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On the local solubility of diophantine systems

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Abstract. Let p be a rational prime number. We refine Brauer's elementary diagonalisation argument to show that any system of r homogeneous polynomials of degree d, with rational coefficients, possesses a non-trivial p-adic solution provided only that the number of variables in this system exceeds $(rd^2)^{2^{d-1}}$. This conclusion improves on earlier results of Leep and Schmidt, and of Schmidt. The methods extend to provide analogous conclusions in field extensions of $\mathbb{Q}_p$, and in purely imaginary extensions of $\mathbb{Q}$. We also discuss lower bounds for the number of variables required to guarantee local solubility.

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1. Introduction

A basic problem in the study of diophantine equations is that of determining conditions which ensure the local solubility of a given system of equations. While the real or complex aspect of this problem is a straightforward issue for algebraic geometry, whether or not there exist non-trivial p-adic zeros is in general a problem of considerable complexity. Given a field for which diagonal equations in sufficiently many variables are soluble, such as the p-adic fields, an elementary inductive argument of Brauer [8] shows that a system of homogeneous polynomials has a non-trivial zero, provided only that the system has sufficiently many variables in terms of the number and degrees of the polynomials. For many years it was thought that the number of variables required in Brauer's method would necessarily be astronomical, but Leep and Schmidt [19] have devised refinements which yield bounds of terrestrial magnitude. Their methods rest in part on the weighty body of work on simultaneous additive equations due to Davenport and Lewis [12–14], and indeed Schmidt [30] has obtained further improvements by introducing substantial extensions to this corpus. In this paper we provide a refinement of Brauer's method which leads to sizeable improvements on the bounds of Leep and Schmidt [19], and of Schmidt [30], and which, moreover, makes use only of the rather easier theory of single additive equations (see [11]). In addition to providing upper bounds for

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the number of variables required to guarantee the local solubility of a system of forms, we are also able to make progress on lower bounds, building modestly on work described in [2, 9, 22].

In order to describe our new results we require some notation, which we adapt from [19]. Given an r-tuple of polynomials $\mathbf{F} = (F_1, \dots, F_r)$ with coefficients in a field k, denote by $\nu(\mathbf{F})$ the number of variables appearing explicitly in $\mathbf{F}$. Define $\mathcal{H}_{d,r}(k)$ to be the set of r-tuples of homogeneous polynomials of degree d, with coefficients in k, which possess no non-trivial zeros over k, and define $\mathcal{D}_{d,r}(k)$ to be the corresponding set of diagonal homogeneous polynomials. Write

$$v_{d,r}(k) = \sup_{\mathbf{g} \in \mathcal{H}_{d,r}(k)} \nu(\mathbf{g}) \quad \text{and} \quad \phi_{d,r}(k) = \sup_{\mathbf{f} \in \mathcal{D}_{d,r}(k)} \nu(\mathbf{f}).$$

For brevity we write $v_d(k)$ for $v_{d,1}(k)$ and $\phi_d(k)$ for $\phi_{d,1}(k)$. Note that whenever $s > v_{d,r}(k)$, any system of r homogeneous polynomial equations of degree d, with coefficients in k, and possessing s variables, has a non-trivial solution over k. Similarly, whenever $s > \phi_d(k)$, and $a_i \in k$ $(1 \leqslant i \leqslant s)$, then the equation $a_1 x_1^d + \cdots + a_s x_s^d = 0$ has a non-trivial solution over k.

The refinement of Brauer's method described in Sect. 2 leads to bounds on the number of variables required to obtain solution sets containing linear subspaces of specified dimension. These results being of somewhat technical interest, we defer such considerations to Sect. 2, and for the moment announce only the simpler bounds recorded in Theorem 1 below.

THEOREM 1. *Let d be a positive integer, and suppose that k is a field satisfying the property that $\phi_i(k) < \infty$ $(2 \leqslant i \leqslant d)$. Then*

$$v_{d,r}(k) \leqslant 2r^{2^{d-1}} \phi_d^{2^{d-2}} \prod_{i=2}^{d-1} (\phi_i + 1)^{2^{i-2}}.$$

For comparison, Leep and Schmidt [19, Thm. 1] have obtained a bound of the shape

$$v_{d,r}(k) \leqslant v_2 v_3^2 \dots v_d^{2^{d-2}} r^{2^{d-1}} (2^{1-2^{d-1}} + O(r^{-1})). \tag{1.1}$$

For most fields of interest, the available upper bounds for v_h are considerably weaker, in practice, than upper bounds for ϕ_h, and so the bound for $v_{d,r}(k)$ provided by Theorem 1 is a substantial sharpening of (1.1).

We discuss several corollaries to Theorem 1 in Sect. 3. When $k = \mathbb{Q}_p$, for example, one can employ the bound $\phi_d(\mathbb{Q}_p) \leqslant d^2$ due to Davenport and Lewis [11] together with Theorem 1 to deduce the following corollary.

COROLLARY 1.1. *For each rational prime number p, one has $v_{d,r}(\mathbb{Q}_p) \leqslant (rd^2)^{2^{d-1}}$, and in particular, $v_d(\mathbb{Q}_p) \leqslant d^{2^d}$.*

We note that previous work of Leep and Schmidt [19, Thm. 3] had shown that for each positive number ε, one has $v_d(\mathbb{Q}_p) \ll_\varepsilon e^{(d!)^2(1+\varepsilon)^d}$, this bound having been improved by Schmidt [30, Thm. 3] to $v_d(\mathbb{Q}_p) = o(e^{2^d d!})$. The superiority of our new estimate is self evident.

It transpires that our conclusions remain strong in number fields. Thus if K is a finite extension of $\mathbb{Q}_p$, one may combine an estimate of Skinner [34] for $\phi_h(K)$ in combination with Theorem 1 to obtain Corollary 1.2 below.

COROLLARY 1.2. *Let d be an integer with $d \geqslant 2$, let p be a prime number, and let K be a finite extension of $\mathbb{Q}_p$. Then $v_{d,r}(K) \leqslant r^{2^{d-1}} e^{2^{d+2}(\log d)^2}$.*

We are also able to obtain an explicit version of a theorem of Peck [25].

COROLLARY 1.3. *Let d be an integer with $d \geqslant 2$, and let L be a purely imaginary field extension of $\mathbb{Q}$. Then $v_{d,r}(L) \leqslant r^{2^{d-1}} e^{2^d d}$.*

Brauer's interest in the local solubility problem for systems of forms seems to have stemmed in part from investigations concerning Hilbert's resolvant problem (see [8, 31]). A problem which naturally arises in this context is that of bounding $v_{d,r}(\mathbb{Q}^{\text{rad}})$, where $\mathbb{Q}^{\text{rad}}$ denotes the radical closure of $\mathbb{Q}$, which is to say the maximal algebraic field extension of $\mathbb{Q}$ with the property that each finite degree subfield is a solvable extension of $\mathbb{Q}$. In Sect. 4 we briefly investigate upper and lower bounds for $v_{d,r}(\mathbb{Q}^{\text{rad}})$, which we record in Theorem 2 below.

THEOREM 2. *Let d be an integer with $d \geqslant 2$. Then $v_{d,r}(\mathbb{Q}^{\text{rad}}) \leqslant (2r^2)^{2^{d-2}}$. Moreover, for infinitely many integers d one has $v_d(\mathbb{Q}^{\text{rad}}) \geqslant d^{\frac{\log 2}{\log 5}}$.*

In Sect. 5 we address the problem of obtaining lower bounds for $v_{d,r}(\mathbb{Q}_p)$, pursuing a line of enquiry originating in a conjecture of Artin [4, p.x] which purports that for every prime number p, and integers d, r, one has $v_{d,r}(\mathbb{Q}_p) = rd^2$. For some time the available evidence seemed to support Artin's Conjecture. The case $r = 1$, $d = 2$ of the conjecture was proved in the last century, and the case $r = 1$, $d = 3$ was established by Demyanov [15] and Lewis [20] around 1950. Furthermore, Ax and Kochen [5] were able to show that for each r and d, there exists a number $p_0(r, d)$ such that whenever $p > p_0(r, d)$, one has $v_{d,r}(\mathbb{Q}_p) = rd^2$. However, Terjanian [35] exhibited an example establishing that $v_4(\mathbb{Q}_2) \geqslant 18$, thus disproving Artin's Conjecture, and indeed later he was able to improve this lower bound to $v_4(\mathbb{Q}_2) \geqslant 20$ (see [36]). The work of Arkhipov and Karatsuba [2, 3], Lewis and Montgomery [22], and Brownawell [9] (see also Alemu [1] for an analogous conclusion in field extensions of $\mathbb{Q}_p$), exhibits forms $F(\mathbf{x})$ for which Artin's Conjecture fails very badly. In order to be precise, we define

$$\psi(d, \varepsilon) = \exp\left(\frac{d}{(\log d)(\log\log d)^{1+\varepsilon}}\right).$$

Then [22, Thm. 1], for example, establishes that for each prime number p and positive number ε, there are infinitely many d such that some form $F(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_n]$ of degree d with $n > \psi(d, \varepsilon)$ has only the trivial zero over $\mathbb{Q}_p$, whence $v_d(\mathbb{Q}_p) > \psi(d, \varepsilon)$. By taking r copies of the form $F(\mathbf{x})$ with disjoint variables, one readily deduces that for each prime number p and positive number ε, there are infinitely many d such that $v_{d,r}(\mathbb{Q}_p) > r\psi(d, \varepsilon)$. However, in common with the examples generated by the argument of the above-mentioned authors, the degrees d corresponding to near-extremal examples belong to an exponentially thin set. By incorporating rather modest modifications into the argument of Lewis and Montgomery, we are able to both sharpen the latter lower bound on $v_{d,r}(\mathbb{Q}_p)$, and establish such bounds for all large d which are divisible by $p-1$ (for odd p). In other words, bad failures of Artin's Conjecture are essentially ubiquitous, and in particular occur for all large even degrees.

THEOREM 3. *Let p be a prime number, and define $q = q(p)$ to be 6 when $p = 2$, and to be $p - 1$ when $p > 2$. Further, let $\alpha_p = (\log p)/(6q)$. Then there exist positive numbers $d_0(\varepsilon)$ and $r_0(d, \varepsilon)$ with the property that for each positive number ε, whenever d is an integer divisible by q with $d > d_0(\varepsilon)$, and $r > r_0(d, \varepsilon)$, one has $v_{d,r}(\mathbb{Q}_p) > re^{(\alpha_p - \varepsilon)d}$.*

While it is notoriously difficult to obtain explicit upper bounds on the number $p_0(r, d)$ arising in the work of Ax and Kochen [5] alluded to above, our methods provide a cheap lower bound which may be of interest.

THEOREM 4. *One has*

$$\lim_{D\to\infty} \sup_{\substack{1\leqslant d\leqslant D\\ r\in\mathbb{N}}} \frac{p_0(r,d)}{d} \geqslant \frac{1}{30}.$$

We note that while the strength of the lower bound recorded in Theorem 4 can doubtless be improved by using more sophisticated methods, it is difficult to imagine any approach which could replace the number $1/30$ occurring in its statement by a number exceeding 1.

2. The reduction argument

In this section we launch our proof of Theorem 1 by establishing the reduction procedure at the heart of our argument. Before describing the details of this argument, we must record some rather general notation. Let $\mathbb{K}$ be a field. We are interested in the existence of solution sets, over $\mathbb{K}$, of systems of homogeneous polynomial equations with coefficients in $\mathbb{K}$. When such a solution set contains a linear subspace of the ambient space, we define its dimension to be that when considered as a projective space. We note that this convention differs from that adopted by Leep

and Schmidt [19]. We denote by $\mathcal{G}_d^{(m)}(r_d,\dots,r_1)$ the set of $(r_d+r_{d-1}+\cdots+r_1)$-tuples of homogeneous polynomials, of which r_i have degree i for $1\leqslant i\leqslant d$, with coefficients in $\mathbb{K}$, which possess no non-trivial linear space of solutions of dimension m over $\mathbb{K}$. We then define $V_d^{(m)}(\mathbf{r})=V_d^{(m)}(r_d,\dots,r_1;\mathbb{K})$ by

$$V_d^{(m)}(r_d,\dots,r_1;\mathbb{K})=\sup_{\mathbf{h}\in\mathcal{G}_d^{(m)}(r_d,\dots,r_1)}\nu(\mathbf{h}).$$

We observe for future reference that $V_d^{(m)}(r_d,\dots,r_1;\mathbb{K})$ is an increasing function of the arguments m and r_i. For the sake of convenience, we abbreviate $V_d^{(0)}(\mathbf{r};\mathbb{K})$ to $V_d(\mathbf{r};\mathbb{K})$. Note that $v_{d,r}(\mathbb{K})=V_d(r,0,\dots,0;\mathbb{K})$.

The proof of the following lemma is motivated by the arguments of [19] leading to [19, equation (2.13)] and [19, equation (3.1)].

LEMMA 2.1. *Let d and r_i $(1\leqslant i\leqslant d)$ be non-negative integers with $d\geqslant 2$ and $r_d>0$. Then on writing ϕ for $\phi_d(\mathbb{K})$, whenever $\phi<\infty$ one has*

$$V_d(r_d,r_{d-1},\dots,r_1;\mathbb{K})\leqslant\phi+V_d(r'_d,r'_{d-1},\dots,r'_1;\mathbb{K}),$$

where $r'_d=r_d-1$, and

$$r'_j=\sum_{i=j}^{d}r_i\binom{\phi+i-j-1}{i-j}\quad(1\leqslant j<d).$$

Proof. Before embarking on the proof proper, we first set the stage. Take N to be any integer with $N>\phi+V_d(\mathbf{r}';\mathbb{K})$. Also, for the sake of convenience, define the integers $\tilde{r}_i$ by

$$\tilde{r}_i=\begin{cases}r_d-1, & \text{when } i=d,\\ r_i, & \text{otherwise.}\end{cases}$$

Consider a form $\mathcal{F}$ of degree d, and forms $\mathcal{G}_{ij}$ of degree i $(1\leqslant j\leqslant\tilde{r}_i,\ 1\leqslant i\leqslant d)$, all having N variables. We claim that when $1\leqslant k\leqslant\phi+1$, there exist k linearly independent vectors $\mathbf{x}_1,\dots,\mathbf{x}_k$ with the property that $\mathcal{F}(t_1\mathbf{x}_1+\cdots+t_k\mathbf{x}_k)$ is a diagonal form in $t_1,\dots,t_k$, and such that $\mathcal{G}_{ij}(t_1\mathbf{x}_1+\cdots+t_k\mathbf{x}_k)$ vanishes identically for $1\leqslant j\leqslant\tilde{r}_i$ and $1\leqslant i\leqslant d$. Since the lemma follows immediately from the case $k=\phi+1$ of this claim, the proof of the latter will suffice to establish the former.

We prove the claim by induction, starting with the observation that since $N>V_d(\mathbf{r}';\mathbb{K})$, the claim holds trivially when $k=1$. Next, when $1\leqslant k\leqslant\phi$, we suppose that $\mathbf{x}_1,\dots,\mathbf{x}_k$ have the claimed property, and seek a vector $\mathbf{x}_{k+1}$ such

that $\mathbf{x}_1, \dots, \mathbf{x}_{k+1}$ also have the claimed property. We define the polynomials $\mathcal{F}_u$ $(0 \leqslant u \leqslant d)$ through the expansion

$$\mathcal{F}(\mathbf{y} + t\mathbf{x}) = \sum_{u=0}^{d} \mathcal{F}_u(\mathbf{y}, \mathbf{x}) t^u, \tag{2.1}$$

valid for each $\mathbf{x}, \mathbf{y} \in \mathbb{K}^N$. Notice that $\mathcal{F}_u$ is a form of degree u in $\mathbf{x}$, and of degree $d-u$ in $\mathbf{y}$. When $0 \leqslant v \leqslant i$, $1 \leqslant j \leqslant \tilde{r}_i$ and $1 \leqslant i \leqslant d$, we define the polynomials $\mathcal{G}_{ijv}$ similarly through the expansion

$$\mathcal{G}_{ij}(\mathbf{y} + t\mathbf{x}) = \sum_{v=0}^{i} \mathcal{G}_{ijv}(\mathbf{y}, \mathbf{x}) t^v, \tag{2.2}$$

and note that $\mathcal{G}_{ijv}$ is a form of degree v in $\mathbf{x}$, and of degree $i-v$ in $\mathbf{y}$. Let T be the subspace of $\mathbb{K}^N$ spanned by $\mathbf{x}_1, \dots, \mathbf{x}_k$, and let S be any subspace of $\mathbb{K}^N$ such that $S \oplus T = \mathbb{K}^N$. Thus, since $k \leqslant \phi$, one has

$$\dim(S) = N - k > V_d(\mathbf{r}'; \mathbb{K}). \tag{2.3}$$

We aim to choose $\mathbf{x} \in S$ so that for each $\mathbf{y} \in T$, one has

$$\mathcal{F}_u(\mathbf{y}, \mathbf{x}) = 0 \quad (1 \leqslant u \leqslant d-1) \tag{2.4}$$

and

$$\mathcal{G}_{ijv}(\mathbf{y}, \mathbf{x}) = 0 \quad (0 \leqslant v \leqslant i,\ 1 \leqslant j \leqslant \tilde{r}_i,\ 1 \leqslant i \leqslant d). \tag{2.5}$$

Having found such an $\mathbf{x}$, on substituting $\mathbf{y} = t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k$ into (2.1) and (2.2), we discover that, by the inductive hypothesis, the form $\mathcal{F}(\mathbf{y} + t\mathbf{x})$ becomes a diagonal form in $t_1, \dots, t_k$ and t, and the forms $\mathcal{G}_{ij}(\mathbf{y} + t\mathbf{x})$ vanish identically. Thus the inductive hypothesis follows with $k+1$ in place of k.

We now establish the existence of the desired element $\mathbf{x} \in S$. Consider an arbitrary element of T, say $\mathbf{y} = s_1\mathbf{x}_1 + \cdots + s_k\mathbf{x}_k$, and substitute this expression into (2.4) and (2.5). We find that $\mathcal{F}_u(\mathbf{y}, \mathbf{x})$ becomes a form of degree $d-u$ in $s_1, \dots, s_k$, whose coefficients are forms of degree u in $\mathbf{x}$. Thus, following a simple counting argument, one finds that the number of such coefficients of degree u is

$$\begin{pmatrix} d-u+k-1 \\ d-u \end{pmatrix} \leqslant \begin{pmatrix} \phi+d-u-1 \\ d-u \end{pmatrix}.$$

Similarly, $\mathcal{G}_{ijv}(\mathbf{y}, \mathbf{x})$ becomes a form of degree $i-v$ in $s_1, \dots, s_k$, whose coefficients are forms of degree v in $\mathbf{x}$. The number of such coefficients of degree v is

$$\begin{pmatrix} i-v+k-1 \\ i-v \end{pmatrix} \leqslant \begin{pmatrix} \phi+i-v-1 \\ i-v \end{pmatrix}.$$

Moreover for each i and j one has $\mathcal{G}_{ij0}(\mathbf{y}, \mathbf{x}) = \mathcal{G}_{ij}(\mathbf{y}) = 0$. Consequently the system of equations (2.4) and (2.5) can be satisfied so long as we can find $\mathbf{x} \in S$ satisfying a system of equations with r_i' of degree i for $1 \leqslant i \leqslant d$. But in view of (2.3), one has $\dim(S) > V_d(\mathbf{r}'; \mathbb{K})$, whence the latter system of equations possesses a non-trivial solution over $\mathbb{K}$. Recalling the concluding remarks of the preceding paragraph, we have completed the induction, and hence also the proof of the lemma.

The following corollary provides a convenient simplification of Lemma 2.1.

COROLLARY. *With the same notation and hypotheses as in the statement of Lemma* 2.1, *one has*

$$V_d(r_d, r_{d-1}, \dots, r_1; \mathbb{K}) \leqslant \phi + V_d(r_d^*, r_{d-1}^*, \dots, r_1^*; \mathbb{K}),$$

where $r_d^* = r_d - 1$, *and*

$$r_j^* = \sum_{i=j}^{d} r_i \phi^{i-j} \quad (1 \leqslant j < d).$$

Repeated application of Lemma 2.1, or its corollary, ultimately yields a bound for $V_d(\mathbf{r}; \mathbb{K})$ in terms of $V_{d-1}(\mathbf{r}'; \mathbb{K})$, for suitable $\mathbf{r}'$. In the next lemma we make this observation precise, making use of an argument strikingly similar to that used in the proof of [19, Lemma 1].

LEMMA 2.2. *With the same notation and hypotheses as in the statement of Lemma* 2.1, *one has*

$$V_d(r_d, \dots, r_1; \mathbb{K}) \leqslant r_d \phi + V_{d-1}(s_{d-1}, \dots, s_1; \mathbb{K}),$$

where

$$s_j = \sum_{i=j}^{d} r_i (r_d \phi)^{i-j} \quad (1 \leqslant j \leqslant d-1). \tag{2.6}$$

Proof. When $1 \leqslant k \leqslant d$, define the integers $r_k^{(0)}$ by taking $r_k^{(0)} = r_k$. Further, when $1 \leqslant j \leqslant r_d$, define the integers $r_k^{(j)}$ inductively by

$$r_d^{(j)} = r_d^{(j-1)} - 1 \tag{2.7}$$

and

$$r_k^{(j)} = \sum_{i=k}^{d} r_i^{(j-1)} \phi^{i-k} \quad (1 \leqslant k < d). \tag{2.8}$$

Then by applying the corollary to Lemma 2.1 repeatedly, we deduce that when $1 \leqslant j \leqslant r_d$, one has

$$V_d(r_d, \dots, r_1; \mathbb{K}) \leqslant j\phi + V_d(r_d^{(j)}, \dots, r_1^{(j)}; \mathbb{K}).$$

Notice that by (2.7) we have

$$r_d^{(j)} = r_d - j, \tag{2.9}$$

and hence, in order to complete the proof of the lemma, it remains only to show that

$$r_k^{(r_d)} \leqslant s_k \quad (1 \leqslant k \leqslant d-1). \tag{2.10}$$

We establish (2.10) by proving that when $1 \leqslant k \leqslant d-1$ and $1 \leqslant j \leqslant r_d$, one has

$$r_k^{(j)} \leqslant \sum_{i=k}^{d} r_i (j\phi)^{i-k}, \tag{2.11}$$

the special case $j = r_d$ of which supplies the desired inequality (2.10). This more general bound we prove by induction. First observe that by (2.8) we have

$$r_k^{(1)} = \sum_{i=k}^{d} r_i \phi^{i-k} \quad (1 \leqslant k < d),$$

whence the inductive hypothesis (2.11) holds when $j = 1$. Suppose next that $1 < J \leqslant r_d$, and that (2.11) holds for each j with $1 \leqslant j < J$ when $1 \leqslant k < d$. Then by (2.8) and (2.11) one has for each k with $1 \leqslant k < d$,

$$r_k^{(J)} \leqslant \sum_{i=k}^{d} \phi^{i-k} \sum_{l=i}^{d} r_l \left((J-1)\phi\right)^{l-i} = \sum_{l=k}^{d} r_l \phi^{l-k} \sum_{i=k}^{l} (J-1)^{l-i}.$$

Consequently the inductive hypothesis (2.11) holds with $j = J$. This completes the induction, and hence also the proof of the lemma.

A comparison of the statement of Lemma 2.2 with [19, Lemma 1] reveals that the term $\phi_d(\mathbb{K})$ in the former replaces $v_d(\mathbb{K})$ in the latter. This observation provides some indication of the nature of our improvements, since the bounds currently available for $v_d(\mathbb{K})$ are substantially weaker than those for $\phi_d(\mathbb{K})$.

We will prove below a theorem somewhat more general than Theorem 1, in that it provides an upper bound for $v_{d,r}^{(m)}(\mathbb{K})$, which we define by

$$v_{d,r}^{(m)}(\mathbb{K}) = V_d^{(m)}(r, 0, \dots, 0; \mathbb{K}).$$

In preparation for the proof of this theorem we require a result due to Leep and Schmidt [19].

LEMMA 2.3. *When m is a positive integer, one has*

$$V_d^{(m)}(r_d,\dots,r_1;\mathbb{K}) \leqslant m + V_d(t_d,\dots,t_1;\mathbb{K}),$$

where

$$t_j = \sum_{i=j}^{d} r_i m^{i-j} \quad (1 \leqslant j \leqslant d).$$

Proof. Bearing in mind the definitions used by Leep and Schmidt, this lemma is nothing other than [19, Eqn (3.1)].

THEOREM 2.4. *Let m, d and r be non-negative integers with $d \geqslant 2$ and $r \geqslant 1$. Write ϕ_i for $\phi_i(\mathbb{K})$ $(2 \leqslant i \leqslant d)$. Then whenever $\mathbb{K}$ is a field for which $\phi_i < \infty$ $(2 \leqslant i \leqslant d)$, one has*

$$v_{d,r}^{(m)}(\mathbb{K}) \leqslant 2(r^2\phi_d + mr)^{2^{d-2}} \prod_{i=2}^{d-1} (\phi_i + 1)^{2^{i-2}}.$$

Proof. We begin by applying Lemma 2.3 to obtain

$$v_{d,r}^{(m)}(\mathbb{K}) = V_d^{(m)}(r,0,\dots,0;\mathbb{K}) \leqslant m + V_d(r,mr,\dots,m^{d-1}r;\mathbb{K}). \tag{2.12}$$

Having eliminated explicit mention of linear subspaces, we next make use of Lemma 2.2 to deduce that

$$v_{d,r}^{(m)}(\mathbb{K}) \leqslant m + r\phi_d + V_{d-1}(s_{d-1},\dots,s_1;\mathbb{K}), \tag{2.13}$$

where

$$s_j = \sum_{i=j}^{d} (rm^{d-i})(r\phi_d)^{i-j} \quad (1 \leqslant j \leqslant d-1). \tag{2.14}$$

Write

$$\psi_d = r^2\phi_d + mr.$$

Then it follows from (2.14) that when $1 \leqslant j \leqslant d-1$,

$$s_j \leqslant r(m + r\phi_d)^{d-j} \leqslant \psi_d^{d-j} \leqslant \psi_d^{2^{d-j-1}},$$

and thus from (2.13) one has

$$v_{d,r}^{(m)}(\mathbb{K}) \leqslant \psi_d + V_{d-1}(\psi_d, \psi_d^2, \dots, \psi_d^{2^{d-2}};\mathbb{K}). \tag{2.15}$$

Having prepared the ground by establishing the estimate (2.15), we now initiate our basic inductive argument. When $2 \leqslant k < d$ we define the integers ψ_k by $\psi_k = \phi_k + 1$, and define further

$$A_i = \prod_{j=0}^{i-1} \psi_{d-j}^{2^{i-j-1}} \quad (1 \leqslant i < d). \tag{2.16}$$

We claim that when $1 \leqslant k < d$ one has

$$v_{d,r}^{(m)}(\mathbb{K}) \leqslant A_k + V_{d-k}(A_k, A_k^2, \dots, A_k^{2^{d-k-1}}; \mathbb{K}). \tag{2.17}$$

Note that since $A_1 = \psi_d$, this claim follows from (2.15) in the case $k = 1$. Suppose next that $K > 1$, and that the proposition holds for $1 \leqslant k < K$. Then (2.17) holds with $k = K - 1$, and so on writing $\delta = d - K + 1$, an application of Lemma 2.2 yields

$$v_{d,r}^{(m)}(\mathbb{K}) \leqslant A_{K-1} + A_{K-1}\phi_\delta + V_{\delta-1}(t_{\delta-1}, \dots, t_1; \mathbb{K}), \tag{2.18}$$

where

$$t_j = \sum_{i=j}^{\delta} A_{K-1}^{2^{\delta-i}} \left(A_{K-1}\phi_\delta\right)^{i-j} \quad (1 \leqslant j \leqslant \delta - 1).$$

But on recalling (2.16) we have

$$t_j \leqslant A_{K-1}^{2^{\delta-j}}(\phi_\delta + 1)^{\delta-j} < A_{K-1}^{2^{\delta-j}}\psi_\delta^{2^{\delta-j-1}} \leqslant A_K^{2^{\delta-j-1}} \quad (1 \leqslant j < \delta).$$

Thus we deduce from (2.18) that (2.17) holds with $k = K$, and so the induction is complete.

Finally, on noting that by (2.16) and (2.17) one has

$$v_{d,r}^{(m)}(\mathbb{K}) \leqslant A_{d-1} + V_1(A_{d-1}; \mathbb{K}) = 2A_{d-1} = 2\prod_{j=0}^{d-2} \psi_{d-j}^{2^{d-j-2}},$$

we complete the proof of the theorem.

We note that Theorem 1 follows immediately from Theorem 2.4 on setting $m = 0$.

3. Several consequences of the reduction argument

We have now ascended to the point from which we may harvest the crop of corollaries stemming from the reduction argument manifesting itself in Theorem

2.4. Since these corollaries will be essentially immediate from suitable bounds on $\phi_i(\mathbb{K})$ $(2 \leqslant i \leqslant d)$, our discussions in this section will be brief.

(a) **p-adic fields.** In order to establish Corollary 1.1 to Theorem 1 we recall Davenport and Lewis [11, Thm. 1], which shows that for each prime p one has $\phi_d(\mathbb{Q}_p) \leqslant d^2$. On substituting the latter bound into Theorem 1, we obtain

$$v_{d,r}(\mathbb{Q}_p) \leqslant 2(rd)^{2^{d-1}} \prod_{i=2}^{d-1} (i^2+1)^{2^{i-2}} \leqslant 2(rd)^{2^{d-1}} \prod_{i=2}^{d-1} d^{2^{i-1}} \leqslant 2r^{2^{d-1}} d^{2^d-1},$$

whence Corollary 1.1 follows whenever $d \geqslant 2$. We note that while a number of refinements on [11, Thm. 1] have been obtained in particular cases (see, for example, [16, 24]), these do not lead to substantial improvements in the quality of Corollary 1.1 to Theorem 1.

(b) **$\mathfrak{p}$-adic fields.** Let K be a finite extension of $\mathbb{Q}_p$, let d be an integer with $d \geqslant 2$, and let $f = \mathrm{ord}_p d$. Then Skinner [34] has shown that

$$\phi_d(K) \leqslant d((d+1)^{\max\{2f,1\}} - 1).$$

It follows in particular that

$$\phi_d(K) \leqslant d((d+1)^{\max\{2\log d/\log p,1\}} - 1), \tag{3.1}$$

and hence, by means of an elementary calculation, that $\phi_d(K) < \exp(C(\log d)^2)$, where $C = \log(18)/(\log 2)^2 < 8$. Consequently an application of Theorem 1, in combination with the latter upper bound, reveals that

$$v_{d,r}(K) < 2r^{2^{d-1}} \exp\left(8(\log d)^2 \sum_{i=0}^{d-2} 2^i\right) < r^{2^{d-1}} \exp\left(2^{d+2}(\log d)^2\right),$$

whence Corollary 1.2 follows whenever $d \geqslant 2$.

We note that modest improvements may be achieved in the latter bound by a more precise analysis. Moreover further refinements may be obtained if one is prepared to accept bounds which depend on the degree of the field extension $K/\mathbb{Q}$.

(c) **Purely imaginary fields.** Let L be a finite field extension of $\mathbb{Q}$, let $s \geqslant 2^d + 1$, and suppose that $b_1, \dots, b_s$ are integers of L. Then by using Siegel's version of the circle method (see [32, 33]), Birch [7, Thm. 3] was able to show that the equation $\sum_{i=1}^{s} b_i x_i^d = 0$ has a non-trivial solution in L provided that it has a non-trivial solution in every real and $\mathfrak{p}$-adic completion of L. Next we observe that if L is purely imaginary, then the real completion of L is simply $\mathbb{C}$, and thus by employing

Skinner [34], in the form (3.1) above, we deduce that the latter equation is soluble over L provided only that

$$s > \max\left\{2^d, d((d+1)^{\max\{2\log d/\log p,1\}} - 1)\right\}. \tag{3.2}$$

The simple bound $\phi_d(L) \leqslant e^{2d}$ is almost immediate from (3.2), and on substituting this inequality into Theorem 1 we obtain Corollary 1.3.

We have aimed above for a uniform bound of simple type. Of course one may sharpen this bound somewhat, and in particular for larger d one may replace the term $e^{2^d d}$ by 2^{d2^d} in the statement of Corollary 1.3. Moreover, forthcoming results of M. Davidson, improving substantially on Birch's estimate (at least for field extensions with degree not too large), will lead to sizeable improvements in the bound recorded in Corollary 1.3.

(d) **Systems of cubic forms.** It transpires that by exploiting Lemma 2.1 more carefully than in our proof of Theorem 1, one may obtain bounds for $v_{3,r}(\mathbb{Q}_p)$ somewhat smaller than have been obtained hitherto, at least for certain intermediate ranges of r. We summarise these new bounds in Lemma 3.1 below.

LEMMA 3.1. *When r is a positive integer, one has*

$$v_{3,r}(\mathbb{Q}_p) \leqslant \begin{cases} \frac{9}{2}r^4 + 12r^3 + \frac{15}{2}r^2 + 3r + 2, \\ \quad \textit{when } p \equiv 2 (\mathrm{mod}\ 3), \quad \textit{or } \ p = 3, \\ 18r^4 + 48r^3 + \frac{57}{2}r^2 + \frac{9}{2}r, \quad \textit{otherwise}. \end{cases}$$

For comparison, Leep and Schmidt [19] have shown that

$$v_{3,r}(\mathbb{Q}_p) \leqslant \tfrac{1}{2}(81r^4 - 108r^3 + 63r^2 - 18r - 8),$$

and Schmidt, in work spanning a series of papers [27–29], has established that

$$v_{3,r}(\mathbb{Q}_p) \leqslant 5300r(3r+1)^2.$$

Thus the bound of Lemma 3.1 supersedes those of Leep and Schmidt, and of Schmidt, when $r \leqslant 10597$ in cases where $p \equiv 2 (\mathrm{mod}\ 3)$, and when $5 \leqslant r \leqslant 2647$ in cases where $p \equiv 1 (\mathrm{mod}\ 3)$. We note that in the case $r = 2$, the bound $v_{3,r}(\mathbb{Q}_p) \leqslant 320$ of Leep and Schmidt is improved by Lemma 3.1 only in cases where $p \equiv 2 (\mathrm{mod}\ 3)$, where we now obtain $v_{3,2}(\mathbb{Q}_p) \leqslant 206$. Meanwhile, when $p \equiv 1 (\mathrm{mod}\ 3)$ the new bounds for $v_{2,r}(\mathbb{Q}_p)$ obtained by Martin [23], improving on previous work of Leep [18] and Schmidt [26], yield, through the methods of Leep and Schmidt [19], the new upper bound $v_{3,2}(\mathbb{Q}_p) \leqslant 308$.

Proof. We note merely that on writing ϕ for $\phi_3(\mathbb{Q}_p)$, the conclusion of Lemma 2.1 implies that when $r_3 \geqslant 1$,

$$V_3(r_3, r_2, r_1; \mathbb{Q}_p) \leqslant \phi + V_3(r_3 - 1, r_2 + \phi r_3, r_1 + \phi r_2 + \tfrac{1}{2}\phi(\phi+1)r_3).$$

On noting the bounds

$$\phi_3(\mathbb{Q}_p) = \begin{cases} 3, & \text{when } p \equiv 2 (\text{mod } 3), \quad \text{or } \ p = 3, \\ 6, & \text{otherwise,} \end{cases}$$

(see Lewis [21]), and making use of the bounds

$$v_{2,r}(\mathbb{Q}_p) \leqslant \begin{cases} 2r^2, & \text{when } r \text{ is even,} \\ 2r^2 + 2, & \text{when } r \text{ is odd,} \end{cases}$$

(see Martin [23]), the desired conclusions follow by a simple induction.

4. Radical zeros of systems of polynomial equations

In this section we briefly consider some diophantine problems over $\mathbb{Q}^{\text{rad}}$, the radical closure of $\mathbb{Q}$. Although a definitive attack on such problems must surely make use of the finer Galois-theoretic properties of $\mathbb{Q}^{\text{rad}}$, we are nonetheless able to provide non-trivial bounds on $v_{d,r}(\mathbb{Q}^{\text{rad}})$ through the use of Theorem 1.

We start by providing the lower bound for $v_{d,r}(\mathbb{Q}^{\text{rad}})$ recorded in Theorem 2, this being essentially immediate from Lemma 4.1 below.

LEMMA 4.1. *Let k be a natural number. Then $v_{5^k}(\mathbb{Q}^{\text{rad}}) \geqslant 2^k$.*

Proof. We begin by observing that the polynomial $f(x) = 2x^5 - 5x^4 + 5$ is irreducible in $\mathbb{Q}[x]$, by Eisenstein's criterion, and has precisely three real roots. It is therefore a straightforward exercise in Galois Theory (see, for example, Garling [17]) to show that the equation $f(x) = 0$ is not soluble by radicals. We write $\phi(x, y) = y^5 f(x/y)$, and note that the equation $\phi(x, y) = 0$ has only the trivial solution $x = y = 0$ over $\mathbb{Q}^{\text{rad}}$.

Next, for each integer k, define the polynomial $\phi_k(\mathbf{x}) = \phi_k(x_1, \dots, x_{2^k})$ by putting $\phi_1(\mathbf{x}) = \phi(x_1, x_2)$, and when $k \geqslant 1$ by using the relation

$$\phi_{k+1}(\mathbf{x}) = \phi(\phi_k(x_1, \dots, x_{2^k}), \phi_k(x_{2^k+1}, \dots, x_{2^{k+1}})).$$

Observe that $\phi_k(\mathbf{x})$ is a polynomial with integral coefficients of degree 5^k in 2^k variables. In order to complete the proof of the lemma, therefore, it suffices to show that for each k the equation $\phi_k(\mathbf{x}) = 0$ has only the solution $\mathbf{x} = \mathbf{0}$ over $\mathbb{Q}^{\text{rad}}$. We establish this proposition by induction, noting that when $k = 1$ the proposition is immediate from the definition of ϕ. Suppose then that $k \geqslant 2$, and that $\boldsymbol{\alpha}$ is a radical 2^k-tuple with $\phi_k(\boldsymbol{\alpha}) = 0$. Write $\xi = \phi_{k-1}(\alpha_1, \dots, \alpha_{2^{k-1}})$ and $\eta = \phi_{k-1}(\alpha_{2^{k-1}+1}, \dots, \alpha_{2^k})$, and observe that the hypothesis $\phi_k(\boldsymbol{\alpha}) = 0$ implies that $\phi(\xi, \eta) = 0$. However, by their definitions, one has $\xi, \eta \in \mathbb{Q}^{\text{rad}}$, whence the definition of ϕ ensures that $\xi = \eta = 0$. Thus

$$\phi_{k-1}(\alpha_1, \dots, \alpha_{2^{k-1}}) = \phi_{k-1}(\alpha_{2^{k-1}+1}, \dots, \alpha_{2^k}) = 0,$$

and so the inductive hypothesis implies that $\boldsymbol{\alpha} = \mathbf{0}$. Then the inductive hypothesis holds for $k+1$ in place of k, and the proof of the lemma is complete.

In order to establish the upper bound of Theorem 3, we have merely to observe that since each equation $ax^k + by^k = 0$, with $a, b \in \mathbb{Q}^{\text{rad}}$, is soluble non-trivially over $\mathbb{Q}^{\text{rad}}$, then $\phi_d(\mathbb{Q}^{\text{rad}}) = 1$ for each $d \in \mathbb{N}$. Thus Theorem 1 implies that

$$v_{d,r}(\mathbb{Q}^{\text{rad}}) \leqslant 2r^{2^{d-1}} \prod_{i=2}^{d-1} 2^{2^{i-2}} \leqslant 2^{2^{d-2}} r^{2^{d-1}}.$$

5. Systems of p-adic forms possessing no non-trivial solutions

Following our main assault on upper bounds for $v_{d,r}(\mathbb{Q}_p)$, in this section we indulge in a diversionary sortie which provides non-trivial lower bounds on $v_{d,r}(\mathbb{Q}_p)$. The sole weapon in our arsenal wielded in the proof of Theorem 3 is a lemma due to Lewis and Montgomery [22]. Throughout the remainder of this section, when s and k are positive integers we write

$$S_{s,k}(\mathbf{x}) = \sum_{i=1}^{s} x_i^k.$$

LEMMA 5.1. *Let p be a prime number, and define $q = q(p)$ as in the statement of Theorem* 3. *Let M be a positive integer, and let $\mathcal{M}$ be a set of K integers in the interval $[M, 2M)$. Suppose that x_i $(1 \leqslant i \leqslant N)$ are integers, not all divisible by p, with the property that*

$$S_{N,qm}(\mathbf{x}) \equiv 0 \pmod{p^{qM}} \quad (m \in \mathcal{M}).$$

Then one has $N \geqslant p^K$.

Proof. The lemma is immediate on conjoining the conclusions of [22, Lemmata 2 and 3].

We are now equipped to prove Theorem 3.

The proof of Theorem 3. Recall the notation of the statement of Theorem 3. Let ε be a positive number, let d be sufficiently large in terms of ε, and let r be sufficiently large in terms of ε and d. Write $M = [d/3]$ and $N = p^{[M/2]-2} - 1$. Further, when $3 \leqslant m < M$ define the polynomial $\phi_m(\mathbf{x})$ by

$$\phi_m(\mathbf{x}) = S_{N,(M+m)q}(\mathbf{x}) S_{N,(d-M-m)q}(\mathbf{x}). \tag{5.1}$$

We note that for each m the polynomial $\phi_m(\mathbf{x})$ has degree dq, and possesses N variables. Suppose that $\mathbf{x}$ is a solution of the system of congruences

$$\phi_m(\mathbf{x}) \equiv 0 \pmod{p^{qd}} \quad (3 \leqslant m < M). \tag{5.2}$$

Let $\mathcal{M}$ denote the set of natural numbers of the form $M+m$, with $1 \leqslant m < M$, for which $S_{N,(M+m)q}(\mathbf{x}) \equiv 0 \pmod{p^{qM}}$. Then since (5.1) and (5.2) together imply that for each m with $3 \leqslant m < M$, at least one of $M+m$ and $d-M-m$ lies in $\mathcal{M}$, we deduce that $\mathrm{card}(\mathcal{M}) \geqslant \frac{1}{2}(M-3)$. But $N < p^{[M/2]-2}$, so that an application of Lemma 5.1 leads to the conclusion that $x_i \equiv 0 \pmod{p}$ $(1 \leqslant i \leqslant N)$. Consequently the system $\phi_m(\mathbf{x}) = 0$ $(3 \leqslant m < M)$ has only the trivial solution over $\mathbb{Q}_p$.

We now construct a system of r forms of degree dq possessing only the trivial solution over $\mathbb{Q}_p$. Let $R = [r/M]$, and consider the system of equations

$$\phi_m(\mathbf{x}_i) = 0 \quad (3 \leqslant m < M,\ 1 \leqslant i \leqslant R), \tag{5.3}$$

together with $r-(M-3)R$ trivial equations, where we write $\mathbf{x}_i = (x_{i1}, \dots, x_{iN})$. The conclusion of the preceding paragraph ensures that the only solution of the system (5.3) over $\mathbb{Q}_p$ is the trivial one, and moreover the number of variables occurring in this system exceeds

$$\left(\frac{r}{M} - 1\right)\left(p^{[M/2]-2} - 1\right) > \frac{r}{M} p^{(1-\varepsilon)M/2} > r p^{(1-2\varepsilon)d/6}.$$

The conclusion of Theorem 3 is immediate from the latter inequality.

The proof of Theorem 4. We are able to establish Theorem 4 cheaply by merely observing that when $p > 2$, the number of variables in the system (5.3) exceeds $r(dq)^2$ so long as

$$\left(\frac{r}{M} - 1\right)\left(p^{[M/2]-2} - 1\right) > r d^2 (p-1)^2.$$

Moreover when $d \geqslant 30$, the latter inequality is satisfied so long as r and p are sufficiently large in terms of d. Thus each exponent $D = 30(p-1)$, for which p is a large prime number, has the property that when r is large, $v_{D,r}(\mathbb{Q}_p) > rD^2$, whence a counterexample to Artin's Conjecture exists for a system of equations of degree D over $\mathbb{Q}_p$. This suffices to complete the proof of the theorem.

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