Dynamics and Control, 9, 39–66 (1999)

Generalized Serret-Andoyer Transformation and Applications for the Controlled Rigid Body

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Received October 17, 1997; Revised May 15, 1998; Accepted June 3, 1998

Abstract. The Serret-Andoyer transformation is a classical method for reducing the free rigid body dynamics, expressed in Eulerian coordinates, to a 2-dimensional Hamiltonian flow. First, we show that this transformation is the computation, in 3-1-3 Eulerian coordinates, of the symplectic (Marsden-Weinstein) reduction associated with the lifted left-action of $SO(3)$ on $T^*SO(3)$—a generalization and extension of Noether's theorem for Hamiltonian systems with symmetry. In fact, we go on to generalize the Serret-Andoyer transformation to the case of Hamiltonian systems on $T^*SO(3)$ with left-invariant, hyperregular Hamiltonian functions. Interpretations of the Serret-Andoyer variables, both as Eulerian coordinates and as canonical coordinates of the co-adjoint orbit, are given. Next, we apply the result obtained to the controlled rigid body with momentum wheels. For the class of Hamiltonian controls that preserve the symmetry on $T^*SO(3)$, the closed-loop motion of the main body can again be reduced to canonical form. This simplifies the stability proof for relative equilibria , which then amounts to verifying the classical Lagrange-Dirichlet criterion. Additionally, issues regarding numerical integration of closed-loop dynamics are also discussed. Part of this work has been presented in [16].

Keywords: Hamiltonian system, canonical transformation, group symmetry, symplectic form, symplectic reduction

1. Introduction

The *Serret-Andoyer transformation* was first introduced by Serret [23] as a canonical transformation for the free rigid body dynamics that results in two ignorable (cyclic) coordinates. Roughly speaking, let the transformation be denoted by

$$(\varphi, \theta, \psi, \Phi, \Theta, \Psi) \mapsto (g, h, l, G, H, L),$$

where $(\varphi, \theta, \psi, \Phi, \Theta, \Psi)$ is the set of 3-1-3 Eulerian coordinates that characterize the motion, and (g, h, l, G, H, L) are the transformed coordinates. Then, the variables g and h are ignorable, i.e., G and H are integral invariants, leaving a Hamiltonian in the remaining variables, $\mathcal{H}(l, L)$. In essence, the transformation reduces the 6-dimensional dynamics of the free rigid body to two dimensions.

* Research partially supported by AFOSR grant F49620-96-1-0100 while at the University of Michigan Department of Aerospace Engineering.
** Research partially supported by National Science Foundation PYI grant DMS-94-96221, AFOSR grant F49620-96-1-0100, a Guggenheim Fellowship and the Institute for Advanced Study.

Serret deduced the transformation by solving the Hamilton-Jacobi equation, where reduction was achieved by requiring the generating function to take on a special form. Andoyer [3] showed using spherical trigonometry that the transformation is really a change of Eulerian coordinates that depends on the angular momentum vector. In effect, choosing a spatial frame such that the 3-axis is parallel to the conserved spatial angular momentum vector **p**, and denoting the set of 3-1-3 Eulerian coordinates in this spatial frame as $(\varphi', \theta', l, \Phi', \Theta', L)$, one finds that the Hamiltonian depends only on Φ', l and L. Moreover, Φ' is a constant of motion (since φ' is cyclic) and in fact equals $\|\mathbf{p}\|$, the magnitude of the angular momentum. Thus, the reduced Hamiltonian above is obtained.

More recently, Deprit and Elipe [11] reconstructed the Serret-Andoyer transformation using differential forms, and explained that the transformation has the meaning of reducing the rigid body dynamics by $SO(3)$ symmetry, although this connection was not explicitly formulated as a quotient map on the phase space, which we do in the present study.

The significance of the transformation is that it allows, through further introduction of a set of action-angle variables, the complete reduction of the free rigid body dynamics onto the 2-torus, that is, in a form that is integrable by quadrature; see [11]. The notion of integrability of Hamiltonian systems is well known.

The main purpose of this paper is to extend the definition of the Serret-Andoyer variables to the controlled rigid body, and to attempt a similar reduction of the closed-loop dynamics for a certain class of Hamiltonian controls.

To do so, we first explicate the connection between the Serret-Andoyer transformation, and the notion of reduction of dynamics with symmetry due to Marsden and Weinstein [2], [20]. In fact, we go on to generalize the Serret-Andoyer transformation to the case of Hamiltonian systems on $T^*SO(3)$ with left-invariant, hyperregular Hamiltonians, and we show that this transformation is the computation in 3-1-3 Eulerian coordinates of the symplectic (Marsden-Weinstein) reduction associated with the lifted left-action of $SO(3)$ on $T^*SO(3)$—a generalization and extension of Noether's theorem on the conservation of momentum maps for Hamiltonian systems with symmetry.

Geometrically, the generalized Serret-Andoyer transformation is rather easy to visualize: it amounts to choosing a spatial frame so that the 3-axis is aligned with the generalized angular momentum, and characterizing the momentum level set with the 3-1-3 Eulerian coordinates. The geometric interpretation of the Serret-Andoyer variables (l, L), i.e., the remaining free variables, is two-fold: l originates as an Euler coordinate due to the above-mentioned choice of the spatial frame, and L as its conjugate momentum; moreover, (l, L) appear as canonical coordinates with respect to the *left Kostant-Kirillov symplectic form* for a local chart on the momentum level set. It is well known that this level set is a leaf of the Poisson manifold $\mathfrak{so}(3)^*$.

The generalized Serret-Andoyer transformation is next applied to the controlled rigid body with momentum wheels. As further investigation of the work of Bloch *et al.* [7], we are in particular interested in control laws that stabilize stationary rotation about the intermediate axis while preserving the group symmetry, i.e., such that the closed-loop motion of the body is again Hamiltonian, left-invariant and hyperregular. A sufficient condition for a symmetry-preserving control is given in Section 6.1.

For the class of Hamiltonian controls that preserve the symmetry on $T^*SO(3)$, Noether's theorem is applicable to the closed-loop motion of the main body. Thus, one can define a *controlled Hamiltonian* which is generally different from the classical Hamiltonian. The generalized Serret-Andoyer transformation for the controlled system then amounts to a judicious choice of Euler coordinates relative to the momentum which, by design, is conserved in space. Reduction is thus achieved, and the closed-loop main body motion is again a 2-dimensional Hamiltonian flow in the generalized Serret-Andoyer variables. A useful feature of the Serret-Andoyer variables analysis of the controlled dynamics is that, since the reduced dynamics are Hamiltonian in canonical form, the stability proof can be done by applying the classical Lagrange-Dirichlet criterion. Finally, we shall investigate the computational properties of the Serret-Andoyer variables representation. To do this, we simulate the reduced closed-loop dynamics using a general integration algorithm (Adams' method) as well as second-order symplectic integrators [9]. We compare the solutions thus obtained with those simulated with other representations, namely, Euler's equations and the full (3-degree-of-freedom) dynamics, focusing on the numerical preservation of energy and momentum. The results show that the reduced representation is more economical by having fewer (two) dynamical equations, naturally preserves momentum, and allows the use of symplectic integrators that are accurate energy-wise.

In this paper, we show how a geometric approach to classical mechanics can be applied to modern control theory. This approach appears to be useful in various settings; see, for example, [8]. Some preliminary results on the topics discussed hereafter have been presented in [16].

2. Representation of Free Rigid Body Motion in Eulerian Coordinates

We consider the motion of a free rigid body in inertial space. By König's theorem in classical mechanics, this amounts to the study of the body's motion about its center of inertia which is fixed in space. Let the *spatial frame* be the set $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ of right-handed orthonormal vectors fixed in space, and let the *body frame* be the set $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ fixed in the body sharing the common origin O with the spatial frame. See Figure 1. In terms of the 3-1-3 Euler angles [14], pp. 354–358 denoted by (φ, θ, ψ), the attitude of the rigid body relative to space results from successive rotations of angle φ about $\mathbf{s}_3$, then of angle θ about the image $\mathbf{l}$ of $\mathbf{s}_1$ by the first rotation, and finally of angle ψ about $\mathbf{b}_3$ which is the image of $\mathbf{s}_3$ by the previous rotations. If (x_1, x_2, x_3) and (X_1, X_2, X_3) are the components of a vector relative to, respectively, the spatial and body frames, then the rotation matrix $\mathbf{R} \in SO(3)$ taking (X_1, X_2, X_3) to (x_1, x_2, x_3) is given by

$$\mathbf{R} = \begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & \sin\theta\sin\varphi \\ \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & -\sin\theta\cos\varphi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}, \tag{1}$$

i.e., $(x_1, x_2, x_3)^{\mathrm{T}} = \mathbf{R}\,(X_1, X_2, X_3)^{\mathrm{T}}$. The classical definition of the *body angular velocity*

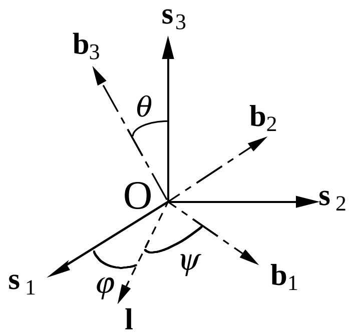


Figure 1. 3-1-3 Euler angles.

$\boldsymbol{\omega} = \dot{\varphi}\,\mathbf{s}_3 + \dot{\theta}\,\mathbf{l} + \dot{\psi}\,\mathbf{b}_3$ yields

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\theta}\cos\psi + \dot{\varphi}\sin\psi\sin\theta \\ -\dot{\theta}\sin\psi + \dot{\varphi}\cos\psi\sin\theta \\ \dot{\varphi}\cos\theta + \dot{\psi} \end{bmatrix}. \tag{2}$$

The *spatial angular velocity*, $\boldsymbol{\Omega} = \mathbf{R}\,\boldsymbol{\omega}$, then has the expression

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\sin\theta \\ \dot{\theta}\sin\varphi + \dot{\psi}\cos\varphi\sin\theta \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{bmatrix}. \tag{3}$$

The Lagrangian of the free rigid body dynamics equals the body's kinetic energy $\mathcal{L}(\boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\boldsymbol{\omega}$, where $\mathbb{I}$ is the matrix of the inertia tensor. Assume that the body axes coincide with the *principal axes of inertia*, $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$. The associated Hamiltonian function in the configuration variables (φ, θ, ψ) and their conjugate momenta (Φ, Θ, Ψ) can be obtained by the Legendre transform of $\mathcal{L}$. Computation yields [19]

$$\begin{aligned} \Phi &= I_1(\dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi)\sin\theta\sin\psi \\ &\quad + I_2(\dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)\sin\theta\cos\psi + I_3(\dot{\varphi}\cos\theta + \dot{\psi})\cos\theta, && \text{(4a)} \\ \Psi &= I_3(\dot{\varphi}\cos\theta + \dot{\psi}), && \text{(4b)} \\ \Theta &= I_1(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\cos\psi - I_2(\dot{\varphi}\cos\varphi\sin\theta - \dot{\theta}\sin\psi)\sin\psi, && \text{(4c)} \end{aligned}$$

and the resulting Hamiltonian function has the expression

$$\begin{aligned} \mathcal{H}(\varphi, \theta, \psi, \Phi, \Theta, \Psi) &= \frac{1}{2}\left(\frac{\sin^2\psi}{I_1} + \frac{\cos^2\psi}{I_2}\right)\left(\frac{\Phi - \Psi\cos\theta}{\sin\theta}\right)^2 \\ &\quad + \frac{\Psi^2}{2I_3} + \frac{1}{2}\left(\frac{\cos^2\psi}{I_1} + \frac{\sin^2\psi}{I_2}\right)\Theta^2 \\ &\quad + \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\left(\frac{\Phi - \Psi\cos\theta}{\sin\theta}\right)\Theta\sin\psi\cos\psi. \end{aligned} \tag{5}$$

The *body angular momentum* is given classically by $\mathbf{m} = \mathbb{I}\,\boldsymbol{\omega}$. Using (4), $\mathbf{m}$ then has the expression

$$m_1 = \left(\frac{\Phi - \Psi\cos\theta}{\sin\theta}\right)\sin\psi + \Theta\cos\psi, \tag{6a}$$

$$m_2 = \left(\frac{\Phi - \Psi\cos\theta}{\sin\theta}\right)\cos\psi - \Theta\sin\psi, \tag{6b}$$

$$m_3 = \Psi. \tag{6c}$$

Finally, the *spatial angular momentum* $\mathbf{p}$ is obtained by transforming $\mathbf{m}$ back to the spatial frame:

$$p_1 = \Theta\cos\varphi + \left(\frac{\Psi - \Phi\cos\theta}{\sin\theta}\right)\sin\varphi, \tag{7a}$$

$$p_2 = \Theta\sin\varphi - \left(\frac{\Psi - \Phi\cos\theta}{\sin\theta}\right)\cos\varphi, \tag{7b}$$

$$p_3 = \Phi. \tag{7c}$$

Remark 1. Note that the variable φ is cyclic in the Hamiltonian $\mathcal{H}$ given by (5), i.e., $\mathcal{H}$ is invariant with respect to coordinate rotations about the axis $\mathbf{s}_3$. It is obvious that this symmetry holds for any arbitrarily chosen $\mathbf{s}_3$. In fact, as shown later, this property is true for a more general class of motions.

3. The Classical Serret-Andoyer Transformation

The Serret-Andoyer transformation results from the following geometric operation. Let the spatial and body frames be denoted as before. Let $\mathbf{i}$ be the image of $\mathbf{s}_1$ by a rotation h about $\mathbf{s}_3$, and let $\mathbf{k}$ be the image of $\mathbf{s}_3$ by a rotation σ about $\mathbf{i}$. See Figure 2(a). The body frame is obtained by successive rotations of angle g about $\mathbf{k}$, then of angle β about the image $\mathbf{j}$ of $\mathbf{i}$ by the first rotation, and finally of angle l about $\mathbf{b}_3$ which is the image of $\mathbf{k}$ by the previous rotations; Figure 2(b). Notice that (g, β, l) are the 3-1-3 Euler angles that locate the body frame *relative to the intermediate frame* formed by $\mathbf{i}$, $\mathbf{k}$ and a third vector. The rotation $\mathbf{R}$ from the spatial frame to the body frame passing through the intermediate frame can be represented by the differential [11]

$$d\mathbf{R} = \mathbf{s}_3 dh + \mathbf{i} d\sigma + \mathbf{k} dg + \mathbf{j} d\beta + \mathbf{b}_3 dl. \tag{8}$$

Instead of relying on spherical trigonometry as did Andoyer [3], Deprit and Elipe [11] deduced the Serret-Andoyer transformation by requiring that the following differential equation hold:

$$\mathbf{m}\cdot d\mathbf{R} = L\,dl + G\,dg + H\,dh, \tag{9}$$

where L, G and H are the momenta canonically conjugate to l, g and h. By choosing $\mathbf{k}$ to be the direction of the angular momentum vector, i.e., $\mathbf{m} = \|\mathbf{m}\|\,\mathbf{k}$ so that $\mathbf{m}\cdot\mathbf{i} = 0$

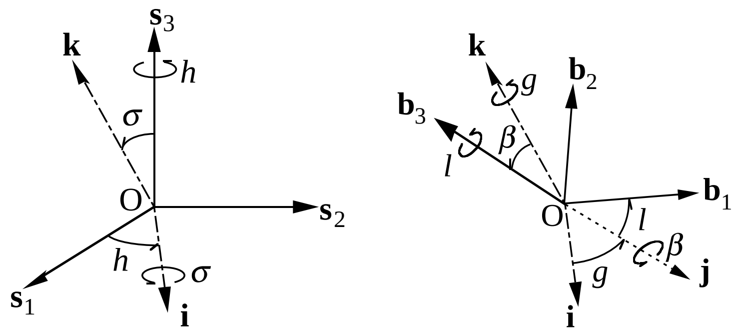


Figure 2. Classical transformation using spherical coordinates.

and $\mathbf{m} \cdot \mathbf{j} = 0$, the transformation $(\varphi, \theta, \psi, \Phi, \Theta, \Psi) \mapsto (l, g, h, L, G, H)$ results in the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)(G^2 - L^2) + \frac{L^2}{2I_3}, \tag{10}$$

with $H = G\cos\sigma = \Phi$, $L = G\cos\beta = \Psi$, and $G = \|\mathbf{m}\|$. Deprit and Elipe [11] argued that the above transformation is canonical and results from a reduction by $SO(3)$. In effect, we notice that the variables g and h are cyclic, and thus G and H are integrals of motion. Moreover, H vanishes in (10), hence h is also constant along the flow of $\mathcal{H}$. An additional integral of motion is obviously the Hamiltonian $\mathcal{H}$. Solutions of $l(t)$ and $L(t)$ can then be obtained by integration by quadrature, yielding

$$\frac{dl}{dt} = G\sqrt{\left(\frac{1}{I_3} - \frac{\sin^2 l}{I_1} - \frac{\cos^2 l}{I_2}\right)\left(\mathcal{A} - \frac{\sin^2 l}{I_1} - \frac{\cos^2 l}{I_2}\right)}, \tag{11a}$$

$$L = G\sqrt{\frac{\mathcal{A} - \frac{1}{I_1}\sin^2 l - \frac{1}{I_2}\cos^2 l}{\frac{1}{I_3} - \frac{1}{I_1}\sin^2 l - \frac{1}{I_2}\cos^2 l}}, \tag{11b}$$

where $\mathcal{A} = 2\mathcal{H}/G^2$ is an integral of motion. Derivation of (11) and further relevant issues regarding integration are discussed in [11].

Remark 2. It is now apparent that the axes $\mathbf{i}$ and $\mathbf{k}$ are fixed in space, since the angle h which locates $\mathbf{i}$ in the space spanned by $\mathbf{s}_1$ and $\mathbf{s}_2$ is a constant of motion, and $\mathbf{k}$ is the direction of the spatial angular momentum which is conserved. Hence, the Serret-Andoyer transformation can be viewed as a change of spatial coordinates that depends on a conserved quantity, namely, the angular momentum in space.

4. Generalities in Hamiltonian Systems with Symmetry

We recall in this section some general notions concerning Hamiltonian systems on Lie groups. We shall employ hereafter standard terminology and methodology of geometric methods in control and dynamics, and we recommend, among others, the texts of Abraham and Marsden [2], Arnold [5] and Marsden and Ratiu [19] for in-depth syntheses of the subject. Nevertheless, we shall highlight here some elements of the theory of Hamiltonian systems on $SO(3)$ that are relevant to the development of our subsequent results.

Consider the Lie group $SO(3)$, i.e., the group of real 3×3 orthogonal matrices with determinant equal to 1. Let $\mathfrak{so}(3)$ denote its Lie algebra, and $\mathfrak{so}(3)^*$ its dual. Recall that elements of $\mathfrak{so}(3)$ are 3×3 skew-symmetric matrices, and the algebra bracket is the usual matrix commutator bracket, $[\mathbf{V}, \mathbf{W}] = \mathbf{VW} - \mathbf{WV}$.

Let the *hat map* $\widehat{\ } : \mathbb{R}^3 \to \mathfrak{so}(3)$ denote the usual Lie algbera isomorphism that identifies $(\mathfrak{so}(3), [\ ,\ ])$ with $(\mathbb{R}^3, \times)$:

$$\mathbf{v} = (v_1, v_2, v_3) \ \mapsto \ \widehat{\mathbf{v}} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} . \tag{12}$$

By duality, $\mathfrak{so}(3)^*$ is also identified with $\mathbb{R}^3$.

With the above identification, the classical definitions of body and spatial angular velocities $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$, respectively, have the following geometric meaning: the pair $(\mathbf{R}, \widehat{\boldsymbol{\omega}})$ is the *body representation*, i.e., left-translation to the identity, of tangent vectors in $TSO(3)$. Indeed, $\dot{\mathbf{R}} \in T_{\mathbf{R}}SO(3)$ has the form $\dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\omega}}$. Thus, left-translation gives $T_{\mathbf{R}}L_{\mathbf{R}}^{-1} \cdot \mathbf{R}\widehat{\boldsymbol{\omega}} = \mathbf{R}^{-1}\mathbf{R}\widehat{\boldsymbol{\omega}} = \widehat{\boldsymbol{\omega}}$. Likewise, the pair $(\mathbf{R}, \widehat{\ })$ is the *spatial representation* of tangent vectors by right-translation to the identity. By duality, the body and angular momentum vectors $\mathbf{m}$ and $\mathbf{p}$ give the body and spatial representations, respectively, of covectors in $T^*SO(3)$. The pairing between tangent vectors and covectors is then given by the usual dot product on $\mathbb{R}^3$: $\langle (\mathbf{R}, \mathbf{m}), (\mathbf{R}, \widehat{\boldsymbol{\omega}}) \rangle = \mathbf{m} \cdot \boldsymbol{\omega}$. Arnold [4] gave a clarification of the various representations for general Lie groups, and showed that they can be applied to fluid mechanics. See also [19] for an exposition.

A Hamiltonian function $\mathcal{H}$ on $T^*SO(3)$ is said to be *left-invariant* if $\mathcal{H} \circ L_{\mathbf{R}}^* = \mathcal{H}$ for all $\mathbf{R} \in SO(3)$, where L^* denotes the cotangent lifted action. In the body representation, left-invariance means that $\mathcal{H}$ depends only on the body angular momentum, i.e., $\mathcal{H} : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R} : (\mathbf{R}, \mathbf{m}) \mapsto \mathcal{H}(\mathbf{m})$. The *fiber derivative* of $\mathcal{H}$ is the map $\mathbb{F}\mathcal{H} : SO(3) \times \mathfrak{so}(3)^* \to SO(3) \times \mathfrak{so}(3) : (\mathbf{R}, \mathbf{m}) \mapsto (\mathbf{R}, \nabla_{\mathbf{m}}\mathcal{H})$. $\mathcal{H}$ is said to be *hyperregular* if $\mathbb{F}\mathcal{H}$ is a diffeomorphism. The following is an important lemma, materials for the proof of which can be found in [2, §4.4].

LEMMA 1 *Let $\mathcal{H} : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R}$ be left-invariant, then the associated Hamiltonian vector field in body coordinates is*

$$X_{\mathcal{H}}(\mathbf{R}, \mathbf{m}) = \left(\mathbf{R} \cdot \widehat{\nabla_{\mathbf{m}}\mathcal{H}} \ , \ \mathbf{m} \times \nabla_{\mathbf{m}}\mathcal{H}\right) . \tag{13}$$

Moreover, if $\mathcal{H}$ is hyperregular, the associated left-invariant Lagrangian $\mathcal{L} : SO(3) \times$

$\mathfrak{so}(3) \to \mathbb{R}$ is given in body coordinates by

$$\mathcal{L}(\boldsymbol{\omega}) = \mathbf{m} \cdot \boldsymbol{\omega} - \mathcal{H}(\mathbf{m}), \tag{14}$$

where $\mathbf{m}$ *is given in terms of* $\boldsymbol{\omega}$ *by the Legendre transform* $\mathbb{F}\mathcal{L} : SO(3) \times \mathfrak{so}(3) \to SO(3) \times \mathfrak{so}(3)^* : (\mathbf{R}, \boldsymbol{\omega}) \mapsto (\mathbf{R}, \mathbf{m}) = (\mathbf{R}, \nabla_\omega \mathcal{L})$*, with* $(\mathbb{F}\mathcal{L})^{-1} = \mathbb{F}\mathcal{H}$*.*

The second component of $X_{\mathcal{H}}$ is sometimes called the *Euler vector field*, or the dynamical equations $\dot{\mathbf{m}} = \mathbf{m} \times \nabla_{\mathbf{m}} \mathcal{H}$ are called *Euler's equation*. In particular, the classical Lagrangian of free rigid body dynamics is given in body coordinates as $\mathcal{L}(\boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega}$. Thus, the body angular momentum is the image by the Legendre transform of the body angular velocity, $\mathbb{F}\mathcal{L} : \boldsymbol{\omega} \mapsto \mathbf{m} = \mathbb{I}\boldsymbol{\omega}$, and the classical spatial angular momentum is its spatial representation.

In general, given a hyperregular Hamiltonian $\mathcal{H}$ (or, equivalently, a hyperregular Lagrangian $\mathcal{L}$), one can define a generalized body angular momentum by the Legendre transform, i.e., $\mathbf{m} = \nabla_\omega \mathcal{L}$, where $\boldsymbol{\omega}$ is the classical body angular velocity. $\mathcal{H}$ in our setting will generally be different from the classical Hamiltonian. For example, for the controlled rigid body, $\mathcal{H}$ can be called the *controlled Hamiltonian*, and $\mathcal{L}$ the *controlled Lagrangian*.

Remark 3. Note that for a left-invariant Lagrangian in general, the variable φ in the 3-1-3 Euler angles (φ, θ, ψ) is necessarily ignorable, since φ does not appear explicitly in the expression of $\boldsymbol{\omega}$ (see (2)). In particular, this is true for the free rigid body dynamics, as mentioned in Remark 1.

We conclude this section with the following property related to the 3-1-3 Euler angle representation. This property is crucial for the generalization of the Serret-Andoyer transformation discussed in the next section.

LEMMA 2 *Let* $\mathcal{H} \in \mathcal{F}(SO(3) \times \mathfrak{so}(3)^*)$ *be a left-invariant, hyperregular Hamiltonian, and let* $\mathcal{L}$ *be the associated Lagrangian. Then, choosing any arbitrary spatial frame and the set* (φ, θ, ψ) *of 3-1-3 Euler angles, the conjugate momenta* (Φ, Θ, Ψ) *associated with* $\mathcal{L}$ *are related to the body representation by*

$$\Phi = (m_1 \sin\psi + m_2 \cos\psi) \sin\theta + m_3 \cos\theta, \tag{15a}$$

$$\Theta = m_1 \cos\psi - m_2 \sin\psi, \tag{15b}$$

$$\Psi = m_3, \tag{15c}$$

for all $(\mathbf{R}, \mathbf{m}) \in SO(3) \times \mathfrak{so}(3)^*$*. Moreover, in the chosen spatial frame and ignoring the singular points corresponding to* $\theta = 0$*, the spatial representation,* $\mathbf{p} = \mathbf{Rm}$*, is then given in terms of these momenta by*

$$p_1 = \Theta \cos\varphi + \left(\frac{\Psi - \Phi \cos\theta}{\sin\theta} \right) \sin\varphi, \tag{16a}$$

$$p_2 = -\Theta \sin\varphi - \left(\frac{\Psi - \Phi \cos\theta}{\sin\theta} \right) \cos\varphi, \tag{16b}$$

$$p_3 = \Phi. \tag{16c}$$

Proof: By definition, $\Phi = \partial\mathcal{L}/\partial\dot{\varphi} = \nabla_\omega \mathcal{L} \cdot \mathbf{D}_{\dot{\varphi}}\omega = \mathbf{m} \cdot \mathbf{D}_{\dot{\varphi}}\boldsymbol{\omega}$. From the expression 2 of the body angular velocity, one easily obtains (15a). (15b) and (15c) can similarly be obtained. Finally, (16) is obtained by inverting (15) to yield $\mathbf{m}$ in terms of (Φ, Θ, Ψ), and transforming to the spatial frame with $\mathbf{R}$ given by (1). ■

Remark 4. The variables $(\varphi, \theta, \psi, \Phi, \Theta, \Psi)$ are a set of local coordinates for $T^*SO(3)$. Hence, Lemma 2 simply relates these coordinates with the vectorial representations, both in the body and in space. Equations (15) and (16) are true for any left-invariant, hyperregular Hamiltonian. In particular, they are true for the free rigid body Hamiltonian, as can be seen by comparing (15) and (16) with (4) and (7).

5. Generalized Serret-Andoyer Transformation

In this section, we shall reconstruct the Serret-Andoyer transformation by employing the notion of *symplectic (Marsden-Weinstein) reduction*. This notion is essentially based on that of *momentum maps*, which are quantities generated by symmetry (group actions) on a Poisson manifold. By *Noether's theorem*, momentum maps are conserved along the trajectories of a Hamiltonian vector field when the Hamiltonian is itself invariant under the symmetry. The conserved quantity defines a 'slice' of the manifold which, under further assumption of *equivariance*, can be projected onto a smooth manifold, the *reduced phase space*, of lesser dimension equipped with a unique symplectic structure. The trajectories of the original Hamiltonian vector field are thus projected onto those of a reduced Hamiltonian vector field on the reduced phase space. The Serret-Andoyer transformation is none other than the computation in Eulerian coordinates of this process of reduction. In fact, it generalizes to rigid motions with left-invariant, hyperregular Hamiltonians. But, first, we shall introduce materials essential to the discussion.

5.1. *Symplectic (Marsden-Weinstein) Reduction of* $T^*SO(3)$

Let G be a Lie group and and let $\mathfrak{g}$ denote its Lie algebra. Moreover, let P be a *Poisson manifold*, i.e., a manifold with a *Poisson bracket* $\{,\}$ on $\mathcal{F}(P) = \mathcal{C}^\infty(P)$ such that $(\mathcal{F}(P), \{,\})$ is a Lie algebra and $\{FG, H\} = \{F, H\}G + F\{G, H\}$ for all F, G and $H \in \mathcal{F}(P)$. Let G act on P (on the left) by Poisson maps, $G \times P \to P : (g, q) \mapsto L_g(q) = g \cdot q$, i.e., L_g preserves the Poisson bracket for all $g \in G$: $\{F, G\} \circ L_g = \{F \circ L_g, G \circ L_g\}$ for all F and $G \in \mathcal{F}(P)$. To this action corresponds an *infinitesimal action* of $\mathfrak{g}$ on P, i.e., the vector field

$$\xi_P(q) = \left.\frac{d}{dt}\right|_{t=0} \left[e^{t\xi} \cdot q\right], \tag{17}$$

$q \in P$, $\xi \in \mathfrak{g}$.

Definition 1. A map $\mathbf{J} : P \to \mathfrak{g}^*$ is called a *momentum map* if $X_{\langle \mathbf{J},\xi\rangle} = \xi_P$ for all $\xi \in \mathfrak{g}$. Moreover, $\mathbf{J}$ is said to be *Ad*-equivariant* if $\mathbf{J} \circ L_g = \mathrm{Ad}_g^* \circ \mathbf{J}$ for all $g \in G$.

The linear map $\mathrm{Ad}^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the *co-adjoint action* of G on $\mathfrak{g}^*$ (see [19] for the definition). For $G = SO(3)$ one has:

LEMMA 3 *The co-adjoint action of $SO(3)$ on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is the usual coordinate transformation in $\mathbb{R}^3$, $\mathrm{Ad}^*_{\mathbf{R}^{-1}}\mathbf{m} = \mathbf{Rm}$ for all $\mathbf{R} \in SO(3)$, $\mathbf{m} \in \mathfrak{so}(3)^*$.*

THEOREM 1 *Let $P = T^*G$ be equipped with the canonical symplectic form and, thus, with the associated Poisson structure. Then, the left-action of G on T^*G is Poisson. Moreover, the Ad*-equivariant momentum mapping of this action is given in body coordinates by $\mathbf{J}(g, \mu) = \mathrm{Ad}^*_{g^{-1}}(\mu)$ [2, pp. 317–318].*

Note in particular that for $G = SO(3)$, Lemma 3 and Theorem 1 imply that the associated momentum map is simply the (generalized) angular momentum represented in the inertia 3-space.

THEOREM 2 (NOETHER'S THEOREM) *If the action of G on P is Poisson and admits a momentum map $\mathbf{J} : P \to \mathfrak{g}^*$, and if the smooth function $\mathcal{H} : P \to \mathbb{R}$ is G-invariant, i.e., $\mathcal{H} \circ L_g = \mathcal{H}$ for all $g \in G$, then $\mathbf{J}$ is a constant of the motion for $X_{\mathcal{H}}$, i.e., $\mathbf{J} \circ \varphi_t = \mathbf{J}$, where φ_t is the flow of $X_{\mathcal{H}}$.*

The following specializes Theorems 1 and 2 to left-invariant Hamiltonian vector fields on cotangent bundles.

COROLLARY 1 *Let $\mathcal{H} : T^*G \to \mathbb{R}$ be left-invariant. Then, the spatial representation of the momentum map $\mathbf{J}$ is invariant along the trajectories of $X_{\mathcal{H}}$.*

Corollary 1 generalizes the classically known fact that the angular momentum of a free rigid body is conserved in space. Now, let $G = SO(3)$, let $\mathbf{p} \in \mathfrak{so}(3)^*$ be given, and let $M_{\mathbf{p}} = \mathbf{J}^{-1}(\mathbf{p})$ denote the *momentum level set* corresponding to $\mathbf{p}$. In this case, $M_{\mathbf{p}}$ is a smooth manifold. Since the momentum map $\mathbf{J}$ is equivariant (Theorem 1), the *stationary subgroup* $G_{\mathbf{p}} \subset SO(3)$ given by

$$G_{\mathbf{p}} = \{\mathbf{R} \in SO(3) : Ad^*_{\mathbf{R}^{-1}}(\mathbf{p}) = \mathbf{p}\}$$

leaves $M_{\mathbf{p}}$ fixed.

PROPOSITION 1 *The stationary subgroup $G_{\mathbf{p}}$ for the rigid body problem is the 1-parameter subgroup of rotations in the direction of the spatial angular momentum $\mathbf{p}$.*

Proof: Let $\mathbf{R} \in G_{\mathbf{p}}$. Then, $\mathrm{Ad}^*_{\mathbf{R}^{-1}}(\mathbf{p}) = \mathbf{p}$, which implies $\mathbf{p} \cdot \mathbf{v} = \mathbf{p} \cdot \mathrm{Ad}^*_{\mathbf{R}^{-1}}(\widehat{\mathbf{v}}) = \mathbf{p} \cdot (\mathbf{R}^{-1}\widehat{\mathbf{v}}\mathbf{R})$. Hence, $\mathbf{p} \cdot \mathbf{v} = \mathbf{p} \cdot \mathbf{R}^{-1}\mathbf{v}$, or $(\mathbf{Rp} - \mathbf{p}) \cdot \mathbf{v} = 0$, for all $\mathbf{v} \in \mathbb{R}^3$. $\mathbf{p}$ is therefore an eigenvector of $\mathbf{R}$. ■

The quotient manifold $F_{\mathbf{p}} = M_{\mathbf{p}}/G_{\mathbf{p}}$ is a symplectic manifold endowed with the unique symplectic form $\omega_{\mathbf{p}}(\alpha, \beta) = \omega(\alpha', \beta')$, where ω is the canonical symplectic form on

$T^*SO(3)$, and the vectors α and β tangent to $F_{\mathbf{p}}$ at $[x] \in F_{\mathbf{p}}$ are obtained by projection of some α' and β' tangent to $M_{\mathbf{p}}$ at x [5], [20]. $F_{\mathbf{p}}$ is known as the *reduced phase space*. Given a left-invariant Hamiltonian on $T^*SO(3)$, define the *reduced Hamiltonian* $\mathfrak{h}_{\mathbf{p}} : F_{\mathbf{p}} \to \mathbb{R}$ by $\mathcal{H}|_{M_{\mathbf{p}}} = \mathfrak{h}_{\mathbf{p}} \circ \pi_{\mathbf{p}}$, where $\pi_{\mathbf{p}}$ is the projection $\pi_{\mathbf{p}} : M_{\mathbf{p}} \to F_{\mathbf{p}}$. Then, the trajectories of $X_{\mathcal{H}}$ project to those of $X_{\mathfrak{h}_{\mathbf{p}}}$. One therefore obtains, as an image by reduction of the original Hamiltonian system, another Hamiltonian system on the reduced phase space with the above-mentioned symplectic structure.

By Lemma 2, given $\mathcal{H}$ left-invariant and hyperregular, $M_{\mathbf{p}}$ can be characterized in Eulerian coordinates by the values of $(\varphi, \theta, \psi, \Phi, \Theta, \Psi)$ satisfying (16) for $\mathbf{p}$ fixed. Moreover, to factor out the action of $G_{\mathbf{p}}$ on $M_{\mathbf{p}}$, one recalls that φ is ignorable in $\mathcal{H}$ (see Remark 3) for an arbitrarily chosen spatial frame. We may always choose a spatial frame such that the axis $\mathbf{s}_3$ is parallel to $\mathbf{p}$. In other words, consider $\mathbf{p} \cong (0, 0, G)$ where $G \in \mathbb{R}$ is a nonzero constant. Substituting into (16) and ignoring the singular points corresponding to $\theta = 0$ yields the following result.

PROPOSITION 2 *Let $\mathcal{H} \in \mathcal{F}(SO(3) \times \mathfrak{so}(3)^*)$ be a left-invariant, hyperregular Hamiltonian, and let $\mathbf{p} \in \mathfrak{so}(3)^*$ be fixed. Choose a spatial frame in which $\mathbf{p} \cong (0, 0, G)$, G being a nonzero constant. Relative to this spatial frame, denote the 3-1-3 Euler angles by (φ, θ, l) and their conjugate momenta by (Φ, Θ, L). Then the momentum level set $M_{\mathbf{p}}$ is locally given by*

$$\Theta = 0, \qquad \cos\theta = L/G, \qquad \Phi = G, \tag{18}$$

$(\varphi, l, L) \in (0, 2\pi) \times (-\pi, \pi) \times (-G, G)$. Moreover, the map $\pi_{\mathbf{p}} : M_{\mathbf{p}} \to F_{\mathbf{p}}$ is the coordinate projection $(\varphi, l, L) \mapsto (l, L)$.

Proof: By the choice of spatial frame, (16c) yields immediately $\Phi = G$. Renaming (ψ, Ψ) (l, L), (16a) and (16b) therefore yield $\Theta = 0$ and $L = G\cos\theta$ since $p_1 = p_2 = 0$. This shows that $M_{\mathbf{p}}$ is a 3-dimensional manifold with local coordinates (φ, l, L). Finally, by Proposition 1, elements of the stationary subgroup $G_{\mathbf{p}}$ are rotations of angle φ about $\mathbf{s}_3$ leaving the variables (l, L) fixed. This shows that $\pi_{\mathbf{p}}$ is the coordinate projection along φ. ∎

Remark 5. Following the nomenclature in [11], we shall call (l, L) the *Serret-Andoyer variables*. Geometrically, they are the third direction cosine and conjugate momentum in the 3-1-3 Euler angle representation associated with the specially chosen spatial frame. In fact, one easily verifies that (l, L) are exactly the same geometric objects as those, by the same names, that result from the classical Serret-Andoyer transformation described in Section 3. Moreover, (18) reproduces the classical transformation.

5.2. The Serret-Andoyer Variables as Canonical Coordinates of the Co-Adjoint Orbit

The reduced phase space $F_{\mathbf{p}}$ can be identified with the orbit $\mathcal{O}_{\mathbf{p}}$ of $\mathbf{p} \in \mathfrak{so}(3)^*$ under the co-adjoint action [5]. Indeed, the projection $\pi_{\mathbf{p}}$ has the following expression in body coordinates:

$$\pi_{\mathbf{p}}(\mathbf{R},\mathbf{m}) = \mathrm{Ad}^*_{\mathbf{R}} \circ \mathbf{J}(\mathbf{R},\mathbf{m}) = \mathrm{Ad}^*_{\mathbf{R}}\mathbf{p} = \mathbf{m},$$

for all $(\mathbf{R},\mathbf{m}) \in M_{\mathbf{p}}$. Hence, $\mathcal{O}_{\mathbf{p}}$ is the body representation of $M_{\mathbf{p}}$. Schematically, this can be represented by

$$\begin{tikzcd} & M_{\mathbf{p}} \arrow[dl, "\pi_{\mathbf{p}}"'] \arrow[dr, "\mathbf{J}"] & \\ F_{\mathbf{p}} \simeq \mathcal{O}_{\mathbf{p}} \subset \mathfrak{so}(3)^* & & \{\mathbf{p}\} \subset \mathfrak{so}(3)^* \arrow[ll, "\text{co-adjoint action}"'] \end{tikzcd}$$

More precisely, $\mathcal{O}_{\mathbf{p}}$ is given by

$$\mathcal{O}_{\mathbf{p}} = \{\mathbf{m} \in \mathbb{R}^3 : \mathbf{m} = \mathbf{R}^{-1}\mathbf{p},\ \ \mathbf{R} \in SO(3)\} = \{\mathbf{m} \in \mathbb{R}^3 : \|\mathbf{m}\| = \|\mathbf{p}\|\},$$

i.e., $\mathcal{O}_{\mathbf{p}}$ is the sphere traced by body angular momentum vectors that have magnitude $\|\mathbf{p}\|$, classically known as the *momentum sphere*. $\mathcal{O}_{\mathbf{p}}$ is a symplectic manifold with the unique symplectic forms $\omega^{\pm}$, called *Kostant-Kirillov* symplectic forms, given by

$$\omega^{\pm}_{\mathbf{m}}(\widehat{\mathbf{v}}_{\mathfrak{so}(3)^*}(\mathbf{m}), \widehat{\mathbf{w}}_{\mathfrak{so}(3)^*}(\mathbf{m})) = \pm\mathbf{m}\cdot(\mathbf{v}\times\mathbf{w})\,, \tag{19}$$

$\mathbf{m} \in \mathcal{O}_{\mathbf{p}}$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, where tangent vectors to $\mathcal{O}_{\mathbf{p}}$ have the form $\widehat{\mathbf{v}}_{\mathfrak{so}(3)^*}(\mathbf{m}) = \mathbf{v}\times\mathbf{m}$. See [19] for an exposition; see also [5].

PROPOSITION 3 *The Serret-Andoyer variables (l, L) given in Proposition 2 define a local chart $\mathcal{U}_{\mathbf{p}} : (-\pi,\pi)\times(-G,G) \to \mathcal{O}_{\mathbf{p}}$ given by*

$$\mathbf{m} = \mathcal{U}_{\mathbf{p}}(l, L) = \left(\sqrt{G^2-L^2}\sin l,\ \sqrt{G^2-L^2}\cos l,\ L\right). \tag{20}$$

Moreover, (l, L) are canonical coordinates with respect to the left (-) Kostant-Kirillov symplectic form.

Proof: As remarked above, $\mathcal{O}_{\mathbf{p}}$ is the body representation of $M_{\mathbf{p}}$. Substituting (18), which defines $M_{\mathbf{p}}$, into (6), which gives the expression of body angular momentum, yields (20) after eliminating θ and renaming (ψ, Ψ) (l, L). Differentiating (20) yields

$$\begin{aligned} \dot{\mathbf{m}}_1 &= -\frac{\sin l}{\sqrt{G^2-L^2}} L\,\dot{L} + \sqrt{G^2-L^2}\ \cos l\,\dot{l} = -\frac{\sin l}{\sqrt{G^2-L^2}}\dot{L}\,\mathbf{m}_3 + \dot{l}\,\mathbf{m}_2 \\ \dot{\mathbf{m}}_2 &= -\frac{\cos l}{\sqrt{G^2-L^2}} L\,\dot{L} - \sqrt{G^2-L^2}\ \sin l\,\dot{l} = -\frac{\cos l}{\sqrt{G^2-L^2}}\dot{L}\,\mathbf{m}_3 - \dot{l}\,\mathbf{m}_1 \\ \dot{\mathbf{m}}_3 &= \dot{L}, \end{aligned}$$

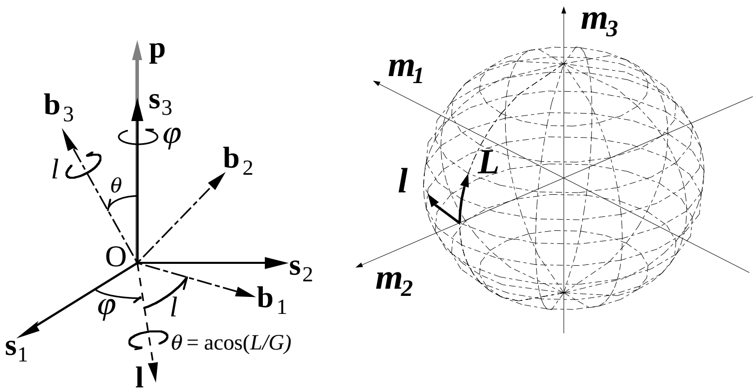


Figure 3. Two-fold interpretation of the Serret-Andoyer variables.

i.e., tangent vectors to $\mathcal{O}_{\mathbf{p}}$ have the form $\dot{\mathbf{m}} = \mathbf{v} \times \mathbf{m}$, where

$$\mathbf{v} = \left(\frac{\cos l}{\sqrt{G^2 - L^2}} \dot{L} \, , \, -\frac{\sin l}{\sqrt{G^2 - L^2}} \dot{L} \, , \, -\dot{l} \right).$$

Substituting into (19) gives

$$\begin{aligned} &\omega^{-}_{\mathbf{m}}(\mathbf{v}_1 \times \mathbf{m}, \mathbf{v}_2 \times \mathbf{m}) \\ &\quad = -\mathbf{m} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \\ &\quad = -\begin{bmatrix} \sqrt{G^2 - L^2} \sin l \\ \sqrt{G^2 - L^2} \cos l \\ L \end{bmatrix} \cdot \left(\begin{bmatrix} \frac{\cos l}{\sqrt{G^2 - L^2}} \dot{L}_1 \\ -\frac{\sin l}{\sqrt{G^2 - L^2}} \dot{L}_1 \\ -\dot{l}_1 \end{bmatrix} \times \begin{bmatrix} \frac{\cos l}{\sqrt{G^2 - L^2}} \dot{L}_2 \\ -\frac{\sin l}{\sqrt{G^2 - L^2}} \dot{L}_2 \\ -\dot{l}_2 \end{bmatrix} \right) \\ &\quad = \dot{l}_1 \dot{L}_2 - \dot{l}_2 \dot{L}_1 \\ &\quad = dl \wedge dL \left((\dot{l}_1, \dot{L}_1), (\dot{l}_2, \dot{L}_2) \right), \end{aligned}$$

which proves that (l, L) are canonical local coordinates. ∎

Remark 6. The Serret-Andoyer variables as geometric objects thus have two meanings. On the one hand, they are Eulerian coordinates as mentioned in Remark 5. On the other hand, by Proposition 3, they locally canonically coordinatize the 2-dimensional symplectic manifold $\mathcal{O}_{\mathbf{p}}$. This geometric interpretation is summarized in Figure 3. In particular, as shown in Figure 3(b), (l, L) can be viewed as the 'longitude' and 'latitude' on the momentum sphere. In addition, the canonical symplectic form can be viewed as the area element $dl \wedge dL$ that is oriented inward, which corresponds to the $(-)$ sign of the left Kostant-Kirillov symplectic form.

5.3. Main Result

As an immediate result of Propositions 2 and 3, the Serret-Andoyer transformation can be generalized for any Hamiltonian system on $T^*SO(3)$ with a left-invariant, hyperregular Hamiltonian. Indeed, identifying the reduced phase space with the co-adjoint orbit, the reduced Hamiltonian system lives on the latter. Since the Serret-Andoyer variables are canonical coordinates of the co-adjoint orbit, dynamics of the reduced system are thus given in canonical symplectic form in these variables.

THEOREM 3 (THE GENERALIZED SERRET-ANDOYER TRANSFORMATION) *Let $\mathcal{H} \in \mathcal{F}(SO(3) \times \mathfrak{so}(3)^*)$ be a left-invariant, hyperregular Hamiltonian, and let $\mathbf{p} \in \mathfrak{so}(3)^*$ be the conserved spatial momentum. Under the conditions of Proposition 2, the reduced Hamiltonian $\mathfrak{h}_\mathbf{p}$ is locally given in the Serret-Andoyer variables by*

$$\mathfrak{h}_\mathbf{p}(l, L) = \mathcal{H} \circ \mathcal{U}_\mathbf{p}(l, L), \tag{21}$$

$(l, L) \in (-\pi, \pi) \times (-L, L)$, where $\mathcal{U}_\mathbf{p}$ is defined by (20). The reduced dynamics are then given in canonical form, that is, $\dot{l} = \partial\mathfrak{h}_\mathbf{p}/\partial L$, $\dot{L} = -\partial\mathfrak{h}_\mathbf{p}/\partial l$. Moreover, relative to the spatial frame defined in Proposition 2, the integral solution in $SO(3)$ is characterized by the 3-1-3 Euler angles (φ, θ, l), with $\cos\theta = L/G$ and $\varphi = \int \partial\mathcal{H}/\partial\Phi|_{M_{\mathbf{p}_p}} dt$.

Remark 7.

1. The last equality, $\varphi = \int \partial\mathcal{H}/\partial\Phi|_{M_\mathbf{p}} dt$, results from the fact that φ is ignorable for $\mathcal{H}$ left-invariant (recall Remark 3). Note that we are taking the restriction of $\partial\mathcal{H}/\partial\Phi$ on the momentum level set, $M_\mathbf{p}$, according to (18). Since $\partial\mathcal{H}/\partial\Phi|_{M_\mathbf{p}}$ is a function solely of (l, L), $\int \partial\mathcal{H}/\partial\Phi|_{M_\mathbf{p}} dt$ is a line integral when $(l(t), L(t))$ are available.

2. The construction leading to Theorem 3 shows that the Serret-Andoyer transformation is the computation in Eulerian coordinates of the symplectic reduction associated with the lifted left-action of $SO(3)$ on $T^*SO(3)$. In particular, the choice by Serret for the axis $\mathbf{k}$ to be in the direction of the spatial angular momentum yields precisely the conditions of Proposition 2.

3. The result of Theorem 3 provides a reduced representation of the class of systems in question. This representation is given in a two-dimensional phase space which one can think of as the unit circle S^1. As we shall discuss in Section 8, this simplifies the numerical integration of the equations of motion to that of S^1 dynamics. One then reconstructs the full dynamics on $T^*SO(3)$ by solving, in closed form, $\theta = \arccos(L/G)$ on the one hand, and taking the line integral $\varphi = \int \partial\mathcal{H}/\partial\Phi|_{M_\mathbf{p}} dt$ on the other hand.

Example. [The Classical Serret-Andoyer transformation] One recovers immediately the results for the free rigid body. Indeed, as mentioned in Remark 5, (l, L) are the same geometric objects as encountered in the classical transformation which is then reproduced by the characterization according to (18) of the reduced phase space. Finally, substituting the free rigid body Hamiltonian, $\mathcal{H} = (\mathbb{I}^{-1}\mathbf{m}) \cdot \mathbf{m}$, into (21) yields (10).

Theorem 3 generalizes the Serret-Andoyer transformation to a large class of rigid motions other than the usual one. In particular, one can consider rigid bodies subject to control by means of internal torques. The presence of control *a priori* breaks the original symmetry of the phase space, which now consists of $T^*SO(3)$ and the shape space. The basic idea, then, is that with Hamiltonian controls that preserve the symmetry on $T^*SO(3)$, so that the motion of the main body or base of the controlled system becomes that of a new (controlled) left-invariant Hamiltonian vector field on $T^*SO(3)$, Noether's theorem still holds. A controlled momentum vector can then be found that is preserved in space. In the case where the Hamiltonian is also hyperregular, the results of Section 5 can then be applied, yielding a set of Serret-Andoyer variables for the controlled motion of the main body.

6. Rigid Body with Single Symmetric Rotor

Consider now a system consisting of a main body (the base) equipped with a single, symmetric rotor aligned with the third principle axis (see Figure 4). Let $J_1 = J_2$ and J_3 be the moments of inertia of the symmetric rotor, and denote by γ the angular position of the rotor relative to the body. The Lagrangian of the free system, i.e., in the absence of control, is given by [7]

$$L_f(\boldsymbol{\omega}, \dot{\gamma}) = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbb{I} + \mathbb{J})\boldsymbol{\omega} + \frac{1}{2} J_3(\omega_3 + \dot{\gamma})^2, \tag{22}$$

where $\mathbb{I} = \operatorname{diag}(I_1, I_2, I_3)$ and $\mathbb{J} = \operatorname{diag}(J_1, J_2, 0)$. The corresponding Legendre's transform is given by

$$\mathbb{F}L_f : \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \dot{\gamma} \end{bmatrix} \longmapsto \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \Gamma \end{bmatrix} = \begin{bmatrix} (J_1 + I_1)\omega_1 \\ (J_2 + I_2)\omega_2 \\ I_3\omega_3 + J_3(\omega_3 + \dot{\gamma}) \\ J_3(\omega_3 + \dot{\gamma}) \end{bmatrix}, \tag{23}$$

which yields the free Hamiltonian

$$\mathcal{H}_f(\mathbf{m}, \Gamma) = \frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{(m_3 - \Gamma)^2}{I_3}\right) + \frac{\Gamma^2}{2\,J_3}, \tag{24}$$

where $\lambda_i = I_i + J_i$, $i = 1, 2$.

The configuration space of the present problem is the Lie Group $SO(3) \times S^1$, with Lie algebra $\mathfrak{so}(3)^* \times \mathbb{R}$. The group action in question is $L_{(\mathbf{R},\gamma)}(\mathbf{S}, \phi) = (\mathbf{RS}, \gamma + \phi)$. As in the case of the free rigid body (recall Section 4), the rigid body with rotor admits a body representation via the mapping $\bar{\lambda} : T^*SO(3) \times T^*S^1 \to (SO(3) \times \mathfrak{so}(3)^*) \times (S^1 \times \mathbb{R}) : (\alpha_{\mathbf{R}}, \gamma, \Gamma) \mapsto (\mathbf{R}, T_e^* L_{\mathbf{R}} \cdot \alpha_{\mathbf{R}}, \gamma, \Gamma)$. As usual, $\mathfrak{so}(3)^*$ is identified with $\mathbb{R}^3$.

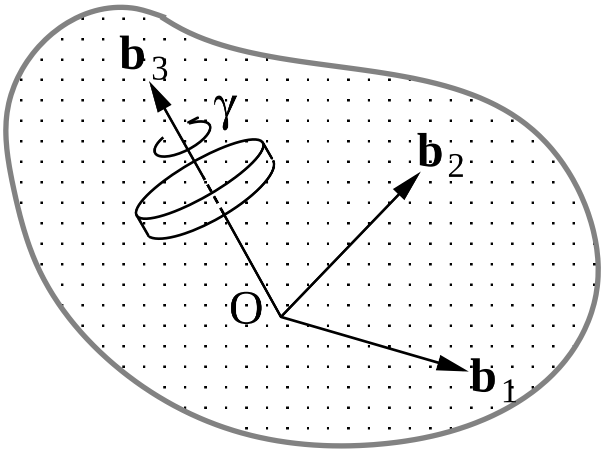


Figure 4. Rigid body with single symmetric rotor.

Let $\mathcal{H} \in \mathcal{F}(\mathfrak{so}(3)^* \times \mathbb{R}))$, in other words, a left-invariant smooth function on the cotangent bundle. It can be shown, by a result analogous to Lemma 1 for the Lie group $G = SO(3) \times S^1$, that the associated Hamiltonian vector field is given in body coordinates by

$$X_{\mathcal{H}}(\mathbf{R}, \mathbf{m}, \gamma, \Gamma) = \left(\mathbf{R} \cdot \widehat{\mathbf{D}_1 \mathcal{H}}\,,\ \mathbf{m} \times \mathbf{D}_1 \mathcal{H}\,,\ \partial \mathcal{H}/\partial \Gamma\,,\ -\partial \mathcal{H}/\partial \gamma\right), \tag{25}$$

where $\mathbf{D}_1$ denotes the derivative with respect to the first argument. In particular, the equations of motion for the free system are obtained with $\mathcal{H} = \mathcal{H}_f$. In addition, let the system be feedback-controlled by applying a torque $u(\mathbf{R}, \mathbf{m}, \gamma, \Gamma)$ to the rotor, which then yields the following controlled equations of motion.

$$\dot{\mathbf{R}} = \mathbf{R} \cdot \widehat{\mathbf{D}_1 \mathcal{H}_f}, \tag{26a}$$
$$\dot{\mathbf{m}} = \mathbf{m} \times \mathbf{D}_1 \mathcal{H}_f, \tag{26b}$$
$$\dot{\gamma} = \partial \mathcal{H}_f / \partial \Gamma, \tag{26c}$$
$$\dot{\Gamma} = u(\mathbf{R}, \mathbf{m}, \gamma, \Gamma). \tag{26d}$$

6.1. Structure Preserving Control

Definition 2. We say that the control $u(\mathbf{R}, \mathbf{m}, \gamma, \Gamma)$ *preserves the canonical structure on $T^*SO(3)$* or *preserves the rigid body structure* if there exists a smooth function $\mathcal{H}_c \in \mathcal{F}(\mathfrak{so}(3)^*)$ such that the closed-loop equations of the base motion have the form

$$\dot{\mathbf{R}} = \mathbf{R} \cdot \widehat{\nabla_{\mathbf{m}} \mathcal{H}_c}, \tag{27a}$$
$$\dot{\mathbf{m}} = \mathbf{m} \times \nabla_{\mathbf{m}} \mathcal{H}_c. \tag{27b}$$

That is, a control that preserves the rigid body structure yields closed-loop Euler's equation that is Hamiltonian with respect to the usual Lie-Poisson structure on $\mathfrak{so}(3)^*$. Moreover, one can easily verify the following lemma which gives a sufficient condition for such a control.

LEMMA 4 *Given the controlled equations of motion (25), a sufficient condition for the control u to preserve the rigid body structure is that, along the flow of the closed-loop system,*

i. *Γ is a function of $\mathbf{m}$, i.e., $\Gamma = \Gamma(\mathbf{m})$, and*

ii.
$$\nabla_{\mathbf{m}}\mathcal{H}_c(\mathbf{m}) = \mathbf{D}_1\mathcal{H}_f(\mathbf{m}, \Gamma(\mathbf{m})). \tag{28}$$

Remark 8. Abesser and Steigenberger [1] discussed Hamiltonian control systems that preserve the structure of the entire phase space. However, for our purpose, Definition 2 suffices.

Bloch *et al*. [7] gave a Hamiltonian control for which the closed-loop reduced equations are Lie-Poisson on $\mathfrak{so}(3)^*$. In fact, they satisfy (28). We shall prove in the following a slightly more general result.

PROPOSITION 4 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Then, the feedback control*

$$u(\mathbf{m}) = \phi'(m_3)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) m_1 m_2 \tag{29a}$$

preserves the rigid body structure with the closed-loop Hamiltonian

$$\mathcal{H}_c(\mathbf{m}) = \frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{m_3^2}{I_3}\right) - \frac{1}{I_3}\int (\phi(m_3) + p)\, dm_3, \tag{29b}$$

where p is a constant.

Proof: Expanding (25b) and (25c), one gets the following:

$$\dot{m}_1 = \left(\frac{1}{I_3} - \frac{1}{\lambda_2}\right) m_2 m_3 - \frac{1}{I_3}\Gamma m_2, \tag{30a}$$

$$\dot{m}_2 = \left(\frac{1}{\lambda_1} - \frac{1}{I_3}\right) m_1 m_3 + \frac{1}{I_3}\Gamma m_1, \tag{30b}$$

$$\dot{m}_3 = \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) m_1 m_2, \tag{30c}$$

$$\dot{\Gamma} = u. \tag{30d}$$

It can thus be seen that, given (29a), the quantity $p = \Gamma - \phi(m_3)$ is conserved and, hence, ***i***. in Lemma 4 is satisfied. Next, by (29b),

$$\begin{aligned}\nabla_{\mathbf{m}}\mathcal{H}_c(\mathbf{m}) &= \left(\frac{m_1}{\lambda_1}, \frac{m_2}{\lambda_2}, \frac{m_3}{I_3} - \frac{1}{I_3}(\phi(m_3) + p)\right) \\ &= \left(\frac{m_1}{\lambda_1}, \frac{m_2}{\lambda_2}, \frac{1}{I_3}(m_3 - \Gamma(m_3))\right) \\ &= \mathbf{D}_1\mathcal{H}_f(\mathbf{m}, \Gamma(\mathbf{m})),\end{aligned}$$

i.e., (28), which completes the proof. ■

Remark 9. Setting $\phi(v) = kv$, where k is a constant, recovers the result in Theorem 5.1 of [7].

6.2. Serret-Andoyer Variables for the Control System

By Proposition 4, the base motion of the system subject to the control (29a) is that of a Hamiltonian system on $T^*SO(3)$ with the left-invariant Hamiltonian $\mathcal{H}_c$. The expression of the Hamiltonian (and the associated Lagrangian) depend ultimately on the definition of the function ϕ. Nevertheless, we are able to proceed implicitly as follows.

THEOREM 4 *Suppose that $\phi'(v) \neq 1$ for all $v \in \mathbb{R}$. Then, the Hamiltonian $\mathcal{H}_c$ given by (29b) is hyperregular, and the closed-loop main body motion of the system (26) with the control (29a) is reduced by the generalized Serret-Andoyer transformation to*

$$\dot{l} = -L\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) + \frac{1}{I_3}\left(L - \phi(L) - p\right), \tag{31a}$$

$$\dot{L} = (G^2 - L^2)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\sin l \cos l \tag{31b}$$

with the reduced Hamiltonian

$$\mathfrak{h}_{\mathbf{p}}(l, L) = \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) + \frac{L^2}{2I_3} - \frac{1}{I_3}\int(\phi(L) + p)\,dL. \tag{32}$$

Moreover, in the spatial frame defined in Proposition 2, the closed-loop motion of the main body is described by the 3-1-3 Euler angles (φ, θ, l), with $\cos\theta = L/G$ and $\varphi = \int \partial\mathcal{H}_c/\partial\Phi|_{M_{\mathbf{p}}}dt$.

Proof: We need only to prove that $\mathcal{H}_c$ is hyperregular, providing which the rest of Theorem 4 is a direct application of Theorem 3. By the inverse Legendre transform for the closed loop, i.e., $\boldsymbol{\omega} = \nabla_{\mathbf{m}}H_c$, one gets

$$\begin{aligned}\omega_1 &= m_1/\lambda_1, \\ \omega_2 &= m_2/\lambda_2, \\ \omega_3 &= (m_3 - \phi(m_3) - p)/I_3.\end{aligned}$$

Since $\phi'(v) \neq 1$ for all $v \in \mathbb{R}$, the above is invertible with differentiable inverse by the implicit function theorem. Hence, the inverse Legendre transform is a diffeomorphism, i.e., $\mathcal{H}_c$ is hyperregular. ■

Example. Let $\phi(v) = kv$, $k \neq 1$. Then,

$$\mathfrak{h}_{\mathbf{p}}(l, L) = \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) + \frac{((1-k)L - p)^2}{2I_3(1-k)}. \tag{34}$$

In particular, it can be verified that the case where the rotor is locked corresponds to $p = 0$ and $k = J_3/(I_3 + J_3)$. Substituting the latter expression of k into (34) recovers the reduced Hamiltonian for the free rigid body with moments of inertia $I_i + J_i$, $i = 1, \ldots, 3$.

6.3. Spin Stabilization About the Intermediate Axis

Suppose in the following that $\lambda_1 > \lambda_2 > I_3 + J_3$, so that the second body axis is the intermediate axis of the locked system. An immediate consequence of Theorem 4 is a simpler stability proof for relative equilibria. First, note that rotation about the intermediate axis, i.e., $\mathbf{m} = (0, G, 0)$, corresponds to an equilibrium point at (l, L) in the reduced phase space. In [7], the energy-Casimir method was used to prove stability of the relative equilibria $\mathbf{m} = (0, G, 0)$ for the closed-loop Lie-Poisson system. However, for Hamiltonian systems in canonical symplectic form, which is the case of the reduced system (31), the classical Lagrange-Dirichlet stability criterion suffices. In effect, the point $(l, L) = (0, 0)$ is a stable equilibrium in the sense of Lyapunov if the partial derivatives of $\mathfrak{h}_{\mathbf{p}}$ vanish at $(0, 0)$, and if the 2×2 matrix $\delta^2\mathfrak{h}_{\mathbf{p}}$ of second partial derivatives evaluated at $(0, 0)$ is either positive- or negative-definite. See [17] for a statement and proof of the Lagrange-Dirichlet criterion. The following generalizes Theorem 5.2 in [7].

THEOREM 5 *Consider the case $\phi(0) + p = 0$ and $\phi'(0) > 1 - I_3/\lambda_2$. Then, the point $(l, L) = (0, 0)$ of the reduced system (31) is stable in the sense of Lyapunov and, hence, the control (29a) stabilizes rotation about the intermediate axis of the body-rotor system.*

Proof: From (32), $\partial\mathfrak{h}_{\mathbf{p}}/\partial l = 0$, and $\partial\mathfrak{h}_{\mathbf{p}}/\partial L = -(\phi(0) + p)/I_3$ which equals zero if $\phi(0) + p = 0$. The point $(0, 0)$ is thus an equilibrium point. Next,

$$\begin{aligned}
\frac{\partial^2\mathfrak{h}_{\mathbf{p}}}{\partial l^2} &= (G^2 - L^2)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)(\sin^2 l - \cos^2 l),\\
\frac{\partial^2\mathfrak{h}_{\mathbf{p}}}{\partial l \partial L} &= 2L\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\sin l \cos l,\\
\frac{\partial^2\mathfrak{h}_{\mathbf{p}}}{\partial L^2} &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\sin^2 l + \left(\frac{1}{I_3} - \frac{1}{\lambda 2}\right) - \frac{1}{I_3}\phi'(L).
\end{aligned}$$

Hence,

$$\delta^2\mathfrak{h}_{\mathbf{p}}(0, 0) = \begin{bmatrix} -G^2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) & 0 \\ 0 & \left(\frac{1}{I_3} - \frac{1}{\lambda 2}\right) - \frac{1}{I_3}\phi'(0) \end{bmatrix},$$

which is (negative) definite for $\phi'(0) > 1 - I_3/\lambda_2$. The Lagrange-Dirichlet criterion is thus satisfied. ■

7. Rigid Body with Three Symmetric Rotors

The Serret-Andoyer analysis can also be applied to a system with three rotors. Indeed, consider now the rigid body equipped with three symmetric rotors, each aligned with a principal axis of inertia of the rotor. The Lie group in question is $SO(3) \times S^3$, with $(SO(3) \times \mathfrak{so}(3)^*) \times (S^3 \times \mathbb{R}^3)$ as cotangent bundle in body representation. The Lagrangian of the free (uncontrolled) system is [7]

$$\mathcal{L}_f(\boldsymbol{\omega}, \dot{\gamma}) = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\Lambda}\boldsymbol{\omega} + \sum_{i=1}^{3} \frac{J_i}{2}(\omega_i + \dot{\gamma}_i) \tag{35}$$

for $\boldsymbol{\omega} \in \mathfrak{so}(3) \simeq \mathbb{R}^3$, $\dot{\gamma} \in \mathbb{R}^3$, where $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ is the locked inertia tensor which is diagonal by the assumption that the rotors are symmetric and aligned with the principal axes. J_i, $i = 1, \ldots, 3$ are the rotors' moments of inertia along their respective axes of rotation. By the Legendre transform, the conjugate momenta are

$$\mathbb{F}\mathcal{L}_f : \begin{bmatrix} \omega_i \\ \dot{\gamma}_i \end{bmatrix} \longmapsto \begin{bmatrix} m_i \\ \Gamma_i \end{bmatrix} = \begin{bmatrix} \lambda_i\omega_i + J_i(\omega_i + \dot{\gamma}_i) \\ J_i(\omega_i + \dot{\gamma}_i) \end{bmatrix}, \qquad i \in \{1, \ldots, 3\}, \tag{36}$$

and the free Hamiltonian is left-invariant and is given by

$$\mathcal{H}_f(\mathbf{m}, \Gamma) = \frac{1}{2}\left(\frac{(m_1 - \Gamma_1)^2}{\lambda_1} + \frac{(m_2 - \Gamma_2)^2}{\lambda_2} + \frac{(m_3 - \Gamma_3)^2}{\lambda_3}\right) + \frac{1}{2}\left(\frac{\Gamma_1^2}{J_1} + \frac{\Gamma_2^2}{J_2} + \frac{\Gamma_3^2}{J_3}\right). \tag{37}$$

Introducing control inputs in the form of torques on the rotors, the generic controlled equations are

$$\dot{\mathbf{R}} = \mathbf{R} \cdot \widehat{\mathbf{D}_1\mathcal{H}_f}, \tag{38a}$$
$$\dot{\mathbf{m}} = \mathbf{m} \times \mathbf{D}_1\mathcal{H}_f, \tag{38b}$$
$$\dot{\gamma} = \mathbf{D}_2\mathcal{H}_f, \tag{38c}$$
$$\dot{\Gamma} = \mathbf{u}. \tag{38d}$$

In the following, we consider feedback control of the form $\mathbf{u} : (SO(3) \times \mathfrak{so}(3)^*) \times (S^3 \times \mathbb{R}^3) \to \mathbb{R}^3$.

As in the case of the system with a single rotor, we are interested in controls that preserve the rigid body structure as defined in Definition 2. We recall below a large class of Hamiltonian controls given in [7] that satisfy the conditions of a straightforward extension of Lemma 4 for the present system.

PROPOSITION 5 *Let* $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ *be a smooth map such that the* 3×3 *matrix* $\mathbf{D}\phi(\mathbf{m})$ *is symmetric for all* $\mathbf{m} \in \mathbb{R}^3$. *Then the feedback controls*

$$\mathbf{u} = \mathbf{D}\phi(\mathbf{m}) \cdot \dot{\mathbf{m}} \tag{39}$$

for the system (38) preserve the rigid body structure in the sense of Lemma 4.

The proof of the above is based on the properties that the given control conserves the quantity $\Gamma - \phi(\mathbf{m})$, and that the symmetry condition guarantees that there exists an $\mathcal{H}_c(\mathbf{m})$ such that $\nabla_{\mathbf{m}}\mathcal{H}_c = \mathbf{D}_1\mathcal{H}_f(\mathbf{m}, \Gamma(\mathbf{m}))$. See [7, Theorem 4.2] for the details.

For the sub-class of (39) where $\phi(\mathbf{m}) = (\phi_1(m_1), \phi_2(m_2), \phi_3(m_3))$, so that the symmetry condition in Proposition 5 is satisfied, the Hamiltonian is quite easily obtained. We shall use this to demonstrate the application of the generalized Serret-Andoyer transformation for the rigid body with three rotors, bearing in mind that even in the general case, the same can be done if the expression of ϕ is given.

COROLLARY 2 *Let $\phi_i : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^1$ functions, $i = 1, \ldots, 3$. Then, the feedback control $\mathbf{u} = (u_1, u_2, u_3)(\mathbf{m})$ for the system (38), defined by*

$$u_i = \phi_i'(m_i)\dot{m}_i, \qquad i \in \{1, 2, 3\} \tag{40}$$

preserves the rigid body structure in the sense of Lemma 4, with the closed-loop Hamiltonian

$$\mathcal{H}_c(\mathbf{m}) = \sum_{i=1}^{3} \frac{1}{\lambda_i} \int (m_i - \phi_i(m_i) - p_i)\, dm_i, \tag{41}$$

where p_1, p_2 and p_3 are constants. Moreover, if $\phi_i'(v) \neq 1$, $i = 1, \ldots, 3$, then $\mathcal{H}_c$ is hyperregular.

Proof: Observe by expanding (38) that the control (40) conserves the quantities $p_i = \Gamma_i - \phi(m_i)$, $i = 1, \ldots, 3$. Then it can easily be verified by taking partial derivatives of (41) that (28) is satisfied. The inverse Legendre transform of the controlled Hamiltonian, $\boldsymbol{\omega} = \nabla_{\mathbf{m}}\mathcal{H}_c$, relates the body controlled momentum $\mathbf{m}$ and the body angular velocity $\boldsymbol{\omega}$ by

$$\omega_i = (m_i - \phi_i(m_i) - p_i)/\lambda_i$$

for $i = 1, \ldots, 3$. Since $\phi_i'(v) \neq 1$ for all $v \in \mathbb{R}$, the above is invertible with differentiable inverse by the implicit function theorem. Hence, $\mathcal{H}_c$ is hyperregular. ■

The following are then immediate applications of Theorem 3.

THEOREM 6 *Suppose $\phi_i'(v) \neq 1$, $i = 1, \ldots, 3$, for all $v \in \mathbb{R}$. Let $\tilde{\phi}_1 = \phi_1 \circ \mathcal{U}_1$ and $\tilde{\phi}_2 = \phi_2 \circ \mathcal{U}_2$. Then the closed-loop main body motion of the system (38) with the control (40) is reduced by the generalized Serret-Andoyer transformation to $\dot{l} = \partial\mathfrak{h}_{\mathbf{p}}/\partial L$, $\dot{L} = -\partial\mathfrak{h}_{\mathbf{p}}/\partial l$ with the reduced Hamiltonian*

$$\begin{aligned}\mathfrak{h}_{\mathbf{p}}(l, L) \;=\; & \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) + \frac{L^2}{2\lambda_3} - \frac{1}{\lambda_3}\int(\phi_3(L) + p_3)dL \\ & - \sqrt{G^2 - L^2}\int\left[(\tilde{\phi}_1(l, L) + p_1)\frac{\cos l}{\lambda_1} - (\tilde{\phi}_2(l, L) + p_2)\frac{\sin l}{\lambda_2}\right]dl. \end{aligned} \tag{42}$$

Moreover, in the spatial frame defined in Proposition 2, the closed-loop motion of the main body is described by the 3-1-3 Euler angles (φ, θ, l)*, with* $\cos\theta = L/G$ *and* $\varphi = \int \partial\mathcal{H}_c/\partial\Phi|_{M_\mathbf{p}} dt$.

For the particular case where $\phi_1 = \phi_2 = 0$, that is, control is applied to rotor 3 only, the momenta Γ_1 and Γ_2 are integrals of motion. We thus have the following.

COROLLARY 3 *Under the conditions of Theorem 6, and choosing* $\phi_1 = \phi_2 = 0$*, the reduced Hamiltonian is then given by*

$$\begin{aligned}\mathfrak{h}_\mathbf{p}(l, L) \;=\; & \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) - \left(\frac{\Gamma_1 \sin l}{\lambda_1} + \frac{\Gamma_2 \cos l}{\lambda_2}\right)\sqrt{G^2 - L^2} \\ & + \frac{L^2}{2\lambda_3} - \frac{1}{\lambda_3}\int(\phi_3(L) + p_3)\,dL, \end{aligned} \tag{43}$$

where Γ_1, Γ_2 *and* p_3 *are constants.*

8. Numerical Integration of Controlled Equations

We conclude this study by presenting some results in numerical integration of the controlled equations. It is well-known that integrating the full 6-dimensional equations of motion presents some distinct disadvantages compared to integrating a reduced set of equations such as the Euler equations, or the 2-dimensional reduced dynamics presented in this paper. In addition to confirming this, we shall also present various ways of integrating the reduced dynamics.

We consider the system of Section 6 with a stabilizing control according to Theorems 4 and 5 and given by:

$$\phi(v) = 2\tan\left(\frac{v}{G}\right), \qquad\qquad p = 0.$$

The closed-loop system is a left-invariant Hamiltonian system with conserved momentum of magnitude $G = 1$ and Hamiltonian $\mathcal{H}_c = 0.47$.

8.1. Modeling and Integration Algorithms

The closed-loop equations of motion of the main body are integrated in the following five approaches, differing in representation and integration algorithm:

1. The full 6-dimensional equations in the variables $(\varphi, \theta, \psi, \Phi, \Theta, \Psi)$, with Adams' integration method.

2. Euler's equations, i.e., (27) and Hamiltonian $\mathcal{H}_c$ given by (29b), and Adams' integration method.

3. The reduced dynamics in the Serret-Andoyer variables, i.e., (31) and reduced Hamiltonian $\mathfrak{h}_\mathbf{p}$ given by (32), with Adams' integration method.

4. The reduced dynamics as above but with a symplectic integrator based on a generating function.

5. The reduced dynamics as above but with a symplectic integrator based on Euler's mid-point rule.

Adams' method is well-known to be *strongly stable* [15, §2.7–§2.8]. However, like most general integration algorithms, Adams' method does not respect the structure of Hamiltonian systems, and long-time simulation may fail to numerically preserve conserved quantities. Integration errors in the Hamiltonian and momentum functions can thus be used to discriminate among the above approaches. Note, however, that the Serret-Andoyer variables already present an advantage by naturally preserving momentum except for some round-off errors. We note that one may alternatively consider the approach of Crouch and Grossman [10] that may help preserve momentum by embedding the momentum level set in some Euclidean space.

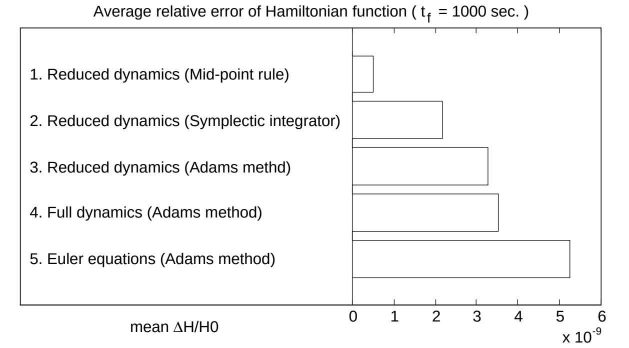
Symplectic integrators, on the other hand, approximate the flow of a Hamiltonian system with a symplectic map. Nevertheless, a theorem found in [13] implies that even such approximations cannot preserve the Hamiltonian function unless the resulting solution is exact up to a reparametrization of time. Here, we apply two second-order symplectic integrators to the reduced dynamics which, as we know, are symplectic. In 4., the symplectic integrator is based on an update law $(l_k, L_k) \mapsto (l_{k+1}, L_{k+1})$ of the form $l_k = \partial S/\partial L_k$, $L_{k+1} = \partial S/\partial l_{k+1}$, where S is a second-order generating function. See [9] for more details. In 5., we discretize the reduced system according to the following mid-point rule:

$$\begin{aligned}\frac{l_{k+1} - l_k}{h} &= \frac{\partial \mathfrak{h}_{\mathbf{p}}}{\partial L}\left(\frac{l_{k+1} + l_k}{2}, \frac{L_{k+1} + L_k}{2}\right), \\ \frac{L_{k+1} - L_k}{h} &= -\frac{\partial \mathfrak{h}_{\mathbf{p}}}{\partial l}\left(\frac{l_{k+1} + l_k}{2}, \frac{L_{k+1} + L_k}{2}\right),\end{aligned}$$

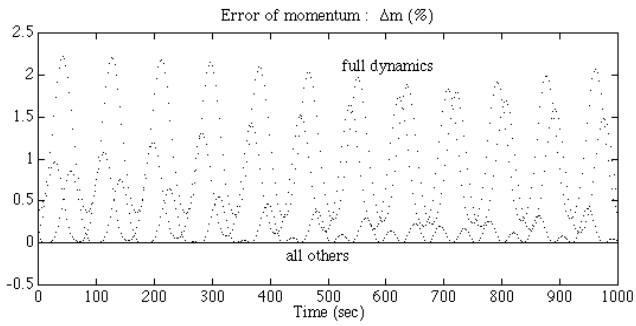
where h is the time step. It was shown in [6] and [12] that Euler's mid-point rule is a symplectic integrator. Moreover, following the standard analysis of [15], one can show that Euler's mid-point rule is a second-order accurate approximation of the differential equations it integrates.

We mention in passing that there exist Lie-Poisson symplectic integrators applicable to the Euler equations representation which we shall omit in the present study; see, for example, [13] and [22]. See also articles in [18] on other implementations of symplectic integrators.

Remark 10. In the case of a hyperregular Hamiltonian, it can be shown that Euler's mid-point rule applied to Hamilton's equations is equivalent to the *discrete Euler-Lagrange* equations. In [21], it was shown that these equations admit a *discrete momentum map* which is conserved with further assumption of invariance under *diagonal action*—a 'discrete Noether's theorem'. For the class of rigid body dynamics considered in this paper, one might want to first discretize the full dynamics on $T^*SO(3)$, then perform the reduction. Unfortunately, when cast in Eulerian coordinates, the resulting discrete equations are not diagonally invariant. However, such an approach appears to be applicable in the quaternionic setting.



(a) Error of Hamiltonian function



(b) Error of angular momentum

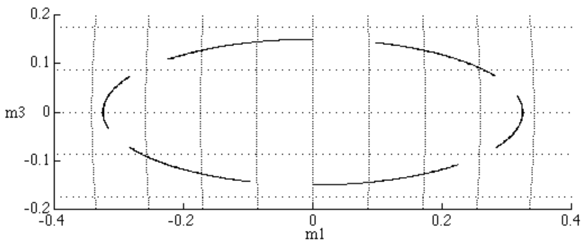
Figure 5. Comparison of errors in numerical values of the Hamiltonian function and angular momentum.

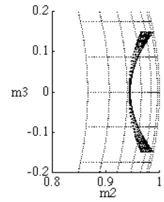
8.2. Numerical Results

The following results are obtained using MATLAB simulation tools. The time step is chosen to be $h = 1$ ms for all five approaches, with round-off tolerance of 1e-6 for Adams' method and 1e-8 for the symplectic integrators. Initial conditions are chosen such that the resulting trajectories are those of stabilized rotation about the intermediate axis.

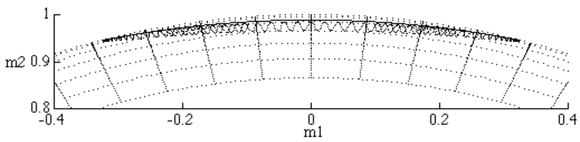
As shown in Figure 5(a), the mid-point rule gives the best performance in preserving energy, whereas the reduced representation generally performs better regardless of algorithm. In addition, one finds in Figure 5(b) that all approaches except the full dynamics preserve momentum. Further evidence of momentum preservation (or failure thereof) is given in Figure 6, where it can be seen, especially in Figures 6(b) and 6(c), that discrepancies in the full dynamics produce trajectories *inside* the sphere of radius 1. Finally, the reduced flows in the Serret-Andoyer variables (post-processed for the Euler equations and full dynamics) are shown in Figure 7.

The above results confirm the numerical advantages presented by the reduced equations in the Serret-Andoyer variables. Moreover, the symplectic nature of these equations allows

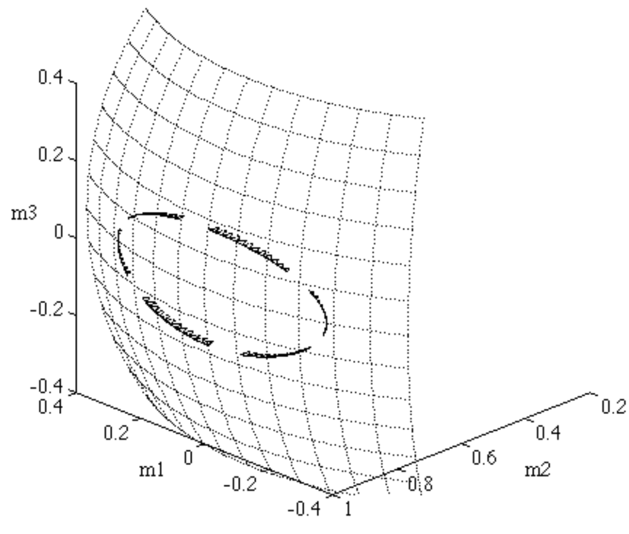




(a) projection along m_2 (b) projection along m_1



(c) projection along m_3



(d) perspective view

Figure 6. Trajectories of body angular momentum: projected and perspective views.

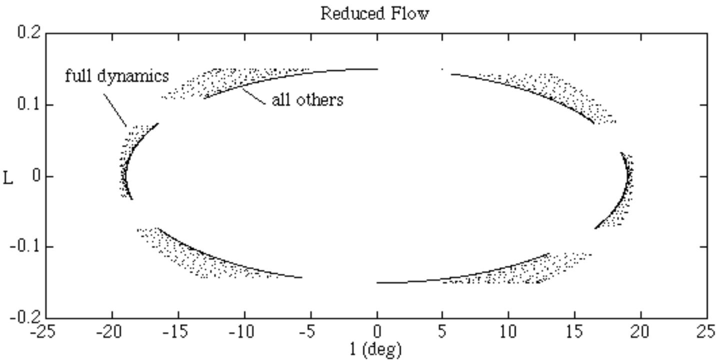


Figure 7. Reduced flow.

simple symplectic integrators such as Euler's mid-point rule to be employed. Although symplectic integrators are usually CPU-intensive, this is compensated by the reduced number of equations—two in the Serret-Andoyer variables.

9. Conclusions

We have shown in this paper that the classical Serret-Andoyer transformation can be understood in the context of geometric mechanics and, in particular, in that of the symplectic (Marsden-Weinstein) reduction of Hamiltonian systems with symmetry. This understanding proves to be useful not only in allowing the transformation to be reconstructed without heavy computations involving spherical trigonometry or differential forms. It is most valuable in enabling us to generalize, in a straightforward fashion, the Serret-Andoyer transformation to a rich class of rigid motions, namely, Hamiltonian systems on $T^*SO(3)$ with left-invariant, hyperregular Hamiltonian functions.

The significance of the analysis is made apparent in Sections 6 and 7, where we examine the dynamics of the controlled rigid body with momentum wheels, for a non-trivial but rich class of Hamiltonian controls that yields closed-loop dynamics with symmetry on $T^*SO(3)$. The key point to note is that, although the presence of feedback control deforms the Hamiltonian structure of the unforced system, if this deformation is such that the symmetry of rigid motions (with a different metric) is preserved, then the generalized Serret-Andoyer transformation is immediately applicable, and reduces the closed-loop motion of the main body to a 2-dimensional Hamiltonian system in canonical form. This computation proves to be useful in that the stability proof of relative equilibria becomes simpler, by verifying the classical Lagrange-Dirichlet criterion. This approach appears to be applicable in various settings, see, e.g., [8]. For asymptotic stability, one can consider a combination of this approach with dissipative feedback of a suitable type. The Serret-Andoyer variables may also be useful for analyzing more complicated coupled rigid body systems.

Additionally, we discuss the numerical integration of the reduced closed-loop dynamics, and compare the solutions with those obtained using other representations. We confirm numerically that the Serret-Andoyer variables naturally preserve momentum; in addition, for moderately long simulation time, energy can also be preserved using a generic integration algorithm, without having to resort to special schemes such as symplectic integrators. Since it requires integration of fewer equations the representation may prove to be advantageous in time-critical implementations.

In adopting a geometrical approach, we not only give the Serret-Andoyer variables a new interpretation, but we establish a link between the Serret-Andoyer variable formulation of classical rigid body motion and geometric control theory. Finally, we note that Deprit and Elipe [11] reformulated the classical Serret-Andoyer transformation in terms of quaternions, a representation that is being used increasingly in the controls community. A similar generalization in the control context also appears to be worth investigating. In view of Remark 5.7, one may also consider incorporating the discretization discussed in [21].

Acknowledgments

The authors wish to thank Jerrold Marsden for various useful comments and suggestions.

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