conditions can be found by using principles of inviscid flow [2], however we take a more conservative approach and assume that the body does not experience any shifts in its shape until this limit is reached. For a general mass distribution there are no clear results concerning the regions over which internal particles will feel a net outwards acceleration. While centrifugal accelerations will vary linearly with distance from the spin axis, the gravitational acceleration will have a more complex variation as a function of its mass distribution. This is a topic for future study.

For a tri-axial ellipsoid the disassembly condition can be reduced to:

$$\omega^2 \geqslant \frac{3}{2}\mathcal{G}M \int_0^\infty \frac{\mathrm{d}u}{(\alpha^2+u)\Delta(u)}, \tag{19}$$

$$= -\frac{3}{2}\frac{\mathcal{G}M\mathcal{I}_\alpha}{\alpha}. \tag{20}$$

When this rotation rate is reached for an ellipsoid the entire interior of the body experiences positive acceleration along the long axis, a situation which leads directly to disruption, unless the body has non-negligible tensile strength.

The actual computation of the resulting motion is difficult and non-trivial, and cannot be understood completely in simple terms. Usual approaches are to simulate the system numerically with an assemblage of point masses [5]. Despite this complexity, we can develop a simple yet practical model for the system energy and angular momentum following disruption.

For tractability, assume the body splits into two disjoint pieces, with the mass ratio between them being a free parameter. Angular momentum is conserved across such a split, and we ideally assume that energy is conserved. If the body is formally taken and split into two rigid bodies, conserving energy and angular momentum, we find that the relationship between kinetic energy and angular momentum in Equation (3) continues to hold across the event. This occurs as all the elements within the body maintain the same rotation rate immediately before and after break-up. The kinetic energy is now split into three parts, however, the rotational kinetic energy of each body and the kinetic energy of translational motion between the two bodies:

$$\mathcal{T}_{r_0} = \mathcal{T}_{r_1} + \mathcal{T}_{r_2} + \mathcal{T}_{12}, \tag{21}$$

where $\mathcal{T}_{r_i} = 1/2 I_{z_i}\omega_i^2$ represents the kinetic energy of rotation and $\mathcal{T}_{12} = 1/2 M_1 M_2/(M_1 + M_2)V^2$ represents the kinetic energy of mutual translational motion, where $V$ is the relative speed between the mass centers.

The potential energy is also partitioned across the split, leading to

$$\mathcal{U}_0 = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_{12}, \tag{22}$$

where $\mathcal{U}_i$ is the self potential of the $i$th body, defined in Equation (4), $\mathcal{U}_{12}$ is the mutual potential between the two mass distributions:

$$\mathcal{U}_{12} = -\mathcal{G}\int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \frac{\mathrm{d}m_1\,\mathrm{d}m_2}{|\mathbf{r} + \rho_{12}|} \tag{23}$$

and $\mathbf{r}$ is again the distance between the two mass centers and $\mathcal{B}_i$ is the mass distribution of the $i$th body.

Balancing the energy before and after disruption we have the basic result:

$$\mathcal{T}_{r_0} + \mathcal{U}_0 = \mathcal{T}_{r_1} + \mathcal{T}_{r_2} + \mathcal{T}_{12} + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_{12}. \tag{24}$$

In the subsequent evolution of the body, the mutual potential $\mathcal{U}_{12}$ can be considered to be 'liberated' energy as it is potential energy that previously was contained in the self-potential of the original body, but is now free to be exchanged with the kinetic energy of the system [7]. Of dynamical interest is the kinetic plus mutual potential energy, $\tilde{E}$, which is conserved in the subsequent dynamical evolution of the system under the full 2-body problem constraints. This 'free' energy $\tilde{E}$ is defined as:

$$\tilde{E} = \mathcal{T}_{r_1} + \mathcal{T}_{r_2} + \mathcal{T}_{12} + U_{12}, \tag{25}$$

$$= \mathcal{T}_{r_0} + \mathcal{U}_0 - \mathcal{U}_1 - \mathcal{U}_2. \tag{26}$$

Whether or not a binary can mutually disrupt is then controlled by the free energy of the system, $\tilde{E}$, which must be positive for there to be sufficient energy for the bodies to escape each other.

## 4. Specifying the Disassembled Bodies

It is fundamentally difficult to compute the free energy of a disassembled system using either Equation (25) or (26). Without a general predictive theory for disruption in the full 2-body problem, we must create parameterized models of these new bodies. Even if we know the initial system in detail, we do not have a generally applicable analytic theory for computing the mutual potentials or self-potentials of a body split arbitrarily into two pieces. This situation also makes it difficult to develop useful models for the moments of inertia of the system after disassembly. The process is inherently difficult and ambiguous, and trying to give detailed specifications of the bodies would involve many free parameters and still require mathematical assumptions. To circumvent these problems we will instead develop bounds on the different quantities of interest and introduce the simplest of parameters to describe the binary system.

If the initial system has a mass $M$, then the binary system is split into two masses $M_1$ and $M_2$ such that $M_1 + M_2 = M$. The mass distribution can then be defined by a single parameter:

$$\mu = \frac{M_1}{M} \tag{27}$$

and subsequently

$$\frac{M_2}{M} = 1 - \mu. \tag{28}$$

Assuming the same mean density for these bodies, $\sigma$, we associate with each body a mean radius defined by Equation (6). If we compare the mean radii of the two new bodies with the original body, we find the relationship:

$$\frac{r_1}{r} = \mu^{1/3}, \tag{29}$$

$$\frac{r_2}{r} = (1 - \mu)^{1/3}. \tag{30}$$

With the parameter $\mu$ we can develop constraints on the self-potentials of the bodies and on the rotational kinetic energy of the binary. First, from the basic inequality stated in Equation (5) we have:

$$\mathcal{U}_1 + \mathcal{U}_2 \geqslant -\frac{3\mathcal{G}M_1^2}{5r_1} - \frac{3\mathcal{G}M_2^2}{5r_2}, \tag{31}$$

$$= -\frac{3\mathcal{G}M^2}{5r}[\mu^{5/3} + (1-\mu)^{5/3}]. \tag{32}$$

Thus, the sum of the self-potentials can be bounded by the total mass, mean radius (or density), and the mass parameter $\mu$. This also allows us to develop a bound on the mutual potential:

$$\mathcal{U}_{12} = \mathcal{U}_0 - \mathcal{U}_1 - \mathcal{U}_2, \tag{33}$$

$$\leqslant \mathcal{U}_0 + \frac{3\mathcal{G}M^2}{5r}[\mu^{5/3} + (1-\mu)^{5/3}]. \tag{34}$$

With the assumption that the disrupted bodies are initially rotating about their maximum moments of inertia, we can also recall the following bound, valid for a general, constant density body:

$$I_z \geqslant \tfrac{2}{5}Mr^2. \tag{35}$$

This result allows us to develop a lower bound on the rotational kinetic energy of our binary system:

$$\mathcal{T}_{r_1} + \mathcal{T}_{r_2} = \tfrac{1}{2}I_{z_1}\omega_1^2 + \tfrac{1}{2}I_{z_2}\omega_2^2, \tag{36}$$

$$\geqslant \tfrac{1}{5}M_1 r_1^2\omega_1^2 + \tfrac{1}{5}M_2 r_2^2\omega_2^2, \tag{37}$$

$$= \tfrac{1}{5}Mr^2[\mu^{5/3}\omega_1^2 + (1-\mu)^{5/3}\omega_2^2]. \tag{38}$$

Thus, with these observations, we can avoid the problem of specifying a detailed mass distribution after disassembly, at the cost of retaining less precision in our results.

We are interested in the case where the system has a positive free energy, $\tilde{E}$, and is Hill unstable, as it will then eventually disrupt [7]. When this happens we have $\mathcal{U}_{12} \to 0$ and $\mathcal{T}_{12} \to 1/2M\mu(1-\mu)V_\infty^2$. Solving for the kinetic energy of rotation we find:

$$\mathcal{T}_{r_1} + \mathcal{T}_{r_2} = \tilde{E} - \tfrac{1}{2}M\mu(1-\mu)V_\infty^2. \tag{39}$$

From this result it is clear that, as a function of $V_\infty$, the system can be given a very low rotational kinetic energy and hence a very slow rotation after disruption. From the constraint $\tilde{E} \geqslant T_{r_1} + T_{r_2} \geqslant 0$, the free parameter $V_\infty$ that defines the escaping orbit must satisfy:

$$0 \leqslant V_\infty^2 \leqslant \frac{2}{M\mu(1-\mu)}\tilde{E}. \tag{40}$$

## 5. Constraints on Rotation Rates

Finally, let us consider our target problem. Assume that the initial body has been spun to disassembly, that the free energy is positive, and that the system is Hill unstable. Then we know that the binary will eventually disrupt and create two uncoupled single bodies, each with their own rotation rate. Our immediate interest is in the kinetic energy of the rotating bodies, thus we recall Equation (39). The inequalities computed previously can be applied to find an upper bound to the free energy (Eq. (32)) and a lower bound on the kinetic energy of rotation (Eq. (38)). Applying these bounds and simplifying we find our main result:

$$\mu^{5/3}\omega_1^2 + (1-\mu)^{5/3}\omega_2^2 \leqslant F(\mu, V_\infty), \tag{41}$$

where

$$F(\mu, V_\infty) = \frac{5}{Mr^2}[\mathcal{T}_{r_0} + \mathcal{U}_0] + \frac{3\mathcal{G}M}{r^3}[\mu^{5/3} + (1-\mu)^{5/3}] - \\ - \frac{5}{2r^2}\mu(1-\mu)V_\infty^2. \tag{42}$$

The right-hand side of the inequality, while somewhat complex, can be evaluated with only two free parameters not specified initially: $V_\infty$ and $\mu$. To enforce greater conservatism, we can always take $V_\infty = 0$, although this is probably too strict as the escape speed will usually be non-zero. At the other extreme, it is always feasible for $V_\infty$ to force the right-hand side to approach zero, if the disruption should occur at a higher energy. We note that to be valid, we require the free energy to be positive, or that:

$$\frac{5}{Mr^2}[\mathcal{T}_{r_0} + \mathcal{U}_0] + \frac{3\mathcal{G}M}{r^3}[\mu^{5/3} + (1-\mu)^{5/3}] \geqslant 0. \tag{43}$$

First, note that each of the bodies has the same maximum constraint on its rotation rate. Applying results from the Appendix we note that $F(\mu, V_\infty) \leqslant F(0, V_\infty) = F(1, V_\infty)$ and, moreover, is independent of $V_\infty$ at this upper limit. We note that the cases of $\mu = 0, 1$ correspond to a in infinitesimal particle splitting from the body, and is a limiting case. Thus we find, for both bodies, a maximum rotation rate:

$$\omega_{\max} = \sqrt{F(0,0)}, \tag{44}$$

where for convenience $V_\infty$ has been set to zero. We note that this limiting spin rate is greater than the original rate $\omega$ of the disrupted system. This implies that one of the bodies may be spun to even faster rates in the ensuing interactions between the binary components.

Second, we note that there is no restriction on either one or both of the spin rates becoming arbitrarily small. Indeed, since this is an inequality, it will never be violated if $\omega_1 = \omega_2 = 0$. Furthermore, if $\mu \neq 0, 1$, the constraint on the $\omega_i$ can be made small by taking $V_\infty$ to its maximum value.

Finally, define the median rotation rate, $\omega_m$:

$$\omega_m = \sqrt{\frac{F(\mu, V_\infty)}{\mu^{5/3} + (1-\mu)^{5/3}}}. \tag{45}$$

Given this definition, we can show that the final spin rates $\omega_1$ and $\omega_2$ cannot be greater than $\omega_m$ at the same time. By definition we have

$$\mu^{5/3}\omega_1^2 + (1-\mu)^{5/3}\omega_2^2 \leqslant [\mu^{5/3} + (1-\mu)^{5/3}]\omega_m^2. \tag{46}$$

If $\omega_1 \geqslant \omega_m$, then the following sequence is true:

$$(1-\mu)^{5/3}\omega_2^2 = [\mu^{5/3} + (1-\mu)^{5/3}]\omega_m^2 - \mu^{5/3}\omega_1^2, \tag{47}$$

$$\leqslant (1-\mu)^{5/3}\omega_m^2, \tag{48}$$

leading directly to the result

$$\omega_2 \leqslant \omega_m. \tag{49}$$

The converse holds as well.

Thus, the final rotation rates of our system must conform to one of these three cases:

$$0 \leqslant \omega_1 \leqslant \omega_m \leqslant \omega_2 \leqslant \omega_{\max}, \tag{50}$$

$$0 \leqslant \omega_2 \leqslant \omega_m \leqslant \omega_1 \leqslant \omega_{\max}, \tag{51}$$

$$0 \leqslant \omega_1, \omega_2 \leqslant \omega_m. \tag{52}$$

Most importantly, we note that one of the rotation rates will always lie in the interval $[0, \omega_m]$. Whether this is relevant or not is an issue, however, as it may be possible for $\omega_m > \omega_{\max}$. Should this occur, then $\omega_m$ does not supply any relevant constraint. We will see, later, that values of $\omega_m \leqslant \omega_{\max}$ are always possible for all values of $\mu \neq 0, 1$, although some may require non-zero values of $V_\infty$ to force $\omega_m \leqslant \omega_{\max}$.

An even stricter and more relevant constraint on $\omega_m$ would be to show that $\omega_m \leqslant \omega$, the initial rotation rate of the ellipsoid. The inequalities we have avaliable for use are, in general, too weak to establish this possibility. However, we are able to show that this stricter constraint occurs for some example computations.

## 6. Normalized Form for an Ellipsoid

If we assume that the initial body is a constant density ellipsoid of dimensions $\alpha \geqslant \beta \geqslant \gamma$, we can further simplify and normalize the result, as then we know specific formulae for the kinetic and self-potential energies and for the disassembly limit. First, the function $F(\mu, V_\infty)$ becomes:

$$\begin{aligned} F(\mu, V_\infty) &= \frac{\alpha^2+\beta^2}{2r^2}\omega^2 - 2\pi\mathcal{G}\sigma r\mathcal{I} + 4\pi\mathcal{G}\sigma[\mu^{5/3} + (1-\mu)^{5/3}] - \\ &- \frac{5}{2r^2}\mu(1-\mu)V_\infty^2. \end{aligned} \tag{53}$$

Since the initial rotation rate must, by definition, be greater than the rotation rate defined in Equation (20), it is reasonable to normalize the function and the inequality by dividing by the quantity:

$$\omega_d^2 = \frac{3}{2}\frac{\mathcal{G}M|\mathcal{I}_\alpha|}{\alpha}. \tag{54}$$

Similarly, we normalize the semi-major axes by the mean radius $r$, leading to $\alpha\beta\gamma = 1$. Applying these definitions the normalized form of $F$ is:

$$\begin{aligned} F(\mu, V_\infty) &= \frac{1}{2}(\alpha^2+\beta^2)\left(\frac{\omega}{\omega_d}\right)^2 - \frac{\alpha\mathcal{I}}{|\mathcal{I}_\alpha|} + \frac{2\alpha}{|\mathcal{I}_\alpha|}[\mu^{5/3} + (1-\mu)^{5/3}] - \\ &- \frac{10\alpha}{3|\mathcal{I}_\alpha|}\mu(1-\mu)\tilde{V}_\infty^2, \end{aligned} \tag{55}$$

$$\tilde{V}_\infty = \frac{V_\infty}{V_{\text{esc}}(r)}, \tag{56}$$

where $V_{\text{esc}}(r) = \sqrt{2\mathcal{G}M/r}$, and is the ideal Keplerian escape speed from the surface of a sphere of equivalent volume and density as the original ellipsoid. It should also be recalled that the term $\mathcal{I}_\alpha < 0$, which is why we specify it with an absolute value sign.

We can derive some relevant inequalities from the fact that $\alpha \geqslant \beta \geqslant \gamma$ and $\alpha\beta\gamma = 1$ along with the identity in Equation (14). They may be of use in developing constraints for specific situations.

$$\beta^2 + \gamma^2 \leqslant 2 \leqslant \alpha^2 + \beta^2, \tag{57}$$

$$\alpha^2 + \beta^2 \leqslant \frac{\alpha\mathcal{I}}{|\mathcal{I}_\alpha|}. \tag{58}$$

The last inequality is significant, as it implies that the initial energy of the system is negative, $E < 0$, for rotation rates up to $\sqrt{2}\omega_d$. Whether the initial system energy is negative or positive has certain implications on our constraints, as are detailed in the following. We also note that, in terms of our normalized system, $\mathcal{I} \leqslant 2$. This, along with the above inequalities, leads to:

$$\frac{1}{\alpha}|\mathcal{I}_\alpha| \leqslant 1. \tag{59}$$

## 7. Functional Dependence of $\omega_m$

The crucial quantity from the above discussions is the median rotation rate, $\omega_m$. We now study some of its properties. Before we proceed, let us define the functions:

$$f(\mu) = \mu^{5/3} + (1-\mu)^{5/3}, \tag{60}$$

$$g(\mu) = \mu(1-\mu). \tag{61}$$

These functions are both symmetric about 1/2, thus we need only consider the interval $\mu \in [0, 1/2]$. It is simple to show that $f$ has maxima at $\mu = 0, 1$ and a minimum at $\mu = 1/2$, while $g$ has minima at $\mu = 0, 1$ and a maximum at $\mu = 1/2$. Both are monotonic between these limits. Thus:

$$f\left(\frac{1}{2}\right) = 2^{-2/3} \leqslant f(\mu) \leqslant 1 = f(0), \tag{62}$$

$$g(0) = 0 \leqslant g(\mu) \leqslant \frac{1}{4} = g\left(\frac{1}{2}\right). \tag{63}$$

The function $F$ can be stated, in general form, as:

$$F(\mu, V_\infty) = A + Bf(\mu) - Cg(\mu)V_\infty^2. \tag{64}$$

We note that $A$ is proportional to the initial system energy, $E$, and can be positive or negative, but that $B, C > 0$. Functionally, we will investigate the properties of

$$\omega_m^2 = \frac{A + Bf(\mu) - Cg(\mu)V_\infty^2}{f(\mu)}. \tag{65}$$

In terms of the escape velocity parameter, $V_\infty$, the system constraints are simple to find:

$$0 \leqslant \omega_m^2 \leqslant \frac{A + Bf(\mu)}{f(\mu)}, \tag{66}$$

where the upper bound on $\omega_m^2$ is monotonic in $V_\infty$ by inspection.

For the variation of $\omega_m$ as a function of $\mu$ let us first consider the case when $V_\infty = 0$. As $f(\mu)$ is monotonic in $\mu$ over the interval [0, 1/2], we can compute the gradient of $\omega_m^2(\mu, 0)$ with respect to $\mu$ to find:

$$\frac{\partial \omega_m^2(\mu, 0)}{\partial \mu} = -\frac{A}{f^2}\frac{\partial f}{\partial \mu}. \tag{67}$$

But $\partial f/\partial \mu < 0$ over the interval being considered, and hence the change in $\omega_m(\mu, 0)$ with $\mu$ has the sign of $A$, which can either be positive or negative.

If the disassembly rotation rate is not too fast, that is, if $E < 0$, then $A < 0$ and $\omega_m$ is minimized by taking $\mu = 1/2$ and maximized by taking $\mu = 0$. This directly implies that $\omega_m(\mu, V_\infty) \leqslant \omega_{\max}$. Thus, as the mass fraction of the binary increases to 1/2, the upper limit on the final rotation rate of at least one of the bodies becomes stricter.

If the initial rotation rate is fast enough, that is, if $E > 0$, then $A > 0$ and the above results switch. Then, $\omega_m$ is maximized by taking $\mu = 1/2$ and minimized with $\mu = 0$. In this case, if $V_\infty = 0$ then $\omega_{\max} \leqslant \omega_m(\mu, 0)$, and thus the bound $\omega_m$ is not physically relevant. The question becomes, does a realizable value of $V_\infty$ exist that will at least force $\omega_m \leqslant \omega_{\max}$?

To answer this, consider the case when $V_\infty > 0$. The partial derivative becomes modified to

$$\frac{\partial \omega_m^2(\mu, V_\infty)}{\partial \mu} = \frac{|\partial f/\partial \mu|}{f^2}[A - C V_\infty^2 \mu(1-\mu)] - \frac{C V_\infty^2}{f}(1-2\mu). \tag{68}$$

For the interval $\mu \in [0, 1/2]$ we see that $V_\infty$ always decreases the partial derivative. For the case when $E < 0$, this does not change anything, and any increase in $V_\infty$ will reduce the value of $\omega_m$ even further. For the case of $E > 0$, we can show that there always exists a viable value of $V_\infty$ which forces $\omega_m < \omega_{\max}$.

Recall from Equation (40) that $V_\infty$ for a Hill unstable binary pair is constrained as:

$$V_\infty^2 \leqslant \frac{2}{M g(\mu)}[E - U_1 - U_2]. \tag{69}$$

We note that the value $V_*^2 = 2E/(M g(\mu))$ will always define an allowed value of $V_\infty$ for $E > 0$, and thus we can always find an admissable value of $V_\infty \geqslant V_*$, for which $\omega_m^2 = F(\mu, V_\infty)/f(\mu) \leqslant F(\mu, V_*)/f(\mu)$. But it can easily be shown that $F(\mu, V_*)/f(\mu) \leqslant F(0, 0) = \omega_{\max}^2$. Thus, we know that, at least, there exist values of $V_\infty$ such that $\omega_m \leqslant \omega_{\max}$.

## 8. Example Computations

It is difficult to go beyond this point analytically, as the inequalities available to us are too weak to enable us to prove the existence of intervals where $\omega_m \leqslant \omega$, although such intervals exist. Thus, in the following we present some examples that show the actual ranges of values taken on by $\omega_m$ for some cases of interest. For the two cases presented we compute results for the case where the free energy of the system, $\tilde{E}$, is positive while the total energy $E$ is negative, and for the case where the initial rotation rate is fast enough so that the total energy of the system, $E$, equals zero. For each of these cases we compute limits that provide the boundaries over which the bound $\omega_m$ is relevant.

CASE 1. Consider an ellipsoid with semi-major axes $\alpha = 3/2$, $\beta = 1$, $\gamma = 2/3$. For the free energy of the resulting system to be positive, the initial rotation rate must be $\omega/\omega_d \geqslant 1.038$. Consider the case when $\omega/\omega_d = 1.04$, then $\omega_m(\mu = 1/2, V_\infty = 0)/\omega_d = 0.107$ and $\omega_m(\mu = 0.18, V_\infty = 0)/\omega_d = 1.04$, the original rotation rate. For the total energy of the initial system to be positive, the initial rotation rate must be $\omega/\omega_d \geqslant 1.774$. For the median rotation rate to be less than or equal to the initial spin rate for this case a $\tilde{V}_\infty \geqslant 0.788$ is required.

CASE 2. Consider an ellipsoid with semi-major axes $\alpha = 3/2$, $\beta = \gamma = \sqrt{2/3}$. For the free energy of the resulting system to be positive, the initial rotation rate must be $\omega/\omega_d \geqslant 1.104$. Consider the case when $\omega/\omega_d = 1.11$, then $\omega_m(\mu = 1/2, V_\infty = 0)/\omega_d = 0.179$ and $\omega_m(\mu = 0.18, V_\infty = 0)/\omega_d = 1.11$, the original rotation rate. For the total energy of the initial system to be positive, the initial rotation rate must be $\omega/\omega_d \geqslant 1.869$. For the median rotation rate to be less than or equal to the initial spin rate for this case a $\tilde{V}_\infty \geqslant 0.713$ is required.

These two cases are somewhat limited, but indicate that non-trivial values of the bound $\omega_m$ do occur for realistic values of spin rates.

## 9. Discussion

Constraints on the final rotation rates of a Hill unstable binary asteroid formed by being spun to disruption are analyzed. These mathematical constraints will be used to guide a future analysis of statistical distributions of NEO spin rates. It is significant to note that the constraints found on rotation rate are scalar, and hence could lead to the linear spin rate distributions found by Harris [3]. Results reported in [8] indicate that a sizable fraction of asteroids may suffer tidal disruption, raising the possibility that this mechanism could be responsible. The next step in this analysis will be to test this hypothesis using a classical Monte Carlo simulation, as reported in [8].

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