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Hecke Algebras of Classical Groups over p-adic Fields II

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Abstract. In the previous part of this paper, we constructed a large family of Hecke algebras on some classical groups G defined over p-adic fields in order to understand their admissible representations. Each Hecke algebra is associated to a pair (J_Σ, ρ_Σ) of an open compact subgroup J_Σ and its irreducible representation ρ_Σ which is constructed from given data $\Sigma = (\Gamma, P'_0, \varrho)$. Here, Γ is a semisimple element in the Lie algebra of G, P'_0 is a parahoric subgroup in the centralizer of Γ in G, and ϱ is a cuspidal representation on the finite reductive quotient of P'_0. In this paper, we explicitly describe those Hecke algebras when P'_0 is a minimal parahoric subgroup, ϱ is trivial and ρ_Σ is a character.

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Introduction

Let k be a p-adic field with odd residue characteristic p, and let G be a connected reductive group over k. In their work on GL_n [HM1, 2], Howe and Moy sketch a scheme for understanding the harmonic analysis on G via the harmonic analysis on Hecke algebras associated to open compact data for G. More recently, Bushnell and Kutzko have generalized this scheme to reductive groups via the theory of types [BK2]. Especially, those Hecke algebras should be in a form such that their harmonic analysis is tractable; in fact, they are expected to be *generalized affine Hecke algebras*. In the stream of this philosophy, in [K1], we constructed a large family of Hecke algebras on some classical groups. Here, we will prove that some of those Hecke algebras are in fact generalized affine Hecke algebras.

We recall the basic situation from [K1]; Let k be a p-adic field with an involution σ and let k_0 be its σ-fixed subfield of k. Let V be a finite dimensional k-linear space equipped with ε-Hermitian form $\langle\,,\rangle$ ($\varepsilon = +1$ or -1). Let G be the connected component of a group of *isometries* on $(V, \langle\,,\rangle)$. In [K1], we constructed a large family of Hecke algebras on G when the residue characteristic of k is big enough

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(see [K1, 3.2.3]). Let $\Sigma = (\Gamma, P'_0, \varrho)$ be given as in Section 1.5.B in [K1], that is, Γ is a semisimple element in the Lie algebra $\mathfrak{g}$ of G as in [K1, 1.3.2], P'_0 is a parahoric subgroup in the centralizer $C_{\mathrm{G}}(\Gamma)$ of Γ in G and ϱ is a cuspidal representation of the finite reductive quotient of P'_0. Associated to such a Σ, we constructed a pair (J_Σ, ρ_Σ) consisting of an open compact subgroup J_Σ and its irreducible representation ρ_Σ. Let $\mathcal{H} = \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ be the Hecke algebra associated to (J_Σ, ρ_Σ). This is the convolution algebra on the space of all compactly supported functions $f: \mathrm{G} \longrightarrow \mathrm{End}(\rho_\Sigma)$, which transform via ρ_Σ under left and right translations by J_Σ. That is, $f(jgj') = \rho_\Sigma(j)f(g)\rho_\Sigma(j')$ for $g \in \mathrm{G}$ and $j, j' \in J_\Sigma$. $\mathcal{H}$ also carries a natural involution $*$ and an inner product (,) (see (5.1.2)).

Assume that P'_0 is a minimal parahoric subgroup I'_0 (see Section 1.5.A) and ϱ is a trivial character of I'_0. Let $\widetilde{W}'$ be the affine Weyl group of $\mathrm{G}' = C_{\mathrm{G}}(\Gamma)$. Then from Proposition 4.2.6 in [K1], we have $\mathrm{Supp}(\mathcal{H}) = J_\Sigma \mathrm{G}' J_\Sigma = J_\Sigma \widetilde{W}' J_\Sigma$ and $\mathcal{H}$ is linearly spanned by functions f_w whose support is a single double coset $J_\Sigma w J_\Sigma$ with $w \in \widetilde{W}'$. In this paper, for the case when ρ_Σ is a character, we will describe the Hecke algebra $\mathcal{H} = \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ by directly finding generators and relations. Moreover, we relate those Hecke algebras to Hecke algebras on $\mathrm{G}' = C_{\mathrm{G}}(\Gamma)$ by establishing an L^2-isomorphism between Hecke algebras:

MAIN THEOREM. *Let k satisfy the assumption in* [K1, 3.2.3] *and let* G *be a classical group considered in* [K1]. *Let* Γ *be a semisimple element in the Lie algebra* $\mathfrak{g}$ *as in* [K1, 1.3.2] *and let* I'_0 *be a minimal parahoric subgroup of* $\mathrm{G}' = C_{\mathrm{G}}(\Gamma)$*, the centralizer of* Γ *in* G. *Let* $\Sigma = (\Gamma, I'_0, 1)$*, where 1 is the trivial character of* I'_0. *Let* (J_Σ, ρ_Σ) *be a pair consisting of an open compact subgroup* J_Σ *and its irreducible representation* ρ_Σ *associated to* Σ *as in Theorem 4.2.9 in* [K1]. *Suppose* ρ_Σ *is a character. Then for some tamely ramified character* χ *of* I'_0*, there is a* $*$*-preserving, support-preserving* L^2*-isomorphism* $\eta: \mathcal{H}' = \mathcal{H}(\mathrm{G}'//I'_0, \chi) \longrightarrow \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma) = \mathcal{H}$ *of* $\mathbb{C}$*-algebras.*

In case of GL_n, in [HM1, 2], Howe and Moy find Hecke algebra isomorphisms by going through certain inductive procedures. On the other hand, in [BK1], Bushnell and Kutzko find them by comparing two Hecke algebras directly. In both cases, the Hecke algebras described are isomorphic to a product of Iwahori Hecke algebras. In our case, we first find generators and relations of $\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ directly and then compare it with a Hecke algebra on a related group G' as in the Main Theorem. Hence the choice of χ in the Main Theorem is made so as to match Hecke algebras. In our case, direct computation is possible because our open compact subgroup behaves well with respect to the root space decomposition (see [K1] for details). Unlike the case of GL_n, where we had only Hecke algebras of Iwahori types, we now see a twisting by tamely ramified characters. This phenomenon can be already found in the work of A. Moy on GSp_4 ([My 2, Cor. 5.8]). Moreover, we find that scaling Γ (e.g. replacing Γ with $\gamma \cdot \Gamma$ in Σ where γ is an element of an extension field E over k; see [K1, 1.3.2]) may yield different shapes of Hecke algebras. We have not found any good explanation for this and hope to return to this point.

When G is split and Γ splits over k_0, A. Roche has an analogous constructions and has proved the above theorem for the cases that he considered (see [R]). However, instead of looking at Hecke algebras on $C_{\mathrm{G}}(\Gamma)$, he finds an appropriate (possibly nonconnected) reductive group $\widetilde{H}$ coming from Langlands parameters and relates $\mathcal{H}$ to the Iwahori Hecke algebra of $\widetilde{H}$.

Describing each Section, we will start by summarizing the idea of the construction of (J_Σ, ρ_Σ) in [K1] for the case $\Sigma = (\Gamma, I_0', 1)$ as in the Main Theorem. However, since details in [K1] are indispensable throughout this paper, rather than repeating things, we will just sketch the idea of the construction. We will also recall parts from [K1] whenever necessary. In Section 5, we introduce some generalities, most of which can be found in [BK2]. Using results from [BK2], we also prove that the computation of our Hecke algebras can be simplified. Roughly speaking, the problem can be reduced either to the case where $\mathrm{G}' = C_{\mathrm{G}}(\Gamma)$ is a general linear group GL defined over a finite extension F of k_0, or to the case where G' is a product of unitary groups (without GL-factors). In Section 6 we compute $\mathcal{H}$ when G' is of the form GL, and in Section 7 we treat the other cases.

Throughout this paper, since we will keep referring to [K1], we keep all the notation and continue with the numbering from it without further reference. In particular, this paper begins with Section 5.

In [K2], we will compute Hecke algebras when ρ_Σ is not necessarily a character for $\Sigma = (\Gamma, I_0', 1)$.

Summary from Part I

We briefly summarize the construction of (J_Σ, ρ_Σ) in [K1]. The following notation and conventions are from [K1]. They are valid throughout this paper.

NOTATION AND CONVENTIONS

Let k be a p-adic field of characteristic 0 with involution σ_0 and let k_0 be the σ_0-fixed subfield. We will denote $\sigma_0(x)$ by x^{σ_0}. Let $\mathcal{O}_{k_0}$ be the ring of integers of k_0 with its maximal ideal $\mathfrak{p}_{k_0}$ and let π_{k_0} be a generator for $\mathfrak{p}_{k_0}$. Let $\mathbb{F}_q = \mathcal{O}_{k_0}/\mathfrak{p}_{k_0}$ be its residue field. For a finite extension E of k_0, let $e_E = e(E/k_0)$ be its ramification index over k_0 and $f_E = f(E/k_0)$ be the residue degree. We also define $\mathcal{O}_E$, π_E, $\mathfrak{p}_E$ and $\mathbb{F}_{q_E}$ similarly. We denote the algebraic closure of k by $\bar{k}$.

Let V be a finite-dimensional vector space over k. If V is equipped with a nondegenerate ε-Hermitian form $\langle\,,\rangle$ such that $\langle v, w\rangle = \varepsilon\langle w, v\rangle^{\sigma_0}$ ($\varepsilon = +1$ or -1), we let G denote $\mathrm{G}(V, \langle\,,\rangle)$, the connected component of the group of isometries of $\langle\,,\rangle$ on V. Let $\mathfrak{g}$ be the Lie algebra of G. Note that *we let* G *and* $\mathfrak{g}$ *act on* V *from the right.* We note that there is an anti-involution σ on $\mathrm{End}(V)$, defined by the equation

$$\langle vx, w\rangle = \langle v, w(x^\sigma)\rangle \quad \text{for } v, w \in V \text{ and } x \in \mathrm{End}_k(V).$$

The group G is characterized as the connected component of

$$\{ g \in \mathrm{GL}(V) \mid \langle vg, wg \rangle = \langle v, w \rangle \text{ for all } v, w \in V \} = \{ g \in \mathrm{GL}(V) \mid g^\sigma = g^{-1} \}$$

and its Lie algebra $\mathfrak{g}$ is characterized as

$$\{ y \in \mathrm{End}(V) \mid \langle vy, w \rangle + \langle v, wy \rangle = 0 \text{ for all } v, w \in V \}$$
$$= \{ y \in \mathrm{End}(V) \mid y^\sigma = -y \}.$$

For $x \in \mathbb{Q}$ and $a \in \mathbb{Z}$, define $\lfloor x \rfloor_a = \lfloor ax \rfloor / a$ where $\lfloor x \rfloor$ is the greatest integer not greater than x, that is, $\lfloor x \rfloor = \max\{y \in \mathbb{Z} \mid y \leqslant x\}$. Let $\lceil x \rceil$ be the least integer not less than x, that is, $\lceil x \rceil = \min\{y \in \mathbb{Z} \mid y \geqslant x\}$. Define also $\lceil x \rceil_a$ as $\lceil ax \rceil / a$.

Remarks. In [K1], we note that a k-linear space V equipped with $\langle\, , \rangle_k$ can also be regarded as an E-linear space with sesquilinear form $\langle\, , \rangle_E$ where E/k is a finite extension. In these cases, we let $\mathrm{G}(V/E)$, $\mathfrak{g}(V/E)$ denote a group and its Lie algebra over E associated to $(V, \langle\, , \rangle_E)$.

HYPOTHESIS. Since the result in [K1] is valid under the assumption

$$\frac{1}{\dim_k(V)} > \frac{2\mathrm{ord}_k(p)}{p-1}\frac{p}{p-1} + \frac{1}{p-1}, \tag{3.2.3}$$

from now on, we assume that k and V satisfy the above inequality.

S1. $\Sigma = (\Gamma, I_0', 1)$

S1.1. SEMISIMPLE ELEMENT Γ AND TAMELY RAMIFIED TORI

Let $\Gamma \in \mathfrak{g}$ be a semisimple element and let $\mathfrak{t}$ be a maximal torus in $\mathfrak{g}$ which is maximally k_0-split among tori in $\mathfrak{g}$ containing Γ. Let T be the torus in G with Lie algebra $\mathfrak{t}$. Let $A[\mathfrak{t}]$ and $A[\mathrm{T}]$ be the subalgebra of $\mathrm{End}_k(V)$ generated by $\mathfrak{t}$ and T respectively. Then $A[\mathfrak{t}] = A[\mathrm{T}]$ and since k satisfies (3.2.3), it can be written as a direct sum of tamely ramified extensions over k. On the other hand, as $\mathfrak{t}$-, T-module, $V \simeq A[\mathfrak{t}] = A[\mathrm{T}]$. Now V can be decomposed as follows:

$$V = \sum_{i=1}^{m} V_i \simeq A[\mathfrak{t}] = A[\mathrm{T}], \tag{1}$$

where $V_i = F_i \oplus \cdots \oplus F_i$ for some tamely ramified extension F_i over k with involution σ_i and where each V_i is equipped with a sesquilinear form f_{V_i} such that $\langle\, , \rangle = \sum \mathrm{Tr}_{F_i/k} \circ f_{V_i}$. We can write V_i with respect to Witt basis (with respect to a fixed ordering) as follows;

$$V_i = V_i^+ \oplus V_i^- \oplus V_i^\delta \oplus V_i^{\delta'},$$

where

$$V_i^+ \oplus V_i^- = F_i^{d_i} \oplus \cdots \oplus F_i^1 \oplus F_i^{-1} \oplus \cdots \oplus F_i^{-d_i} \tag{2}$$

with V_i^+ a maximal isotropic subspace in V and V_i^- its dual with respect to f_{V_i} and where

$$V_i^\delta = 0,\ F_i^\delta \quad \text{or} \quad F_i^{\delta_1} \oplus F_i^{\delta_2}, \quad \text{and} \quad V_i^{\delta'} = 0,\ F_i^{\delta'} \quad \text{or} \quad F_i^{\delta'_1} \oplus F_i^{\delta'_2}.$$

We refer to Section 1.4 for details and notation. Then under the above identifications (1) and (2), $\Gamma \in \mathfrak{t}$ can be written as $\Gamma = (\cdots, \gamma_i, \ldots, \gamma_i, -\gamma_i^{\sigma_i}, \ldots, -\gamma_i^{\sigma_i}, \cdots)$ with $\gamma_i \in F_i$ (see (1.3.5) and (1.4.1)). Moreover, $\mathrm{G}' = C_{\mathrm{G}}(\Gamma)$ can be written as $\prod_{i=1}^m \mathrm{G}'_i$ where G'_i is either isomorphic to $\mathrm{GL}(V_i^+/F_i)$ or to the group of isometries on (V_i, f_{V_i}). That is,

$$\mathrm{G}' = \prod_{i=1}^m \mathrm{G}'_i \quad \text{where} \quad \mathrm{G}'_i = \mathrm{GL}(V_i^+/F_i) \quad \text{or} \quad \mathrm{G}(V_i, f_{V_i}). \tag{3}$$

From now on, we assume $(\Gamma, \mathfrak{t})$ satisfies (P) (recall it is defined in (1.3.2)).

The construction of (J_Σ, ρ_Σ) is based on the data Σ which consists of three ingredients (Γ, P'_0, ϱ) (see Section 1.5); Γ is a semisimple element in $\mathfrak{g}$ with $(\Gamma, \mathfrak{t})$ satisfying (P), P'_0 is a parahoric subgroup in $C_{\mathrm{G}}(\Gamma)$ and ϱ is a cuspidal representation of the reductive quotient of P'_0. Here, we restrict our attention to the special cases considered in this paper. From now on, we let $\Sigma = (\Gamma, I'_0, 1)$ be as follows:

$$(\mathrm{H}_\Sigma) \qquad \begin{aligned} \Gamma &= \text{ a semisimple element in } \mathfrak{g} \text{ with } (\Gamma, \mathfrak{t}) \text{ satisfying (P)}, \\ I'_0 &= \text{ a minimal parahoric subgroup of } C_{\mathrm{G}}(\Gamma) \text{ as in Section 1.5}, \\ 1 &= \text{ the trivial character of } I'_0. \end{aligned}$$

Note that we label such $\Sigma = (\Gamma, I'_0, 1)$ as (H_Σ).

S2. Construction of (J_Σ, ρ_Σ)

Recall that we have a useful list of notation and definitions in (2.1.1). We will use them throughout this paper.

Decomposing $\mathfrak{g}$ as a sum of irreducible $\mathfrak{t}$-modules (see (2.2.9) for details and notation),

$$\mathfrak{g} = \bigoplus_{(v,\tau)\sim(v_\sigma,\tau_\sigma)} \widetilde{\mathrm{M}}_v^\tau.$$

On each $\mathfrak{t}$-root space $\widetilde{\mathrm{M}}_v^\tau$, we have a lattice structure induced from fractional ideals in $\tilde{F}_v^{(\tau)}$. However, to produce a lattice in $\mathfrak{g}$, we need to work with 'shifted' (by $\frac{1}{2}a_v$) lattices as in (2.3.3) due to nonself-duality of lattices associated to the parahoric subgroups P'_0. That is, for any $s \in \mathbb{Q}$, the lattice $\widetilde{\mathrm{M}}_v^\tau(s) = \widetilde{\mathrm{M}}_v'^\tau(s + \frac{1}{2}a_v)$ corresponds to $\mathfrak{p}^{n_s}_{\tilde{F}_v^{(\tau)}}$ where $n_s = \lceil e(\tilde{F}_v^{(\tau)}/k_0)\cdot(s+\frac{1}{2}a_v)\rceil$ with a_v defined in (2.1.1). Then the following lattices defined in (2.3.9) are normalized by I'_0:

$$\mathcal{A}_\Gamma(s) = \bigoplus_{\substack{v\in\Upsilon\\ \tau\in\mathrm{Gal}_v^\sigma}} \widetilde{\mathrm{M}}_v^\tau(s) = \bigoplus_{\substack{v\in\Upsilon\\ \tau\in\mathrm{Gal}_v^\sigma}} \widetilde{\mathrm{M}}_v^{\prime\tau}\big(s+\tfrac{1}{2}a_v\big),$$

$$\mathcal{A}_\Gamma(s^+) = \bigoplus_{\substack{v\in\Upsilon\\ \tau\in\mathrm{Gal}_v^\sigma}} \widetilde{\mathrm{M}}_v^\tau(s^+) = \bigoplus_{\substack{v\in\Upsilon\\ \tau\in\mathrm{Gal}_v^\sigma}} \widetilde{\mathrm{M}}_v^{\prime\tau}\big(s+\tfrac{1}{2}a_v\big)^+.$$

Let $\mathcal{Y}_\Gamma$ and $\mathcal{Y}'_\Gamma$ be defined as in (3.3.3). For our cases when Σ is as in (H_Σ) and ρ_Σ is a character, we note that $\mathcal{Y}_\Gamma = \mathcal{Y}'_\Gamma$. More explicitly, we have

$$\mathcal{Y}_\Gamma = \mathcal{Y}'_\Gamma = \mathcal{K}'_1 + \sum_{\widetilde{\mathrm{M}}_v^\tau \not\subseteq \mathfrak{g}'} \widetilde{\mathrm{M}}_v^\tau\big(\tfrac{1}{2}(-1-\mathrm{ord}(\gamma_i^\tau - \gamma_j))\big),$$

where $\mathcal{K}'_1 = \mathfrak{g}'\cap\mathcal{A}_\Gamma(0^+)$. Then in (3.3.3), the open compact subgroup J_Σ is defined as $J_\Sigma = I'_0\cdot Y_\Gamma = I'_0\cdot Y'_\Gamma$, where $Y_\Gamma = \exp(\mathcal{Y}_\Gamma)$ and $Y'_\Gamma = \exp(\mathcal{Y}'_\Gamma)$.

On Y'_Γ, Γ defines a character as $\chi_\Gamma(y) = \theta(\mathrm{Tr}(\Gamma\log(y)))$ for $y\in Y'_\Gamma$ where θ is the additive character with the conductor $\mathcal{O}_{k_0}$ fixed in (2.4.1). Now we extend this to the whole J_Σ as a character. For a given Γ, we fix a character χ_Γ° of the maximal compact subgroup T_0 of T, which coincides with χ_Γ on $T_0\cap Y'_\Gamma$ and which is extended to a character of I'_0 factoring through I'_0/I'_1. Here I'_1 is the maximal pro-p subgroup of I'_0. We still denote this extended character by χ_Γ°. Define the extended character $\widetilde{\chi}_\Gamma$ of χ_Γ to J_Σ as follows;

$$\widetilde{\chi}_\Gamma(t\cdot b) = \chi_\Gamma^\circ(t)\chi_\Gamma(b) \quad \text{for} \quad t\cdot b\in I'_0\cdot Y'_\Gamma. \tag{1}$$

In our case when ρ_Σ is a character with $\Sigma = (\Gamma, I'_0, 1)$, we have $\rho_\Sigma = \widetilde{\chi}_\Gamma$

PROPOSITION 4.2.6. *Let* $\Sigma = (\Gamma, I'_0, 1)$ *be as in* (H_Σ). *Let* $\widetilde{W}'$ *be the affine Weyl group of* G′. *Then we have* $\mathrm{Supp}(\mathcal{H}) = J_\Sigma \mathrm{G}' J_\Sigma = J_\Sigma \widetilde{W}' J_\Sigma$ *and* $\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ *is spanned by functions* f_w *whose support is a single double coset* $J_\Sigma w J_\Sigma$ *with* $w\in\widetilde{W}'$. *Moreover,* f_w *is unique up to multiplication by a constant.*

5. Preliminaries

Let $C_\mathrm{G}(\Gamma) = \mathrm{G}' = \prod \mathrm{G}'_i$ be as in (1.4.3) (or S1). In this section, we will show that for any Σ as in (H_Σ), $\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ is isomorphic to a smaller Hecke algebra $\mathcal{H}(\mathrm{M}_\Sigma//(J_\Sigma\cap\mathrm{M}_\Sigma), \rho_\Sigma|\mathrm{M}_\Sigma)$ for an appropriate Levi subgroup M_Σ of G. In Section 5.1, we recall results from [BK2]. We remark that the results summarized in (5.1.3) [BK2, R] are valid for any connected reductive group defined over a p-adic field.

5.1. SOME RESULTS OF BUSHNELL AND KUTZKO

5.1.1. *Notation and Definitions*

Let M be a proper Levi subgroup of G and let P be a parabolic subgroup with its Levi decomposition $\mathrm{P} = \mathrm{MN}$. Let $\underline{\mathrm{N}}$ be the opposite unipotent radical of N relative to M and let $\underline{\mathrm{P}} = \mathrm{M}\underline{\mathrm{N}}$. For any subgroup J of G and its representation ρ, we denote

$$J_{\mathrm{N}} = J \cap \mathrm{N}, \quad J_{\underline{\mathrm{N}}} = J \cap \underline{\mathrm{N}}, \quad J_{\mathrm{M}} = J \cap \mathrm{M}. \quad \rho_{\mathrm{M}} = \rho | J_{\mathrm{M}}.$$

(1) If we have $J = J_{\underline{\mathrm{N}}} J_{\mathrm{M}} J_{\mathrm{N}}$, we say that J is *decomposed with respect to* (M, P).
(2) We also say (J, ρ) is *decomposed with respect to* (M, P) if J is decomposed with respect to (M, P) and the groups $J_{\underline{\mathrm{N}}}$, J_{N} are both contained in the kernel of ρ.

5.1.2. Let J be an open compact subgroup of G and let ρ be its irreducible representation. Let $\mathcal{H} = \mathcal{H}(\mathrm{G}//J, \rho)$ be the Hecke algebra associated to (J, ρ). This is the convolution algebra on the space of all compactly supported functions $f : \mathrm{G} \longrightarrow \mathrm{End}(\rho)$, which transform via ρ under left and right translations by J. That is, $f(jgj') = \rho(j) f(g) \rho(j')$ for $g \in \mathrm{G}$ and $j, j' \in J$. The convolution $\star$ is defined by

$$f_1 \star f_2(g) = \int_{\mathrm{G}} f_1(x) f_2(x^{-1} g)\, \mathrm{d}x \qquad \text{for} \quad f_1, f_2 \in \mathcal{H}. \tag{1}$$

It also carries a natural involution $*$ and an inner product $(,)$. They are defined as follows: for $f, f_1, f_2 \in \mathcal{H}$,

$$f^*(g) = (f(g^{-1}))^*, \qquad (f_1, f_2) = \mathrm{Tr}(f_1 \star f_2^*(1)) \tag{2}$$

where $(f(g^{-1}))^*$ is the adjoint of $f(g^{-1})$ in the sense of Hilbert space operators.

Convention. For any Hecke algebra $\mathcal{H}(\mathrm{G}//J, \rho)$ of the above form, we assume the convolution $\star$ is defined with respect to a normalized Haar measure with $\mathrm{vol}(J) = 1$.

The following results can be found in [BK2; Theorem 7.2 (ii)] and [R; Proposition 5.1]:

THEOREM 5.1.3 [BK2, R]. *Let* (J, ρ) *be a pair of an open compact subgroup* J *of* G *and its irreducible representation* ρ. *Suppose that there is a proper Levi subgroup* M *such that* (J, ρ) *is decomposed with respect to* (M, P) *and suppose also that* $\mathrm{Supp}(\mathcal{H}(\mathrm{G}//J, \rho)) \subset J\mathrm{M}J$. *Let the Haar measures on* G *(resp.* M*) be normalized such that the volume of* J *(resp.* J_{M}*) is 1.*

(1) *Let* t *be the map from* $\mathcal{H}(\mathrm{M}//J_{\mathrm{M}}, \rho_{\mathrm{M}})$ *to* $\mathcal{H}(\mathrm{G}//J, \rho)$ *defined by* $t(f)(jmj') = \rho(j) f(m) \rho(j')$ *for* $f \in \mathcal{H}(\mathrm{M}//J_{\mathrm{M}}, \rho_{\mathrm{M}})$. *Then* t *is an algebra isomorphism and* t *preserves supports of functions in the sense that* $\mathrm{Supp}(t(f)) = J \cdot \mathrm{Supp}(f) \cdot J$.

(2) *Let* δ_{P} *be the modulus function defined by* $\delta_{\mathrm{P}}(m) = |\det(\mathrm{Ad}(m)|\mathrm{Lie}(\mathrm{N}))|$ *for* $m \in \mathrm{M}$. *Then* $\tilde{t}: \mathcal{H}(\mathrm{M}//J_{\mathrm{M}}, \rho_{\mathrm{M}}) \longrightarrow \mathcal{H}(\mathrm{G}//J, \rho)$ *defined by*

$$\tilde{t}(f) = t(\delta_{\mathrm{P}}^{\frac{1}{2}} \cdot f) \quad \textit{for} \quad f \in \mathcal{H}(\mathrm{M}//J_{\mathrm{M}}, \rho_{\mathrm{M}})$$

is a $*$*-preserving, support-preserving* L^2*-isomorphism of* $\mathbb{C}$*-algebras.*

5.2. REDUCING TO SIMPLER COMPUTATIONS

In this section, we will find appropriate Levi subgroups and parabolic subgroups of G satisfying the conditions in (5.1.3) to simplify our computation. Basically, we will show that $\mathcal{H}$ is isomorphic to a tensor product of smaller Hecke algebras. Then it will be enough to describe each smaller Hecke algebra.

5.2.1. Let $\mathrm{G}' = \prod \mathrm{G}'_i$ as in (1.4.3). By rearranging factors in (1.4.3), we can write

$$V = V_+ \oplus V_0 \oplus V_-$$

such that G' acts as a product of GL-factors on V_+ and V_- and as a product of nontrivial (i.e. they are not GL) unitary groups on V_0. For example, we can put $V_\pm = \sum' V_i^\pm$, $V_0 = \sum'' V_i$ where $\sum'$ is over i's with G'_i isomorphic to $\mathrm{GL}_{d_i}(F_i)$ and $\sum''$ is over i's with G'_i not of GL type. Then we note that $\langle\, , \rangle$ is trivial on V_+ and V_-, and it is nondegenerate on $V_+ \oplus V_-$ and V_0. Let P_a be the parabolic subgroup associated to the flag $\mathcal{F}: \mathcal{F}_+ = V \supset \mathcal{F}_0 = V_0 \oplus V_- \supset \mathcal{F}_- = V_-$ and let M_a and N_a be its Levi subgroup and its unipotent radical respectively. Then we have

$$\mathrm{M}_a = \{ g \in \mathrm{G} \mid V_\varepsilon \cdot g \subset V_\varepsilon, \quad \text{for } \varepsilon = -, 0, + \},$$
$$\mathrm{P}_a = \mathrm{M}_a\mathrm{N}_a = \{ g \in \mathrm{G} \mid \mathcal{F}_\varepsilon \cdot g \subset \mathcal{F}_\varepsilon, \quad \text{for } \varepsilon = -, 0, + \}.$$

Note that P_a is a proper subgroup only when V_+ and V_- are nontrivial.

PROPOSITION 5.2.2. (J_Σ, ρ_Σ) is decomposed with respect to $(\mathrm{M}_a, \mathrm{P}_a)$.

Proof. Let $\mathfrak{p}_a$, $\mathfrak{m}_a$, $\mathfrak{n}_a$ and $\underline{\mathfrak{n}}_a$ be the Lie algebras of P_a, M_a, N_a and $\underline{\mathrm{N}}_a$ respectively. Note that $\mathfrak{g}' = \mathrm{Lie}(\mathrm{G}') \subset \mathfrak{m}_a$ and $\mathfrak{n}_a, \underline{\mathfrak{n}}_a \subset \mathfrak{g}'^{\perp}$. Since $\log((J_\Sigma)_{\mathrm{N}_a}) \subset \mathfrak{n}_a \subset \mathfrak{g}'^{\perp}$, from the construction of ρ_Σ (see S2 or (3.4.2)), we see $(J_\Sigma)_{\mathrm{N}_a} \subset \ker(\rho_\Sigma)$. Similarly, $(J_\Sigma)_{\underline{\mathrm{N}}_a} \subset \ker(\rho_\Sigma)$. Now we show that J_Σ is decomposable with respect to $(\mathrm{M}_a, \mathrm{P}_a)$. Let $J_\Sigma = I'_0 \cdot Y_\Gamma$ as in (3.3.3). Then from (3.3.2), we can write

$$J_\Sigma = I'_0 \cdot Y_{r_0} \cdots Y_1, \quad \text{where } Y_i = \exp(\mathcal{Y}_i).$$

Since $I'_0 \subset \mathrm{G}' \subset \mathrm{M}_a$, we can write $I'_0 = (I'_0)_{\underline{\mathrm{N}}_a}(I'_0)_{\mathrm{M}_a}(I'_0)_{\mathrm{N}_a}$ with $(I'_0)_{\underline{\mathrm{N}}_a} = (I'_0)_{\mathrm{N}_a} = 1$ and hence I'_0 is decomposed with respect to $(\mathrm{M}_a, \mathrm{P}_a)$. We claim each Y_r is also decomposed with respect to $(\mathrm{M}_a, \mathrm{P}_a)$:

LEMMA 5.2.3. $Y_r = (Y_r)_{\underline{\mathrm{N}}_a}(Y_r)_{\mathrm{M}_a}(Y_r)_{\mathrm{N}_a}$.

Proof. For $y \in Y_r$, write $\log(y) = X_{\underline{\mathfrak{n}}_a} + X_{\mathfrak{m}_a} + X_{\mathfrak{n}_a}$ with $X_{\underline{\mathfrak{n}}_a} \in \underline{\mathfrak{n}}_a \cap Y_r, X_{\mathfrak{m}_a} \in \mathfrak{m}_a \cap \mathcal{Y}_r$ and $X_{\mathfrak{n}_a} \in \mathfrak{n}_a \cap \mathcal{Y}_r$. Then $y = z_1 y_1 = \tilde{z}_1 \tilde{y}_1$ with $z_1 = \tilde{z}_1 = \exp(X_{\underline{\mathfrak{n}}_a})$ and $y_1 = \tilde{y}_1 = \exp(-X_{\underline{\mathfrak{n}}_a})y$. Again, writing $\log(\tilde{y}_1) = X^1_{\underline{\mathfrak{n}}_a} + X^1_{\mathfrak{m}_a} + X^1_{\mathfrak{n}_a}$, we note that $X^1_{\underline{\mathfrak{n}}_a}$ is closer to 0 than $X_{\underline{\mathfrak{n}}_a}$. Let $\tilde{y}_1 = \tilde{z}_2 \tilde{y}_2$ with $\tilde{z}_2 = \exp(X^1_{\underline{\mathfrak{n}}_a})$ and $\tilde{y}_2 = \exp(-X^1_{\underline{\mathfrak{n}}_a})y_1$ and write $y = z_2 y_2$ with $z_2 = \tilde{z}_1 \tilde{z}_2$ and $y_2 = \tilde{y}_1 \tilde{y}_2$. Repeating above process,we see z_j (resp. y_j) converges to an element of $(Y_r)_{\underline{\mathrm{N}}_a}$ (resp. $Y_r \cap \mathrm{P}_a$). It can be easily checked that $Y_r \cap \mathrm{P}_a = (Y_r)_{\mathrm{M}_a}(Y_r)_{\mathrm{N}_a}$. Hence we have $Y_r = (Y_r)_{\underline{\mathrm{N}}_a}(Y_r)_{\mathrm{M}_a}(Y_r)_{\mathrm{N}_a}$. □

Going back to the proof of (5.2.2), since Y_i's are normalized by I'_0, we can write J_Σ as $(I'_0)_{\underline{\mathrm{N}}_a}(I'_0)_{\mathrm{M}_a} Y_{r_0} \dots Y_1 (I'_0)_{\mathrm{N}_a}$. Inductively, since Y_i is normalized by Y_t for $i \leqslant t$, we can write J_Σ as

$$(I'_0)_{\underline{\mathrm{N}}_a} \prod_i (Y_i)_{\underline{\mathrm{N}}_a} \cdot (I'_0)_{\mathrm{M}_a} \prod_i (Y_i)_{\mathrm{M}_a} \cdot (I'_0)_{\mathrm{N}_a} \prod_i (Y_i)_{\mathrm{N}_a}.$$

Now we have $(J_\Sigma)_\alpha = (I'_0)_\alpha \prod (Y_i)_\alpha$ for $\alpha \in \{\underline{\mathrm{N}}_a, \mathrm{M}_a, \mathrm{N}_a\}$ and J_Σ is decomposed with respect to $(\mathrm{M}_a, \mathrm{P}_a)$. □

5.2.4. From (4.2.1), we have $\mathrm{Supp}(\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)) \subset J_\Sigma \mathrm{G}' J_\Sigma$. Since $\mathrm{G}' \subset \mathrm{M}_a$ and thus $\mathrm{Supp}(\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)) \subset J_\Sigma \mathrm{M}_a J_\Sigma$, we can apply (5.1.3) and define $\tilde{t}: \mathcal{H}(\mathrm{M}_a//(J_\Sigma)_{\mathrm{M}_a}, (\rho_\Sigma)_{\mathrm{M}_a}) \longrightarrow \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ as in (5.1.3)–(2). Moreover, $\tilde{t}$ is a $*$-preserving, support-preserving L^2-isomorphism of $\mathbb{C}$-algebras. Hence we can reduce the computation of $\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ to that of $\mathcal{H}(\mathrm{M}_a//(J_\Sigma)_{\mathrm{M}_a}, (\rho_\Sigma)_{\mathrm{M}_a})$. Note that M_a is a direct product

$$\mathrm{M}_a = \mathrm{M}_s \times \mathrm{M}_0 = \mathrm{GL}^*(V_+/k) \times \mathrm{G}(V_0), \tag{1}$$

where

$$\mathrm{M}_s = \mathrm{GL}^*(V_+/k) = \mathrm{M}_a \cap \Big(\mathrm{GL}(V_+/k) \times \{1_{V_0}\} \times \mathrm{GL}(V_-/k)\Big)$$

and

$$\mathrm{M}_0 = \{1_{V_+}\} \times \mathrm{G}(V_0) \times \{1_{V_-}\}$$

(recall $\mathrm{G}(V_i)$ is defined in (1.4.3)). Note that M_a is embedded in $\mathrm{GL}(V_+/k) \times \mathrm{G}(V_0) \times \mathrm{GL}(V_-/k)$, however, the third component is determined by the first component in $\mathrm{GL}(V_+/k)$. From construction, we observe

$$(J_\Sigma)_{\mathrm{M}_a} \simeq (J_\Sigma)_{\mathrm{M}_s} \times (J_\Sigma)_{\mathrm{M}_0},$$

where $(J_\Sigma)_{\mathrm{M}_0} = (J_\Sigma)_{\mathrm{M}_a} \cap \mathrm{M}_0$ and $(J_\Sigma)_{\mathrm{M}_s} = (J_\Sigma)_{\mathrm{M}_a} \cap \mathrm{M}_s$. We can also write

$$(\rho_\Sigma)_{\mathrm{M}_a} \simeq (\rho_\Sigma)_s \otimes (\rho_\Sigma)_0$$

for some irreducible representations $(\rho_\Sigma)_s$, $(\rho_\Sigma)_0$ of $(J_\Sigma)_{\mathrm{M}_s}$, $(J_\Sigma)_{\mathrm{M}_0}$ respectively. Consider the map

$$\eta' : \mathcal{H}(\mathrm{M}_s//(J_\Sigma)_{\mathrm{M}_s}, (\rho_\Sigma)_s) \otimes \mathcal{H}(\mathrm{M}_0//(J_\Sigma)_{\mathrm{M}_0}, (\rho_\Sigma)_0) \longrightarrow \mathcal{H}(\mathrm{M}_a//(J_\Sigma)_{\mathrm{M}_a}, (\rho_\Sigma)_{\mathrm{M}_a})$$

$$f_s \otimes f_0 \longmapsto f \qquad (2)$$

with f defined by $f(\mathrm{m}_s\mathrm{m}_0) = f_s(\mathrm{m}_s)f_0(\mathrm{m}_0) = f_0(\mathrm{m}_0)f_s(\mathrm{m}_s) = f(\mathrm{m}_0\mathrm{m}_s)$. Then it can be easily checked that η' is a $*$-preserving, support-preserving L^2-isomorphism of $\mathbb{C}$-algebras. Hence it is enough to describe each factor $\mathcal{H}(\mathrm{M}_s//(J_\Sigma)_{\mathrm{M}_s}, (\rho_\Sigma)_s)$ and $\mathcal{H}(\mathrm{M}_0//(J_\Sigma)_{\mathrm{M}_0}, (\rho_\Sigma)_0)$.

5.2.5. Here, we will decompose $\mathcal{H}(\mathrm{M}_s//(J_\Sigma)_{\mathrm{M}_s}, (\rho_\Sigma)_s)$ even further. Recall that we can write V_+ as

$$V_+ = \sum_{i=1}^{m'} V_i^+, \quad \text{where } \sum \text{ runs over } i \text{ with } \mathrm{G}'_i \text{ of GL-type}$$

and V_- can be written in a similar way. Moreover, $V_i^+ = d_iF_i$ for some tamely ramified extension F_i over k and $\mathrm{G}'_i = \mathrm{GL}^*_{d_i}(F_i)$. Let P_b be the parabolic subgroup in M_s associated to the flag $\mathcal{F}: V_\pm = \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_{m'+1} = 0$ with $\mathcal{F}_j = \oplus_{i=j}^{m'} V_i^+$. Let M_b and N_b be its Levi subgroup and unipotent radical of P_b respectively. Then

$$\begin{aligned} \mathrm{P}_b &= \{ g \in \mathrm{M}_s \mid \mathcal{F}_j \cdot g \subset \mathcal{F}_j, \ \text{for } j = 1, \ldots, m' \}, \\ \mathrm{M}_b &= \{ g \in \mathrm{M}_s \mid V_i^+ \cdot g \subset V_i^+, \ \text{for } i = 1, \ldots, m' \}, \\ \mathrm{N}_b &= \{ g \in \mathrm{P}_b \mid (\mathcal{F}_j/\mathcal{F}_{j+1}) \cdot g = \mathrm{Id}_{\mathcal{F}_j/\mathcal{F}_{j+1}}, \ \text{for } j = 1, \ldots, m' \}. \end{aligned} \qquad (1)$$

Note that G'_i is contained in M_b. From the construction of (J_Σ, ρ_Σ), we see that $J_\Sigma \cap \mathrm{N}_b$ and $J_\Sigma \cap \mathrm{N}_{\underline{\mathrm{P}}_b}$ are contained in the kernel of ρ_Σ. Proceeding as in (5.2.2)–(5.2.4), we can prove $((J_\Sigma)_{\mathrm{M}_s}, \rho_s)$ is decomposed with respect to $(\mathrm{M}_b, \mathrm{P}_b)$. To prove (5.2.3) in this situation, we can apply the proof of (5.2.3) inductively. Hence $(J_\Sigma)_{\mathrm{M}_s} = \prod_{i=1}^{m'} (J_\Sigma)_{\mathrm{M}_i}$ where $\mathrm{M}_i = \mathrm{GL}^*(V_i^+/k)$, $(J_\Sigma)_{\mathrm{M}_i} = (J_\Sigma)_{\mathrm{M}_s} \cap \mathrm{M}_i$ and $\rho_\Sigma | \mathrm{M}_b = (\rho_\Sigma)_1 \otimes \cdots \otimes (\rho_\Sigma)_{m'}$ for some character $(\rho_\Sigma)_i$ of $(J_\Sigma)_{\mathrm{M}_i}$. Hence we have

$$\mathcal{H}(\mathrm{M}_s//(J_\Sigma)_{\mathrm{M}_s}, (\rho_\Sigma)_s) \simeq \bigotimes_{i=1}^{m'} \mathcal{H}(\mathrm{M}_i//(J_\Sigma)_{\mathrm{M}_i}, (\rho_\Sigma)_i). \qquad (2)$$

5.2.6. Summarizing (5.2.4)–(5.2.5), there is a $*$-preserving, support-preserving L^2-isomorphisms of $\mathbb{C}$-algebras:

$$\begin{aligned} &\tilde{\imath}: \mathcal{H}(\mathrm{M}_\Sigma//(J_\Sigma)_{\mathrm{M}_\Sigma}, (\rho_\Sigma)_{\mathrm{M}_\Sigma}) \longrightarrow \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma), \\ &\tilde{\imath}': \bigotimes_{i=0}^{m'} \mathcal{H}(\mathrm{M}_i//(J_\Sigma)_{\mathrm{M}_i}, (\rho_\Sigma)_i) \longrightarrow \mathcal{H}(\mathrm{M}_\Sigma//(J_\Sigma)_{\mathrm{M}_\Sigma}, (\rho_\Sigma)_{\mathrm{M}_\Sigma}). \end{aligned}$$

Here $\tilde{\imath}$ is defined as in (5.1.3)–(2). More precisely, since $\mathrm{Supp}(\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)) \subset J_\Sigma \mathrm{G}' J_\Sigma$ from (4.2.1) and $\mathrm{G}' \subset \mathrm{M}_\Sigma$, we can apply (5.1.3). Hence we can define $\tilde{\imath}$

as in (5.1.3)–(2) with $(\mathrm{M}, \mathrm{P}) = (\mathrm{M}_\Sigma, \mathrm{P}_\Sigma)$, where P_Σ is the parabolic subgroup with its Levi subgroup $\mathrm{M}_\Sigma = \prod_{i=0}^{m'} \mathrm{M}_i$ and unipotent radical $\mathrm{N}_\Sigma = \mathrm{N}_a\mathrm{N}_b$. Let $\tilde{\eta}$ be $\tilde{\imath} \circ \tilde{\imath}'$. Then

$$\tilde{\eta}: \bigotimes_{i=0}^{m'} \mathcal{H}\big(\mathrm{M}_i//(J_\Sigma)_{\mathrm{M}_i}, (\rho_\Sigma)_i\big) \longrightarrow \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$$

is given by $\tilde{\eta}(f_0 \otimes f_1 \otimes \cdots f_{m'})(\mathrm{m}) = \delta_{\mathrm{P}_\Sigma}^{\frac{1}{2}}(\mathrm{m}) \prod_i f_i(\mathrm{m}_i)$ for $\mathrm{m} = \mathrm{m}_0\mathrm{m}_1 \cdots \mathrm{m}_{m'}$, $\mathrm{m}_i \in \mathrm{M}_i$ and $\mathrm{Supp}(f) = J_\Sigma \mathrm{m} J_\Sigma$. Hence, it is enough to describe each Hecke algebra $\mathcal{H}(\mathrm{M}_i//(J_\Sigma)_{\mathrm{M}_i}, (\rho_\Sigma)_i)$.

To prove the main theorem, we claim that it is enough to prove that there is a $*$-preserving, support-preserving L^2-isomorphism $\tilde{\eta}_i$ between $\mathcal{H}(\mathrm{M}_i//(J_\Sigma)_{\mathrm{M}_i}, (\rho_\Sigma)_i)$ and $\mathcal{H}(\mathrm{G}'_i//I_0'^{\,i}, \xi_i)$ for some *tamely ramified character* ξ_i of $I_0'^{\,i}$ (see (7.3.1) for definition).

Suppose there exist such $\tilde{\eta}_i$ and ξ_i. Then defining a character χ of $I'_0 = \prod_i I_0'^{\,i}$ by $\otimes_i \xi_i$, it is obvious that the map $\tilde{\eta}': \mathcal{H}(\mathrm{G}'//I'_0, \chi) \longrightarrow \otimes_i \mathcal{H}(\mathrm{G}'_i//I_0'^{\,i}, \xi_i)$ defined by $f \longmapsto \otimes_i (f|I_0'^{\,i})$ is a $*$-preserving, support-preserving L^2-isomorphism. Composing $\tilde{\eta}'$, $\otimes_i\tilde{\eta}_i$ and $\tilde{\eta}$, we will see that η defined by $\tilde{\eta} \circ \big(\otimes_i\tilde{\eta}_i\big) \circ \tilde{\eta}'$ is a $*$-preserving, support-preserving L^2-isomorphism of $\mathbb{C}$-algbras from $\mathcal{H}(\mathrm{G}'//I'_0, \chi)$ to $\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$.

Note that $\mathrm{M}_0 = \mathrm{G}(V_0)$ and for $i = 1, \ldots, m'$, M_i is isomorphic to $\mathrm{GL}(V_i^+/k)$ and is a proper Levi subgroup of $\mathrm{G}(V_i)$ (see (1.4.3) for notation). Hence to describe each $\mathcal{H}(\mathrm{M}_i//(J_\Sigma)_{\mathrm{M}_i}, (\rho_\Sigma)_i)$, we may assume that we have one of the following cases: Let $\Sigma = (\Gamma, I'_0, 1)$ as in (H_Σ) and M_Σ is the Levi subgroup associated to Σ as above.

Case 1: $\mathrm{M}_\Sigma \simeq \mathrm{GL}(V^+/k)$. Equivalently, G' is isomorphic to $\mathrm{GL}(V^+/F)$.

Case 2: $\mathrm{M}_\Sigma \simeq \mathrm{G}(V)$. Equivalently, G' is isomorphic to a product of unitary groups, $\prod_{m_i}(F_i/k_i)$.

Remark. 5.2.7. Note that M_Σ is the smallest Levi subgroup in G containing G'. From (5.2.6), we see M_Σ is a proper subgroup of G unless $V = V_0$, that is, it is proper when there is G'_i isomorphic to $\mathrm{GL}_d(F)$. In this case, $\mathrm{G}'/Z_{\mathrm{G}}$ does not have compact center. Hence via the L^2-isomorphism η of $\mathbb{C}$-algebras in (5.2.6), we see that there is no discrete series containing (J_Σ, ρ_Σ).

6. Computation: $G' = \mathrm{GL}_d(F)$

Let $\Sigma = (\Gamma, I'_0, 1)$ be as in (H_Σ). In this chapter, we consider the case $\mathrm{G}' = \mathrm{GL}^*(V^+/F) \simeq \mathrm{GL}(V^+/F) = \mathrm{GL}_d(F)$. To simplify notation, we will identify G' with $\mathrm{GL}(V^+/F)$. We also drop 1's from $(1, 1, a, b) \in \Upsilon$. For computation, we need to describe root spaces more explicitly. In the following section, we describe root spaces and the action of the affine Weyl group of G' on those root spaces.

6.1. AFFINE WEYL GROUPS AND ROOT SPACES IN $\mathfrak{gl}(V^+/k)$

6.1.1. From now on, we fix the order of the basis over F as follows:

$$V^+ = F^1 \oplus \cdots \oplus F^d \quad \text{with } F = F^1 = \cdots = F^d.$$

Note that this ordering is reverse to what we have in Section 1.5.A (or S1.1). Recall from Section 1.5, we have chosen an Iwahori subgroup in G' as a stabilizer of the following slice of lattices on V^+; for $r = 0, \dots, d-1$,

$$\begin{aligned} L_0 &= \bigoplus_{i=1}^{d} \mathcal{O}_F \supset \cdots \supset L_r \\ &= \bigoplus_{i=1}^{r} \mathfrak{p}_F \oplus \bigoplus_{i=r+1}^{d} \mathcal{O}_F \supset \cdots \supset L_{d-1} = \bigoplus_{i=1}^{d-1} \mathfrak{p}_F \oplus \mathcal{O}_F. \end{aligned}$$

From (2.2.8), we have the following decomposition of $\mathfrak{gl}(\mathfrak{V}^+/\mathfrak{k})$:

$$\mathfrak{gl}(V^+/k) = \mathfrak{t} \oplus \sum_{\substack{a,b=1,\dots,d \\ \tau \in \mathrm{Gal}_v \\ (a,b,\tau)\neq(a,a,1)}} \mathrm{M}_{ab}^{\tau}$$

More explicitly, we have

$$\mathrm{M}_{ab}^{\tau} = F \cdot \tau_{ab} \cdot F \simeq (F\tau)F = \tilde{F}_v^{(\tau)}$$

where $\tau_{ab}\colon F^a \to F^b$ is defined as in (2.2.3) and F acts on F via multiplications. Then

$$\mathfrak{gl}(V^+/k) = \sum_{a,b} \mathrm{Hom}_k(F^a, F^b) = \sum_{a,b} \sum_{\tau \in \mathrm{Gal}_v} F \cdot \tau_{ab} \cdot F.$$

Recall each $\mathrm{M}_{ab}^{\tau} = F \cdot \tau_{ab} \cdot F$ is a $\mathfrak{t}$-root space defined over k where $\mathfrak{t}$ acts via the adjoint action as

$$\mathrm{ad}(t)(x) = (t_a \cdot x - x \cdot t_b) \quad \text{for } t = (t_1, \dots, t_n) \in \mathfrak{t}, \quad x \in \mathrm{M}_{ab}^{\tau}.$$

Let Φ', Φ'_+ and Δ' be the sets of roots, positive roots and simple roots respectively. We will use same notations Φ', Φ'_+ and Δ' for those sets of corresponding root spaces in $\mathfrak{g}'$. Let Φ be the set of k-rational root spaces in $\mathrm{GL}(V^+/k)$. Then we can find Φ, Φ', Φ'_+, Δ' as follows:

$$\begin{aligned} \Phi &= \left\{ \mathrm{M}_{ab}^{\tau} \mid a, b = 1, \dots, d,\ \tau \in \mathrm{Gal}_v \right\}, \\ \Phi' &= \left\{ \mathrm{M}_{ab}^{1} \in \Phi \right\}, \\ \Phi'_+ &= \left\{ \mathrm{M}_{ab}^{1} \in \Phi' \mid a < b \right\}, \\ \Delta' &= \left\{ \mathrm{M}_{ab}^{1} \in \Phi' \mid b = a+1, \quad a = 1, \dots, d-1 \right\}. \end{aligned}$$

We recall affine root systems for $\mathrm{GL}_d(F)$ [BT,IM]. Let Φ'_{aff} and Δ'_{aff} be the sets of affine roots and affine simple roots respectively. We again use the same notation for the sets in $\mathfrak{g}'$. Let $1/e_F\mathbb{Z}$ be the value group of F/k. Then we have

$$\begin{aligned} \Phi'_{\mathrm{aff}} &= \left\{ \mathrm{M}_{ab}^{1}(\beta) \mid \mathrm{M}_{ab}^{1} \in \Phi', \quad \beta \in \frac{1}{e_F}\mathbb{Z} \right\}, \\ \Delta'_{\mathrm{aff}} &= \left\{ \mathrm{M}_{a,a+1}^{1}(0), \quad \mathrm{M}_{d1}^{1}\left(\frac{1}{e_F}\right) \right\}. \end{aligned}$$

For each $\mathrm{M}_{ab}^{1}(\beta) \subset \mathfrak{g}'$, we have the corresponding subgroup $\mathrm{N}_{ab}^{1}(\beta) = \exp(\mathrm{M}_{ab}^{1}(\beta))$ in

G'. Then the Iwahori subgroup I_0' (see (1.5.1)) can be written as

$$I_0' = T_0 \cdot \prod_{a<b} N_{ab}^1(0) \cdot \prod_{a>b} N_{ab}^1\left(\frac{1}{e_F}\right),$$

where T_0 is the maximal compact subgroup of T.

6.1.2. Let $\widetilde{W}' = N_{G'}(T)/T_0$ be the affine Weyl group of $GL_d(F)$. For $a = 1, \ldots, d-1$, let s_a be the simple reflection with its corresponding affine space $M_{a,a+1}^1(0)$ and let s_d be the extended Weyl element corresponding to an affine space $M_{d1}^1(1/e_F)$. That is, s_a is the elementary transposition in G' which switches rows a and $a+1$ and s_d can be written as a matrix in $GL(V^+/F)$ as follows:

$$s_d = \begin{pmatrix} 0 & 0 & \pi_F^{-1} \\ 0 & \mathrm{Id}_{d-2} & 0 \\ \pi_F & 0 & 0 \end{pmatrix}.$$

Let $\widetilde{W}_0'$ be the group generated by the images of $S = \{s_i \mid i = 1, \ldots, d\}$ in $\widetilde{W}'$. Let Ω be the subgroup of $\widetilde{W}'$ normalizing I_0'. That is, $\Omega = \left\{ w \in \widetilde{W}' \mid \mathrm{Ad}\dot{w}(I_0') = I_0' \right\}$ where $\dot{w}$ is a representative in $N_{G'}(T)$ of w. It is generated by

$$t = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \vdots & & 0 & \ddots & \\ 0 & & & 0 & 1 \\ \pi_F & 0 & & & 0 \end{pmatrix}.$$

Then $\mathrm{Ad}t(s_a) = s_{a-1}$, $\mathrm{Ad}t(s_1) = s_d$, and we have a semi-direct product decomposition $\widetilde{W}' = \widetilde{W}_0' \rtimes \Omega$.

Notation. From now on, if there is no confusion, we will use the same notation w for both an element w of $\widetilde{W}'$ and its representative $\dot{w}$ in $N_{G'}(T)$.

6.1.3. On each k-rational root space M_{ab}^τ for $a \neq b$ and on each $M_{ab}^\tau(\beta)$ with $a \neq b$ or $a = b$ and $\beta > 0$, the exponential map is well defined. Denote

$$N_{ab}^\tau = \exp(M_{ab}^\tau), \qquad N_{ab}^\tau(\beta) = \exp(M_{ab}^\tau(\beta)). \tag{1}$$

If $a \neq b$, N_{ab}^τ and $N_{ab}^\tau(\beta)$ become subgroups. If $\tau = 1$, N_{ab}^1 (resp. $N_{ab}^1(\beta)$) is a usual root subgroup (resp. an affine root subgroup) in G' with respect to T.

Let $T_{x,y}^\tau = x \cdot \tau_{ab} \cdot y$ be an element of $M_{ab}^\tau(\beta)$ for $x, y \in F$ with $\mathrm{ord}(x) + \mathrm{ord}(y) \geqslant \beta$ (see (2.2.3)). Let s be an element in S, then we have $\mathrm{Ad}s(x \cdot \tau_{ab} \cdot y) \in M_{s(a),s(b)}^\tau$ as follows:

$$\mathrm{Ad}s(x \cdot \tau_{ab} \cdot y) = \begin{cases} x \cdot \tau_{s(a),s(b)} \cdot y, & \text{if } s \in S \setminus \{s_d\}, \\ (\pi_F^{\delta_{1a} - \delta_{da}} x) \cdot \tau_{s(a),s(b)} \cdot (y \pi_F^{\delta_{db} - \delta_{1b}}), & \text{if } s = s_d, \end{cases} \tag{2}$$

where δ_{ab} is the Kronecker's delta function and where $s \in S$ acts on $\{1, \ldots, d\}$ as a permutation.

6.2. IWAHORI HECKE ALGEBRA OF GL_n

Since we will establish an isomorphism between $\mathcal{H} = \mathcal{H}\big(\mathrm{M}_\Sigma//J_\Sigma, (\rho_\Sigma)_{\mathrm{M}_\Sigma}\big)$ and the Iwahori Hecke algebra $\mathcal{H}' = \mathcal{H}(\mathrm{G}'//I_0', 1)$ of $\mathrm{G}' = \mathrm{GL}_d(F)$, we briefly describe the Iwahori Hecke algebra $\mathcal{H}'$ of $\mathrm{GL}_d(F)$. Let $\widetilde{W}' = \widetilde{W}_0' \rtimes \Omega$ be the affine Weyl group of $\mathrm{GL}_d(F)$ in (6.1.2). Let l be the length function defined on $\widetilde{W}'$; for $w \in \widetilde{W}'$, $l(w)$ is defined such that $[I_0'wI_0' \colon I_0'] = q_F^{l(w)}$. Note that $\mathcal{H}'$ is linearly spanned by $\langle e_w \mid w \in \widetilde{W}' \rangle$ where e_w is the unique function in $\mathcal{H}'$ with support $I_0'wI_0'$ and $e_w(w) = 1$.

The following result describes $\mathcal{H}'$ in terms of generators and relations.

THEOREM 6.2.1 [IM]. *The algebra $\mathcal{H}'$ is generated by*

$$e_s, \quad s \in S = \{s_1, \ldots, s_d, t\}.$$

The elements e_w, $w \in \widetilde{W}'$ satisfy the relations

$$\text{(L)} \quad e_w \star e_{w'} = e_{ww'} \ \textit{if} \ l(ww') = l(w) + l(w'),$$
$$\text{(Q)} \quad e_s \star e_s = q_F e_1 + (q_F - 1)e_s, s \in S.$$

Here, q_F denotes the cardinality of the residue field of F.

We note that the following two relations are resulted from (L):

$$\text{(B)} \quad \text{(i) } e_{s_i} \star e_{s_j} = e_{s_j} \star e_{s_i} \quad \text{if} \quad |i-j| > 1 \ (\mathrm{mod}\ d),$$
$$\text{(ii) } e_{s_i} \star e_{s_{i+1}} \star e_{s_i} = e_{s_{i+1}} \star e_{s_i} \star e_{s_{i+1}}, \quad i \ (\mathrm{mod}\ d);$$

$$\text{(T)} \quad \text{(i) } e_{t^i} \star e_{t^j} = e_{t^{i+j}},$$
$$\text{(ii) } e_t \star e_{s_i} = e_{s_{i-1}} \star e_t, \quad i \,(\mathrm{mod}\ d).$$

In the following theorem, let μ (resp. μ') denote a normalized Haar measure on M_Σ (resp. G') with $\mu((J_\Sigma)_{\mathrm{M}_\Sigma}) = 1$ (resp. $\mu'(I_0') = 1$).

THEOREM 6.2.2. *Let* $\Sigma = (\Gamma, I_0', 1)$ *be as in* (H_Σ), *suppose* $\mathrm{G}' = C_{\mathrm{G}}(\Gamma) \simeq \mathrm{GL}_d(F)$ *for some tamely ramified extension F over k. Let $\widetilde{W}'$ be the affine Weyl group of* G' *with* $\mathrm{G}' = I_0'\widetilde{W}'I_0'$. *For $w \in \widetilde{W}'$, let*

$$C_w = \frac{\mu((J_\Sigma)_{\mathrm{M}_\Sigma} w (J_\Sigma)_{\mathrm{M}_\Sigma})}{\mu'(I_0'wI_0')}$$

and let $e_w \in \mathcal{H}'$ with $e_w(w) = 1$ and $\mathrm{Supp}(e_w) = I_0'wI_0'$. *Let $f_w \in \mathcal{H}$ with $f_w(w) = 1$ and*

$\mathrm{Supp}(f_w) = (J_\Sigma)_{\mathrm{M}_\Sigma} w (J_\Sigma)_{\mathrm{M}_\Sigma}$. *Define a map* $\eta: \mathcal{H}' \longrightarrow \mathcal{H}$ *as follows:*

$$\eta(e_w) = \left(\tfrac{1}{C_w}\right)^{\frac{1}{2}} \varepsilon^{l(w)} f_w.$$

Here $\varepsilon = \widetilde{\chi}_\Gamma^\circ(-1)$ *where* $\widetilde{\chi}_\Gamma^\circ$ *is a character of* $\mathcal{O}_F^\times$ *such that* $\rho_\Sigma | I_0' = \widetilde{\chi}_\Gamma^\circ \circ \det$ *(recall that* ρ_Σ *factors through determinant on* I_0'*). Then* η *is a* $*$*-preserving, support-preserving* L^2*-isomorphism of* $\mathbb{C}$*-algebras.*

Notation. To find an isomorphism between $\mathcal{H}(\mathrm{M}_\Sigma // (J_\Sigma)_{\mathrm{M}_\Sigma}, (\rho_\Sigma)_{\mathrm{M}_\Sigma})$ and $\mathcal{H}(\mathrm{G}' // I_0', 1)$ where $\mathrm{M}_\Sigma \simeq \mathrm{GL}(V^+/k)$, we identify M_Σ with $\mathrm{GL}(V^+/k)$ and regard $(J_\Sigma)_{\mathrm{M}_\Sigma}$ as a subgroup of $\mathrm{GL}(V^+/k)$. From now on, in case there is no worry about confusion, we will drop the subscript ${}_{\mathrm{M}_\Sigma}$. For example,we will just write (J_Σ, ρ_Σ) for $((J_\Sigma)_{\mathrm{M}_\Sigma}, (\rho_\Sigma)_{\mathrm{M}_\Sigma})$.

Proof of Theorem 6.2.2. Note that from (4.2.6), η in Theorem 6.2.2 is a linear isomorphism. Since we have

$$e_w^* = e_{w^{-1}}, \qquad f_w^* = f_{w^{-1}} \quad \text{and} \quad C_w = C_{w^{-1}},$$

we see η is $*$-preserving. Since

$$\begin{aligned}(\eta(e_w), \eta(e_{w'}))\\ &= \varepsilon^{l(w)+l(w')} \left(\tfrac{1}{C_w C_{w'}}\right)^{\frac{1}{2}} \int_{\mathrm{G}} f_w(g) \overline{f_{w'}(g)}\, dg\\ &= \delta_{w,w'} \mu'(I_0' w I_0') = (e_w, e_{w'}),\end{aligned}$$

η is an L^2-isomorphism. From the following Lemma, we see that η is support-preserving, that is, $\mathrm{Supp}(\eta(e_w)) = (J_\Sigma)_{\mathrm{M}_\Sigma} \mathrm{Supp}(e_w) (J_\Sigma)_{\mathrm{M}_\Sigma}$.

LEMMA 6.2.3. *For* $w \in \widetilde{W}'$, $(J_\Sigma w J_\Sigma) \cap \mathrm{G}' = I_0' w I_0'$.

Proof. Assume that $I_0' w I_0'$ is strictly contained in $(J_\Sigma w J_\Sigma) \cap \mathrm{G}'$. Then since $\mathrm{G}' = I_0' \widetilde{W}' I_0'$ and $I_0' \subset J_\Sigma$, there should be $w' \in \widetilde{W}'$ with $w' \neq w$ such that $I_0' w' I_0' \subset J_\Sigma w J_\Sigma$. Then we can write

$$w' = j_1 w j_2, \quad \text{for some} \quad j_1, j_2 \in J_\Sigma. \tag{1}$$

For any $t \in T_0$, we have $\mathrm{Ad}w'(t) = \mathrm{Ad}(j_1 w j_2)(t)$ and thus

$$(w t^{-1} w^{-1}) j_1^{-1} (w' t w'^{-1}) j_1 = w((t^{-1} j_2 t) j_2^{-1}) w^{-1}. \tag{2}$$

Since $(w t^{-1} w^{-1}) j_1^{-1} (w' t w'^{-1}) j_1 \in J_\Sigma$ for any $t \in T_0$, we also have $w((t^{-1} j_2 t) j_2^{-1}) w^{-1} \in J_\Sigma$ for all $t \in T_0$. Now, observing the Ad action of the torus T_0 and $\widetilde{W}'$ on J_Σ, we see that $w j_2 w^{-1} \in J_\Sigma$. Combining with (1), $w' w^{-1} = j_1 (w j_2 w^{-1}) \in (J_\Sigma \cap \mathrm{G}' \cap N_{\mathrm{G}'}(T_0)) = T_0$. Hence, $w = w'$ and it contradicts the assumption. □

Now, it is left to prove that η is an algebra isomorphism. In Section 6.3, we will show it by verifying the relations (L), (Q), (B) and (T) for $\mathcal{H}$ corresponding to those for $\mathcal{H}'$ in (6.2.1).

6.3. $\mathcal{H} = \mathcal{H}(\mathrm{M}_\Sigma//(J_\Sigma)_{\mathrm{M}_\Sigma}, \rho_\Sigma)$

Recall that since our ρ_Σ is a character, we have $Y_\Gamma = Y'_\Gamma$, $J_\Sigma = J'_\Sigma$ (see (3.3.3) and (3.4.2)). Let $e = e(F/k_0)$ be the ramification index of F over k_0. For $\tau \in \mathrm{Gal}_v$, let

$$\beta_\tau = \begin{cases} \frac{1}{2}(-1 - \mathrm{ord}(\gamma^\tau - \gamma)), & \text{if } \tau \neq 1, \\ \frac{1}{2e}, & \text{if } \tau = 1. \end{cases}$$

Here, γ^τ denotes the Galois conjugate of γ under τ. Since ρ_Σ is a character and $Y_\Gamma = Y'_\Gamma$, for any $\tau \in \mathrm{Gal}_v$, we have $\mathrm{M}_v^\tau(\beta_\tau) = \mathrm{M}_v^\tau(\beta_\tau^+)$.

PROPOSITION 6.3.1.

(*L*) *If* $l(ww') = l(w) + l(w')$ *for* $w, w' \in \widetilde{W}'$,

$$f_w \star f_{w'} = \left(\frac{\mu(J_\Sigma w J_\Sigma)\mu(J_\Sigma w' J_\Sigma)}{\mu(J_\Sigma ww' J_\Sigma)}\right)^{\frac{1}{2}} f_{ww'} = \left(\frac{C_w C_{w'}}{C_{ww'}}\right)^{\frac{1}{2}} f_{ww'}.$$

(*Q*) *Let* $\widetilde{\chi}_\Gamma^\circ$ *be a character of* $\mathcal{O}_F^\times$ *as in* (6.2.2) (*note that* $\widetilde{\chi}_\Gamma^\circ(-1) = 1$ *or* -1).

$$f_{s_i} \star f_{s_i} = q_F f_1 + \widetilde{\chi}_\Gamma^\circ(-1)(q_F - 1) f_{s_i}, \quad s_i \in S \setminus \{s_d\},$$

$$f_{s_d} \star f_{s_d} = C_{s_d} q_F f_1 + \widetilde{\chi}_\Gamma^\circ(-1) \cdot C_{s_d}^{\frac{1}{2}} \cdot (q_F - 1) f_{s_d}.$$

(B) (i) $\tilde{f}_{s_i} \star \tilde{f}_{s_j} = \tilde{f}_{s_j} \star \tilde{f}_{s_i}$ *if* $|i - j| > 1 \pmod d$

(ii) $\tilde{f}_{s_i} \star \tilde{f}_{s_{i+1}} \star \tilde{f}_{s_i} = \tilde{f}_{s_{i+1}} \star \tilde{f}_{s_i} \star \tilde{f}_{s_{i+1}}$, $i \pmod d$.

(T) (i) $\tilde{f}_{t^i} \star \tilde{f}_{t^j} = \tilde{f}_{t^{i+j}}$

(ii) $\tilde{f}_t \star \tilde{f}_{s_i} = \tilde{f}_{s_{i-1}} \star \tilde{f}_t$, $i \pmod d$

where

$$C_w = \frac{\mu(J_\Sigma w J_\Sigma)}{\mu'(I'_0 w I'_0)} \quad \textit{and} \quad \tilde{f}_w = \left(\frac{1}{C_w}\right)^{\frac{1}{2}} f_w$$

for any $w \in \widetilde{W}'$.

We first note that (B) and (T) follow from (L). In (6.3.2)–(6.3.4), we will prove the relation (L) in the Proposition 6.3.1.

LEMMA 6.3.2. *Let* $w, w' \in \widetilde{W}'$. *If* $l(ww') = l(w) + l(w')$, *then* $(J_\Sigma w J_\Sigma w' J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma) = J_\Sigma ww' J_\Sigma$.

Proof. We will find an open compact subgroup $\tilde{J}$ containing J_Σ, which behaves similarly as I'_0 does under the action of Weyl group. Let $\tilde{\mathcal{J}}_p$ be the $\mathcal{O}_k$-lattice defined as follows:

$$\tilde{\mathcal{J}}_p = \mathcal{I}'_1 + \sum_{a \geqslant b, \tau} \mathrm{M}^{\tau}_{ab}(\beta_\tau) + \sum_{a<b, \tau} \mathrm{M}^{\tau}_{ab}(\beta_\tau - \tfrac{1}{e}). \tag{1}$$

We note that $\tilde{\mathcal{J}}_p$ is closed under Lie bracket. From the assumption on k, we can define $\tilde{J}_p = \exp(\tilde{\mathcal{J}}_p)$. Then $\tilde{J} = I'_0 \cdot \tilde{J}_p$. Note that $J_\Sigma \subset \tilde{J}$. Since $I'_0 \subset \tilde{J}$ and $\mathrm{G}' \cap \tilde{J} = I'_0$, $\tilde{J}\mathrm{G}'\tilde{J} = \cup_{w \in \widetilde{W}'} \tilde{J}w\tilde{J}$. Moreover, we can also check that for any $w \in \widetilde{W}'$, $(\tilde{J}w\tilde{J}) \cap \mathrm{G}' = I'_0 w I'_0$ as in (6.2.3).

For $\tau \in \mathrm{Gal}_v$, let

$$\tilde{\mathcal{J}}_\tau = \sum_{a \geqslant b} \mathrm{M}^{\tau}_{ab}(\beta_\tau) + \sum_{a<b} \mathrm{M}^{\tau}_{ab}(\beta_\tau - \tfrac{1}{e}).$$

Then we note that $\tilde{\mathcal{J}}_1 = \mathcal{I}'_1$ and from (6.1.3)–(6.1.2), $\tilde{\mathcal{J}}_\tau$ behaves in a similar way as $\mathcal{I}'_1$ does with respect to the Weyl group action. If $l(ww') = l(w) + l(w')$ as in Lemma, we have $I'_0 w I'_0 w' I'_0 = I'_0 ww' I'_0$ and equivalently, for any M^1_{ab}, we have $\mathrm{Ad}w(\mathrm{M}^1_{ab} \cap \tilde{\mathcal{J}}_1) \subset \tilde{\mathcal{J}}_1$ or $\mathrm{Ad}(w'^{-1})(\mathrm{M}^1_{ab} \cap \tilde{\mathcal{J}}_1) \subset \tilde{\mathcal{J}}_1$. Similarly,

$$\mathrm{Ad}w(\mathrm{M}^{\tau}_{ab} \cap \tilde{\mathcal{J}}_\tau) \subset \tilde{\mathcal{J}}_\tau \subset \tilde{\mathcal{J}}_p \quad \text{or} \quad \mathrm{Ad}(w'^{-1})(\mathrm{M}^{\tau}_{ab} \cap \tilde{\mathcal{J}}_\tau) \subset \tilde{\mathcal{J}}_\tau \subset \tilde{\mathcal{J}}_p$$

and, hence, $\tilde{J}w\tilde{J}w'\tilde{J} \subset \tilde{J}ww'\tilde{J}$. Now we have

$$J_\Sigma w J_\Sigma w' J_\Sigma \subset \tilde{J}w\tilde{J}w'\tilde{J} = \tilde{J}ww'\tilde{J}$$

and

$$\begin{aligned}(J_\Sigma w J_\Sigma w' J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma) &\subseteq \tilde{J}w\tilde{J}w'\tilde{J} \cap (J_\Sigma \mathrm{G}' J_\Sigma) \\ &= (\tilde{J}ww'\tilde{J}) \cap (J_\Sigma \mathrm{G}' J_\Sigma) = J_\Sigma ww' J_\Sigma.\end{aligned}$$

Hence

$$(J_\Sigma w J_\Sigma w' J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma) = J_\Sigma ww' J_\Sigma. \qquad \square$$

The following is an immediate consequence of (6.3.2).

COROLLARY 6.3.3. *If* $l(ww') = l(w) + l(w')$ *for* $w, w' \in \widetilde{W}'$, $f_w \star f_{w'}$ *is a constant multiple of* $f_{ww'}$.

6.3.4. Going back to the proof of (6.3.1)–(L), from (6.3.3), we see that $f_w \star f_{w'} = c \cdot f_{ww'}$ for some constant c. We can write $J_\Sigma w J_\Sigma$ and $J_\Sigma w' J_\Sigma$ as a disjoint sum of J_Σ-cosets

$$J_\Sigma w J_\Sigma = \bigcup_{j_\Sigma} j_\Sigma w J_\Sigma, \qquad J_\Sigma w' J_\Sigma = \bigcup_{j'_\Sigma} J_\Sigma w' j'_\Sigma,$$

where each j_Σ and j'_Σ varies over coset representatives

$$\lambda_w = J_\Sigma \big/ (J_\Sigma \cap w J_\Sigma w^{-1}), \qquad \lambda_{w'} = (J_\Sigma \cap w'^{-1} J_\Sigma w') \big\backslash J_\Sigma, \tag{1}$$

respectively. Then

$$\mu(J_\Sigma w J_\Sigma) = \sharp(\lambda_w), \qquad \mu(J_\Sigma w' J_\Sigma) = \sharp(\lambda_{w'}).$$

Note that we may assume that j_Σ and j'_Σ are unipotent. Now we can find c as follows:

$$\begin{aligned} c &= f_w \star f_{w'}(ww') \\ &= \int_{\mathrm{GL}(V^+/k)} f_w(x) f_{w'}(x^{-1}ww')\mathrm{d}x \\ &= \sum_{j_\Sigma \in \lambda_w} \int_{J_\Sigma} f_w(j_\Sigma wx) f_{w'}(x^{-1}w^{-1}j_\Sigma^{-1}ww')\mathrm{d}x \\ &= \sum_{j_\Sigma \in \lambda_w} f_{w'}(w^{-1}j_\Sigma^{-1}ww') = \sum_{\lambda_{w,w'}} \rho_\Sigma(w^{-1}j_\Sigma^{-1}ww'j'^{-1}_\Sigma w'^{-1}) = \sharp\big(\lambda_{w,w'}\big) \end{aligned}$$

where

$$\lambda_{w,w'} = \Big\{(j_\Sigma, j'_\Sigma) \in \lambda_w \times \lambda_{w'} \mid w^{-1}j_\Sigma^{-1}ww'j'^{-1}_\Sigma w'^{-1} \in J_\Sigma\Big\}.$$

Here, $\rho_\Sigma(w^{-1}j_\Sigma^{-1}ww'j'^{-1}_\Sigma w'^{-1}) = 1$ since $\det(w^{-1}j_\Sigma^{-1}ww'j'^{-1}_\Sigma w'^{-1}) = 1$. Computing $\sharp\big(\lambda_{w,w'}\big)$, if $w^{-1}j_\Sigma^{-1}ww'j'^{-1}_\Sigma w'^{-1} = \mathbf{j} \in J_\Sigma$, from (6.3.2), we should have $\mathbf{j} = xy$ where $x, y \in J_\Sigma$ with wxw^{-1}, $w'^{-1}yw' \in J_\Sigma$. Then we have

$$w^{-1}j_\Sigma(wxw^{-1})ww'(w'^{-1}yw')j'^{-1}_\Sigma w'^{-1} = 1$$

and, hence, from (1), we may assume that $x = y = 1$. Then we see that

$$\frac{\sharp(\lambda_w)\sharp(\lambda_{w'})}{\big(\sharp\big(\lambda_{w,w'}\big)\big)^2} = \mu(J_\Sigma ww' J_\Sigma).$$

Since $\sharp(\lambda_w) = \mu(J_\Sigma w J_\Sigma)$ and $\sharp(\lambda_{w'}) = \mu(J_\Sigma w' J_\Sigma)$, we have

$$\sharp\big(\lambda_{w,w'}\big) = \left(\frac{\mu(J_\Sigma w J_\Sigma)\mu(J_\Sigma w' J_\Sigma)}{\mu(J_\Sigma ww' J_\Sigma)}\right)^{\frac{1}{2}}$$

and the relation (6.3.1)–(L) is proved. □

In rest of this section, we will basically prove (6.3.1)–(Q).

6.3.5. We first consider the case s_d. We can write $J_\Sigma s_d J_\Sigma$ as a disjoint sum of J_Σ-cosets

$$J_\Sigma s_d J_\Sigma = \bigcup_{j_\Sigma} j_\Sigma s_d J_\Sigma, \qquad J_\Sigma s_d J_\Sigma = \bigcup_{j'_\Sigma} J_\Sigma s_d j'_\Sigma,$$

where j_Σ and j'_Σ vary over

$$\begin{aligned} \lambda_{s_d} &= J_\Sigma/(J_\Sigma \cap s_d J_\Sigma s_d) = (J_\Sigma \cap s_d J_\Sigma s_d)\big\backslash J_\Sigma \\ &= (J_\Sigma)_p\Big/((J_\Sigma)_p \cap s_d(J_\Sigma)_p s_d) = ((J_\Sigma)_p \cap s_d(J_\Sigma)_p s_d)\Big\backslash(J_\Sigma)_p \\ &\simeq (\mathcal{J}_\Sigma)_p\Big/((\mathcal{J}_\Sigma)_p \cap s_d(\mathcal{J}_\Sigma)_p s_d), \end{aligned}$$

where $(\mathcal{J}_\Sigma)_p = \log(I_1') + Y_\Gamma$ with I_1' the maximal pro-p subgroup of I_0'. More explicitly, we have

$$\lambda_{s_d} = \frac{\mathrm{N}^1_{d1}\left(\frac{1}{e}\right)}{\mathrm{N}^1_{d1}(2e)} \times \\ \times \frac{\exp\left(\sum_{\tau\neq 1} \mathrm{M}^\tau_{d1}(\beta_\tau) + \sum_{i=2}^{d-1}\sum_{\tau\neq 1} \mathrm{M}^\tau_{i1}(\beta_\tau) + \sum_{j=2}^{d-1}\sum_{\tau\neq 1} \mathrm{M}^\tau_{dj}(\beta_\tau)\right)}{\exp\left(\sum_{\tau\neq 1} \mathrm{M}^\tau_{d1}\left(\beta_\tau + \frac{2}{e}\right) + \sum_{i=2}^{d-1}\sum_{\tau\neq 1} \mathrm{M}^\tau_{i1}\left(\beta_\tau + \frac{1}{e}\right) + \sum_{j=2}^{d-1}\sum_{\tau\neq 1} \mathrm{M}^\tau_{dj}\left(\beta_\tau + \frac{1}{e}\right)\right)} \tag{q1}$$

For convenience of notation, we used long division instead of $\backslash$ or $/$. Since each denominator is normalized by its numerator, our notation is harmless. We can choose $j_\Sigma, j'_\Sigma \in \lambda_{s_d}$ such that they are of the following form:

$$\begin{aligned} j_\Sigma &= \exp(Z) \cdot \exp\left(\sum_{i=2}^{d-1}\sum_{\tau\neq 1} a^{(\tau)}_{i1} + \sum_{j=1}^{d-1}\sum_{\tau\neq 1} a^{(\tau)}_{dj}\right), \\ j'_\Sigma &= \exp(Z') \cdot \exp\left(\sum_{i=2}^{d-1}\sum_{\tau\neq 1} b^{(\tau)}_{i1} + \sum_{j=1}^{d-1}\sum_{\tau\neq 1} b^{(\tau)}_{dj}\right), \end{aligned} \tag{q2}$$

where $a^{(\tau)}_{ij}, b^{(\tau)}_{ij} \in \mathrm{M}^\tau_{ab}, Z, Z' \in \mathfrak{p}_F \cap \mathrm{Hom}_k(F^d, F^1)$ and where $a^{(\tau)}_{ij}, b^{(\tau)}_{ij} \in \mathrm{M}^\tau_{ab}(\beta_\tau)$ mod $\mathrm{M}^\tau_{ab}(\beta_\tau) \cap s_d(J_\Sigma)_p s_d$. In a matrix form, j_Σ (mod $J_\Sigma \cap s_d J_\Sigma s_d$) can be written as

$$j_\Sigma \equiv \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \sum a^{(\tau)}_{21} & 1 & & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \sum a^{(\tau)}_{d-1,1} & & & 1 & 0 \\ \begin{matrix} Z + \sum a^{(\tau)}_{d1} \\ +A \end{matrix} & \sum a^{(\tau)}_{d2} & \cdots & \sum a^{(\tau)}_{d,d-1} & 1 \end{pmatrix} \tag{q3}$$

where $A = \sum_{j=2}^{d-1}\left(\sum_{\tau\neq 1} a^{(\tau)}_{dj} \cdot \sum_{\tau\neq 1} a^{(\tau)}_{jd}\right)$. We can write j'_Σ in a similar way.

Now finding the support of $f_{s_d} \star f_{s_d}$, we first note that $\mathrm{Supp}(f_{s_d} \star f_{s_d}) \subset J_\Sigma s_d J_\Sigma s_d J_\Sigma$.

LEMMA 6.3.6. $(J_\Sigma s_d J_\Sigma s_d J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma) = J_\Sigma \cup J_\Sigma s_d J_\Sigma$.

Proof. If $(J_\Sigma s_d J_\Sigma s_d J_\Sigma) \underset{+}{\supset} (J_\Sigma \cup J_\Sigma s_d J_\Sigma)$, we have $w \in \widetilde{W}' \setminus \{1, s_d\}$ such that

$$s_d j_\Sigma s_d j_2 = j_1 w \tag{q4}$$

for some $j_1, j_2 \in J_\Sigma$ and j_Σ in the above form (q3). For simplicity, denote $s_d j_\Sigma s_d$ by j_s. Moreover, since we know $I_0' s_d I_0' s_d I_0' = I_0' \cup I_0' s_d I_0'$, we may assume $Z = 0$ in j_Σ.

Now for any $t \in T_0$, we have $\mathrm{Ad}(j_s j_2)(t) = \mathrm{Ad}(j_1 w)(t)$ and

$$j_s \cdot \mathrm{Ad}t(j_s{}^{-1}) \cdot (t \cdot \mathrm{Ad}j_s(t^{-1} j_2 t j_2^{-1})) = j_1 \cdot \mathrm{Ad}w(t) \cdot j_1^{-1}. \tag{1}$$

Observing that j_s normalizes $(J_\Sigma)_p$ from direct computation, we see that

$$\left(t \cdot \mathrm{Ad}j_s(t^{-1} j_2 t j_2^{-1})\right) \in J_\Sigma, \quad \text{for all} \quad t \in T_0.$$

Since we also have $j_1 \cdot \mathrm{Ad}w(t) \cdot j_1^{-1} \in J_\Sigma$, from (1), we have $j_s \cdot \mathrm{Ad}t(j_s{}^{-1}) \in J_\Sigma$ for all $t \in T_0$, which implies $j_s \in J_\Sigma$. From $w = j_1^{-1} j_s j_2 \in J_\Sigma \cap \widetilde{W}'$, $w = 1$ which is a contradiction. Hence the Lemma follows. □

Now since $\mathrm{Supp}(f_{s_d} \star f_{s_d}) \subset (J_\Sigma s_d J_\Sigma s_d J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma)$, (6.3.6) implies that $f_{s_d} \star f_{s_d}$ is a linear combination of f_1 and f_{s_d}, that is,

$$f_{s_d} \star f_{s_d} = c_1 f_1 + c_2 f_{s_d} \tag{q5}$$

for some constants c_1 and c_2.

Since $\mu'(I_0' s_d I_0') = q_F$, we can find c_1 and c_2 as follows;

$$\begin{aligned}
c_1 &= f_{s_d} \star f_{s_d}(1) = \int_{\mathrm{GL}(V^+/k)} f_{s_d}(x) f_{s_d}(x^{-1})\, \mathrm{d}x \\
&= \int_{J_\Sigma s_d J_\Sigma} f_{s_d}(x) f_{s_d}(x^{-1}) \mathrm{d}x = \int_{J_\Sigma s_d J_\Sigma} 1\, \mathrm{d}x \\
&= \mu(J_\Sigma s_d J_\Sigma) = C_{s_d} \cdot q_F \\
c_2 &= f_{s_d} \star f_{s_d}(s_d) = \int_{\mathrm{GL}(V^+/k)} f_{s_d}(x) f_{s_d}(x^{-1} s_d) \mathrm{d}x \\
&= \sum_{j_\Sigma \in \lambda_{s_d}} \int_{J_\Sigma} f_{s_d}(j_\Sigma s_d x) f_{s_d}(x^{-1} s_d j_\Sigma^{-1} s_d) \mathrm{d}x \\
&= \sum_{j_\Sigma \in \lambda_{s_d}} f_{s_d}(s_d j_\Sigma^{-1} s_d) = \sum_{j_\Sigma \in \lambda_{s_d}} f_{s_d}(s_d j_\Sigma s_d)
\end{aligned} \tag{q6}$$

Since $f_{s_d}(s_d j_\Sigma s_d) \neq 0$ (if and) only if $s_d j_\Sigma s_d \in J_\Sigma s_d J_\Sigma$,

$$c_2 = \sum_{\substack{(j_\Sigma, j'_\Sigma) \in \lambda_{s_d} \times \lambda_{s_d} \\ s_d j_\Sigma s_d j'_\Sigma s_d \in J_\Sigma}} \rho_\Sigma(s_d j_\Sigma s_d j'_\Sigma s_d). \tag{q7}$$

To find the condition on j_Σ, j'_Σ such that $s_d j_\Sigma s_d j'_\Sigma s_d \in J_\Sigma s_d J_\Sigma$, we compute $s_d j_\Sigma s_d j'_\Sigma s_d$

explicitly. For simplicity of notation, let $Z = Z_0\pi_F$, $Z' = Z'_0\pi_F$. We will write down only terms of nontrivial contribution:

$$s_d j_\Sigma s_d j'_\Sigma s_d \overset{\text{mod}}{\equiv}$$

$$\begin{pmatrix} \begin{matrix} Z_0+ \\ \pi_F^{-1}\sum_\tau a_{d1}^{(\tau)} \end{matrix} & \cdots & \begin{matrix} \sum_\tau \pi_F^{-1}(a_{di}^{(\tau)}+ \\ Z_0 b_{di}^{(\tau)})\tau \end{matrix} & \cdots & \begin{matrix} \pi_F^{-1}(1+Z_0Z'_0)+ \\ \sum_\tau \pi_F^{-1}(Z_0 b_{d1}^{(\tau)}+ \\ a_{d1}^{(\tau)} Z'_0)\pi_F^{-1} \end{matrix} \\ \vdots & \ddots & * & * & \vdots \\ \sum_\tau a_{i1}^{(\tau)} & * & \begin{matrix} 1+\left(\sum a_{i1}^{(\tau)}\pi_F^{-1}\right) \\ \cdot\left(\sum b_{di}^{(\tau)}\right) \end{matrix} & * & \begin{matrix} \sum_\tau (b_{i1}^{(\tau)}+ \\ a_{i1}^{(\tau)} Z'_0)\pi_F^{-1} \end{matrix} \\ \vdots & * & * & \ddots & \vdots \\ \pi_F & \cdots & \sum_\tau b_{di}^{(\tau)} & \cdots & \begin{matrix} Z'_0+ \\ \left(\sum_\tau b_{d1}^{(\tau)}\pi_F^{-1}\right) \end{matrix} \end{pmatrix} \tag{q8}$$

From (q8), $s_d j_\Sigma s_d j'_\Sigma s_d \in J_\Sigma$ if and only if

$$\begin{aligned} &1 + Z_0 Z'_0 \equiv 0, \quad Z_0 b_{d1}^{(\tau)} + a_{d1}^{(\tau)} Z'_0 \equiv 0, \\ &a_{d1}^{(\tau)} \in \mathrm{M}_n^\tau u\left(\beta_\tau + \frac{1}{e}\right) \left(\operatorname{mod} \mathrm{M}_n^\tau u\left(\beta_\tau + \frac{2}{e}\right)\right), \\ &a_{di}^{(\tau)} + Z_0 b_{di}^{(\tau)} \equiv 0, \qquad b_{i1}^{(\tau)} + a_{i1}^{(\tau)} Z'_0 \equiv 0, \qquad \text{for } i = 2, \ldots, d-1. \end{aligned} \tag{q9}$$

Hence j'_Σ is determined by j_Σ, that is,

$$\begin{aligned} &Z'_0 \equiv -\frac{1}{Z_0}, \quad b_{d1}^{(\tau)} \equiv \frac{1}{Z_0} a_{d1}^{(\tau)} \frac{1}{Z_0} \in \mathrm{M}_{d1}^{(\tau)}\left(\beta_\tau + \frac{1}{e}\right) \left(\operatorname{mod}\ \mathrm{M}_{d1}^{(\tau)}\left(\beta_\tau + \frac{2}{e}\right)\right), \\ &b_{di}^{(\tau)} \equiv -\frac{1}{Z_0} a_{di}^{\tau}, \quad b_{i1}^{(\tau)} \equiv a_{i1}^{\tau} \frac{1}{Z_0}, \qquad \text{for } i = 2, \ldots, d-1. \end{aligned} \tag{q10}$$

Then (q8) becomes

$$s_d j_\Sigma s_d j'_\Sigma s_d \overset{\text{mod}}{\equiv} \begin{pmatrix} Z_0 & & \\ & \mathrm{Id} & \\ & & -\frac{1}{Z_0} \end{pmatrix} \cdot$$

$$\begin{pmatrix} \begin{matrix} 1+ \\ \frac{\pi_F^{-1}}{Z_0} \sum_\tau a_{d1}^{(\tau)} \end{matrix} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & * & * & \vdots \\ \sum_\tau a_{i1}^{(\tau)} & * & \begin{matrix} 1+\left(\sum a_{i1}^{(\tau)} \pi_F^{-1}\right) \\ \cdot\left(\sum b_{di}^{(\tau)}\right) \end{matrix} & * & \vdots \\ \vdots & * & * & \ddots & \vdots \\ -Z_0\pi_F & \cdots & -Z_0 \cdot \sum_\tau b_{di}^{(\tau)} & \cdots & \begin{matrix} 1- \\ Z_0\left(\sum_\tau b_{d1}^{(\tau)} \pi_F^{-1}\right) \end{matrix} \end{pmatrix}$$

Recall that θ is defined in (2.4.1). Then

$$\begin{aligned} \rho_\Sigma(s_d j_\Sigma s_d j'_\Sigma s_d) =& \widetilde{\chi}_\Gamma^\circ(-1) \cdot \theta\left(\mathrm{Tr}_{k/k_0} \circ \mathrm{Tr}\left(2\gamma \cdot \left(\sum_\tau \frac{1}{Z_0 \pi_F} a_{d1}^{(\tau)} + \sum_\tau \frac{1}{Z'_0} b_{d1}^{(\tau)} \frac{1}{\pi_F} \right)\right)\right) \\ & \cdot \prod_{i=2}^{d-1} \theta\left(\mathrm{Tr}_{k/k_0} \circ \mathrm{Tr}\left(2\gamma \cdot \left(\sum_\tau a_{i1}^{(\tau)} \frac{1}{\pi_F} \right) \cdot \left(\sum_{\tau'} b_{di}^{(\tau')} \right)\right)\right) \\ =& \widetilde{\chi}_\Gamma^\circ(-1) \cdot \prod_{i=2}^{d-1} \theta\left(\mathrm{Tr}_{k/k_0} \circ \mathrm{Tr}\left(2\gamma \cdot \left(\sum_\tau a_{i1}^{(\tau)} \frac{1}{\pi_F} b_{di}^{(\tau^{-1})} \right)\right)\right). \end{aligned}$$

where we write

$$\rho_\Sigma \begin{pmatrix} Z_0 & & \\ & \mathrm{Id} & \\ & & -\frac{1}{Z_0} \end{pmatrix} = \widetilde{\chi}_\Gamma^\circ(-1).$$

Second equality follows from (2.4.3). Note that 2 comes from components of Γ and M_s in $\mathrm{GL}(V^-)$. Then (q7) becomes

$$\begin{aligned} c_2 &= \widetilde{\chi}^{\circ}_{\Gamma}(-1)\cdot\sum\prod_{i=2}^{d-1}\theta\left(\mathrm{Tr}_{k/k_0}\circ\mathrm{Tr}\left(2\gamma\cdot\left(\sum_{\tau} a^{(\tau)}_{i1}\frac{1}{\pi_F}b^{(\tau^{-1})}_{di}\right)\right)\right)\\ &= \widetilde{\chi}^{\circ}_{\Gamma}(-1)\cdot\sharp\left(\prod_{\tau}\mathrm{N}^{\tau}_{d1}\left(\beta^{\tau}_{v_0}+\frac{1}{e}\right)\Big/\mathrm{N}^{\tau}_{d1}\left(\beta^{\tau}_{v_0}+\frac{2}{e}\right)\right)\\ &\qquad\cdot\sum\prod_{i=2}^{d-1}\theta\left(\mathrm{Tr}_{k/k_0}\circ\mathrm{Tr}\left(2\gamma\cdot\left(\sum_{\tau} a^{(\tau)}_{i1}\frac{1}{\pi_F}\frac{1}{Z_0}\left(-a^{(\tau^{-1})}_{di}\right)\right)\right)\right)\\ &= \widetilde{\chi}^{\circ}_{\Gamma}(-1)\cdot C^{\frac{1}{2}}_{s_d}\cdot(q_F-1) \end{aligned} \qquad \text{(q11)}$$

where the first $\sum$ runs over (j_Σ, j'_Σ) satisfying (q9) and (q10), and second $\sum$ runs over $a^{(\tau)}_{di}$, $a^{(\tau)}_{i1}$ and Z_0 satisfying (q9). From (q6) and (q11), we get the quadratic relation (Q) for s_d.

6.3.7. *Proof of (6.3.1)–(Q) continued.* Now assume $i \neq d$. The computation is similar to the case $i = d$, but it is simpler. Note that $C_{s_i} = 1$ for $s_i \in S \setminus \{s_d\}$. We also note that $\mathrm{Ad}s_i(Y_\Gamma) = Y_\Gamma$. Then since $I'_0 s_i I'_0 s_i I'_0 = I'_0 \cup I'_0 s_i I'_0$ and $I'_0 \subset J_\Sigma$, we have $J_\Sigma s_i J_\Sigma s_i J_\Sigma = J_\Sigma s_i I'_0 s_i J_\Sigma = J_\Sigma \cup J_\Sigma s_i J_\Sigma$ and $(J_\Sigma s_i J_\Sigma s_i J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma) = J_\Sigma \cup J_\Sigma s_i J_\Sigma$. Hence we have

$$(J_\Sigma s_i J_\Sigma s_i J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma) = J_\Sigma \cup J_\Sigma s_i J_\Sigma.$$

Then, again we can write

$$f_{s_i} \star f_{s_i} = c_1 f_1 + c_2 f_{s_i}$$

for some constants c_1, c_2. In this case,

$$\lambda_{s_i} = J_\Sigma \big/ (J_\Sigma \cap s_i J_\Sigma s_i) = (J_\Sigma \cap s_i J_\Sigma s_i)\big\backslash J_\Sigma = \mathrm{N}^1_{i,i+1}(0)\Big/\mathrm{N}^1_{i,i+1}\left(\frac{1}{e}\right).$$

and we can similarly (but simpler) compute

$$f_{s_i} \star f_{s_i} = q_F f_1 + \widetilde{\chi}^{\circ}_{\Gamma}(-1)(q_F - 1) f_{s_i}. \qquad \square$$

6.3.8. Finally, comparing (6.2.1) and (6.3.1), we see that η defined in Theorem 6.2.2 is an algebra isomorphism. Combining this with the previous remarks in the beginning of the proof of (6.2.2), now Theorem 6.2.2 is proved. □

7. Computation: $\mathbf{G}' = \prod_{i=1}^{m} \mathbf{U}_{m_i}(F_i/k_i)$

In this Section, we assume that $\mathrm{G}' = \prod_{i=1}^{m} \mathrm{U}_{m_i}(F_i/k_i)$ without GL-factors where k_i is a fixed subfield of F_i under its involution σ_i. Then $[F_i : k_i] = 1$ or 2. In the first two sections, we describe affine root systems and affine Weyl groups more explicitly. Those are necessary in order to see our computation in Section 7.4 explicitly. In Section 7.3, we give a brief description on *tamely ramified* Hecke algebras. In Section 7.4, we show that our Hecke algebra $\mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$ is isomorphic to some *tamely ramified* Hecke algebra on G'.

7.1. AFFINE ROOT SYSTEMS

7.1.1. Let $\mathrm{G}(V_i)$ be a subgroup of G which consists of isometries on $(V_i, \langle\, ,\, \rangle|_{V_i})$ and let $\mathfrak{g}(V_i)$ be its Lie algebra. Let $\mathfrak{t}_i = \mathfrak{g}(V_i) \cap \mathfrak{t}$ and $\mathrm{T}_i = \mathrm{G}(V_i) \cap \mathrm{T}$ with $V_i = m_i F_i$. Then we can write $\mathfrak{t}$ and Γ as follows:

$$\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2 \oplus \cdots \oplus \mathfrak{t}_m,$$
$$\Gamma = \Gamma^{(1)} + \Gamma^{(2)} + \cdots + \Gamma^{(m)}$$

where $\Gamma^{(i)} = (\gamma_i, \ldots, \gamma_i) \in \mathfrak{t}_i$. From (1.5.0), we can write

$$V = \bigoplus_{i=1}^{m} V_i = \bigoplus_{i=1}^{m} (V_i^+ \oplus V_i^\delta \oplus V_i^- \oplus V_i^{\delta'}).$$

Let d_i be $\dim_{F_i}(V_i^+) = \dim_{F_i}(V_i^-)$. Then we decompose V_i further as in (1.5.0), that is,

$$V_i = V_i^+ \oplus V_i^\delta \oplus V_i^- \oplus V_i^{\delta'} = F_i^{d_i} \oplus \cdots \oplus F_i^1 \oplus V_i^\delta \oplus F_i^{-1} \oplus \cdots \oplus F_i^{-d_i} \oplus V_i^{\delta'}$$

where $F_i^{i'}$ denotes i'-th component in V_i regarded as a vector space over F_i and where V_i^δ and $V_i^{\delta'}$ are as in (1.4.6)–(1.4.7). Moreover, $F_i^{i'}$ and $F_i^{-i'}$ are dual to each other with respect to the ε-Hermitian form f_{V_i} defined on V_i.

7.1.2. Let

$$\mathfrak{g} = \sum \mathfrak{t}_i \oplus \sum \widetilde{\mathrm{M}}_v^\tau = \mathfrak{t} \oplus \sum \widetilde{\mathrm{M}}_v^\tau \tag{1}$$

be the decomposition as in (2.2.8) where $\sum$ runs over nontrivial $\mathfrak{t}$-spaces $\widetilde{\mathrm{M}}_v^\tau$. Recall from (2.1.1)–(5),

$$\Upsilon = \left\{ v = (i, j, i', j') \,\middle|\, \begin{matrix} i, j = 1, \ldots, m, \\ i' \in \mathrm{Ix}_i, \quad j' \in \mathrm{Ix}_j \end{matrix} \right\} \Big/ v \sim v_\sigma. \tag{2}$$

We will find k_0-rational roots in each factor $\mathfrak{g}_i' = \mathfrak{u}_{m_i}$ of $\mathfrak{g}'$ and $\mathfrak{g}(V_i)$. Restricting to each $\mathfrak{g}(V_i)$, (1) becomes

$$\mathfrak{g}(V_i) = \mathfrak{t}_i \oplus \sum \widetilde{\mathrm{M}}_v^\tau \tag{3}$$

where $\sum$ runs over $v = (i, i, i', j') \in \Upsilon$, $\tau \in \mathrm{Gal}_v^\sigma$ with $(v, \tau) \neq ((i, i, i', i'), 1)$.

7.1.3. Let v be (i, j, i', j') as before. To each k_0-rational root space $\widetilde{\mathrm{M}}_v^\tau$ with $i \neq j$ or $i' \neq j'$, we define N_v^τ as follows;

$$\mathrm{N}_v^\tau = \exp\left(\widetilde{\mathrm{M}}_v^\tau\right).$$

Recall that a_v is defined in (2.1.1). Note that exp is well defined. To each $\widetilde{\mathrm{M}}_v^\tau(\beta)$ with $i \neq j$ or $i' \neq j'$, we associate $\mathrm{N}_v^\tau(\beta)$ as follows;

$$\mathrm{N}_v^\tau(\beta) = \exp\left(\widetilde{\mathrm{M}}_v^\tau(\beta)\right) = \exp\left(\widetilde{\mathrm{M}}_v'^\tau\left(\beta + \frac{a_v}{2}\right)\right).$$

When $i = j$ and $i' = j'$, for $\beta > 0$, this is again well defined due to the assumption (3.2.3) on residue characteristic of k. As before, if $\tau = 1$, $i = j$ and $i' \neq j'$, then N_v^1 and $\mathrm{N}_v^1(\beta)$ are usual root subgroups.

7.1.4. Write $\mathrm{G}_i' = \mathrm{G}(V_i, f_{V_i})$. Then f_{V_i} belongs to one of the following cases. Recall notation from Section 1.4:

(A) f_{V_i} is ε-Hermitian with F_i/k_i unramified,
  (A1) $(V_i)_0 = 0$.
  (A2) $d_{i0} = 1$ and $(V_i)_0 = V_i^\delta \neq 0$.
  (A3) $d_{i0} = 1$ and $(V_i)_0 = V_i^{\delta'} \neq 0$.
  (A4) $(V_i)_0 \neq 0$ with $d_{i0} = 2$.

(B) f_{V_i} is ε-Hermitian with F_i/k_i ramified,
  (B1) $(V_i)_0 = 0$.
  (B2) $(V_i)_0 \neq 0$ with $d_{i0} = 1$.
  (B3) $(V_i)_0 \neq 0$ with $d_{i0} = 2$.

(C) f_{V_i} is $+1$-symmetric with $\sigma_0 = 1$.
  (C1) $(V_i)_0 = 0$.
  (C2) $V_i^\delta \neq 0$ and $V_i^{\delta'} = 0$.
  (C3) $V_i^\delta = 0$ and $V_i^{\delta'} \neq 0$.
  (C4) $V_i^\delta \neq 0$ and $V_i^{\delta'} \neq 0$.

(D) f_{V_i} is -1-symmetric, i.e., symplectic with $\sigma_0 = 1$.

In (7.1.5) and (7.1.6), we will explicitly describe affine roots in $\mathfrak{g}_i'$ with respect to T_i in terms of $\widetilde{\mathrm{M}}_v^\tau$. For general discussions, we refer to [BT] and [T].

7.1.5. Let $\mathrm{T}_i^s \subset \mathrm{T}_i$ be the maximal k_i-split torus of G_i'. Let Φ_i' and Δ_i' be the set of roots and simple roots of G_i' respectively. We use the same notation for the sets of corresponding root spaces in $\mathfrak{g}_i'$. We also define Φ_{i+}' such that the Iwahori subgroup

$I_0'^{\,i}$ (see Section 1.5.A) in G_i' can be written as

$$I_0'^{\,i} = (Z_{\mathrm{G}_i'}(\mathrm{T}_i^s))_0 \cdot \prod_{\tilde{\mathrm{M}}_v^1 \in \Phi_i' \setminus \Phi_{i+}'} \mathrm{N}_v^1\left(\frac{1}{e_F}\right) \cdot \prod_{\tilde{\mathrm{M}}_v^1 \in \Phi_{i+}'} \mathrm{N}_v^1(0)$$

where $(Z_{\mathrm{G}_i'}(\mathrm{T}_i^s))_0$ is the maximal compact subgroup of the centralizer $Z_{\mathrm{G}_i'}(\mathrm{T}_i^s)$ of T_i^s in G_i'. We can explicitly find $(Z_{\mathrm{G}_i'}(\mathrm{T}_i^s))_0$ as $T_{i0} \cdot \mathrm{G}'((V_i)_0)$ where T_{i0} is the maximal compact subgroup of T_i and $\mathrm{G}'((V_i)_0)$ is the group of isometries on $((V_i)_0, f_{V_i}|(V_i)_0)$. We can find Φ_i', Φ_{i+}', Δ_i' as follows:

7.1.6. Let $\frac{1}{e_{F_i}}\mathbb{Z}$ be the value group of F_i where $e_{F_i} = e(F_i/k_0)$. In all cases, we have

$$\Phi_i' = \left\{ \widetilde{\mathrm{M}}_v^\tau \mid v = (i, i, i', j') \in \Upsilon \text{ with } i' \neq j',\ \tau = 1 \right\},$$

Then the Lie algebra $\mathfrak{g}_i'$ of G_i' can be written as

$$\mathfrak{g}_i' = \mathfrak{t}_i \oplus \sum_{\widetilde{\mathrm{M}}_v^1 \in \Phi_i'} \widetilde{\mathrm{M}}_v^1.$$

For simplicity of notation, we will abbreviate $v = (i, i, i', j')$ by (i', j') or $i'j'$ if there is no confusion and we will identify v with its representative in Υ.

(A) f_{V_i} is ε-Hermitian with F_i/k_i unramified,

(A1) $(V_i)_0 = 0$.

$$\Phi_{i+}' = \left\{ \widetilde{\mathrm{M}}_v^1 \in \Phi_i' \mid i' = 1, \ldots, d_i,\ -i' \leqslant j' < i' \right\}$$
$$\Delta_i' = \left\{ \widetilde{\mathrm{M}}^1_{(-d_i, d_i)}\left(\frac{1}{e_{F_i}}\right),\ \widetilde{\mathrm{M}}_v^1(0) \;\middle|\; \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } v = (1, -1) \end{array} \right\}$$

(A2) $d_{i0} = 1$ and $(V_i)_0 = V_i^\delta \neq 0$.

$$\Phi_{i+}' = \left\{ \widetilde{\mathrm{M}}_v^1 \in \Phi_i' \middle| \, i' \in \{1, \ldots, d_i\},\ j' \in \nabla_i \cup \{-i', \ldots, i'-1\} \right\}$$
$$\Delta_i' = \left\{ \widetilde{\mathrm{M}}^1_{(-d_i, d_i)}\left(\frac{1}{e_{F_i}}\right),\ \widetilde{\mathrm{M}}_v^1(0) \middle| \, v = (i', i'-1),\ i' > 1, \text{ or } v = (1, \delta) \right\}$$

(A3) $d_{i0} = 1$ and $(V_i)_0 = V_i^{\delta'} \neq 0$.

$$\Phi_{i+}' = \left\{ \widetilde{\mathrm{M}}_v^1 \in \Phi_i' \middle| \begin{array}{l} i' \in \{1, \ldots, d_i\},\ j' \in \nabla_i \cup \{-i', \ldots, i'-1\} \\ \text{or } j' = \delta',\ i' = -d_i, \ldots, d_i \end{array} \right\}$$
$$\Delta_i' = \left\{ \widetilde{\mathrm{M}}_v^1(0) \middle| \, v = (i', i'-1),\ i' > 1, \text{ or } (1, -1),\ (-d_i, \delta') \right\}$$

(A4) $(V_i)_0 \neq 0$ with $d_{i0} = 2$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \,\middle|\, \begin{array}{l} i' \in \{1, \ldots, d_i\},\ j' \in \nabla_i \cup \{-i', \ldots, i'-1\} \\ \text{or } j' = \delta,\ i' = 1, \ldots, d_i \\ \text{or } j' = \delta',\ i' = -d_i, \ldots, d_i, \delta \end{array} \right\}$$

$$\Delta'_i = \left\{ \widetilde{\mathrm{M}}^1_v(0) \,\middle|\, v = (i', i'-1),\ i' > 1,\ \text{ or } (1, \delta),\ (-d_i, \delta') \right\}$$

(B) f_{V_i} is ε-Hermitian with F_i/k_i ramified,

(B1) $(V_i)_0 = 0$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \mid i' = 1, \ldots, d_i,\ -i' \leqslant j' < i' \right\}$$

$$\Delta'_i = \begin{cases} \left\{ \begin{array}{l} \widetilde{\mathrm{M}}^1_{(-d_i,d_i)}\left(\dfrac{1}{e_{F_i}}\right), \\ \widetilde{\mathrm{M}}^1_v(0) \end{array} \,\middle|\, \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } (2, -1) \end{array} \right\} & \text{if } d_i \geqslant 2 \\ \left\{ \widetilde{\mathrm{M}}^1_{(-1,1)}\left(\dfrac{1}{e_{F_i}}\right),\ \widetilde{\mathrm{M}}^1_{(1,-1)}\left(\dfrac{1}{e_{F_i}}\right) \right\} & \text{if } d_i = 1. \end{cases}$$

(B2)–(B3) $(V_i)_0 \neq 0$ with $d_{i0} = 1$ or $d_{i0} = 2$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \,\middle|\, \begin{array}{l} i' \in \nabla_i \cup \{1, \ldots, d_i\} \\ j' \in \nabla_i \cup \{-i', \ldots, i'-1\} \end{array} \right\}$$

$$\Delta'_i = \left\{ \widetilde{\mathrm{M}}^1_{(-d_i,d_i)}\left(\frac{1}{e_{F_i}}\right),\ \widetilde{\mathrm{M}}^1_v(0) \,\middle|\, \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } (1, i') \text{ with } i' \in \nabla_i \end{array} \right\}$$

(C) f_{V_i} is $+1$-symmetric with $\sigma_0 = 1$. Note that we have $e_{F_i} = 1$ for this case.

(C1) $(V_i)_0 = 0,\ d_i \geqslant 2$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \,\middle|\, i' = 1, \ldots, d_i,\ -i' \leqslant j' < i' \right\}$$

$$\Delta'_i = \begin{cases} \left\{ \widetilde{\mathrm{M}}^1_{(-d_i+1,d_i)}(1),\ \widetilde{\mathrm{M}}^1_v(0) \,\middle|\, \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } (2, -1) \end{array} \right\} & \text{if } d_i > 2 \\ \left\{ \widetilde{\mathrm{M}}^1_{(2,1)}(0),\ \widetilde{\mathrm{M}}^1_{(2,-1)}(0),\ \widetilde{\mathrm{M}}^1_{(1,2)}(1),\ \widetilde{\mathrm{M}}^1_{(-1,2)}(1) \right\} & \text{if } d_i = 2 \end{cases}$$

(C2) $V_i^{\delta} \neq 0,\ V_i^{\delta'} = 0$ and $d_i \geqslant 2$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \,\middle|\, i' \in \{1, \ldots, d_i\},\ j' \in \nabla_i \cup \{-i', \ldots, i'-1\} \right\}$$

$$\Delta'_i = \left\{ \widetilde{\mathrm{M}}^1_{(-d_i+1,d_i)}(1),\ \widetilde{\mathrm{M}}^1_v(0) \,\middle|\, \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } (1, i') \text{ with } i' \in \nabla_i \end{array} \right\}$$

(C3) $V_i^{\delta} = 0$, $V_i^{\delta'} \neq 0$ and $d_i \geqslant 2$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \,\middle|\, \begin{array}{l} i' \in \{1, \dots, d_i\},\ j' \in \{-i', \dots, i'-1\} \\ \text{or } j' \in \nabla_i,\ i' \in \nabla_i \cup \{-d_i, \dots, d_i \text{ with } i' \neq j'\} \end{array} \right\}$$

$$\Delta'_i = \left\{ \widetilde{\mathrm{M}}^1_v(0) \,\middle|\, \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } (2, -1) \\ \text{or } (-d_i, j') \text{ with } j' \in \nabla_i \end{array} \right\}$$

(C4) $V_i^{\delta} \neq 0$ and $V_i^{\delta'} \neq 0$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \,\middle|\, \begin{array}{l} i' \in \{1, \dots, d_i\},\ j' \in \{-i', \dots, i'-1, \delta_1, \delta_2\} \\ \text{or } j' = \delta'_1, \delta'_2,\ i' \in \nabla_i \cup \{-d_i, \dots, d_i\} \text{ with } i' \neq j' \end{array} \right\}$$

$$\Delta'_i = \left\{ \widetilde{\mathrm{M}}^1_v(0) \,\middle|\, \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } (2, -1) \\ \text{or } (-d_i, j') \text{ with } j' = \delta'_1, \delta'_2 \end{array} \right\}$$

In all (C1)–(C4), if $d_i = 1$, we have $\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_{(1,i)} \mid i \in \nabla_i \right\} = \Delta'_i$.

(D) f_{V_i} is -1-symmetric, i.e., symplectic with $\sigma_0 = 1$. Note that we have $e_{F_i} = 1$.

$$\Phi'_{i+} = \left\{ \widetilde{\mathrm{M}}^1_v \in \Phi'_i \,\middle|\, i' = 1, \dots, d_i,\ -i' \leqslant j' < i' \right\}$$

$$\Delta'_i = \left\{ \widetilde{\mathrm{M}}^1_{(-d_i, d_i)}(1),\ \widetilde{\mathrm{M}}^1_v(0) \,\middle|\, \begin{array}{l} v = (i', i'-1),\ i' > 1, \\ \text{or } (1, -1) \end{array} \right\}$$

We also find the set of affine roots $(\Phi'_i)_{\mathrm{aff}}$ as follows:

$$(\Phi'_i)_{\mathrm{aff}} = \left\{ \widetilde{\mathrm{M}}^1_v(\beta) \mid \widetilde{\mathrm{M}}^1_v \in \Phi'_i, \quad \beta \in \frac{1}{e_{F_i}} \mathbb{Z} \right\}$$

In the following lemma, we note that when ρ_Σ is a character, some of the cases in (7.1.4) do not occur as G'_i under certain situations.

LEMMA 7.1.7.

(1) *Suppose* G *itself is not a group of type* (*A3*) *or* (*A4*). *If one of* G'_i *is of type* (*A3*) *or* (*A4*) *with* $d_i \geqslant 1$, *then* ρ_Σ *is a Heisenberg representation.*

(2) *Let* G *be of type* (*A*) *with* $V_i^{\delta'} \neq 0$ *or of type* (*C*) *with* $V_i^{\delta'} \neq 0$. *Suppose one of* G'_i *is again of the same type with* $d_i \geqslant 1$, $V_i^{\delta'} \neq 0$ *and* $F_i = k$ *(that is,* G'_i *corresponds to* $\gamma_i = 0$*). If* ρ_Σ *is a character, then* e_{F_j} *is even for all* $j \neq i$.

Proof. (1) Assume ρ_Σ is a character. Then we have

$$\widetilde{\mathrm{M}}'^{\tau}_v\left(\beta^\tau_v + \tfrac{1}{2}a_v\right) = \widetilde{\mathrm{M}}'^{\tau}_v\left(\beta^\tau_v + \tfrac{1}{2}a_v\right)^+ \quad \text{where} \quad \beta^\tau_v = \tfrac{1}{2}(-1 - \mathrm{ord}(\gamma^\tau_i - \gamma_j)). \qquad (*)$$

Recall that $a_v = a'_v + a''_v$ is defined in (2.1.1)–(2.1.4). In our case, since P'_0 is Iwahori, we have $a''_v = 0$. Let $v = (i, i, d_i, d_i)$ and $\tau \neq 1$. Then note that $a_v = 0$. From $(*)$, we should have

$$\beta^{\tau}_{v} + \frac{1}{2}a_v = \frac{1}{2}(-1 - \mathrm{ord}(\gamma_i - \gamma^{\tau}_i)) \in \frac{1}{2e_{F_i}}\mathbb{Z} \setminus \frac{1}{e_{F_i}}\mathbb{Z}.$$

Now, for $v' = (i, i, \delta', d_i)$, we have $a_{v'} = 1/e_{F_i}$ and $\beta^{\tau}_{v'} + \frac{1}{2}a_{v'} = \frac{1}{2}(-1 - \mathrm{ord}(\gamma_i - \gamma^{\tau}_i)) + 1/2e_{F_i} \in 1/e_{F_i}\mathbb{Z}$. Hence we have $\widetilde{\mathrm{M}}'^{\tau}_{v}(\beta^{\tau}_{v'} + \frac{1}{2}a_{v'}) \neq \widetilde{\mathrm{M}}'^{\tau}_{v}(\beta^{\tau}_{v'} + \frac{1}{2}a_{v'})^{+}$, contradicting $(*)$.

(2) We will prove the case of type (A). The other case can be proved similarly. Since ρ_Σ is a character, we will have $(*)$ above. Let $v = (i, j, d_i, j')$ and $v' = (i, j, \delta'_i, j')$ with $j \neq i$ and $\delta'_i \in \nabla_i$. From $(*)$, we should have

$$\beta^{\tau}_{v} + \frac{1}{2}a_v = \frac{1}{2}(-1 - \mathrm{ord}(\gamma_j)) + \frac{1}{2}a_v \in \frac{1}{2e_{F_j}}\mathbb{Z} \setminus \frac{1}{e_{F_j}}\mathbb{Z}.$$

If e_{F_j} is odd, then $\beta^{\tau}_{v'} + \frac{1}{2}a_{v'} = \beta^{\tau}_{v} + \frac{1}{2}a_v + \frac{1}{2} \in (1/e_{F_j})\mathbb{Z}$ and $\widetilde{\mathrm{M}}'^{\tau}_{v}(\beta^{\tau}_{v'} + \frac{1}{2}a_{v'}) \neq \widetilde{\mathrm{M}}'^{\tau}_{v}(\beta^{\tau}_{v'} + \frac{1}{2}a_{v'})^{+}$, contradicting $(*)$. Hence e_{F_j} should be even. □

7.2. AFFINE WEYL GROUPS

7.2.1. Let $\widetilde{W}'_i = N_{\mathrm{G}'_i}(\mathrm{T}^s_i)/(Z_{\mathrm{G}'_i}(\mathrm{T}^s_i))_0$ be the affine Weyl group in each G'_i. Let $\Omega_i = \left\{ w \in \widetilde{W}'_i \,\middle|\, \mathrm{Ad}\dot{w}(I'^{i}_0) = I'^{i}_0 \right\}$. Here $\dot{w}$ denotes a representative of $w \in \widetilde{W}'_i$ in $N_{\mathrm{G}'_i}(\mathrm{T}^s_i)$. Then

$$\widetilde{W}'_i = \Omega_i \ltimes \widetilde{W}'^{0}_i$$

where $\widetilde{W}'^{0}_i$ is the Coxeter group generated by simple reflections $s_{ii'}$ for $i' = 0, 1, \ldots, d_i$, corresponding to affine simple roots. For the rest of this section, we again drop the index i from $(i, i, i', j') \in \Upsilon$ for simplicity. We also use the same notation w for both $w \in \widetilde{W}'_i$ and its representative $\dot{w} \in N_{\mathrm{G}'_i}(\mathrm{T}^s_i)$ in case there is no confusion.

Describing $\widetilde{W}'_i$ more explicitly, we first consider the case $d_i \geqslant 2$. For $i' = 1, \ldots, d_i - 1$, let $s_{i'}$ be the simple reflection corresponding to a root space $\widetilde{\mathrm{M}}^1_{(i', i'-1)}$. Then $s_{i'}$ is the permutation which interchanges rows i' and $i' - 1$ (and hence it also interchanges $-i'$, $-i' + 1$). Recall that we index rows and columns by $\mathrm{Ix}_i = \{d_i, \ldots, -d_i\} \cup \nabla_i$.

Let s_0 be the one corresponding to $\widetilde{\mathrm{M}}^1_{(2,-1)}$ in cases (B1), (C1) and (C3). Otherwise, we let s_0 correspond to $\widetilde{\mathrm{M}}^1_{(1,-1)}$. Then in the former case, s_0 interchanges rows 2 and -1 (hence also -2 and 1) and in the latter case, s_0 is the one interchanging rows 1 and -1. In a matrix form, $s_{i'}$ can be written as a monomial matrix with entries $0, \pm 1$.

Let s_{d_i} be the extended Weyl element in G_i', then it can be written in a matrix form as

$$\begin{pmatrix} 0 & 0 & \overline{\pi}_{F_i}^{-1} & \\ 0 & \mathrm{Id} & 0 & \\ \pi_{F_i} & 0 & 0 & \\ & & & \mathrm{Id}_{V_i^{\delta'}} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \overline{\pi}_{F_i}^{-1}\mathrm{Id}_2 & \\ 0 & \mathrm{Id} & 0 & \\ \pi_{F_i}\mathrm{Id}_2 & 0 & 0 & \\ & & & \mathrm{Id}_{V_i^{\delta'}} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -\pi_k^{-1} \\ 0 & \mathrm{Id} & 0 \\ \pi_k & 0 & 0 \end{pmatrix}$$

in cases (A), (B), (C3), (C4) — (C1) with $d_i > 2$, (C2) — (D)

where $^{-}$ denotes the Galois conjugation over the quadratic extension F_i/k_i. In case (C1) with $d_i = 2$, we have two extended Weyl elements

$$s_2 = \begin{pmatrix} & \pi^{-1} & & \\ \pi & & & \\ & & & \pi^{-1} \\ & & \pi & \end{pmatrix} \qquad s_2' = \begin{pmatrix} & & \pi^{-1} & \\ & & & \pi^{-1} \\ \pi & & & \\ & \pi & & \end{pmatrix}$$

In all above cases, $\widetilde{W}_i'^{0}$ is generated by $S_i = \{s_{i0}, s_{i1}, s_{i2}, s_{i2}'\}$ in case (C1) and by $S_i = \{s_{i0}, s_{i1}, \ldots, s_{id_i}\}$ otherwise.

7.2.2. Describing the action of $\widetilde{W}_i'^{0}$ on $\mathrm{M}^{\tau}_{i'j'}$, let $x \cdot \tau_{i'j'} \cdot y$ be an element of $\mathrm{M}^{\tau}_{i'j'}$ for $x, y \in F_i$ (see (2.2.3)). Let s be an element in S_i, then we have $\mathrm{Ad}s(x \cdot \tau_{i'j'} \cdot y) \in \mathrm{M}^{\tau}_{s(i'),s(j')}$ as follows;

$$\mathrm{Ad}s(x \cdot \tau_{i'j'} \cdot y) = \begin{cases} x \cdot \tau_{s(i'),s(j')} \cdot y & \text{if } s \in S \setminus \{s_{d_i}\}, \\ \pi_F^{\delta_{d_i i'} + \delta_{-d_i, j'} - \delta_{-d_i, i'} - \delta_{d_i j'}} x \cdot \tau_{s(i'),s(j')} \cdot y & \text{if } s = s_{d_i} \end{cases}$$

where $\delta_{i'j'}$ is the Kronecker's delta function and where $s(i')$, $s(j')$ denotes the permutation induced by s on $\{d_i, \ldots, -d_i\}$.

7.2.3. If $d_i = 1$, S_i can be found in the same way except for cases (B1), (C1)–(C3). In case (B1) with $d_i = 1$, we have

$$S_i = \left\{ s_{i0} = \begin{pmatrix} & \overline{\pi}_{F_i} \\ \pi_{F_i}^{-1} & \end{pmatrix}, \quad s_{i1} = \begin{pmatrix} & \overline{\pi}_{F_i}^{-1} \\ \pi_{F_i} & \end{pmatrix} \right\}$$

In case (C1) with $d_i = 1$, $\widetilde{W}_i' = \Omega_i$. For (C2)–(C3), $\widetilde{W}_i'$ is generated by s_0 switching rows 1 and -1.

7.2.4. Let l be the length function defined on $\widetilde{W}' = \prod \widetilde{W}_i'$: if $w_i \in \widetilde{W}_i'$, $l(w_i)$ is defined such that $[I_0'^{i} w_i I_0'^{i} : I_0'^{i}] = [I_0' w_i I_0' : I_0'] = q_F^{l(w_i)}$, and if $w_i \in \widetilde{W}_i'$, $w_j \in \widetilde{W}_j'$ with $i \neq j$, $l(w_i w_j) = l(w_i) + l(w_j)$. We observe that w can be written as $w = w_1 w_2 \cdots w_m$ with $w_i \in \widetilde{W}_i'$ and $l(w) = \sum_i l(w_i)$.

7.3. TAMELY RAMIFIED HECKE ALGEBRAS

Since we will build an isomorphism between $\mathcal{H} = \mathcal{H}(G//J_\Sigma, \rho_\Sigma)$ and the Hecke algebra $\mathcal{H}' = \mathcal{H}(G'//I'_0, \chi)$ of G' for some *tamely ramified character* χ of I'_0, in this section, we introduce such Hecke algebras and account the ones that we need.

DEFINITION 7.3.1 [G]. Let F_0 be a p-adic field, let G be the group of F_0-points of a reductive group defined over F_0 and let I_0 be a Iwahori subgroup of G. Then a character χ of I_0 is called a *tamely ramified character* if it is trivial on the maximal pro-p subgroup I_1 of I_0. We also call the Hecke algebra $\mathcal{H}(\mathsf{G}//I_0, \chi)$ associated to (I_0, χ) a tamely ramified Hecke algebra,

We describe these Hecke algebras for cases (A)–(D). This can be summarized on the *indexed affine Dynkin diagram*. For more details, we refer to [G, L, Mo] and for some examples of explicit computation, we refer to [My1, 2].

7.3.1. TAMELY RAMIFIED HECKE ALGEBRAS AND INDEXED AFFINE DYNKIN DIAGRAMS

7.3.2. Let F be a p-adic field with an involution σ and $F^\sigma = F_0$. Let (V, f) be one of types (A)–(D) with $\dim_F(V) = n = 2d + d_0$. Let $\mathsf{G} = \mathsf{G}(V, f)$ be the group of isometries on (V, f). Let T^s be a maximal F_0-split torus and let I_0 be an Iwahori subgroup. Let χ be a tamely ramified character of I_0, and let $\mathcal{H} = \mathcal{H}(\mathsf{G}//I_0, \chi)$ be the Hecke algebra associated to (I_0, χ). Assume first that $\mathrm{Supp}(\mathcal{H}) = \mathsf{G}$. Then since we have an Iwahori decomposition, we can rewrite $\mathrm{Supp}(\mathcal{H}) = \mathsf{G} = I_0 \widetilde{W} I_0$ where $\widetilde{W} = N_{\mathsf{G}}(\mathrm{T}^s)/(Z_{\mathsf{G}}(\mathrm{T}^s))_0$ is the affine Weyl group of G with the generating set $S = \{s_0, s_1, \ldots, s_d\}$ (see (7.2.1)). For $w \in \widetilde{W}$, let $\hat{e}_w$ be $\mathcal{H}(\mathsf{G}//I_0, \chi)$ with $\mathrm{Supp}(\hat{e}_w) = I_0 w I_0$ and $\hat{e}_w(w) = 1$. As a linear space, $\mathcal{H}$ is spanned by elements $\hat{e}_w$, $w \in \widetilde{W}$. We can normalize each $\hat{e}_w$ properly, say, $e_w = c_w \hat{e}_w$ for some constant c_w's so that $\mathcal{H}(\mathsf{G}//I_0, \chi)$ can be described as follows (see (7.3.4) for some explicit values of c_w); as an algebra, it is generated by $\{e_s \mid s \in S\}$ subject to the following three relations:

(L) $e_w \star e_{w'} = e_{ww'}$ if $l(ww') = l(w) + l(w')$,

(Q) For $s \in S$ and for some $\mathrm{wt}(s) \in \mathbb{Z}^+ \cup \{0\}$,

$$e_s \star e_s = q_{F_0}^{\mathrm{wt}(s)} e_1 + (q_{F_0}^{\mathrm{wt}(s)} - 1) e_s.$$

Here q_{F_0} denotes the cardinality of the residue field of F_0.

(B) $e_{s_i} \star e_{s_j} \star e_{s_i} \star \cdots = e_{s_j} \star e_{s_i} \star e_{s_j} \star \cdots$

where we have $m_{ij} = \mathrm{ord}(s_i s_j)$ factors on each side.

Note that (B) follows from (L). We can represent this Hecke algebra $\mathcal{H}(\mathsf{G}//I_0, \chi)$ on the affine Dynkin diagram by attaching $\mathrm{wt}(s)$ to each vertex corresponding to s.

We will call such an affine Dynkin diagram an *indexed affine Dynkin diagram*. For simplicity, we abbreviate it as IADD. The function $\mathrm{wt}\colon S \to \mathbb{Z}^+ \cup \{0\}$ above will be called a *weight*.

EXAMPLE 7.3.3. *Iwahori Hecke algebra* [IM].
This is a Hecke algebra associated to the trivial representation of I_0, which consists of I_0 bi-invariant functions. Then it is linearly spanned by functions e_w supported on $I_0 w I_0$ with $e_w(w) = 1$ for $w \in \widetilde{W}$. The elements e_w satisfy (L), (Q), (B) in (7.3.2) with $\mathrm{wt}(s_i) = \log_{q_{F_0}}(\mu(I_0 s_i I_0))$.

7.3.4. We continue to assume that $\mathrm{Supp}(\mathcal{H}) = \mathsf{G}$ as in (7.3.2) and follow the notation in (7.3.2). For each case (V, f) and G from (7.1.4), we will list possible IADD for $\mathcal{H}$:

Explanation.

(1) For each diagram, except for the case (C1) with $d_i = 2$, the cardinality of dots is $d+1$ and one of the left most dots will correspond to the extended affine root. In case (C1) with $d_i = 2$, we have two extended affine roots.

(2) The first row of indices right above dots represent the weight function wt_0 corresponding to the trivial character of I_0, hence it is associated to an Iwahori Hecke algebra for each case. The other rows correspond to nontrivial tamely ramified characters which have different IADD. We have put down only numbers which are different from the first row.

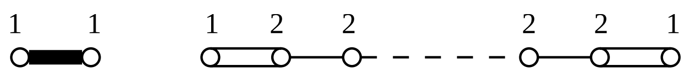
Notation. For later use, we denote the character corresponding to i-th row by χ_{i-1}. For example, χ_0 is the trivial character corresponding to the first row.

(3) (See an example in (A2) below) Let wt_0 be the weight function corresponding to the Iwahori Hecke algebra $\mathcal{H}(\mathsf{G}//I_0, 1)$. For each (I_0, χ), let $\hat{e}_w \in \mathcal{H}(\mathsf{G}//I_0, \chi)$ be as in (7.3.2) and let wt be its weight function. Then for $s \in S$, we can find normalization such that

$$e_s = \pm\left(q_{F_0}^{\mathrm{wt}_0(s)-\mathrm{wt}(s)}\right)^{-\frac{1}{2}}\hat{e}_s.$$

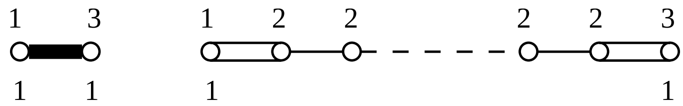
(A) f_V is ε-Hermitian with F/F_0 unramified,

(A1) $V_0 = 0$.



A thick line means that s_0 and s_1 have no relations, that is, $s_0 s_1$ has an infinite order.

(A2) $d_0 = 1$ and $V_0 = V^\delta \neq 0$.



Giving some examples of Hecke algebras with the weight function in the second row, let χ_1 be a character defined as follows:

$$\chi_1(t) = \chi_{11}(t_d \cdots t_2 t_1)\chi_{12}(t_\delta)$$

for $t = (t_d, \ldots, t_1, t_\delta, t_{-1}, \ldots, t_{-d}) \in I_0/I_1$ where χ_{11} is a character of $F^\times$ and χ_{12} is a character of $\ker(N_{F/F_0})$. To have $\mathrm{Supp}(\mathcal{H}) = \mathsf{G}$, it is necessary that $\chi_{11}(z) = \chi_{11}(\frac{1}{\bar{z}})$ (hence $\chi_{11}(z\bar{z}) = 1$). Now assume $\chi_{11}(z)\chi_{12}(\frac{\bar{z}}{z}) \neq 1$. For $w \in \widetilde{W}$, let $\hat{e}_w$ be as in (7.3.2). Then $\mathcal{H}(\mathsf{G}//I_0, \chi_1)$ is generated by $\{\hat{e}_{s_i} \mid i = 0, \ldots, d\}$ subject to the following relations:

(L) $\hat{e}_w \star \hat{e}_{w'} = \hat{e}_{ww'}$ if $l(ww') = l(w) + l(w')$,

(Q) $$\hat{e}_s \star \hat{e}_s = \begin{cases} q_{F_0}\hat{e}_1 + \chi_{11}(\sqrt{\zeta})(q_{F_0} - 1)\hat{e}_s & \text{if } s = s_d \\ q_{F_0}^2\hat{e}_1 + \chi_{11}(-1)(q_{F_0}^2 - 1)\hat{e}_s & \text{if } s = s_1, \ldots, s_{d-1} \\ q_{F_0}^3\hat{e}_1 + \chi_{11}(\sqrt{\zeta})q_{F_0}(q_{F_0} - 1)\hat{e}_s & \text{if } s = s_0. \end{cases}$$

Here q_{F_0} is the cardinality of the residue field of F_0 and ζ is nonsquare in $\mathbb{F}_{q_{F_0}}$.

(B) $\hat{e}_{s_i} \star \hat{e}_{s_j} \star \hat{e}_{s_i} \star \cdots = \hat{e}_{s_j} \star \hat{e}_{s_i} \star \hat{e}_{s_j} \star \cdots$
where we have $m_{ij} = \mathrm{ord}(s_i s_j)$ factors on each side.

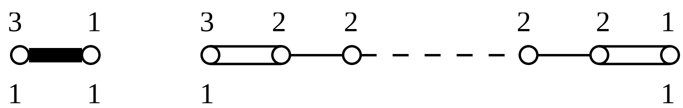
From the assumption that $\chi_{11}(z\bar{z}) = 1$, we see that $\chi_{11}(\sqrt{\zeta}) = +1$ or -1. If we put

$$e_s = \begin{cases} \frac{\chi_{11}(\sqrt{\zeta})}{q_{F_0}}\hat{e}_{s_0} & \text{if } s = s_0 \\ \chi_{11}(-1)\hat{e}_s & \text{if } s = s_1, \ldots, s_{d-1} \\ \chi_{11}(\sqrt{\zeta})\hat{e}_{s_d} & \text{if } s = s_d, \end{cases}$$

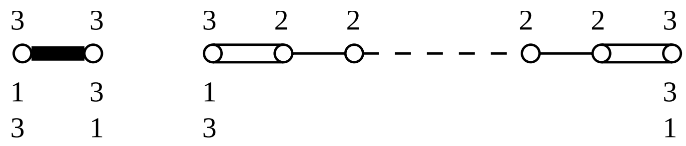
$$e_w = e_{s_{i_1}} \star e_{s_{i_2}} \star \cdots \star e_{s_{i_l}} \quad \text{for} \quad w = s_{i_1}s_{i_2}\cdots s_{i_l} \in \widetilde{W} \text{ with } l(w) = l,$$

above relations can be normalized as in (7.3.2) with the weight function in the second row of the above IADD.

(A3) $d_0 = 1$ and $V_0 = V^{\delta'} \neq 0$.

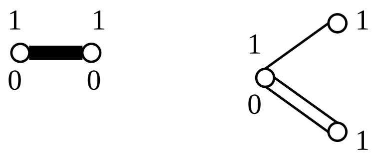


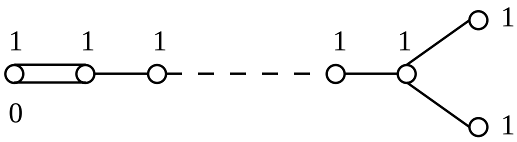
(A4) $V_0 \neq 0$ with $d_0 = 2$.



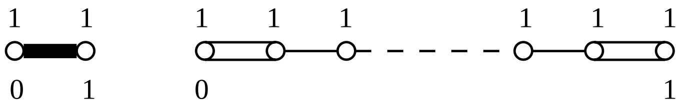
(B) f_V is ε-Hermitian with F/F_0 ramified,

(B1) $V_0 = 0$.

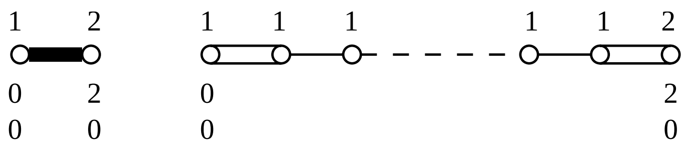




(B2) $V_0 \neq 0$ with $d_0 = 1$.

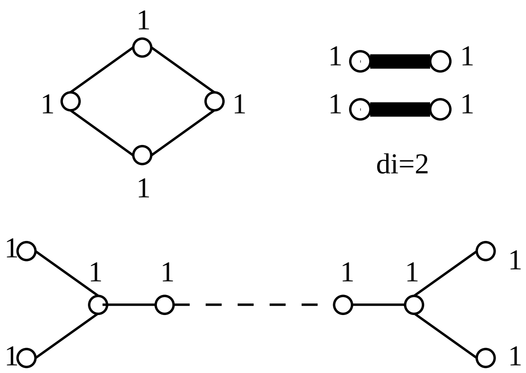


(B3) $V_0 \neq 0$ with $d_0 = 2$.

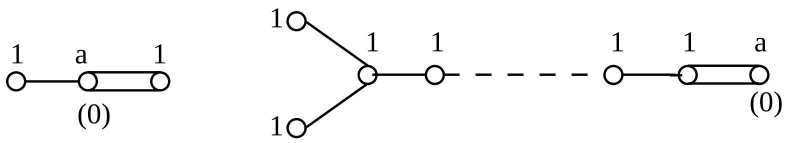


(C) f_V is $+1$-symmetric. Then $F = F_0$. Let $a = \dim(V^{\delta})$ and $b = \dim(V^{\delta'})$. In (C1)–(C3), we assume $d_i \geqslant 2$.

(C1) $V_0 = 0,\ d_i \geqslant 2.$

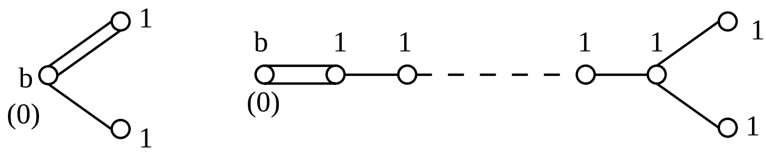


(C2) $V^{\delta} \neq 0$ and $V^{\delta'} = 0.$



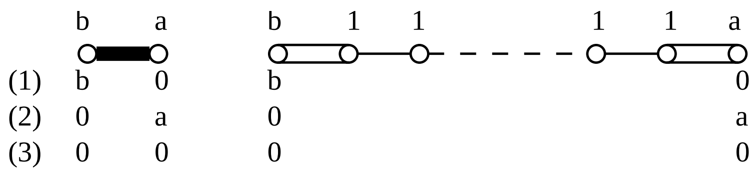
If $a = 2$, we can have the second row in parentheses associated to $\mathcal{H}(\mathsf{G}//I_0, \chi)$ for some χ.

(C3) $V^{\delta} = 0$ and $V^{\delta'} \neq 0.$



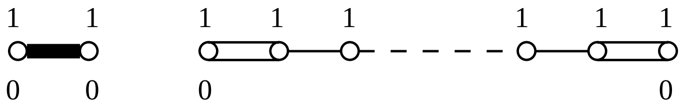
If $b = 2$, we can have the second row in parentheses.

(C4) $V^{\delta} \neq 0$ and $V^{\delta'} \neq 0.$

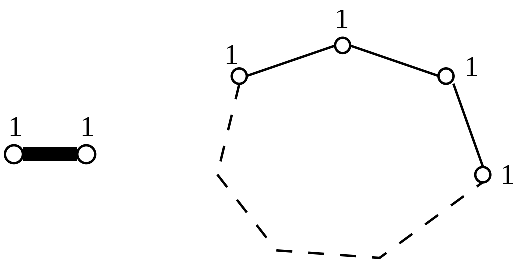


If $a = 2$, we can have row (1), if $b = 2$, we can have row (2) and if $a = b = 2$, we can have all (1)–(3).

(D) f_V is -1-symmetric, i.e., symplectic with $F = F_0$,



Similarly, we denote the Iwahori Hecke algebras of $\mathrm{GL}_n(F_0)$ as IADD;



7.3.5. Tamely ramified Hecke algebras: General Cases.
Let χ be a tamely ramified character of I_0 and let $\mathcal{H} = \mathcal{H}(\mathsf{G}//I_0, \chi)$ be its associated Hecke algebra. Then $\mathrm{Supp}(\mathcal{H}) = I_0 \widetilde{W}' I_0$ where $\widetilde{W}' < \widetilde{W}$ is an affine Weyl group of $\prod \mathsf{G}_i$ with G_i isomorphic to either $\mathrm{GL}_m(F)$ or a group of type (A)–(D) in (7.1.4). Write $\widetilde{W}' = \prod \widetilde{W}_i$ where $\widetilde{W}_i = \widetilde{W}_i^0 \rtimes \Omega_i$ is the affine Weyl group of G_i with its generating set $S_i = \{s_{ij}\}$. Then it is isomorphic to a tensor product of tamely ramified Hecke algebras of G_i's, that is, there are tamely ramified characters χ_i of I_0^i such that $\mathcal{H} \simeq \otimes \mathcal{H}_i$ where $\mathcal{H}_i = \mathcal{H}(\mathsf{G}_i//I_0^i, \chi_i)$. Hence we can represent $\mathcal{H}$ as a sum of IADD's corresponding to $\mathcal{H}_i$.

7.4. $\mathcal{H} = \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma)$

We fix a Haar measure μ on G (resp. μ' on G$'$) such that $\mu(J_\Sigma) = 1$ (resp. $\mu'(I_0') = 1$).

THEOREM 7.4.1. *For a given* $\Sigma = (\Gamma, I_0', 1)$ *as in* (H_Σ)*, suppose* $\mathrm{G}' = C_{\mathrm{G}}(\Gamma) = \prod_{i=1}^m \mathrm{U}_{m_i}(F_i/k_i)$ *for some tamely ramified extensions* F_i*,* k_i *over* k*. Then there is a tamely ramified character* χ *of* I_0' *such that there is a* $*$*-preserving, support-preserving* L^2*-isomorphism*

$$\eta\colon \mathcal{H}' = \mathcal{H}(\mathrm{G}'//I_0', \chi) \longrightarrow \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma) = \mathcal{H}.$$

defined as follows: For $w \in \widetilde{W}'$*, let* $\hat{e}_w \in \mathcal{H}'$ *with* $\hat{e}_w(w) = 1$ *and* $\mathrm{Supp}(\hat{e}_w) = I_0' w I_0'$*, and let* $f_w \in \mathcal{H}$ *with* $f_w(w) = 1$ *and* $\mathrm{Supp}(f_w) = J_\Sigma w J_\Sigma$*. Then*

$$\eta(\hat{e}_w) = \left(\frac{1}{C_w}\right)^{\frac{1}{2}} f_w \qquad \textit{with} \qquad C_w = \frac{\mu(J_\Sigma w J_\Sigma)}{\mu'(I_0' w I_0')}$$

where $\dot{f}_w \in \mathcal{H}$ is properly normalized with $\dot{f}_w = f_w$ or $-f_w$. Moreover, $\mathcal{H}$ is L^2-isomorphic to $\bigotimes \mathcal{H}(\mathrm{G}'_i//I'^i_0, \chi|I'^i_0)$ as an $\mathbb{C}$-algebra via a $*$-preserving, support-preserving map.

Proof of Theorem 7.4.1. From (4.2.6), we see η in Theorem 7.4.1 is a linear isomorphism. It can be proved similarly as in Theorem 6.2.2 that η is a $*$-preserving L^2-isomorphism. From the following Lemma, we see that η is support-preserving, that is, $\mathrm{Supp}(\eta(\hat{e}_w)) = J_\Sigma \mathrm{Supp}(\hat{e}_w) J_\Sigma$.

LEMMA 7.4.2. *For* $w \in \widetilde{W}'$, $(J_\Sigma w J_\Sigma) \cap \mathrm{G}' = I'_0 w I'_0$.

Lemma can be proved similarly as in (6.2.3) replacing T_0 with $(Z_{\mathrm{G}'}(\mathrm{T}^s))_0$ where $\mathrm{T}^s = \prod \mathrm{T}^s_i$.

Rest of this section is devoted to proving η is an algebra isomorphism. We find generators and relations in $\mathcal{H}$. Recall (see the proof of (7.1.7)) that when ρ_Σ is a character, we have

$$\widetilde{\mathrm{M}}^\tau_v\big(\beta^\tau_v\big) = \widetilde{\mathrm{M}}^\tau_v\big(\beta^{\tau+}_v\big) \quad \text{where} \quad \beta^\tau_v = \tfrac{1}{2}(-1 - \mathrm{ord}(\gamma^\tau_i - \gamma_j)).$$

For all cases (A)–(D) in (7.1.4), we have the following relations:

PROPOSITION 7.4.3.

(L) Length preserving relation. *If* $l(ww') = l(w) + l(w')$ *for* $w, w' \in \widetilde{W}'$,

$$f_w \star f_{w'} = \left(\frac{C_w C_{w'}}{C_{ww'}}\right)^{\frac{1}{2}} f_{ww'}.$$

In particular, if $w_i \in \widetilde{W}'_i$ *and* $w_j \in \widetilde{W}'_j$ *for* $i \neq j$, *we have* $f_{w_i} \star f_{w_j} = f_{w_j} \star f_{w_i}$.

(B) Braid relation. *If* $s_{ii'}s_{jj'}$ *is of order* m_v,

$$\tilde{f}_{s_{ii'}} \star \tilde{f}_{s_{jj'}} \star \cdots = \tilde{f}_{s_{jj'}} \star \tilde{f}_{s_{ii'}} \star \cdots$$

where each side has m_v factors and where $\tilde{f}_s = \left(\frac{1}{C_s}\right)^{\frac{1}{2}} f_s$. In particular, if $i \neq j$, $f_{s_{ii'}} \star f_{s_{jj'}} = f_{s_{jj'}} \star f_{s_{ii'}}$.

(Q) Quadratic relation. *Let* q_i *be the cardinality of the residue field of* σ_i*-fixed subfield* k_i *of* F_i. *Then*

$$f_{s_{ii'}} \star f_{s_{ii'}} = q_i^{\mathrm{v}_{ii'}} C'_{s_{ii'}} f_1 \pm C'^{\frac{1}{2}}_{s_{ii'}} (q_i^{\mathrm{v}_{ii'}} - 1) f_{s_{ii'}}$$

for some ${}_{ii'} \in \mathbb{Z}^+$ and for $C'_{s_{ii'}} = \mu(J_\Sigma s_{ii'} J_\Sigma)/q_i^{\mathrm{v}_{ii'}}$.

Proof. Note that (B) follows from (L). To prove (L), we first claim that $(J_\Sigma w J_\Sigma w' J_\Sigma) \cap \mathrm{G}' = (J_\Sigma ww' J_\Sigma) \cap \mathrm{G}'$. Consider first the case $w \in \widetilde{W}'_{i_0}$ and $w' \in \widetilde{W}'_{j_0}$

with $i_0 \neq j_0$. Regarding Γ as an element in $\mathfrak{gl}(V)$, we first construct a representation $\hat{\rho}_\Gamma$ on some open compact subgroup $\hat{J}$ in $\mathrm{GL}(V)$ as in [K1] such that $J_\Sigma = \mathrm{G} \cap \hat{J}$ and $\mathrm{Supp}(\mathcal{H}(\mathrm{GL}(V)//\hat{J}, \hat{\rho}_\Gamma)) = \hat{J} C_{\mathrm{GL}(V)}(\Gamma)\hat{J}$ as follows: Let $\hat{P}_0$ be the parahoric subgroup in $C_{\mathrm{GL}(V)}(\Gamma)$ associated to lattice chains in (1.5.2) and let $\hat{\mathcal{P}}_1$ be the lattice in the Lie algebra corresponding to the maximal pro-p subgroup $\hat{P}_1$ of $\hat{P}_0$. Define a lattice $\hat{\mathcal{J}}_p$ in $\mathrm{GL}(V)$ as

$$\hat{\mathcal{J}}_p = \hat{\mathcal{P}}_1 + \sum \mathrm{M}'^{\tau}_{v}(-1 - \mathrm{ord}(\gamma_i^\tau - \gamma_j) + \tfrac{1}{2} a_v)$$

where $\sum$ runs over $v = (i, j, i', j')$ with $i, j = 1, \ldots, m$, $i' \in \mathrm{Ix}_i$, $j' \in \mathrm{Ix}_j$ and $\tau \in \mathrm{Gal}_v$ (recall that Gal_v is defined in (2.1.1)). Then following the construction and proof in [K1], we can define $\hat{\rho}_\Gamma$ on $\hat{J} = \hat{P}_0 \cdot \exp(\hat{\mathcal{J}}_p)$ satisfying above property. Then (5.2.6)–(2) and similar argument in (6.3.2)–(6.3.4) will imply that $(\hat{J} w \hat{J} w' \hat{J}) \cap C_{\mathrm{GL}(V)} = (\hat{J} w w' \hat{J}) \cap C_{\mathrm{GL}(V)}$. Moreover, since $J_\Sigma \subset \hat{J}$ and $\mathrm{G}' \subset C_{\mathrm{GL}(V)}$, we have $(J_\Sigma w J_\Sigma w' J_\Sigma) \cap \mathrm{G}' = (J_\Sigma w w' J_\Sigma) \cap \mathrm{G}' = I_0' w w' I_0'$.

If w, $w' \in \widetilde{W}_i'$, it can be proved as in (6.3.2)–(6.3.4) with

$$\tilde{\mathcal{J}}_p = \sum_{\widetilde{\mathrm{M}}_v^\tau \in \Phi'_{i+}} \widetilde{\mathrm{M}}_v^\tau\left(\beta_v^\tau - \frac{1}{e_{F_i}}\right) + \sum_{\widetilde{\mathrm{M}}_v^\tau \in \Phi'_i \setminus \Phi'_{i+}} \widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau) + (\log(I_1') \cap \mathfrak{g}(V_i))$$

where $\beta_v^\tau = \frac{1}{2}(-1 - \mathrm{ord}(\gamma_i^\tau - \gamma_i))$, and where $\widetilde{\mathrm{M}}_v^\tau \in \Phi_i'$ means that $v \in \Upsilon$, $\tau \in \mathrm{Gal}_v^\sigma$ with $\widetilde{\mathrm{M}}_v^1 \in \Phi'_{i+}$. Note that β_v^τ is the same for any $v \in \Phi_i'$.

General cases will follow from combining the above two cases and the claim is proved.

Now the coefficient $(C_w C_{w'}/C_{ww'})^{\frac{1}{2}}$ can be computed as in (6.3.4).

To prove the quadratic relations (Q), we first find $\mathrm{Supp}(f_s \star f_s)$. The following Lemma can be proved exactly as in (6.3.6):

LEMMA 7.4.4. *For any* $s \in S_i$, $(J_\Sigma s J_\Sigma s J_\Sigma) \cap (J_\Sigma \mathrm{G}' J_\Sigma) = J_\Sigma \cup J_\Sigma s J_\Sigma$. *Moreover,* $f_s \star f_s = c_1 f_1 + c_2 f_s$ *for some constants* c_1 *and* c_2.

7.4.5. For each case (A)–(D), c_1 and c_2 in (7.4.4) can be found as in (q5)–(q6). That is,

$$\begin{aligned}
c_1 &= f_s \star f_s(1) = \int_{\mathrm{G}} f_s(x) f_s(x^{-1})\, \mathrm{d}x \\
&= \int_{J_\Sigma s J_\Sigma} f_s(x) f_s(x^{-1})\, \mathrm{d}x = \mu(J_\Sigma s J_\Sigma) \\
c_2 &= f_s \star f_s(s) = \int_{\mathrm{G}} f_s(x) f_s(x^{-1} s)\, \mathrm{d}x \\
&= \sum_{j_\Sigma \in \lambda_s} \int_{J_\Sigma} f_s(j_\Sigma s x) f_s(x^{-1} s^{-1} j_\Sigma^{-1} s)\, \mathrm{d}x
\end{aligned}$$

where $\lambda_s = J_\Sigma/(J_\Sigma \cap sJ_\Sigma s) = (J_\Sigma \cap sJ_\Sigma s)\backslash J_\Sigma$. Now we will compute c_1 and c_2 more explicitly.

7.4.6. Quadratic relation for $s_{ii'}$, $i' \neq d_i$
Assume G'_i is not of type (B1) with $d_i = 1$. Let q_i be the cardinality of the residue field of the σ_i-fixed field $F_i^{\sigma_i}$. Then for $s_{ii'} \in S_i$ with $i' = 1, \ldots d_i - 1$, we have

$$f_{s_{ii'}} \star f_{s_{ii'}} = q_i^{\mathrm{V}_{ii'}} f_1 + \varepsilon_{ii'}(q_i^{\mathrm{V}_{ii'}} - 1) f_{s_{ii'}}$$

where $\mathrm{V}_{ii'}$ coincides with the weight of $s_{ii'}$ for the Iwahori Hecke algebras (we refer to those numbers in (7.3.4)) and $\varepsilon_{ii'} = +1$ or -1. In fact, $\varepsilon_{ii'} = \tilde{\chi}^i_\Gamma(\det(s_{ii'}))$ where $\tilde{\chi}^i_\Gamma$ is the character of $\mathcal{O}^\times_{F_i}$ such that $\rho_\Sigma|I_0'^{\,i} = \tilde{\chi}^i_\Gamma \circ \det$.

Let $s_{ii'} \in S_i$ with $i' \neq d_i$. Denote the image of $v \in \Upsilon$ under the action of $w \in \widetilde{W}'$ by $w(v)$. Since we have $\beta_v = \beta_{s_{ii'}(v)}$ and $a_v = a_{s_{ii'}(v)}$,

$$\begin{aligned}
\mathrm{Ad}(s_{ii'})(\widetilde{\mathrm{M}}^\tau_v \cap \mathcal{Y}_\Gamma) &= \mathrm{Ad}(s_{ii'})\Big(\widetilde{\mathrm{M}}^\tau_v\big(\beta^\tau_v\big)\Big) \\
&= \widetilde{\mathrm{M}}^\tau_{s_{ii'}(v)}\big(\beta^\tau_v\big) = \widetilde{\mathrm{M}}^\tau_{s_{ii'}(v)}\big(\beta^\tau_{s_{ii'}(v)}\big) \\
&= \widetilde{\mathrm{M}}^\tau_{s_{ii'}(v)} \cap \mathcal{Y}_\Gamma
\end{aligned}$$

and thus $\mathrm{Ad}(s_{ii'})(\mathcal{Y}_\Gamma) = \mathcal{Y}_\Gamma$. Hence the computation occurs in G' and the relation (Q) is inherited from the quadratic relation in a tamely ramified Hecke algebra $\mathcal{H}(\mathrm{G}'//I'_0, \chi^\circ_\Gamma)$, where χ°_Γ is as in S2-(1) or (3.4.2), from which (7.4.6) follows. For example, for $s_{ii'}$ with $i' \neq 0$, d_i, we have $\varepsilon_{ii'} = 1$ and

$$f_{s_{ii'}} \star f_{s_{ii'}} = \begin{cases} q_i f_1 + (q_i - 1) f_{s_{ii'}} & \text{if } F_i/F_i^{\sigma_i} \text{ is ramified or } \sigma_i = 1, \\ q_i^2 f_1 + (q_i^2 - 1) f_{s_{ii'}} & \text{if } F_i/F_i^{\sigma_i} \text{ is quadratic unramified.} \end{cases}$$

Now let $i' = d_i$. We first consider the case (A2). Then we have the following quadratic relations:

7.4.7. Quadratic relations for s_{id_i} in case (A2)

$$f_{s_{id_i}} \star f_{s_{id_i}} = q_i C_{s_{id_i}} f_1 \pm C^{\frac{1}{2}}_{s_{id_i}} (q_i - 1) f_{s_{id_i}}$$

Denote s_{id_i} by s for simplification. Let

$$\beta_v^\tau = \tfrac{1}{2}(-1 - \operatorname{ord}(\gamma_i^\tau - \gamma_j))$$

$$v_0 = (i, i, -d_i, d_i) \in \Upsilon$$

$$\iota_s = \left\{ (v,\tau) \;\middle|\; \begin{array}{c} v = (j, i, j', d_i) \in \Upsilon \text{ with } i \neq j, \\ \text{or } v = (i, i, i', d_i) \text{ with } i' \neq d_i, -d_i \\ \tau \in \mathrm{Gal}_v^\sigma, \quad \mathrm{Ad}(s)(\widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau)) \underset{+}{\subset} \mathcal{Y}_\Gamma \end{array} \right\}$$

$$\iota_s' = \left\{ (v_0, \tau) \quad | \quad 1 \neq \tau \in \mathrm{Gal}_{v_0}^\sigma \right\} \tag{q1}$$

$$\lambda_s = J_\Sigma/(J_\Sigma \cap sJ_\Sigma s) = (J_\Sigma \cap sJ_\Sigma s)\backslash J_\Sigma$$

$$= \frac{\mathrm{N}_{v_0}^1\left(\dfrac{1}{e_{F_i}}\right)}{\mathrm{N}_{v_0}^1\left(\dfrac{2}{e_{F_i}}\right)} \cdot \frac{\exp\left(\displaystyle\sum_{(v_0,\tau)\in\iota_s'} \widetilde{\mathrm{M}}_{v_0}^\tau(\beta_{v_0}^\tau) + \sum_{(v,\tau)\in\iota_s} \widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau)\right)}{\exp\left(\displaystyle\sum_{(v_0,\tau)\in\iota_s'} \widetilde{\mathrm{M}}_{v_0}^\tau\left(\beta_{v_0}^\tau + \frac{2}{e_{F_i}}\right) + \sum_{(v,\tau)\in\iota_s} \widetilde{\mathrm{M}}_v^\tau\left(\beta_v^\tau + \frac{1}{e_{F_i}}\right)\right)}$$

Note that for $(v,\tau)\notin\iota_s \cup \iota_s'$, $\mathrm{Ad}(s)\Big(\widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau)\Big) \subset Y_\Gamma$. Continuing from (7.4.5),

$$c_1 = \sharp(\lambda_s), \qquad c_2 = \sum_{j_\Sigma\in\lambda_s} f_s(s^{-1}j_\Sigma^{-1}s) = \sum_{j_\Sigma\in\lambda_s} f_s(s^{-1}j_\Sigma s). \tag{q2}$$

Since $f_s(s^{-1}j_\Sigma s) \neq 0$ if and only if $s^{-1}j_\Sigma s \in J_\Sigma s J_\Sigma$, we have

$$c_2 = \sum_{\substack{(j_\Sigma, j_\Sigma')\in\lambda_s\times\lambda_s \\ sj_\Sigma s^{-1}j_\Sigma' s\in J_\Sigma}} \rho_\Sigma(sj_\Sigma s^{-1}j_\Sigma' s). \tag{q3}$$

We can write $j_\Sigma, j_\Sigma' \in \lambda_s$ as

$$
\begin{aligned}
j_\Sigma &= n^1_{v_0}(Z) \cdot \exp\left(\sum_{v \in \iota_S \cup \iota'_S} \left(a^{(\tau)}_v - \sigma\left(a^{(\tau)}_v \right) \right) \right), \\
j'_\Sigma &= n^1_{v_0}(Z') \cdot \exp\left(\sum_{v \in \iota_S \cup \iota'_S} \left(b^{(\tau)}_v - \sigma\left(b^{(\tau)}_v \right) \right) \right)
\end{aligned}
\tag{q4}
$$

where $n^1_{v_0}(Z) = \exp(Z)$, $n^1_{v_0}(Z') = \exp(Z')$ with $Z, Z' \in \ker(\mathrm{Tr}_{F_i/k_i}) \cap \mathfrak{p}_{F_i} \subset \tilde{F}^{(1)}_{v_0} = F_i \subset \mathrm{Hom}_k(F_i^{-d_i}, F_i^{d_i})$ and $a^{(\tau)}_v$, $b^{(\tau)}_v \in \mathrm{M}^\tau_v(\beta^\tau_v)$. Note that $a^{(\tau)}_v - \sigma(a^{(\tau)}_v)$, $b^{(\tau)}_v - \sigma(b^{(\tau)}_v) \in \widetilde{\mathrm{M}}^\tau_v(\beta^\tau_v)$. In a matrix form, j_Σ can be written as

$$
j_\Sigma = \begin{pmatrix}
\ddots & & & \vdots & & & & & & & & \\
& 1 & & \sum a^{(\tau)}_v & & & & & & & & \\
& & \ddots & \vdots & & & & & & & & \\
& & & 1 & 0 & \cdots & & \cdots & 0 & & & \\
& & & \vdots & 1 & & & & \vdots & & & \\
& & & \sum a^{(\tau)}_v & & & 1 & & \vdots & & & \\
& & & \vdots & & & & \ddots & \vdots & & & \\
\cdots & -\sum \sigma(a^{(\tau)}_v) & \cdots & \begin{matrix} Z + \sum a^{(\tau)}_{v_0} \\ -\frac{1}{2}\sigma(U)U \end{matrix} & \cdots & & -\sum \sigma(a^{(\tau)}_v) & \cdots & 1 & \cdots & -\sum \sigma(a^{(\tau)}_v) & \cdots \\
& & & \vdots & & & & & & \ddots & & \\
& & & \sum a^{(\tau)}_v & & & & & & & 1 & \\
& & & \vdots & & & & & & & & \ddots
\end{pmatrix}
\tag{q5}
$$

and j'_Σ can be written in the same way. For simplicity of notation, let

$$
\varpi = \pi_{F_i}, \quad Z = Z_0 \varpi, \quad Z' = Z'_0 \varpi, \quad \epsilon = \frac{\varpi}{\overline{\varpi}}.
$$

To find the condition on j_Σ, j'_Σ such that $s j_\Sigma s^{-1} j'_\Sigma s \in J_\Sigma s J_\Sigma$, we compute $s j_\Sigma s^{-1} j'_\Sigma s$ rather explicitly in a matrix form. Again, we will write down only terms of nontrivial contribution for computation;

$sj_\Sigma s^{-1} j'_\Sigma s \overset{\text{mod}}{\equiv}$

$$\begin{pmatrix}
\ddots & \vdots & & & & \vdots & \\
\cdots & \begin{matrix} Z_0\varepsilon + \\ \frac{1}{\varpi}\sum_\tau a^{(\tau)}_{v_0} \\ -\frac{1}{2}\frac{1}{\varpi}\sigma(U)U \end{matrix} & \cdots & \begin{matrix} -\sum_\tau \frac{1}{\varpi}(\sigma(a^{(\tau)}_v)+ \\ Z_0\sigma(b^{(\tau)}_v))\tau \end{matrix} & \cdots & \begin{matrix} \frac{1}{\varpi}(1+Z_0Z'_0\varepsilon)+ \\ \sum_\tau \frac{1}{\varpi}(Z_0 b^{(\tau)}_{v_0}\frac{1}{\varpi}+ \\ a^{(\tau)}_{v_0}\frac{Z'_0}{\varpi}) \end{matrix} & \cdots \\
& \vdots & \ddots & * & * & \vdots & \\
& \sum_\tau a^{(\tau)}_v & * & \begin{matrix} 1-\left(\sum a^{(\tau)}_v \frac{1}{\varpi}\right) \\ \cdot\left(\sum \sigma(b^{(\tau)}_{\hat{v}})\right) \end{matrix} & * & \begin{matrix} \sum_\tau(b^{(\tau)}_v + \\ a^{(\tau)}_v Z'_0)\frac{1}{\varpi} \end{matrix} & \\
& \vdots & * & * & \ddots & \vdots & \\
\cdots & \varpi & \cdots & -\sum_\tau \sigma(b^{(\tau)}_v) & \cdots & \begin{matrix} Z'_0\varepsilon + \\ \left(\sum_\tau b^{(\tau)}_{v_0}\frac{1}{\varpi}\right) \\ -\frac{1}{2}\sigma(U')U'\frac{1}{\varpi} \end{matrix} & \cdots \\
& \vdots & & & & \vdots & \ddots
\end{pmatrix} \tag{q6}$$

Here, for $v = (\alpha, \beta, \alpha', \beta')$, define $\hat{v}$ as follows;

$$\hat{v} = \begin{cases} (\alpha, \beta, -\alpha', \beta') & \text{if } \alpha' \notin \nabla_i \\ v & \text{if } \alpha' \in \nabla_i. \end{cases},$$

Recall ∇_j is defined in (2.1.1)–(1). Then (j, j, j', j')-diagonal component for $j' > 0$ or $j' \in \nabla_j$ is given by

$$d_{jj'} = 1 - \left(\sum_\tau a^{(\tau)}_{v_{jj'}} \frac{1}{\varpi}\right)\left(\sum_\tau \sigma\left(b^{(\tau)}_{\hat{v}_{jj'}}\right)\right) \quad \text{where} \quad v_{jj'} = (j, i, j', d_i). \tag{q7}$$

From $(q6)$, $sj_\Sigma s^{-1} j'_\Sigma s \in J_\Sigma$ if and only if

$$\begin{aligned} &1 + Z_0 Z'_0 \varepsilon \equiv 0, \quad Z_0 b^{(\tau)}_{v_0} + a^{(\tau)}_{v_0}\varepsilon Z'_0 \equiv 0, \quad a^{(\tau)}_{v_0} \in \widetilde{\mathrm{M}}^\tau_{v_0}\left(\beta^\tau_{v_0} + \frac{1}{e_{F_i}}\right), \\ &\sigma(a^{(\tau)}_v) + Z_0\sigma(b^{(\tau)}_v) \equiv 0, \qquad b^{(\tau)}_v + a^{(\tau)}_v Z'_0 \equiv 0, \qquad \text{for } v \in \imath_s. \end{aligned} \tag{q8}$$

Hence j'_Σ is determined by j_Σ, that is,

$$\begin{aligned} &Z'_0 \equiv -\frac{1}{Z_0}\varepsilon, \qquad b^{(\tau)}_{v_0} \equiv \frac{1}{Z_0} a^{(\tau)}_{v_0} \frac{1}{Z_0} \in \widetilde{\mathrm{M}}^{\tau}_{v_0}\left(\beta^{\tau}_{v_0} + \frac{1}{e_{F_i}}\right), \\ &b^{(\tau)}_{v} \equiv a^{(\tau)}_{v} \frac{\varepsilon}{Z_0} \qquad \text{for } v \in \iota_s. \end{aligned} \tag{q9}$$

In these cases, if we let D_{Z_0} be an element $(1, \ldots, 1, Z_0\varepsilon, 1, \ldots, 1, -\frac{1}{Z_0}\varepsilon, 1, \ldots, 1)$ in the torus T and let $\widetilde{\chi}^i_\Gamma$ be as in (7.4.6), we can compute

$$\begin{aligned} \rho_\Sigma(sj_\Sigma s^{-1} j'_\Sigma s) &= \chi^\circ_\Gamma(D_{Z_0})\chi_\Gamma(D^{-1}_{Z_0} s j_\Sigma s^{-1} j'_\Sigma s) \\ &= \widetilde{\chi}^i_\Gamma(-1)\cdot\theta\left(\mathrm{Tr}_{k/k_0}\circ\mathrm{Tr}\left(\gamma_i\cdot\left(\sum_\tau \frac{1}{Z_0\varpi} a^{(\tau)}_{v_0} + \sum_\tau \frac{1}{Z'_0\varepsilon} b^{(\tau)}_{v_0}\frac{1}{\varpi}\right)\right)\right) \\ &\quad\cdot \prod_{\substack{j,j' \\ j'\in\{1,\ldots,d_j\}\cup\nabla_j, \\ j'\neq d_i}} \theta\left(\mathrm{Tr}_{k/k_0}\circ\mathrm{Tr}\left(\varepsilon_{j'}\gamma_j\cdot\left(\sum_\tau a^{(\tau)}_{v_{jj'}}\frac{1}{\varpi}\right)\cdot\left(-\sum_{\tau'}\sigma\left(b^{(\tau')}_{\hat{v}_{jj'}}\right)\right)\right)\right) \\ &= \widetilde{\chi}^i_\Gamma(-1) \prod_{\substack{j,j' \\ j'\in\{1,\ldots,d_j\}\cup\nabla_j, \\ j'\neq d_i}} \theta\left(\mathrm{Tr}_{k/k_0}\circ\mathrm{Tr}\left(\varepsilon_{j'}\gamma_j\cdot\left(-\sum_\tau a^{(\tau)}_{v_{jj'}}\frac{1}{\varpi}\sigma\left(b^{(\tau^{-1})}_{\hat{v}_{jj'}}\right)\right)\right)\right) \end{aligned} \tag{q10}$$

where

$$\varepsilon_{j'} = \begin{cases} 1 & \text{if } j' \in \nabla_j \\ 2 & \text{otherwise.} \end{cases}$$

Now $(q3)$ becomes

$$\begin{aligned} c_2 &= \widetilde{\chi}^i_\Gamma(-1)\cdot\sum \prod_{\substack{j,j' \\ j'\in\{1,\ldots,d_j\}\cup\nabla_j, \\ j'\neq d_i}} \theta\left(\mathrm{Tr}_{k/k_0}\circ\mathrm{Tr}\left(\varepsilon_{j'}\gamma_j\cdot\left(-\sum_\tau a^{(\tau)}_{v_{jj'}}\frac{1}{\overline{\varpi}}\sigma\left(b^{(\tau^{-1})}_{\hat{v}_{jj'}}\right)\right)\right)\right) \\ &= \widetilde{\chi}^i_\Gamma(-1)\cdot\sharp\left(\prod_{(v_0,\tau)\in\iota'_s}(\lambda_s)^\tau_{v_0}\right)^{\frac{1}{2}} \\ &\quad\cdot\sum \prod_{\substack{j,j' \\ j'\in\{1,\ldots,d_j\}\cup\nabla_j, \\ j'\neq d_i}} \theta\left(\mathrm{Tr}_{k/k_0}\circ\mathrm{Tr}\left(\varepsilon_{j'}\gamma_j\cdot\left(-\sum_\tau a^{(\tau)}_{v_{jj'}}\frac{1}{\overline{\varpi}}\sigma\left(a^{(\tau^{-1})}_{\hat{v}_{jj'}}\frac{\varepsilon}{Z_0}\right)\right)\right)\right) \end{aligned} \tag{q11}$$

where the first $\sum$ runs over (j_Σ, j'_Σ) satisfying $(q7)$, $(q8)$ and the second $\sum$ runs over $a^{(\tau)}_{v_{jj'}}$ and Z_0 satisfying $(q7)$ and where $(\lambda_s)^\tau_{v_0} = \mathrm{N}^\tau_{v_0}(\beta^\tau_{v_0})\big/\mathrm{N}^\tau_{v_0}(\beta^\tau_{v_0} + \frac{2}{e_{F_i}})$. Let

$(\lambda_s)_v^\tau = \mathrm{N}_v^\tau(\beta_v^\tau)\Big/\mathrm{N}_v^\tau(\beta_v^\tau + \frac{1}{e_{F_i}})$. Then for $j' \notin \nabla_j$, since $v_{jj'} \neq \hat{v}_{jj'}$, we have

$$\sum_{a_{v_{jj'}}, a_{\hat{v}_{jj'}}} \theta\left(\mathrm{Tr}_{k/k_0} \circ \mathrm{Tr}\left(2\gamma_j \cdot \left(-\sum_\tau a_{v_{jj'}}^{(\tau)} \frac{1}{\varpi} \sigma\left(a_{\hat{v}_{jj'}}^{(\tau^{-1})} \frac{\varepsilon}{Z_0}\right)\right)\right)\right) = \left(\sharp\left((\lambda_s)_{v_{jj'}}^\tau (\lambda_s)_{\hat{v}_{jj'}}^\tau\right)\right)^{\frac{1}{2}}. \tag{q12}$$

For the case $j' \in \nabla_j$, we need the following results on Gaussian sums over finite fields:

LEMMA 7.4.8 [S, IR]. *Let ψ be a character of $\mathbb{F}_q$. Let $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{F}_q$ and let $Q(v)$ be a nondegenerate quadratic form on $\mathbb{V}$ over $\mathbb{F}_q$. Let $G(\psi, Q) = \sum_{v \in \mathbb{V}} \psi(Q(v))$. Then*

(1) $G(\psi, Q)^2 = q^n = \sharp(\mathbb{V})$, *hence* $G(\psi, Q) = \pm(\sqrt{q})^n = \pm\sharp(\mathbb{V})^{\frac{1}{2}}$,
(2) $G(\psi, xQ) = \mathrm{sgn}(x)^n G(\psi, Q)$

where sgn *is the unique nontrivial quadratic character of $\mathbb{F}_q$.*

Let $j' \in \nabla_j$, then we have $v_{jj'} = \hat{v}_{jj'} \in \Upsilon$. Then computing rest of factors,

$$\begin{aligned} &\sum_{a_{v_{jj'}}, a_{\hat{v}_{jj'}}} \prod_{\substack{(v_{jj'}, \tau) \in \iota_s \\ j' \in \nabla_j}} \theta\left(\mathrm{Tr}_{k/k_0} \circ \mathrm{Tr}\left(\gamma_j \cdot \left(-\sum_\tau a_{v_{jj'}}^{(\tau)} \frac{1}{\varpi} \sigma\left(a_{\hat{v}_{jj'}}^{(\tau^{-1})} \frac{\varepsilon}{Z_0}\right)\right)\right)\right) \\ &\quad = \sum_{a_{v_{jj'}}} \prod_{\substack{(v_{jj'}, \tau) \in \iota_s \\ j' \in \nabla_j}} \theta\left(\mathrm{Tr}_{k/k_0} \circ \mathrm{Tr}\left(\sum_\tau \frac{1}{Z_0} \sigma\left(a_{v_{jj'}}^{(\tau)}\right) \gamma_j a_{v_{jj'}}^{(\tau)} \frac{1}{\varpi}\right)\right). \end{aligned} \tag{q13}$$

We first note that $(\lambda_s)_{v_{jj'}}^\tau$ is a right $\mathbb{F}_{q_{F_i}}$-vector space. Let $l_{jj'}$ be its dimension over $\mathbb{F}_{q_{F_j}}$. From (2.2.5)–(2), (2.2.4)–(5) and (2.2.7), we can define a nondegenerate quadratic form $Q_{jj'}$ on $(\lambda_s)_{v_{jj'}} = \sum_{(v_{jj'}, \tau) \in \iota_s} (\lambda_s)_{v_{jj'}}^\tau$ as follows;

$$\begin{aligned} Q_{jj'}: (\lambda_s)_{v_{jj'}} = \sum_{(v_{jj'}, \tau) \in \iota_s} (\lambda_s)_{v_{jj'}}^\tau &\longrightarrow \mathbb{F}_{q_{F_i}} \cap \mathrm{End}(\mathbb{F}_{q_{F_i}}) \\ \sum_\tau a_{v_{jj'}}^\tau &\longrightarrow \sum_\tau \left(\sigma\left(a_{v_{jj'}}^{(\tau)}\right) \gamma_j a_{v_{jj'}}^{(\tau)} \frac{1}{\varpi}\right)_{F_i} \in \mathbb{F}_{q_{F_i}} \end{aligned} \tag{q14}$$

where $\left(\sigma\left(a_{v_{jj'}}^{(\tau)}\right) \gamma_j a_{v_{jj'}}^{(\tau)} \frac{1}{\varpi}\right)_{F_i}$ is the projection on F_i (see [K1; (0.2)]) regarding

$\sigma\left(a_{v_{jj'}}^{(\tau)}\right)\gamma_j a_{v_{jj'}}^{(\tau)}\frac{1}{\varpi}$ as an element of $\tilde{F}_{v_{jj'}}^{(\tau)}$. Then by the above Lemma 7.4.8,

$$\begin{aligned}(q13) &= \sum_{Z_0\in\mathbb{F}_{q_i}^{\times}} \prod_{\substack{(v_{jj'},\tau)\in\iota_s\\ j'\in\nabla_j}} G\left(\widetilde{\chi}_{\Gamma}, \frac{1}{Z_0}Q_{jj'}\right)\\ &= \sum_{Z_0\in\mathbb{F}_{q_i}^{\times}} \prod_{\substack{(v_{jj'},\tau)\in\iota_s\\ j'\in\nabla_j}} \mathrm{sgn}(Z_0)^{l_{jj'}} G(\widetilde{\chi}_{\Gamma}, Q_{jj'})\\ &= \sum_{Z_0\in\mathbb{F}_{q_i}^{\times}} \mathrm{sgn}(Z_0)^{\sum l_{jj'}} \prod_{\substack{(v_{jj'},\tau)\in\iota_s\\ j'\in\nabla_j}} G(\widetilde{\chi}_{\Gamma}, Q_{jj'})\end{aligned} \tag{q15}$$

where $\widetilde{\chi}_{\Gamma}$ is the character of $\mathbb{F}_{q_i}^{\times}$ induced from χ_{Γ}. Since $\mathrm{sgn}(Z_0)=1$ for any $Z_0\in\mathbb{F}_{q_i}^{\times}$,

$$(q13) = \widetilde{\chi}_{\Gamma}^{i}(-1)(q_i-1)\prod_{\substack{(v_{jj'},\tau)\in\iota_s\\ j'\in\nabla_j}} G(\widetilde{\chi}_{\Gamma}, Q_{jj'}) = \pm\widetilde{\chi}_{\Gamma}^{i}(-1)(q_i-1)\prod_{\substack{(v_{jj'},\tau)\in\iota_s\\ j'\in\nabla_j}}\left(\sharp(\lambda_s)_{v_{jj'}}\right)^{\frac{1}{2}}. \tag{q16}$$

Here, last equality follows from $G(\widetilde{\chi}_{\Gamma}, Q_{jj'}) = \pm\left(\sharp(\lambda_s)_{v_{jj'}}\right)^{\frac{1}{2}}$. Since $\widetilde{\chi}_{\Gamma}^{i}(-1)$ is also ± 1, combining all together,

$$c_2 = \pm(q_i-1)\prod\left(\sharp(\lambda_s)_v^{\tau}\right)^{\frac{1}{2}} = \pm(q_i-1)\cdot\sharp(\lambda_s)^{\frac{1}{2}} = \pm C_{s_{id_i}}^{\frac{1}{2}}(q_i-1). \tag{q17}$$

Hence we have (7.4.7) for $i'=d_i$. □

7.4.9. *Quadratic relations for* (A)

(A1) $f_{s_{id_i}}\star f_{s_{id_i}} = q_i C_{s_{id_i}} f_1 \pm C_{s_{id_i}}^{\frac{1}{2}}(q_i-1) f_{s_{id_i}}$

(A3)–(A4) *These cases happen only when* G *and* G_i' *are related as in (7.1.7)–(2).* $f_{s_{id_i}}\star f_{s_{id_i}} = q_i^3 C_{s_{id_i}} f_1 \pm C_{s_{id_i}}^{\frac{1}{2}}(q_i^3-1) f_{s_{id_i}}$

In case (A1), we can compute c_1, c_2 in (7.4.5) exactly in the same way as in (A2) by putting $Y=Y'=0$. Now, since (A3) and (A4) are similar, we will consider only the case (A3). Let $s=s_{id_i}$. Note that we have $F_i=k$, $k_i=k_0$, $e_{F_i}=1$ in this case.

Moreover, since k is unramified over k_0, we may choose $\pi = \pi_k = \pi_{k_0}$.

$$\begin{aligned}
&\beta_v^\tau = \frac{1}{2}(-1 - \mathrm{ord}(\gamma_i^\tau - \gamma_j))\\
&v_{\delta'} = (i, i, -d_i, \delta'), \quad v_0 = (i, i, -d_i, d_i) \in \Upsilon\\
&\iota_s = \left\{ (v, \tau) \;\middle|\; \begin{array}{l} (j, i, j', d_i) \in \Upsilon \text{ with } i \neq j, \\ \text{or } v = (i, i, i', d_i) \text{ with } i' \neq d_i, -d_i \\ \tau \in \mathrm{Gal}_v^\sigma, \quad \mathrm{Ad}(s)(\widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau)) \underset{+}{\subset} \mathcal{Y}_\Gamma \end{array} \right\} \qquad \text{(q18)}\\
&\lambda_s = J_\Sigma/(J_\Sigma \cap sJ_\Sigma s) = (J_\Sigma \cap sJ_\Sigma s)\backslash J_\Sigma\\
&= \frac{\mathrm{N}_{v_0}^1(1)}{\mathrm{N}_{v_0}^1(2)} \cdot \frac{\mathrm{N}_{v_{\delta'}}^1(0)}{\mathrm{N}_{v_{\delta'}}(1)} \cdot \frac{\exp\Big(\sum_{(v,\tau)\in\iota_s} \widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau)\Big)}{\exp\Big(\sum_{(v,\tau)\in\iota_s} \widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau + 1)\Big)}
\end{aligned}$$

Then we put

$$\begin{aligned}
j_\Sigma &= n_{v_{\delta'}}^1(Y) \cdot n_{v_0}^1(Z_0\pi_k) \cdot \exp\left(\sum_{(v,\tau)\in\iota_s} \Big(a_v^{(\tau)} - \sigma\Big(a_v^{(\tau)}\Big)\Big)\right),\\
j'_\Sigma &= n_{v_{\delta'}}^1(Y') \cdot n_{v_0}^1\big(Z'_0\pi_k\big) \cdot \exp\left(\sum_{(v,\tau)\in\iota_s} \Big(b_v^{(\tau)} - \sigma\Big(b_v^{(\tau)}\Big)\Big)\right),
\end{aligned} \qquad \text{(q19)}$$

where $n_{v_{\delta'}}^1(Y) = \exp(Y)$ and $n_{v_0}^1(Z_0\pi_k) = \exp(Z_0\pi_k)$ with $Y \in \mathcal{O}_{F_i} \subset \tilde{F}_{v_0}^{(1)} = F_i \subset \mathrm{Hom}_k(F_i^{-d_i}, F_i^{\delta'})$ and $Z_0 \in \ker(\mathrm{Tr}_{F_i/k_i}) \cap \mathcal{O}_{F_i} \subset \mathrm{Hom}_k(F_i^{-d_i}, F_i^{d_i})$ respectively and where $a_v^{(\tau)} \in \mathrm{M}_v^\tau(\beta_v^\tau)$ and $a_v^{(\tau)} - \sigma(a_v^{(\tau)}) \in \widetilde{\mathrm{M}}_v^\tau(\beta_v^\tau)$. j'_Σ is written in a similar way. In a matrix form, j_Σ can be written as follows;

$$\begin{aligned}
j_\Sigma &\overset{\mathrm{mod}}{\equiv} \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ \pi_k Z & & & & 1_{\delta'} & Y \\ -\pi_k \overline{Y} & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & & & & \\ & 1 & & & & \\ -\sigma(U) & & \ddots & & & \\ & & & \ddots & & \\ -\frac{1}{2}U\sigma(U) & U & & & 1_{\delta'} & \\ & & & & & 1 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 & & & & \\ & 1 & & & & \\ -\sigma(U) & & \ddots & & & \\ & & & \ddots & & \\ \pi_k Z - \frac{1}{2}U\sigma(U) & U & & & 1_{\delta'} & Y \\ -\pi_k \overline{Y} & & & & & 1 \end{pmatrix}
\end{aligned} \qquad \text{(q20)}$$

where

$$Z = (Z_0 - \tfrac{1}{2} Y\overline{Y}) \qquad \text{with} \quad \overline{Z}_0 = -Z_0$$
$$U = \sum_{(v,\tau)\in\iota_s} a_v^{(\tau)} \in \mathrm{Hom}_k\left(F_i^{-d_i}, \bigoplus_{(j,j')\neq(i,-d_i)} F_j^{j'}\right) \tag{q21}$$

We can write j'_Σ analogously. To find the condition on j_Σ, j'_Σ such that $sj_\Sigma s^{-1}j'_\Sigma s \in J_\Sigma s J_\Sigma$, we find $sj_\Sigma s^{-1}j'_\Sigma s$ in a matrix form;

$sj_\Sigma s^{-1}j'_\Sigma s =$

$$\begin{pmatrix} Z - \frac{1}{\pi_k}\frac{1}{2}U\sigma(U) & \begin{matrix}\frac{1}{\pi_k}U + \\ (Z - \frac{1}{\pi_k}\frac{1}{2}U\sigma(U))\frac{1}{\pi_k}U'\end{matrix} & \begin{matrix}\frac{1}{\pi_k} - \frac{1}{\pi_k}U\sigma(U')\frac{1}{\pi_k} \\ +(Z - \frac{1}{\pi_k}\frac{1}{2}U\sigma(U)) \\ \cdot(Z' - \frac{1}{\pi_k}\frac{1}{2}U'\sigma(U'))\frac{1}{\pi_k} \\ -\frac{1}{\pi_k}Y\overline{Y}'\end{matrix} & \begin{matrix}(Z - \frac{1}{\pi_k}\frac{1}{2}U\sigma(U))\frac{1}{\pi_k}Y' \\ +\frac{1}{\pi_k}Y\end{matrix} \\ -\sigma(U) & \mathrm{Id} - \sigma(U)\frac{1}{\pi_k}U' & \begin{matrix}-\sigma(U')\frac{1}{\pi_k} \\ -\sigma(U)(Z' - \frac{1}{\pi_k}\frac{1}{2}U'\sigma(U'))\frac{1}{\pi_k}\end{matrix} & -\sigma(U)\frac{1}{\pi_k}Y'^{(\dagger)} \\ \pi_k & U' & Z' - \frac{1}{2}U'\sigma(U')\frac{1}{\pi_k} & Y' \\ -\overline{Y}\pi_k & -\overline{Y}U'^{(\dagger)} & \begin{matrix}-\overline{Y}(Z' - \frac{1}{2}U'\sigma(U')\frac{1}{\pi_k}) \\ -\overline{Y}'\end{matrix} & 1 - \overline{Y}Y' \end{pmatrix} \tag{q22}$$

From $(q22)$, $sj_\Sigma s^{-1}j'_\Sigma s \in J_\Sigma$ only if

$$Z \in \mathcal{O}_{F_i}^\times, \qquad Z' = -\frac{1}{\overline{Z}}, \qquad Y' = -\frac{Y}{\overline{Z}}U' = -\frac{1}{Z}U. \tag{q23}$$

Moreover, from the terms (†) in $(q22)$, for $sj_\Sigma s^{-1}j'_\Sigma s$ to be in J_Σ, we should have

$$U \in \sum_{(v,\tau)\in\iota_s} \widetilde{\mathrm{M}}_v^\tau\left(\beta_v^\tau + \tfrac{1}{2}\right).$$

Then $(q22)$ becomes

$$\mathbf{j}\cdot\begin{pmatrix} Z & & & \\ & 1 & & \\ & & -\frac{1}{\overline{Z}} & \\ & & & \frac{\overline{Z}}{Z} \end{pmatrix}$$

for some $\mathbf{j} \in \ker\rho_\Sigma$. Hence $\rho_\Sigma(sj_\Sigma s^{-1}j'_\Sigma s) = \widetilde{\chi}_\Gamma^i(-1)$. By counting the number of

Y, Z, U satisfying (q23), we can find

$$c_2 = \widetilde{\chi}^i_\Gamma(-1)C_s^{\frac{1}{2}}(q_i^3 - 1). \tag{q24}$$

7.4.12. Quadratic relations for (B)

(1) In *cases* (Bl) with $d_i \geqslant 2$, (B2) and (B3),

$$f_{s_{id_i}} \star f_{s_{id_i}} = \begin{cases} q_i C_{s_{id_i}} f_1 \pm C^{\frac{1}{2}}_{s_{id_i}}(q_i - 1) f_{s_{id_i}} & \textit{if } \log_{q_i}(C_{s_{id_i}}) \equiv 0 \pmod 2 \\ q_i C_{s_{id_i}} f_1 & \textit{if } \log_{q_i}(C_{s_{id_i}}) \equiv 1 \pmod 2. \end{cases}$$

(2) In *case* (Bl) with $d_i = 1$, let $i' = 0, 1$.

If $\log_{q_i}(C_{s_{i0}}) \equiv \log_{q_i}(C_{s_{id_i}}) \equiv 0 \pmod 2$,

$$f_{s_{ii'}} \star f_{s_{ii'}} = q_i C_{s_{ii'}} f_1 \pm C^{\frac{1}{2}}_{s_{ii'}}(q_i - 1) f_{s_{ii'}}.$$

If $\log_{q_i}(C_{s_{i0}}) \equiv \log_{q_i}(C_{s_{id_i}}) \equiv 1 \pmod 2$,

$$f_{s_{ii'}} \star f_{s_{ii'}} = q_i C_{s_{ii'}} f_1.$$

In (2) above, by counting, it is not difficult to see that $\log_{q_i}(C_{s_{i0}}) \equiv \log_{q_i}(C_{s_{id_i}})$ (mod2). For all (B1)–(B3), if $i' = d_i$, the computation can be done following the same procedure (q1)–(q17) as in the case (A2) with $i' = d_i$. In (q15), if $\sum l_{jj'}$ is even, $\prod \operatorname{sgn}(Z_0)^{l_{jj'}} = 1$ and we get c_2 as in (q17). If $\sum l_{jj'}$ is odd, $\prod \operatorname{sgn}^{l_{jj'}} = \operatorname{sgn}$ is a nontrivial quadratic character of $\mathbb{F}_{q_i}^\times$ and thus we get $c_2 = 0$ in (q16). Finally since $\log_{q_i} \sharp(\lambda_s)_v$ is even for any $v \in \iota'_s$,

$$\begin{aligned} \log_{q_i}(C_{s_{id_i}}) &\equiv \log_{q_i} \frac{\sharp(\lambda_s)}{q_i} \\ &\equiv \sum_{(v_{jj'},\tau) \in \iota'_s, j' \in \nabla_j} l_{jj'} + 2 \sum_{v \in \iota'_s, j' \notin \cup \nabla_j} \log_{q_i} \sharp(\lambda_s)_v + \sum_{v \in \iota'_s} \log_{q_i} \sharp(\lambda_s)_v \equiv \sum l_{jj'} \pmod 2, \end{aligned} \tag{q25}$$

and hence (1) and part of (2) follow. Now we consider the case $i' = 0$ for (B1) with $d_i = 1$. Then

$$s_{i0} = \begin{pmatrix} 0 & \overline{\pi}_{F_i} \\ \pi_{F_i}^{-1} & 0 \end{pmatrix}$$

and the computation is similar to s_{id_i} case. In this case, we have

$$c_2 = \begin{cases} C^{\frac{1}{2}}_{s_{i0}}(q_i - 1) & \text{if } \log_{q_i}(C_{s_{i0}}) \equiv 0 \pmod 2 \\ 0 & \text{if } \log_{q_i}(C_{s_{i0}}) \equiv 1 \pmod 2. \end{cases} \tag{q26}$$

Lastly, the quadratic relations for (C)–(D) can be computed similarly. We summarize the result as follows;

7.4.13. Quadratic relations for (C1)–(D)

In all *cases* (C1)–(C4),

$$f_{s_{id_i}} \star f_{s_{id_i}} = q_i^{\mathrm{wt}(s_{id_i})} C_{s_{id_i}} f_1 \pm C_{s_{id_i}}^{\frac{1}{2}} (q_i^{\mathrm{wt}(s_{id_i})} - 1) f_{s_{id_i}}$$

where $\mathrm{wt}(s_{id_i})$ coincides to the value of the weight function corresponding to Iwahori Hecke algebras in IADD (*See* (*7.3.4*)).

In *case* (D), $$f_{s_{id_i}} \star f_{s_{id_i}} = q_i C_{s_{id_i}} f_1 \pm C_{s_{id_i}}^{\frac{1}{2}} (q_i - 1) f_{s_{id_i}}.$$

We notice that in all (C1)–(D), when ρ_Σ is a character, $\log_{q_i}(C_{s_{id_i}})$ is even. Now, following the computation as in (q1)–(q17) and (q25)–(q26), we get (7.4.13).

7.4.14. Concluding Theorem 7.4.1.

Let $\dot{f}_{s_{ii'}} = \pm f_{s_{ii'}}$ where $\pm$ coincides with the sign of the coefficient c_2 of $f_{s_{ii'}}$ in (7.4.6)–(7.4.13). Based on the results (7.3.4) and (7.4.3)–(7.4.13), we can choose a tamely ramified character χ of $I_0' = \prod_i I_0'^{\,i}$ as follows. Let ξ_i be a tamely character of $I_0'^{\,i}$ defined as

$$\xi_i = \begin{cases} \chi_0 & \text{if } \mathrm{G}_i' \text{ is of type (A), (C) or (D),} \\ \chi_0 & \text{if } \mathrm{G}_i' \text{ is of type (B) and } \log_{q_i}(C_{s_{i0}}) \equiv 0 \pmod 2. \\ \chi_1 & \text{if } \mathrm{G}_i' \text{ is of type (B) and } \log_{q_i}(C_{s_{i0}}) \equiv 1 \pmod 2 \end{cases}$$

Recall from (7.3.4) that in each case (A1)–(D), χ_0 is the trivial character of $I_0'^{\,i}$ and χ_1 is the character of $I_0'^{\,i}$ corresponding to the second row of each IADD. Then

$$\chi = \otimes_i \xi_i.$$

Now from the choice of χ and (7.3.4), (7.4.3)–(7.4.13), we see that η defined as in (7.4.1) is an algebra isomorphism.

Considering the map $\eta' : \mathcal{H}' \longrightarrow \otimes\, \mathcal{H}(\mathrm{G}_i' // I_0'^{\,i}, \xi_i)$ defined by $f \mapsto \otimes_i (f | I_0'^{\,i})$, it is obvious that η' is a $*$-preserving, support-preserving L^2-isomorphism of $\mathbb{C}$-algebras. Composing η and η'^{-1}, we see that $\eta \circ \eta'^{-1} : \mathcal{H}(\mathrm{G}_i' // I_0'^{\,i}, \xi_i) \longrightarrow \mathcal{H}$ is a $*$-preserving, support-preserving L^2-isomorphism of $\mathbb{C}$-algebras.

Combining above with the previous remark at the start of the proof of (7.4.1), now Theorem 7.4.1 is proved. □

Conclusion

Via the reduction step carried out in Section 5 (especially in (5.2.6)), now Theorems 6.2.2 and 7.4.1 imply the main theorem. Here, we restate the theorem with more explicit description of η. In the following, we keep all the previous notation and also recall the Haar measure on G' (resp. G) are normalized such that the volume of I'_0 (resp. J_Σ) is 1.

MAIN THEOREM. *Let k satisfy (3.2.3) and let* $\Sigma = (\Gamma, I'_0, 1)$ *be as in* (H_Σ). *Let* (J_Σ, ρ_Σ) *be a pair consisting of an open compact subgroup* J_Σ *and its irreducible representation* ρ_Σ *associated to* Σ *as in Theorem 4.2.9. Suppose* ρ_Σ *is a character. Then for some tamely ramified character* χ *of* I'_0, *there is a* $*$*-preserving, support-preserving* L^2*-isomorphism*

$$\eta: \mathcal{H}' = \mathcal{H}(\mathrm{G}'//I'_0, \chi) \longrightarrow \mathcal{H}(\mathrm{G}//J_\Sigma, \rho_\Sigma) = \mathcal{H}$$

of $\mathbb{C}$*-algebras. More explicitly,* η *is defined as follows: Let* P_Σ *be the parabolic subgroup associated to* Σ *in (5.2.6) and let* $\delta_{\mathrm{P}_\Sigma}$ *be the modulus function associated to* P_Σ. *For* $w \in \widetilde{W}'$, *where* $\widetilde{W}'$ *is an affine Weyl group of* G' *with* $\mathrm{G}' = I'_0 \widetilde{W}' I'_0$, *let* $\hat{e}_w \in \mathcal{H}'$ *with* $\hat{e}_w(w) = 1$ *and* $\mathrm{Supp}(\hat{e}_w) = I'_0 w I'_0$, *and let* $f^\delta_w \in \mathcal{H}$ *with* $f^\delta_w(w) = \delta^{\frac{1}{2}}_{\mathrm{P}_\Sigma}(w)$ *and* $\mathrm{Supp}(f^\delta_w) = J_\Sigma w J_\Sigma$. *Then*

$$\eta(\hat{e}_w) = \left(\frac{1}{C_w}\right)^{1/2} \dot{f}^\delta_w \quad \text{with} \quad C_w = \frac{\mu(J_\Sigma w J_\Sigma)}{\mu'(I'_0 w I'_0)},$$

where $\dot{f}^\delta_w$ *is properly normalized with* $\dot{f}^\delta_w = f^\delta_w$ *or* $-f^\delta_w$.

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