Journal of Elasticity **62:** 217–237, 2001.

Torsion of an Elastomeric Cylinder Undergoing Microstructural Changes

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Received 8 May 2000; in revised form 22 May 2001

Abstract. A constitutive theory which accounts for scission and cross linking processes in polymers during deformation is used to analyze the torsion of a circular bar. In each increment of deformation at a material element of the torsion bar, some volume fraction of material undergoes scission and then re-cross links to form a new network with a new reference state. The scission process reduces the ability of the material to transmit stress. The newly formed networks restore the ability of the material to transmit stress. The total stress is assumed to be the superposition of the stress in the remainder of the original material, determined by its deformation from its original configuration, and the stress in each newly formed network, determined by the deformation in that network from the configuration at which it formed.

The interaction of this material response with the inhomogeneous deformation during torsion is studied. The analysis shows the evolution of regions of original and modified material, the softening effects associated with the process of scission and re-cross linking and the occurrence of residual stress and deformation on removal of load.

Key words: torsion, elastomer, scission, residual stress and deformation.

1. Introduction

The general form of the constitutive equation for nonlinear elasticity, used to represent the mechanical response of rubbery materials, is expressed in terms of a strain energy function. Although this strain energy function can be discussed using only phenomenological considerations, it is often related to models which describe the assumed response of macromolecules in cross linked rubber networks. Han et al. [2] considered strain energy functions for several such macromolecular models and compared their implications for uniaxial response with experiment.

Each of these models is associated with a single mechanism of macromolecular response. Tobolsky [9] showed that another macromolecular mechanism can occur during the response of rubber which causes a change in the microstructure. This mechanism consists of scission and subsequent cross linking of macromolecular networks and leads to substantial softening and permanent set on removal of the applied load. Fong and Zapas [1] outlined a constitutive theory which is motivated by the work of Tobolsky and which allows for continuous recruitment of new cross links, scission of existing cross links and their reformation or healing.

The implications of this constitutive theory for several homogeneous and non-homogenous deformations have been explored by Wineman and Rajagopal [10], Rajagopal and Wineman [7], Huntley [3], Wineman and Huntley [11], Huntley et al. [4, 5]. For homogeneous simple shear, uniaxial and equal biaxial extension, they showed that scission can lead to a reduction in the slope of a stress vs. strain graph (softening) and scission-healing can result in the graph being nonmonotonic. The reforming of cross-links also leads to permanent set on release of external loads. For nonhomogeneous deformations of circular shear and spherical inflation, the interaction of spatial variation of deformation with deformation induced scission can lead to regions of highly localized deformations, the reduction of load carrying capacity, and the presence of residual stresses and residual deformations on release of load.

Although the preceding discussion has been in terms of a scission-healing micromechanism, the underlying modeling concept can be applied to other micromechanisms, such as de-attachment and subsequent re-attachment of macromolecular bonds with inclusions such as filler particles. In this sense, the scission-healing micromechanism provides a convenient intuitive model for the purpose of developing a constitutive theory.

The present work explores the implications of the constitutive theory for the torsion of a rubber cylinder. It extends a previous treatment by Wineman and Rajagopal [10], in which attention was restricted to a discrete scission process leading to a two-network theory. Here we consider a process of continuous scission of existing cross links and their reformation or healing into new networks. The constitutive theory is outlined in Section 2. The kinematics, stress expressions and equilibrium considerations for torsion are presented in Section 3. A discussion of the implications of scission on the softening of the shear stress-shear deformation response curve is discussed in Section 4. It is shown when there is a sufficient amount of scission, the shear stress-shear deformation response curve can become nonmonotonic. The resultant moment and axial force on the cylinder are discussed in Section 5. It is shown that if the shear stress-shear deformation curve becomes nonmonotonic, then the moment-twist curve can also become nonmonotonic. Sections 4 and 5 discussed a process of increasing twist of the cylinder at a fixed length. The response upon reversal of twist is treated in Section 6. When the moment is reduced to zero, there is a residual twist and also an axial force. Section 7 discusses the case when both the moment and axial force are reduced to zero, resulting in permanent elongation and twist. When the twist is reversed, the shear at each particle is reversed. It is shown in Section 8 that there is a discontinuity in the slope of the shear stress-shear plot at each particle where scission has occurred. This is shown to result in a corresponding discontinuity in the slope of the twisting moment-twist plot.

2. Constitutive Equation

A detailed development of the constitutive equation can be found in the previously cited references. For the sake of brevity, only the essential details are given here.

Consider a sample of material which is assumed to be homogeneous and in a stress free configuration, which is taken as a reference configuration. The material is subjected to a homogeneous deformation described by $\mathbf{x} = \chi(\mathbf{X}, t)$, in which $\mathbf{x}$ is the current position of a particle located at $\mathbf{X}$ in the reference configuration. The associated deformation gradient is $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$.

It is assumed that there is a regime of deformations from the reference configuration in which the mechanical response is that of an incompressible, isotropic, nonlinear elastic solid. The constitutive equation has the form

$$\mathbf{T} = -p\mathbf{I} + 2\left[W_1^{(1)}\mathbf{B} - W_2^{(1)}\mathbf{B}^{-1}\right], \tag{2.1}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $-p\mathbf{I}$ is the isotropic stress associated with incompressibility. $W^{(1)} = W^{(1)}(I_1, I_2)$ is the strain energy density function with $I_1 = \mathrm{tr}(\mathbf{B})$ and $I_2 = \mathrm{tr}(\mathbf{B}^{-1})$ being the first two invariants of $\mathbf{B}$. Also, $W_\alpha^{(1)} = \partial W^{(1)}/\partial I_\alpha,\ \alpha = 1, 2$.

It is assumed that there is a set of deformations at which a new micromechanism is activated. This event is characterized by introducing an activation criterion, a scalar-valued function of $\mathbf{F}$. This function vanishes when $\mathbf{F}$ corresponds to a configuration in the deformed state at which the new micromechanism is activated. Material frame indifference, isotropy and incompressibility imply that the activation criterion can be expressed as $A(I_1, I_2) = 0$.

Microstructural change or transformation of the original network is assumed to occur continuously with increasing deformation. In order to describe this process, a scalar parameter s is introduced, called the deformation state parameter, which is determined by the extent of deformation. It is assumed that it can be expressed in terms of the invariants of $\mathbf{B}$: $s = s(I_1, I_2)$. At the reference configuration, $\mathbf{F} = \mathbf{I}$ and $s = 0$. The value of s increases as the deformation increases. As this work is concerned with torsion, in which a particle deforms in simple shear, it is clear what is meant by increasing deformation. Microstructural transformation is initiated when the state parameter s first reaches the activation value s_{a}. The activation criterion can be recast in terms of the state parameter: $A(I_1, I_2) = s(I_1, I_2) - s_{\mathrm{a}}$.

For $s < s_{\mathrm{a}}$, no conversion has yet occurred, the material has the original microstructure and the stress is given by (2.1). Consider a value of the deformation state parameter $\hat{s} > s_{\mathrm{a}}$. During an increment of increasing deformation, some volume fraction of network junctions of the original material is broken. The newly broken network junctions immediately reform to produce a new network. Its reference configuration is the configuration of the original material at state $\hat{s}$ and is assumed to be an unstressed configuration. Define the deformation gradient for the material formed at state $\hat{s}$ as $\widehat{\mathbf{F}} = \partial\mathbf{x}/\partial\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the position of a particle in the configuration corresponding to deformation state $\hat{s}$. The associated left Cauchy–Green tensor is $\widehat{\mathbf{B}} = \widehat{\mathbf{F}}\widehat{\mathbf{F}}^{\mathrm{T}}$. It is assumed, for the sake of simplicity, that there is no scission of newly formed networks. During this process, the material is said to be transforming to new networks. In the remainder of this article, the terminology 'network' is generalized to refer to either the original material or any newly transformed material.

The network formed at state $\hat{s}$ is assumed to be elastic, isotropic and incompressible. The extra Cauchy stress at state s in a network formed at deformation state $\hat{s}$ then becomes

$$\mathcal{T}^{(2)} = 2\big[W_1^{(2)}\widehat{\mathbf{B}} - W_2^{(2)}\widehat{\mathbf{B}}^{-1}\big]. \tag{2.2}$$

Here, $W^{(2)} = W^{(2)}(\hat{I}_1, \hat{I}_2)$ is the strain energy function for the material formed at state $\hat{s}$ and subsequently deformed to state s. Further, $\hat{I}_1$ and $\hat{I}_2$ are the appropriate invariants of $\widehat{\mathbf{B}}$. The strain energy functions $W^{(1)}$ and $W^{(2)}$ may each have any form. It is assumed, for convenience, that $W^{(2)}$ governs the strain energy in each newly formed network.

There is no relative motion between the networks, so that all networks have the same current configuration. The total stress at each stage of deformation is defined to be the superposition of contributions from the remaining portion of the original material and from each network formed at deformation states $\hat{s} \in [s_a, s]$. During a process of increasing deformation, the Cauchy stress $\mathbf{T}$ at the deformation state corresponding to state parameter s is given by

$$\begin{aligned}\mathbf{T} \;=\; & -p\mathbf{I} + 2b(s)\big[W_1^{(1)}\mathbf{B} - W_2^{(1)}\mathbf{B}^{-1}\big] \\ & +2\int_{s_a}^{s} a(\hat{s})\big[W_1^{(2)}\widehat{\mathbf{B}} - W_2^{(2)}\hat{\mathbf{B}}^{-1}\big]\,\mathrm{d}\hat{s}.\end{aligned} \tag{2.3}$$

In (2.3), $a(s)$ is a scalar-valued function which determines the rate of network transformation induced by increasing deformation. It is subject to the conditions $a(s) = 0,\ s < s_a$ and $a(s) \geqslant 0,\ s \geqslant s_a$. The latter condition ensures that an increase in deformation is associated with additional microstructural change. The function $b(s)$ is the volume fraction of the remaining portion of the original material at state s, with $b(s) = 1,\ s < s_a$ and $b(s) \in [0, 1],\ s \geqslant s_a$. $b(s)$ decreases as state parameter s increases.

Let $C(s)$ denote the volume fraction of material formed by the new networks at state s. Then

$$C(s) = \int_{s_a}^{s} a(\hat{s})\,\mathrm{d}\hat{s}. \tag{2.4}$$

For the sake of simplicity, the rate of decrease of volume fraction of original material is assumed to equal the rate of increase of volume fraction of new material,

$$b(s) = 1 - C(s) = 1 - \int_{s_a}^{s} a(\hat{s})\,\mathrm{d}\hat{s}. \tag{2.5}$$

It is further assumed that at a deformation corresponding to $s^* > s_a$, there is a sequence of deformations for which the state parameter can decrease, and

(1) there is no further conversion of original material, (2) there is no reversal of microstructural deformation. Then, $a(s) = 0$, and the upper limits of the integrals in (2.3)–(2.5) are fixed at s^*. According to (2.5), $b(s)$ is fixed at the value $b(s^*)$ given by

$$b(s^*) = 1 - \int_{s_a}^{s^*} a(\hat{s})\,d\hat{s}, \tag{2.6}$$

and (2.3) becomes

$$\begin{aligned}\mathbf{T} &= -p\mathbf{I} + 2b(s^*)\left[W_1^{(1)}\mathbf{B} - W_2^{(1)}\mathbf{B}^{-1}\right] \\ &\quad + 2\int_{s_a}^{s^*} a(\hat{s})\left[W_1^{(2)}\widehat{\mathbf{B}} - W_2^{(2)}\widehat{\mathbf{B}}^{-1}\right] d\hat{s}.\end{aligned} \tag{2.7}$$

Equations (2.3)–(2.7) represent the complete statement of the constitutive equation.

3. Formulation

3.1. KINEMATICS OF DEFORMATION

Consider a solid circular cylinder of a homogeneous isotropic incompressible elastomeric material which undergoes deformation induced microstructural change.The undeformed cylinder has length L_o and radius R_o. The Z-axis of a cylindrical polar coordinate system coincides with the cylinder's centerline and the origin is located at one end. The coordinates of a particle in the reference configuration are denoted by (R, Θ, Z) and in the current configuration by (r, θ, z).

Twisting moments M are applied to the ends of the cylinder and its curved surface is traction free. The deformation is assumed to be axisymmetric and is described by

$$r = R, \qquad \theta = \Theta + \psi Z, \qquad z = Z, \tag{3.1}$$

in which there is no length change and ψ denotes the angle of twist per unit length. The deformation gradient in cylindrical polar coordinates is

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & R\psi \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

It is seen from (3.2) that each material element of the cylinder is in a state of local simple shear $k = k(R) = R\psi$. The current left Cauchy–Green tensor and is inverse are, respectively,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+k^2 & k \\ 0 & k & 1 \end{bmatrix}, \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -k \\ 0 & -k & 1+k^2 \end{bmatrix}. \tag{3.3}$$

The invariants are $I_1 = I_2 = 3 + k^2$.

Recall the requirement that the deformation state parameter $s(I_1, I_2)$ increase with deformation. It can be seen that I_1 and I_2 increase monotonically with the shear k, which acts as a parameter for the deformation. Thus the deformation state parameter reduces to $s(k)$, and is a monotonically increasing function of k. Let k_a denote the shear when $s = s_a$, that is, $s_a = s(k_a)$. The material particle at radius R undergoes scission and forms a new network when the local shear k corresponds to values of the deformation state parameter exceeding the activation value s_a.

The activation criterion is met at different material elements at different twists ψ and different volume fractions of new networks have formed at different material particles at different ψ. The cylinder consists of three sets of material particles, those that have not undergone microstructural change, those that are undergoing microstructural change and those for which the activation criterion is satisfied at the current ψ. The current configuration of the cylinder is not a natural configuration associated with every network as different networks have been formed at different twists ψ. (For a discussion of the notion of a 'natural configuration', see [6]).

Consider a particle where the shear is $\hat{k} > k_a$.This corresponds to a state parameter $\hat{s} = s(\hat{k})$. A new network is formed at this state. On subsequent deformation to a value $k > \hat{k}$, the new network has a relative deformation gradient denoted as $\widehat{\mathbf{F}} = \widehat{\mathbf{F}}(k, \hat{k})$. In order to construct this deformation gradient, we follow the discussion presented by Rajagopal and Wineman [8]. Let a thought experiment be conducted in which every material particle of a body is in the same state of shear as the particle of interest and undergoes the same deformation. The configuration of the body at shear $\hat{k}$ can be used as a natural configuration and serves as a reference configuration for use in defining a deformation gradient. Let the deformation gradient in the original material at this state be denoted by $\mathbf{F}(\hat{k})$. At $k > \hat{k}$, the new network has the relative deformation gradient $\widehat{\mathbf{F}} = \widehat{\mathbf{F}}(k, \hat{k})$, which can be constructed as

$$\widehat{\mathbf{F}}(k, \hat{k}) = \mathbf{F}(k)\mathbf{F}^{-1}(\hat{k}). \tag{3.4}$$

From (3.2) it follows that

$$\widehat{\mathbf{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k - \hat{k} \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.5}$$

The quantity $\hat{k}$ is the shear of a particle of original material when a new network is formed at state $\hat{s}$. The quantity $k - \hat{k}$ is thus the current shear of the new network relative to its reference configuration. For the new network, the relative left Cauchy–Green tensor and its inverse are, respectively,

$$\widehat{\mathbf{B}} = \widehat{\mathbf{F}}\widehat{\mathbf{F}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + (k - \hat{k})^2 & k - \hat{k} \\ 0 & k - \hat{k} & 1 \end{bmatrix}, \tag{3.6}$$

$$\widehat{\mathbf{B}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -(k-\hat{k}) \\ 0 & -(k-\hat{k}) & 1+(k-\hat{k})^2 \end{bmatrix}. \tag{3.7}$$

The invariants of $\widehat{\mathbf{B}}$ are $\hat{I}_1 = \hat{I}_2 = 3 + (k - \hat{k})^2$.

3.2. STRESSES

When $s < s_{\rm a}$, the stresses are obtained from (2.1) and (3.3). When $s > s_{\rm a}$ and twist is increasing, the stress is given by (2.3), (3.3), (3.6) and (3.7). When twist is decreasing, the stress is given by (2.7), (3.3), (3.6) and (3.7).

First consider the shear stresses. It is seen from (2.1), (2.3), (2.7) and the form of (3.3), (3.6) and (3.7) that $T_{r\theta} = T_{rz} = 0$. For $s < s_{\rm a}$,

$$T_{z\theta} = \mu^{(1)} k, \tag{3.8}$$

where $\mu^{(1)} = 2(W_1^{(1)} + W_2^{(1)})$, a shear modulus which depends on k^2 through the expressions for the invariants I_1, I_2. For $s \geqslant s_{\rm a}$, and increasing twist,

$$T_{z\theta} = b(s)\mu^{(1)} k + \int_{s_{\rm a}}^{s} a(\hat{s})\mu^{(2)}\bigl(k - \hat{k}\bigr)\,{\rm d}\hat{s}, \tag{3.9}$$

where $\mu^{(2)} = 2(W_1^{(2)} + W_2^{(2)})$, a shear modulus which depends on $(k-\hat{k})^2$ through the expressions for the invariants $\hat{I}_1$, $\hat{I}_2$. For decreasing twist,

$$T_{z\theta} = b\bigl(s^*\bigr)\mu^{(1)} k + \int_{s_{\rm a}}^{s^*} a(\hat{s})\mu^{(2)}\bigl(k - \hat{k}\bigr)\,{\rm d}\hat{s}. \tag{3.10}$$

Next, consider the normal stresses. These are written in terms of extra stresses,

$$T_{rr} = -p + \mathcal{T}_{rr}, \qquad T_{\theta\theta} = -p + \mathcal{T}_{\theta\theta}, \qquad T_{zz} = -p + \mathcal{T}_{zz}. \tag{3.11}$$

For the purpose of brevity, the expressions for the extra stresses will be given only for the case of $s \geqslant s_{\rm a}$ and increasing twist. The expressions when $s < s_{\rm a}$ can be recovered by setting $a(\hat{s}) = 0$ and $b(s) = 1$. The expressions for reversal of twist can be recovered by setting $s = s^*$ in the upper limit of the integral and letting $b(s) = b(s^*)$. Thus,

$$\mathcal{T}_{rr} = 2b(s)\bigl[W_1^{(1)} - W_2^{(1)}\bigr] + 2\int_{s_{\rm a}}^{s} a(\hat{s})\bigl[W_1^{(2)} - W_2^{(2)}\bigr]\,{\rm d}\hat{s}, \tag{3.12a}$$

$$\begin{aligned}\mathcal{T}_{\theta\theta} = {} & 2b(s)\bigl[W_1^{(1)}\bigl(1+k^2\bigr) - W_2^{(1)}\bigr] \\ & + 2\int_{s_{\rm a}}^{s} a(\hat{s})\bigl[W_1^{(2)}\bigl(1+(k-\hat{k})^2\bigr) - W_2^{(2)}\bigr]\,{\rm d}\hat{s},\end{aligned} \tag{3.12b}$$

$$\begin{aligned}\mathcal{T}_{zz} = {} & 2b(s)\bigl[W_1^{(1)} - W_2^{(1)}\bigl(1+k^2\bigr)\bigr] \\ & + 2\int_{s_{\rm a}}^{s} a(\hat{s})\bigl[W_1^{(2)} - W_2^{(2)}\bigl(1+(k-\hat{k})^2\bigr)\bigr]\,{\rm d}\hat{s}.\end{aligned} \tag{3.12c}$$

3.3. EQUILIBRIUM

Since $k = R\psi$, the deformation state parameter, the shear stress and the extra stresses depend on R. When body forces are neglected, the equilibrium equations in cylindrical polar coordinates reduce to the statements that $p = p(R)$ and

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}R} + \frac{T_{rr} - T_{\theta\theta}}{R} = 0, \tag{3.13}$$

where use is made of the relation $r = R$. Since the surface $R = R_{\mathrm{o}}$ is traction free, $T_{rr}(R_{\mathrm{o}}) = 0$ and it follows from (3.13) that

$$-p = -\mathcal{T}_{rr} + \int_R^{R_{\mathrm{o}}} \frac{\mathcal{T}_{rr} - \mathcal{T}_{\theta\theta}}{\overline{R}} \,\mathrm{d}\overline{R}. \tag{3.14}$$

The normal stress distribution on the end of the cylinder becomes

$$T_{zz} = \mathcal{T}_{zz} - \mathcal{T}_{rr} + \int_R^{R_{\mathrm{o}}} \frac{\mathcal{T}_{rr} - \mathcal{T}_{\theta\theta}}{\overline{R}} \,\mathrm{d}\overline{R}. \tag{3.15}$$

The resultant twisting moment and axial force are, respectively,

$$M = 2\pi \int_0^{R_{\mathrm{o}}} T_{z\theta} \overline{R}^2 \,\mathrm{d}\overline{R}, \tag{3.16}$$

$$N = 2\pi \int_0^{R_{\mathrm{o}}} T_{zz} \overline{R} \,\mathrm{d}\overline{R}. \tag{3.17}$$

With (3.15) and an integration by parts, this last expression can be written as

$$N = 2\pi \int_0^{R_{\mathrm{o}}} \left(\mathcal{T}_{zz} - \mathcal{T}_{rr}\right) \overline{R} \,\mathrm{d}\overline{R} + 2\pi \int_0^{R_{\mathrm{o}}} \left(\mathcal{T}_{rr} - \mathcal{T}_{\theta\theta}\right) \frac{\overline{R}}{2} \,\mathrm{d}\overline{R}. \tag{3.18}$$

4. Stresses – Increasing Twist

As was shown in Section 3, each particle undergoes a simple shear deformation, with local shear $k = R\psi$. The deformation state parameter becomes $s = s(R\psi)$. At fixed twist ψ, the shear increases linearly and s increases monotonically with radius R. At fixed R, the shear increases linearly and s increases monotonically with twist ψ. Activation occurs when $R\psi = k_{\mathrm{a}}$.

For any twist ψ, the maximum shear in the cylinder is $R_{\mathrm{o}}\psi$. If $R_{\mathrm{o}}\psi < k_{\mathrm{a}}$, then $s(R\psi) < s(k_{\mathrm{a}}) = s_{\mathrm{a}}$ for $0 \leqslant R \leqslant R_{\mathrm{o}}$ and the activation condition is not met at any particle. Activation occurs initially at the outer radius at a twist ψ_{a} which is such that $R_{\mathrm{o}}\psi_{\mathrm{a}} = k_{\mathrm{a}}$. Then, $R\psi_{\mathrm{a}} < k_{\mathrm{a}}, s(R\psi_{\mathrm{a}}) < s_{\mathrm{a}}, 0 \leqslant R < R_{\mathrm{o}}$. If $\psi > \psi_{\mathrm{a}}$, then $R_{\mathrm{o}}\psi > R_{\mathrm{o}}\psi_{\mathrm{a}}$ and activation occurs at an interior radius $R_{\mathrm{a}} < R_{\mathrm{o}}$ where $R_{\mathrm{a}}\psi = k_{\mathrm{a}} = R_{\mathrm{o}}\psi_{\mathrm{a}}$, or

$$R_{\mathrm{a}} = \frac{\psi_{\mathrm{a}}}{\psi} R_{\mathrm{o}}. \tag{4.1}$$

Then

$$s(R\psi) < s(R_a\psi) = s_a, \quad 0 \leqslant R < R_a,$$
$$s(R\psi) \geqslant s(R_a\psi) = s_a, \quad R_a \leqslant R < R_o. \tag{4.2}$$

Thus, there is an inner core of original material and an outer layer of material undergoing microstructural change.

By (3.8), the shear stress in the inner core, $0 \leqslant R < R_a$, is

$$T_{z\theta} = \mu^{(1)}\big[(R\psi)^2\big]R\psi. \tag{4.3}$$

Next consider the particles in the outer layer, $R_a \leqslant R < R_o$. The particle at radius R has undergone an amount of transformation corresponding to a state parameter value of $s(R\psi)$. Hence,

$$T_{z\theta} = b\big(s(R\psi)\big)\mu^{(1)}\big[(R\psi)^2\big]R\psi + \int_{s_a}^{s(R\psi)} a(\hat{s})\mu^{(2)}\big[\big(R\psi - \hat{k}\big)^2\big]\big(R\psi - \hat{k}\big)\,\mathrm{d}\hat{s}. \tag{4.4}$$

A numerical example shows how scission and cross-linking affect the shear stress. To this end, several assumptions are made about the material response:

(1) the original material and the newly formed material are each neo-Hookean. Then, $\mu^{(1)} = 2W_1^{(1)} = c^{(1)}$, a constant, $\mu^{(2)} = 2W_1^{(2)} = c^{(2)}$, another constant and $W_2^{(1)} = W_2^{(2)} = 0$;

(2) $s(k) = k$;

(3) the conversion process occurs over a finite interval of shear strains $k_a \leqslant k \leqslant k_c$ and the conversion rate function $a(s) = a(k)$ has the form

$$a(k) = \begin{cases} 0, & k < k_a \\ \dfrac{6C}{(k_c - k_a)^3}(k_c - k)(k - k_a) & k \in [k_a, k_c] \\ 0, & k_c < k \end{cases}, \tag{4.5}$$

where C, by (2.5), is the total volume fraction of material converted to new networks. The function $b(s) = b(k)$ is found using (2.6). These assumptions were used in by [3–5, 10] and motivation for these choices can be found there.

When these assumptions are used in (4.3) and (4.4), the shear stress in the inner core, $0 \leqslant R < R_a$, can be written as

$$\frac{T_{z\theta}}{c^{(1)}} = R\psi, \tag{4.6}$$

and the shear stress in the outer layer, $R_a \leqslant R < R_o$, can be written as

$$\frac{T_{z\theta}}{c^{(1)}} = b(R\psi)R\psi + \int_{k_a}^{R\psi} a(\hat{s})\frac{c^{(2)}}{c^{(1)}}\big(R\psi - \hat{k}\big)\,\mathrm{d}\hat{s}. \tag{4.7}$$

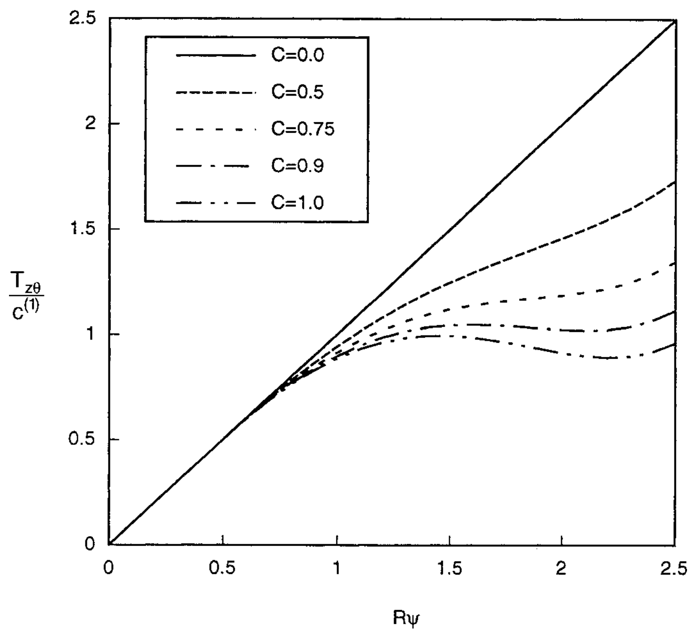

Figure 1. $T_{z\theta}/c^{(1)}$ versus shear $R\psi$ for several values of the volume fraction of conversion C.

Figure 1 shows a plot of $T_{z\theta}/c^{(1)}$ versus $R\psi$ for several values of the volume fraction of conversion C. Calculations were carried out with $k_a = 0.5$, $k_c = 2.65$ and $c^{(2)}/c^{(1)} = 1$. The plots coincide and are linear for $k \in [0, k_a]$, since the relation is given by (4.6). Each value of $R\psi$ can be regarded as the maximum shear in the cylinder, $R_o\psi$ at R_o, for some twist ψ. Then, the portion of the plot from the origin to this value of $R\psi$ shows the radial shear stress distribution corresponding to this twist. When $C = 0$, there is no conversion, the response is neo-Hookean, and the plot is the straight line given by (4.6). As C increases, scission reduces the stress transmitted by the original material (softening), and the curves become shallower. For shear near k_a, the shear stress decreases due to scission of the original material. As shear increases, the new networks begin to carry more stress and the curves become steeper (stiffening). When C is small, initial softening occurs slower than stiffening and the plots are monotonic. As C increases, initial softening occurs faster than stiffening and the curves become nonmonotonic. Thus, the maximum shear stress in the cylinder, at radius R_o for small twist, decreases with increasing C. The radial variation of the shear stress may become nonmonotonic and the maximum shear stress may occur in the interior of the cylinder.

The ratio of the shear modulus of the new material to that of the original material, $c^{(2)}/c^{(1)}$, also affects the plot of $T_{z\theta}/c^{(1)}$ vs. $R\psi$. According to Figure 1, for the case in which $C = 0.75$ and $c^{(2)}/c^{(1)} = 1$, the plot is monotonic but has a portion with a very small slope. It can be shown that when $c^{(2)}/c^{(1)} = 2$, the curve is monotonic, but significantly steeper. When $c^{(2)}/c^{(1)} = 0.5$, the curve becomes nonmonotonic. A detailed discussion of the influence of different properties on the monotonicity of the plot, using the general equation (4.4), has been presented by Huntley [3].

5. Moment and Axial Force – Increasing Twist

For sufficiently small twist, $R\psi < k_a$ at all radii. The moment–twist relation is obtained by substituting (4.3) into (3.16). When the twist is large enough that there is an outer layer of particles undergoing conversion, the integrand of (3.16) is given by (4.3) for $R \in [0, R_a]$ and by (4.4) for $R \in [R_a, R_o]$, where R_a is determined from (4.1). The moment–twist relation becomes

$$\begin{aligned} M &= 2\pi \int_0^{R_a} \{\mu^{(1)}[(\overline{R}\psi)^2]\overline{R}\psi\}\overline{R}^2\,\mathrm{d}\overline{R} \\ &\quad + 2\pi \int_{R_a}^{R_o} \Big\{ b(s(\overline{R}\psi))\mu^{(1)}[(\overline{R}\psi)^2]\overline{R}\psi \\ &\quad + \int_{s_a}^{s(\overline{R}\psi)} a(\hat{s})\mu^{(2)}[(\overline{R}\psi - \hat{k})^2](\overline{R}\psi - \hat{k})\,\mathrm{d}\hat{s} \Big\} \overline{R}^2\,\mathrm{d}\overline{R}. \end{aligned} \quad (5.1)$$

It was seen in Section 4 that when the twist and/or volume fraction of conversion become sufficiently large, the shear stress can become nonmonotonic in the radius. Since this occurs in the outer layer of the cylinder, which contributes more to the resultant moment than does the inner core, it is possible that the moment–twist relation can become nonmonotonic. In order to show this more precisely, note that the derivative of the relation (3.16) with respect to the twist is given by,

$$\frac{\mathrm{d}M}{\mathrm{d}\psi} = 2\pi \int_0^{R_o} \left.\frac{\mathrm{d}T_{\vartheta z}(k)}{\mathrm{d}k}\right|_{k=R\psi} \overline{R}^3\,\mathrm{d}\overline{R}. \quad (5.2)$$

As Figure 1 suggests, the slope of the $T_{\theta z}$ vs. k graph can be negative in the outer layer of the cylinder where $\overline{R}$ is largest, in which case it is possible that $\mathrm{d}M/\mathrm{d}\psi < 0$.

Let the response be neo-Hookean as in Section 4. When $C = 0$, the moment–twist relation can be written as

$$\frac{M}{c^{(1)}(I_z/R_o)} = R_o\psi, \quad (5.3)$$

where $I_z = \pi R_o^4/2$. When $C > 0$, the moment–twist relation is given by (5.3) when $R_o\psi \leqslant k_a$. When $R_o\psi \geqslant k_a$, the moment–twist relation is given by

$$\begin{aligned} \frac{M}{c^{(1)}} &= 2\pi \int_0^{R_a} (\overline{R}\psi)\overline{R}^2\,\mathrm{d}\overline{R} \\ &\quad + 2\pi \int_{R_a}^{R_o} \Big\{ b(s(\overline{R}\psi))\overline{R}\psi + \int_{s_a}^{s(\overline{R}\psi)} a(\hat{s})\frac{c^{(2)}}{c^{(1)}}(\overline{R}\psi - \hat{k})\,\mathrm{d}\hat{s} \Big\} \overline{R}^2\,\mathrm{d}\overline{R}. \end{aligned} \quad (5.4)$$

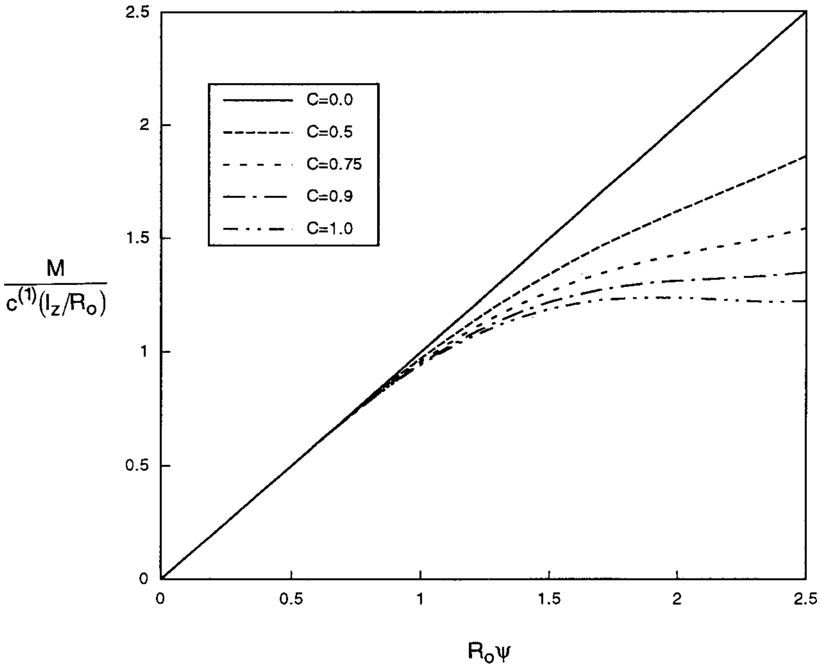

Figure 2. $M/(c^{(1)}I_z/R_o)$ versus shear $R_o\psi$ for several values of the volume fraction of conversion C.

With the use of (2.6) and introducing $x = \overline{R}/R_o$, this can also be written in the form

$$\frac{M}{c^{(1)}(I_z/R_o)} = R_o\psi + 4\int_{R_a/R_o}^{1}\left\{\int_{s_a}^{s(xR_o\psi)} a(\hat{s}) \times\left[\frac{c^{(2)}}{c^{(1)}}\left(xR_o\psi - \hat{k}\right) - xR_o\psi\right] d\hat{s}\right\}x^2\,dx. \quad (5.5)$$

Figure 2 shows a plot of $M/(c^{(1)}(I_z/R_o))$ versus $R_o\psi$ for the same conditions as used in the numerical example of Section 4. As C increases, the curve changes from being a straight line, as in the neo-Hookean case, to one with a flatter portion at larger twist. As C approaches unity, the curve becomes nonmonotonic, as discussed above. Thus, the scission and cross-linking process leads to a softening of the moment–twist response.

Consider next the resultant axial force on the ends of the cylinder. This can be calculated using (3.18) and the expressions for the extra stresses in (3.12a)–(3.12c) which are appropriate to the regions $R \in [0, R_a]$ and $R \in [R_a, R_o]$. The axial force is given by

$$\frac{N}{2\pi} = -\int_0^{R_a}\left[W_1^{(1)} + 2W_2^{(1)}\right]\overline{R}\left(\overline{R}\psi\right)^2 d\overline{R}$$

$$- \int_{R_a}^{R_o} \left\{ b\big(s(\overline{R}\psi)\big)\big[W_1^{(1)} + 2W_2^{(1)}\big]\big(\overline{R}\psi\big)^2 + \int_{s_a}^{s(\overline{R}\psi)} a\big(\hat{s}\big)\big[W_1^{(2)} + 2W_2^{(2)}\big]\big(\overline{R}\psi - \hat{k}\big)^2 \,\mathrm{d}\hat{s} \right\} \overline{R}\,\mathrm{d}\overline{R}. \tag{5.6}$$

If both the original and newly formed networks are neo-Hookean, then $W_2^{(1)} = W_2^{(2)} = 0$, and $W_1^{(1)}$ and $W_1^{(2)}$ are positive constants. The axial force on the cylinder is then compressive.

6. Reversal of Twist

Let the twist be increased to ψ^* and then be reversed. At the maximum twist ψ^*, the boundary between the inner core of original material and the outer layer of material having undergone microstructural change has radius R_a^* , where by (4.1),

$$R_a^* = \frac{\psi_a}{\psi^*} R_o. \tag{6.1}$$

As the twist decreases, the deformation is assumed to be given by (3.1). At each material element of the original material, the deformation gradient is given by (3.2). The material element is undergoing simple shear, and the shear is decreasing. The deformation state parameter is then decreasing and there is no microstructural transformation. For the networks in the outer layer that formed as the twist increased from ψ_a to ψ^*, the deformation gradient is given by (3.5). As the twist decreases, there is a reduction and possibly reversal of shear. For simplicity of discussion, it is assumed that no further transformation occurs. Thus, there is no change in the radius R_a^* of the boundary between the inner core and the outer layer, and in the material in the outer layer.

Consider the material element in the inner core at radius $R \in [0, R_a^*]$. At any twist ψ during the reversal, the shear stress distribution is still given by (4.3). Next, consider the material element in the outer layer at radius $R \in [R_a^*, R_o]$. The volume fraction of original material remains at $b(s(R\psi^*))$. For the networks formed as the twist increased from ψ_a to ψ^*, the transformation rate $a(s) = 0$. By (3.10) and (4.4), the shear stress in the outer layer, $R_a^* \leqslant R < R_o$, is

$$T_{z\theta} = b\big(s\big(R\psi^*\big)\big)\mu^{(1)}\big[\big(R\psi\big)^2\big]R\psi + \int_{s_a}^{s(R\psi^*)} a\big(\hat{s}\big)\mu^{(2)}\big[\big(R\psi - \hat{k}\big)^2\big]\big(R\psi - \hat{k}\big)\,\mathrm{d}\hat{s}. \tag{6.2}$$

When the original and newly formed networks are neo-Hookean, the shear stress distribution in the inner core, $0 \leqslant R < R_a^*$, is given by (4.6). The shear stress in the outer layer, (6.2), reduces to

$$T_{z\theta} = b\big(s\big(R\psi^*\big)\big)c^{(1)}R\psi + \int_{s_a}^{s(R\psi^*)} a\big(\hat{s}\big)c^{(2)}\big(R\psi - \hat{k}\big)\,\mathrm{d}\hat{s}. \tag{6.3}$$

This can be written as

$$T_{z\theta} = R\psi\left[b\big(s\big(R\psi^*\big)\big)c^{(1)} + c^{(2)}\int_{s_a}^{s(R\psi^*)} a(\hat{s})\,d\hat{s}\right] - c^{(2)}\int_{s_a}^{s(R\psi^*)} a\big(\hat{s}\big)\hat{k}\,d\hat{s}. \tag{6.4}$$

Thus, at each fixed radius, the shear stress decreases linearly with twist ψ. If $c^{(1)} = c^{(2)}$, it follows from (2.6) that

$$\frac{T_{z\theta}}{c^{(1)}} = R\psi - \int_{s_a}^{s(R\psi^*)} a\big(\hat{s}\big)\hat{k}\,d\hat{s}. \tag{6.5}$$

and the stress-twist graph is a straight line parallel to that in the inner core.

The twisting moment is found from (3.16), (4.3) and (6.2) to be

$$\begin{aligned} M \;=\; & 2\pi\int_0^{R_a^*}\{\mu^{(1)}[(\overline{R}\psi)^2](\overline{R}\psi)\}\overline{R}^2\,d\overline{R} \\ & +2\pi\int_{R_a^*}^{R_o}\Big\{b\big(s\big(\overline{R}\psi^*\big)\big)\mu^{(1)}\big[\big(\overline{R}\psi\big)^2\big]\overline{R}\psi \\ & +\int_{s_a}^{s(\overline{R}\psi^*)} a\big(\hat{s}\big)\mu^{(2)}\big[\big(\overline{R}\psi-\hat{k}\big)^2\big]\big(\overline{R}\psi-\hat{k}\big)\,d\hat{s}\Big\}\overline{R}^2\,d\overline{R}. \end{aligned} \tag{6.6}$$

When the original and newly formed networks are neo-Hookean, this becomes, using (6.4),

$$\begin{aligned} M \;=\; & c^{(1)}\psi\left[2\pi\int_0^{R_a^*}\overline{R}^3\,d\overline{R}\right] \\ & +2\pi\int_{R_a^*}^{R_o}\Big\{\overline{R}\psi\Big[b\big(s\big(\overline{R}\psi^*\big)\big)c^{(1)} + c^{(2)}\int_{s_a}^{s(\overline{R}\psi^*)} a\big(\hat{s}\big)\,d\hat{s}\Big] \\ & -c^{(2)}\int_{s_a}^{s(\overline{R}\psi^*)} a(\hat{s})\hat{k}\,d\hat{s}\Big\}\overline{R}^2\,d\overline{R}. \end{aligned} \tag{6.7}$$

If $c^{(1)} = c^{(2)}$, it follows from (2.6) that

$$\frac{M}{c^{(1)}\,(I_z/R_o)} = R_o\psi - \frac{4}{R_o^3}\int_{R_a^*}^{R_o}\left\{\int_{s_a}^{s(\overline{R}\psi^*)} a\big(\hat{s}\big)\hat{k}\,d\hat{s}\right\}\overline{R}^2\,d\overline{R}. \tag{6.8}$$

The moment–twist graph is a straight line parallel to that during the increasing twist, as seen from (5.3). According to (6.8), when $M = 0$, there is a residual twist ψ_{res} given by

$$\psi_{res} = \frac{2\pi}{I_z}\int_{R_a^*}^{R_o}\left\{\int_{s_a}^{s(\overline{R}\psi^*)} a\big(\hat{s}\big)\hat{k}\,d\hat{s}\right\}\overline{R}^2\,d\overline{R}, \tag{6.9}$$

and is positive.

Although $M = 0$, there can be a nonzero shear stress at each material element. Since each original material element and the newly formed networks are in a sheared state, there will also be normal stresses. A resultant axial force will still be required to maintain the cylinder at its original lengh. This is given by (5.6), making use of the discussion preceding (3.12),

$$\begin{aligned}\frac{N}{2\pi} &= -\int_0^{R_a^*}\left[2W_2^{(1)} + W_1^{(1)}\right]\overline{R}\left(\overline{R}\psi\right)^2 \mathrm{d}\overline{R} \\ &\quad - \int_{R_a^*}^{R_o}\Bigg\{b\left(s\left(\overline{R}\psi^*\right)\right)\left[2W_2^{(1)} + W_1^{(1)}\right]\left(\overline{R}\psi\right)^2 \\ &\quad + \int_{s_a}^{s(\overline{R}\psi^*)} a\left(\hat{s}\right)\left[2W_2^{(2)} + W_1^{(2)}\right]\left(\overline{R}\psi - \hat{k}\right)^2 \mathrm{d}\hat{s}\Bigg\}\overline{R}\,\mathrm{d}\overline{R}. \qquad (6.10)\end{aligned}$$

When the original and newly formed networks are neo-Hookean, $W_2^{(1)} = W_2^{(2)} = 0$ and

$$\begin{aligned}\frac{N}{2\pi} &= -c^{(1)}\int_0^{R_a^*}\overline{R}\left(\overline{R}\psi\right)^2 \mathrm{d}\overline{R} - \int_{R_a^*}^{R_o}\Bigg\{c^{(1)}b\left(s\left(\overline{R}\psi^*\right)\right)\left(\overline{R}\psi\right)^2 \\ &\quad + c^{(2)}\int_{s_a}^{s(\overline{R}\psi^*)} a\left(\hat{s}\right)\left(\overline{R}\psi - \hat{k}\right)^2 \mathrm{d}\hat{s}\Bigg\}\overline{R}\,\mathrm{d}\overline{R}. \qquad (6.11)\end{aligned}$$

The resultant axial force is compressive.

7. Residual Deformation when Moment and Axial Force are Zero

It was shown in Section 6 that when the cylinder is twisted to $\psi^*$and then the moment is reduced to zero, while the length is held fixed, an axial force is required. In this section, the case is considered when the cylinder is twisted to ψ^* and then both the moment and axial force are reduced to zero. It is assumed, for the sake of simplicity of analysis, that there is no further scission. Then the original network and all networks which are subsequently formed up to ψ^* respond elastically. By a discussion similar to that in Section 6, it can be concluded that the interface radius R_a^* does not change.

Let (r^*, θ^*, z^*) denote coordinates at the maximum twist ψ^* of the point at (R, Θ, Z) in the reference configuration. By (3.1)

$$r^* = R, \qquad \theta^* = \Theta + \psi^* Z, \qquad z^* = Z. \qquad (7.1)$$

Let $(\tilde{r}, \tilde{\theta}, \tilde{z})$ denote the coordinates of the point in the state in which the moment and axial force are zero. It is assumed that the cylinder arrives at this state by a change of length, radius and twist described by

$$\tilde{r} = \frac{1}{\sqrt{\mu}}r^*, \qquad \tilde{\theta} = \theta^* + \widetilde{\psi}\mu z^*, \qquad \tilde{z} = \mu z^*. \qquad (7.2)$$

Then, the point in the reference configuration at (R, Θ, Z) has coordinates $(\tilde{r}, \tilde{\theta}, \tilde{z})$ in the residual configuration given by

$$\tilde{r} = \frac{1}{\sqrt{\mu}}R, \qquad \tilde{\theta} = \Theta + \left(\psi^* + \widetilde{\psi}\mu\right)Z, \qquad \tilde{z} = \mu Z. \tag{7.3}$$

The residual length is then μL_o, the residual radius is $R_o/\sqrt{\mu}$ and the residual twist is $(\psi^* + \widetilde{\psi}\mu)L_o$.

For the original material in the inner core and the remaining portion of original material in the outer layer, the deformation gradient found from (7.3) is denoted by $\mathbf{F}(\tilde{\mathbf{x}}, \mathbf{X})$. Then

$$\mathbf{F}(\tilde{\mathbf{x}}, \mathbf{X}) = \begin{bmatrix} \frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\mu}} & \tilde{r}\left(\psi^* + \widetilde{\psi}\mu\right) \\ 0 & 0 & \mu \end{bmatrix}, \tag{7.4}$$

and

$$\mathbf{B}(\tilde{\mathbf{x}}, \mathbf{X}) = \begin{bmatrix} \frac{1}{\mu} & 0 & 0 \\ 0 & \frac{1}{\mu} + \tilde{r}^2\left(\psi^* + \widetilde{\psi}\mu\right)^2 & \tilde{r}\mu\left(\psi^* + \widetilde{\psi}\mu\right) \\ 0 & \tilde{r}\mu\left(\psi^* + \widetilde{\psi}\mu\right) & \mu^2 \end{bmatrix}. \tag{7.5}$$

Suppose that the cylinder is at twist $\widehat{\psi}$, and consider the particle where the activation condition is satisfied and a new network is formed. If the particle is at radius R, the shear in the original network is $\hat{k} = R\widehat{\psi}$. As discussed in Section 3, let a thought experiment be conducted in which the configuration of the cylinder at twist $\widehat{\psi}$ is a natural configuration for every particle of the cylinder. This configuration serves as a reference cofigutration for use in determining the deformation gradient for any R when the twist increases to $\psi > \widehat{\psi}$. Evaluating at the particle of interest gives the deformation gradient at that particle. Let its deformation gradient in the residual configuration be denoted by $\mathbf{F}(\tilde{\mathbf{x}}, \hat{\mathbf{x}})$. Then

$$\mathbf{F}(\tilde{\mathbf{x}}, \hat{\mathbf{x}}) = \begin{bmatrix} \frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\mu}} & \tilde{r}\left(\psi^* + \widetilde{\psi}\mu - \widehat{\psi}\right) \\ 0 & 0 & \mu \end{bmatrix}, \tag{7.6}$$

and

$$\mathbf{B}(\tilde{\mathbf{x}}, \hat{\mathbf{x}}) = \begin{bmatrix} \frac{1}{\mu} & 0 & 0 \\ 0 & \frac{1}{\mu} + \tilde{r}^2\left(\psi^* + \widetilde{\psi}\mu - \widehat{\psi}\right)^2 & \tilde{r}\mu\left(\psi^* + \widetilde{\psi}\mu - \widehat{\psi}\right) \\ 0 & \tilde{r}\mu\left(\psi^* + \widetilde{\psi}\mu - \widehat{\psi}\right) & \mu^2 \end{bmatrix}. \tag{7.7}$$

In the residual state, it follows from (7.3) that the radius of the interface is $\tilde{R}_a^* = R_a^*/\sqrt{\mu}$ and radius of the cylinder is $R_o/\sqrt{\mu}$. The resultant moment is

$$M = 2\pi \int_0^{\tilde{R}_o} T_{z\theta} \tilde{r}^2 \, d\tilde{r}, \tag{7.8}$$

which reduces by (7.3) to

$$M = \frac{2\pi}{\mu\sqrt{\mu}} \int_0^{R_o} T_{z\theta} \overline{R}^2 \, d\overline{R}. \tag{7.9}$$

For the sake of simplicity of presentation, both the original and newly formed networks are assumed to be neo-Hookean. The shear stress distribution $T_{z\theta}$ is given by:

Residual inner core: $0 \leqslant \tilde{r} \leqslant \widetilde{R}_a^*$, $T_{z\theta} = c^{(1)} \tilde{r} \mu (\psi^* + \widetilde{\psi}\mu)$ or, using (7.3),

$$0 \leqslant R \leqslant R_a^*, \quad T_{z\theta} = c^{(1)} \sqrt{\mu} R \left(\psi^* + \widetilde{\psi}\mu\right), \tag{7.10}$$

Outer layer: $R_a^* \leqslant R \leqslant R_o$

$$\begin{aligned} T_{z\theta} \; = \; & b\left(R\psi^*\right) c^{(1)} \sqrt{\mu} R \left(\psi^* + \widetilde{\psi}\mu\right) \\ & + \int_{k_a}^{R\psi^*} a\left(\hat{k}\right) c^{(2)} \left[\mu\sqrt{\mu} R \widetilde{\psi} + \sqrt{\mu}\left(R\psi^* - \hat{k}\right)\right] d\hat{k}. \end{aligned} \tag{7.11}$$

Let (7.10) and (7.11) be substituted into (7.9). On letting $M = 0$ and simplifying, the resultant expression becomes

$$\begin{aligned} & \int_0^{R_a^*} c^{(1)} \overline{R}^3 \left(\psi^* + \widetilde{\psi}\mu\right) d\overline{R} \\ & \quad + \int_{R_a^*}^{R_o} \left\{ b\left(\overline{R}\psi^*\right) c^{(1)} \overline{R}^3 \left(\psi^* + \widetilde{\psi}\mu\right) \right. \\ & \quad \left. + \overline{R}^2 \int_{k_a}^{\overline{R}\psi^*} a\left(\hat{k}\right) c^{(2)} \left[\mu \overline{R} \widetilde{\psi} + \left(\overline{R}\psi^* - \hat{k}\right)\right] d\hat{k} \right\} d\overline{R} = 0. \end{aligned} \tag{7.12}$$

Using (2.6), this can be re-written in the form

$$\begin{aligned} & \left(\psi^* + \widetilde{\psi}\mu\right) \left[c^{(1)} \int_0^{R_a^*} \overline{R}^3 \, d\overline{R} + c^{(2)} \int_{R_a^*}^{R_o} \overline{R}^3 \, d\overline{R} \right. \\ & \qquad \left. + \left(c^{(1)} - c^{(2)}\right) \int_{R_a^*}^{R_o} \overline{R}^3 b\left(\overline{R}\psi^*\right) d\overline{R} \right] \\ & \quad = c^{(2)} \int_{R_a^*}^{R_o} \left\{ \overline{R}^2 \int_{k_a}^{\overline{R}\psi^*} a\left(\hat{k}\right) \hat{k} \, d\hat{k} \right\} d\overline{R}. \end{aligned} \tag{7.13}$$

If $c^{(1)} = c^{(2)}$, this reduces to

$$\big(\psi^* + \widetilde{\psi}\mu\big) = \frac{1}{I_z}\int_{R_\mathrm{a}^*}^{R_\mathrm{o}} \left\{\overline{R}^2 \int_{k_\mathrm{a}}^{\overline{R}\psi^*} a\big(\hat{k}\big)\hat{k}\,\mathrm{d}\hat{k}\right\} \mathrm{d}\overline{R}. \tag{7.14}$$

This is an expression for the residual twist, which is seen to be positive.

Now consider the resultant axial force in the residual state. This is given by

$$N = 2\pi \int_0^{\widetilde{R}_\mathrm{o}} T_{zz}\tilde{r}\,\mathrm{d}\tilde{r}, \tag{7.15}$$

which reduces by (7.3) to

$$N = \frac{2\pi}{\mu}\int_0^{R_\mathrm{o}} T_{zz}\overline{R}\,\mathrm{d}\overline{R}. \tag{7.16}$$

As in Section 3, this can be written in terms of the extra stresses defined in (3.11),

$$N = \frac{2\pi}{\mu}\int_0^{R_\mathrm{o}} \left[\big(\mathcal{T}_{zz} - \mathcal{T}_{rr}\big) + \frac{1}{2}\big(\mathcal{T}_{rr} - \mathcal{T}_{\theta\theta}\big)\right]\overline{R}\,\mathrm{d}\overline{R}. \tag{7.17}$$

When the material is neo-Hookean, the terms in the integrand of (7.17) are as follows:

Residual inner core: $0 \leqslant R \leqslant R_\mathrm{a}^*$,

$$\mathcal{T}_{zz} - \mathcal{T}_{rr} = c^{(1)}\left(\mu^2 - \frac{1}{\mu}\right), \tag{7.18}$$

$$\mathcal{T}_{rr} - \mathcal{T}_{\theta\theta} = -\frac{c^{(1)}}{\mu}R^2\big(\psi^* + \widetilde{\psi}\mu\big)^2. \tag{7.19}$$

Outer layer: $R_\mathrm{a}^* \leqslant R \leqslant R_\mathrm{o}$

$$\mathcal{T}_{zz} - \mathcal{T}_{rr} = b\big(R\psi^*\big)c^{(1)}\left(\mu^2 - \frac{1}{\mu}\right) + \int_{k_\mathrm{a}}^{R\psi^*} a\big(\hat{k}\big)c^{(2)}\left(\mu^2 - \frac{1}{\mu}\right)\mathrm{d}\hat{k}, \tag{7.20}$$

$$\begin{aligned}\mathcal{T}_{rr} - \mathcal{T}_{\theta\theta} \;=\; & -b\big(R\psi^*\big)\frac{c^{(1)}}{\mu}R^2\big(\psi^* + \widetilde{\psi}\mu\big)^2 \\ & + \int_{k_\mathrm{a}}^{R\psi^*} a\big(\hat{k}\big)\big(-c^{(2)}\big)\left[\frac{R}{\sqrt{\mu}}\widetilde{\psi}\mu + \frac{1}{\sqrt{\mu}}\big(R\psi^* - \hat{k}\big)\right]^2 \mathrm{d}\hat{k}.\end{aligned} \tag{7.21}$$

On using (7.18)–(7.21) in (7.17), making use of (2.6), and then setting $N = 0$, one obtains

$$\begin{aligned}&\big(\mu^3 - 1\big)\left[c^{(1)}\int_0^{R_\mathrm{a}^*}\overline{R}\,\mathrm{d}\overline{R} + c^{(2)}\int_{R_\mathrm{a}^*}^{R_\mathrm{o}}\overline{R}\,\mathrm{d}\overline{R} + \big(c^{(1)} - c^{(2)}\big)\int_{R_\mathrm{a}^*}^{R_\mathrm{o}} b\big(\overline{R}\psi^*\big)\overline{R}\,\mathrm{d}\overline{R}\right] \\ &\quad - \frac{c^{(1)}}{2}\big(\psi^* + \widetilde{\psi}\mu\big)^2\left[\int_0^{R_\mathrm{a}^*}\overline{R}^3\,\mathrm{d}\overline{R} + \int_{R_\mathrm{a}^*}^{R_\mathrm{o}} b\big(\overline{R}\psi^*\big)\overline{R}^3\,\mathrm{d}\overline{R}\right] \\ &\quad - \frac{c^{(2)}}{2}\int_{R_\mathrm{a}^*}^{R_\mathrm{o}}\overline{R}\int_{k_\mathrm{a}}^{\overline{R}\psi^*} a\big(\hat{k}\big)\big[\overline{R}\big(\psi^* + \widetilde{\psi}\mu\big) - \hat{k}\big]^2\,\mathrm{d}\bar{k}\,\mathrm{d}\overline{R} = 0.\end{aligned} \tag{7.22}$$

The residual twist $\psi^* + \widetilde{\psi}\mu$ is known from (7.13), the condition that $M = 0$. The above is an equation for the residual stretch μ. If $c^{(1)} = c^{(2)}$, (7.22) reduces to

$$\mu^3 - 1 = \frac{1}{R_o^2}\left(\psi^* + \widetilde{\psi}\mu\right)^2\left[\int_0^{R_a^*} \overline{R}^3\, \mathrm{d}\overline{R} + \int_{R_a^*}^{R_o} b\left(\overline{R}\psi^*\right)\overline{R}^3\, \mathrm{d}\overline{R}\right] + \frac{1}{R_o^2}\int_{R_a^*}^{R_o} \overline{R}\int_{k_a}^{\overline{R}\psi^*} a\left(\hat{k}\right)\left[\overline{R}\left(\psi^* + \widetilde{\psi}\mu\right) - \hat{k}\right]^2 \mathrm{d}\hat{k}\, \mathrm{d}\overline{R}, \tag{7.23}$$

and it is seen that $\mu > 1$.

8. Discontinuity of Slope on Reversal of Twist

When the twist is reversed at ψ^*, the shear at each particle is reversed. There is no discontinuity in the slope of the shear stress vs. shear plot in the elastic core. It is shown here that there is a discontinuity in the slope of the shear stress vs. shear plot at each particle where new networks have formed, that is, in the outer layer. This is shown to result in a corresponding discontinuity in the slope of the twisting moment vs. twist plot.

Since $k = R\psi$, shear increases as twist increases. At a particle in the outer layer, the shear stress vs. shear relation during increasing shear is given by (3.9). Then,

$$\frac{\mathrm{d}T_{z\theta}}{\mathrm{d}k} = \frac{\mathrm{d}b(s)}{\mathrm{d}s}\frac{\mathrm{d}s(k)}{\mathrm{d}k}\mu^{(1)}k + b(s)\frac{\mathrm{d}}{\mathrm{d}k}\left[\mu^{(1)}k\right] + \frac{\mathrm{d}s}{\mathrm{d}k}(k)\left[a\left(\hat{s}\right)\mu^{(2)}\left(k - \hat{k}\right)\right]\Big|_{\hat{s}=s,\hat{k}=k} + \int_{s_a}^{s} a\left(\hat{s}\right)\frac{\mathrm{d}}{\mathrm{d}(k - \hat{k})}\left[\mu^{(2)}\left(k - \hat{k}\right)\right]\mathrm{d}\hat{s}. \tag{8.1}$$

The third term in (8.1) vanishes. Let (8.1) be evaluated at maximum twist ψ^*, where $k = k^*$ and $s = s^*$. The slope of the shear stress vs. shear plot corresponding to increasing shear, evaluated at the maximum shear, is then

$$\left.\frac{\mathrm{d}T_{z\theta}}{\mathrm{d}k}\right|_{\mathrm{inc}} = \frac{\mathrm{d}b(s^*)}{\mathrm{d}s}\frac{\mathrm{d}s(k^*)}{\mathrm{d}k}\left[\mu^{(1)}k\right]_{k=k^*} + b\left(s^*\right)\frac{\mathrm{d}}{\mathrm{d}s}\left[\mu^{(1)}k\right]_{k=k^*} + \left.\int_{s_a}^{s} a\left(\hat{s}\right)\frac{\mathrm{d}}{\mathrm{d}(k - \hat{k})}\left[\mu^{(2)}\left(k - \hat{k}\right)\right]\mathrm{d}\hat{s}\right|_{k=k*}. \tag{8.2}$$

The slope of the shear stress vs. shear plot corresponding to decreasing shear, evaluated at the maximum shear, is found by differentiating (3.10) with respect to the shear, and evaluating at $k = k^*$ and $s = s^*$,

$$\left.\frac{\mathrm{d}T_{z\theta}}{\mathrm{d}k}\right|_{\mathrm{dec}} = b\left(s^*\right)\frac{\mathrm{d}}{\mathrm{d}s}\left[\mu^{(1)}k\right]_{k=k^*} + \left.\int_{s_a}^{s^*} a\left(\hat{s}\right)\frac{\mathrm{d}}{\mathrm{d}(k - \hat{k})}\left[\mu^{(2)}(k - \hat{k})\right]\mathrm{d}\hat{s}\right|_{k=k*}. \tag{8.3}$$

It then follows that

$$\left.\frac{\mathrm{d}T_{\theta z}}{\mathrm{d}k}\right|_{\mathrm{dec}} - \left.\frac{\mathrm{d}T_{\theta z}}{\mathrm{d}k}\right|_{\mathrm{inc}} = -\frac{\mathrm{d}b(s^*)}{\mathrm{d}s}\frac{\mathrm{d}s(k^*)}{\mathrm{d}k}\big[\mu^{(1)}k\big]_{k=k^*}. \tag{8.4}$$

The deformation state parameter increases monotonically with the shear and the volume fraction of original material decreases with increasing deformation. Thus, $\mathrm{d}b(s^*)/\mathrm{d}s < 0$ and $\mathrm{d}s(k^*)/\mathrm{d}k > 0$. The factor $[\mu^{(1)}k]_{k=k^*}$ is the shear stress at $k = k^*$ and is positive. According to (8.4), at reversal of the shear, the slope of the shear stress vs. shear plot is steeper on decreasing shear than on increasing shear, as would be expected.

The twisting moment is related to the shear stress by (3.16). The slope of the twisting moment–twist relation, for increasing or decreasing twist, is given by

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{M}{2\pi}\right) = \frac{\mathrm{d}}{\mathrm{d}\psi}\int_0^{R_\mathrm{o}} \overline{R}^2 T_{z\theta}(\overline{R}\psi)\,\mathrm{d}\overline{R} = \int_0^{R_\mathrm{o}} \overline{R}^3 \frac{\mathrm{d}T_{z\theta}(\overline{R}\psi)}{\mathrm{d}k}\,\mathrm{d}\overline{R}, \tag{8.5}$$

where use has been made of the relation $k = R\psi$. Let this be evaluated at maximum twist ψ^* for both increasing and decreasing twist. Then

$$\left.\left[\left.\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{M}{2\pi}\right)\right|_{\mathrm{dec}} - \left.\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{M}{2\pi}\right)\right|_{\mathrm{inc}}\right]\right|_{\psi=\psi^*}$$
$$= \int_0^{R_\mathrm{o}} \overline{R}^3 \left[\left.\frac{\mathrm{d}T_{z\theta}(\overline{R}\psi)}{\mathrm{d}k}\right|_{\mathrm{dec}} - \left.\frac{\mathrm{d}T_{z\theta}(\overline{R}\psi)}{\mathrm{d}k}\right|_{\mathrm{inc}}\right] \mathrm{d}\overline{R}\,\Bigg|_{\psi=\psi^*}. \tag{8.6}$$

Since there is no discontinuity in the slope of the shear stress vs. shear plot in the elastic core,

$$\left.\frac{\mathrm{d}T_{z\theta}(\overline{R}\psi)}{\mathrm{d}k}\right|_{\mathrm{dec}} - \left.\frac{\mathrm{d}T_{z\theta}(\overline{R}\psi)}{\mathrm{d}k}\right|_{\mathrm{inc}} = 0, \tag{8.7}$$

$0 \leqslant \overline{R} \leqslant R_\mathrm{a}$. (8.6) then reduces to

$$\left.\left[\left.\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{M}{2\pi}\right)\right|_{\mathrm{dec}} - \left.\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{M}{2\pi}\right)\right|_{\mathrm{inc}}\right]\right|_{\psi=\psi^*}$$
$$= \int_{R_\mathrm{a}}^{R_\mathrm{o}} \overline{R}^3 \left[\left.\frac{\mathrm{d}T_{z\theta}(\overline{R}\psi)}{\mathrm{d}k}\right|_{\mathrm{dec}} - \left.\frac{\mathrm{d}T_{z\theta}(\overline{R}\psi)}{\mathrm{d}k}\right|_{\mathrm{inc}}\right] \mathrm{d}\overline{R}\,\Bigg|_{\psi=\psi^*}. \tag{8.8}$$

By (8.4), the integrand is positive for $R_\mathrm{a} \leqslant \bar{R} \leqslant R_\mathrm{o}$. It follows that

$$\left.\left[\left.\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{M}{2\pi}\right)\right|_{\mathrm{dec}} - \left.\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{M}{2\pi}\right)\right|_{\mathrm{inc}}\right]\right|_{\psi=\psi^*} > 0. \tag{8.9}$$

At ψ^* the slope of the moment vs. twist plot is steeper on decreasing twist than on increasing twist, as would be expected.

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