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Technical Report No. 2

THE PLANE ELASTIC CHARACTERISTICS OF CORD-RUBBER LAMINATES

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NOMENCLATURE

English Letters:

a_{ij}, c_{ij}	Constants associated with generalized Hooke's law, using properties based on cord tension.
a'_{ij}	Constants associated with generalized Hooke's law, using properties based on cord compression.
E,F,G	Elastic constants for orthotropic laminates with cords in tension.
E',F',G'	Elastic constants for orthotropic laminates with cords in compression.
ℓ	Direction cosine.
u,v,w	Displacements in the x, y, z directions, respectively.
W	Elastic strain energy.
x,y,z	Orthogonal co-ordinates aligned along and normal to the cord direction.

Greek Letters:

α	One-half the included angle between cords in adjoining plies in a two-ply laminate.
ϵ	Strain
μ	Poisson's ratio
ξ, η, ζ	Orthogonal co-ordinates aligned along and normal to the principal axes of elasticity, or orthotropic axes, in an orthotropic laminate.
σ	Stress
σ'	Interply stress
σ^*	External stress acting on a ply whose cords are in tension.
σ^{**}	External stress acting on a ply whose cords are in compression.

I. STATEMENT

The behavior of a tire cannot be understood without first understanding the character and nature of the cord-rubber structure which makes up its carcass, or main load-carrying portion. Efforts have been made in the past to derive equations or to propose physical ideas which would lead to an understanding of the action of cords and rubber. These generally have not been successful since in most cases the presence and importance of interply shear stresses between laminations of cord-rubber sheets was not recognized. It is hoped that this projected series of reports will remedy that defect.

This report investigates and attempts to provide a coherent theory for the various elastic moduli of cord-rubber laminations, and for the stresses which exist in these laminations. It differs from previous work done using cords alone in that here we visualize load-carrying capacity in directions perpendicular to the cord, as well as in the cord directions themselves.

It is anticipated that the results of a study such as this will be useful in forming a basis for the accurate calculation of deformations and cord loads in cord-rubber structures in general, with particular application to pneumatic tires.

II. SUMMARY

An analysis is presented for the state of stress and deformation in a laminated cord-rubber sheet based on the principals of orthotropic materials. Expressions are derived for the elastic moduli of cord-rubber laminations and, as a by-product, for the stresses existing in these laminations. Experiments have been performed on test specimens in order to verify these theories. The agreement between experiment and theory for the elastic properties appears to be quite good, with one exception, over a wide range of physical characteristics.

It is shown in this report that a total of 14 possible elastic constants exist for a two-ply cord rubber laminate with arbitrary cord angle. Four of these are sufficient to describe the elastic nature of the laminate when the cords in both plies are in tension; a different set of four apply to the same body when it is loaded in such a way that the cords in both plies are in compression; and six still different elastic constants must be used to describe the elastic nature of the body when the loads are such that the cords of one ply are in tension while the cords of the other are in compression.

It is shown that in those cases where both plies are in either tension or compression, the deformation of the body is described by a statically determinate set of equations whose solution yields elastic constants as well as interply shear stresses.

For bodies subjected to combinations of external loads which force the cords of one ply into compression while those of the other ply remain in ten-

sion, it may be shown that deformation is now described by a statically indeterminate set of equations for which only approximate solutions are obtained at this time.

As mentioned earlier, interply shear stresses enter into the calculations. These are used here only insofar as they influence the elastic characteristics of this material, but a detailed discussion of these stresses by themselves is given in another report.

It may generally be concluded from the work reported here that the elastic properties of any type of cord-rubber lamination may be derived from the elastic properties of a single sheet of its constituent material.

IV. ELASTIC PROPERTIES OF CORD-RUBBER COMBINATIONS

In this section the elastic properties of a certain class of laminated structure will be expressed in terms of the elastic properties of a single sheet of parallel textile fibers imbedded in rubber and forming one lamina. It is hoped that, in the future, it will be possible to show how the elastic properties of these single sheets can be calculated from the known elastic properties of the rubber and cord which make up these sheets, and from the geometry of their combination.

A portion of the material presented in this section was previously worked out in Ref. 1. It is given again here for completeness and continuity.

The class of laminated structure considered here is that of a two-ply combination in which the cords are separated angularly in the two plies by an included angle 2α, as shown in Fig. 1. In that illustration, the heavy diagonal

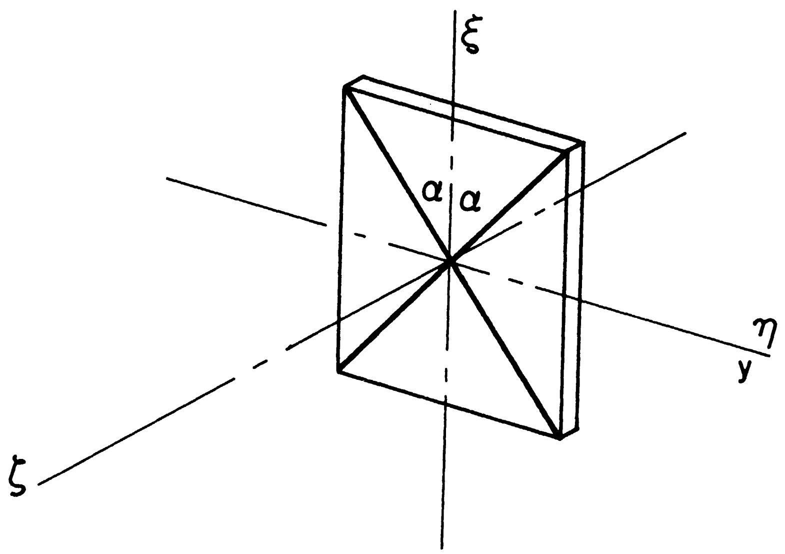

Fig. 1. Schematic view of a small section of two-ply laminate.

lines representing typical textile cords are shown being bisected by a coordinate axes ξ and η, two arms of the orthogonal ξ, η, ζ system.

This report will be confined to those structures in which stresses in the ζ direction are negligibly small compared with those in the ξ and η directions. This assumption will generally be true since this report is limited to structures which are large in the ξ and η directions compared to their thickness in the ζ direction.

While discussion in this report is limited to the characteristics of a two-ply system, the ideas and techniques developed allow fairly easy extension to any number of plies provided that the structure remains orthotropic, as defined below.

Neglecting unsymmetrical effects through the thickness of the two-ply structure (ζ direction), one may define an orthotropic material as one possessing two planes of symmetry at every point. For example, in Fig. 1 the planes of symmetry are ζ - η and ζ - ξ. From this it is seen that not only will the two-ply construction be orthotropic, but so will a multi-ply construction made up of a series of similar two-ply structures, and so will a number of special cases such as illustrated in Fig. 2. For the present only the two-ply construction of Fig. 1 is considered, for reasons to be brought out later.

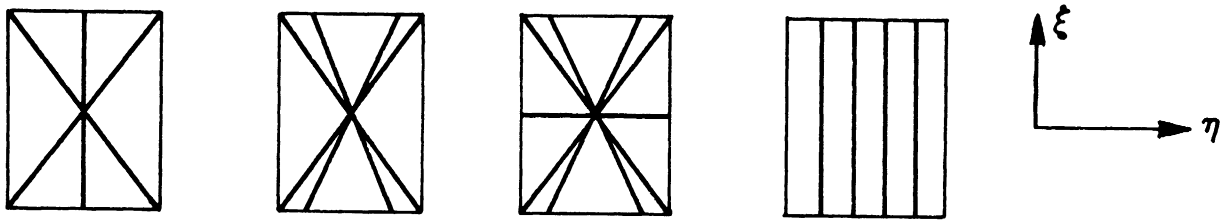

Fig. 2. Typical orthotropic laminates.

From the nature of the structure of Fig. 1, it is seen that pure normal stresses acting in either the ξ or η directions do not cause distortion (shear deformation) of the element. Similarly, shearing stresses distributed along its edges do not cause extensions in these directions, so that the usual form of Hooke's Law is seen to apply for an orthotropic material, except that the elastic constants will have different values in the different directions. This law is, of course, valid only when written in the ξ and η directions since these are the only directions in which, from symmetry, coupling between normal stresses and shearing strains vanishes. The equations relating stress and strain are

$$\epsilon_\xi = \frac{\sigma_\xi}{E_\xi} - \frac{\mu_{\eta\xi}}{E_\eta}\sigma_\eta$$

$$\epsilon_\eta = -\frac{\mu_{\xi\eta}}{E_\xi}\sigma_\xi + \frac{\sigma_\eta}{E_\eta}$$

$$\epsilon_{\xi\eta} = \frac{1}{G_{\xi\eta}}\sigma_{\xi\eta} \qquad (1)$$

In Eqs. (1), the elastic constants are expressed in the usual fashion, where the E terms represent relations between normal stress and strain while the ratio of lateral contraction to longitudinal extension is given by μ. The shear modulus is given by G. Note that $\mu_{\xi\eta}$ is not necessarily equal to $\mu_{\eta\xi}$.

Equations (1) may be, and in fact are, obtained directly by utilizing the properties of the planes of symmetry of an orthotropic sheet to write a conventional set of stress-strain relations with some modifications necessary to account for the anisotropic nature of the body. The same results may be obtained more exactly, and additional information is also gained, by use of the strain-energy function. The form of this function is adequately discussed in numerous reference works,[2,3] and will not be repeated here. It represents the work necessary to bring a unit element of the body in question from a stress-free state to one in which stresses act, and hence physically is the quantity of elastic energy stored. Its form is

$$W = \frac{1}{2}(\sigma_\xi \epsilon_\xi + \sigma_\eta \epsilon_\eta + \sigma_{\xi\eta} \epsilon_{\xi\eta}) \tag{a}$$

The generalized form of Hooke's Law is usually written

$$\begin{aligned} \sigma_\xi &= c_{11}\epsilon_\xi + c_{12}\epsilon_\eta + c_{16}\epsilon_{\xi\eta} \\ \sigma_\eta &= c_{21}\epsilon_\xi + c_{22}\epsilon_\eta + c_{26}\epsilon_{\xi\eta} \\ \sigma_{\xi\eta} &= c_{61}\epsilon_\xi + c_{62}\epsilon_\eta + c_{66}\epsilon_{\xi\eta} \end{aligned} \tag{b}$$

If these expressions for the stresses are substituted into the immediately preceding expression for strain energy, one obtains

$$\begin{aligned} 2W = {} & c_{11}\epsilon_\xi^2 + (c_{12} + c_{21})\epsilon_\xi\epsilon_\eta \\ & + (c_{16} + c_{61})\epsilon_\xi\epsilon_{\xi\eta} + c_{22}\epsilon_\eta^2 \\ & + (c_{26} + c_{62})\epsilon_\eta\epsilon_{\xi\eta} + c_{66}\epsilon_{\xi\eta}^2 \ . \end{aligned} \tag{2}$$

Equation (2) may be simplified somewhat by drawing upon the specific properties of an orthotropic material. In particular, referring to Figs. 1 or 2, the form of the strain-energy function must not change if the positive ξ or η coordinate directions are reversed. The consequences of reversal of the coordinate directions may be seen by examining the definitions of strain,

$$\epsilon_\xi = \frac{\partial u}{\partial \xi} \qquad \epsilon_\eta = \frac{\partial v}{\partial \eta} \qquad \epsilon_{\xi\eta} = \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \eta} \tag{c}$$

so that one may finally write the stress-strain relations for any orthotropic body in the form

$$\begin{aligned} \epsilon_\xi &= \frac{\sigma_\xi}{E_\xi} - \frac{\sigma_\eta}{F_{\xi\eta}} \\ \epsilon_\eta &= -\frac{\sigma_\xi}{F_{\xi\eta}} + \frac{\sigma_\eta}{E_\eta} \\ \epsilon_{\xi\eta} &= \frac{1}{G_{\xi\eta}} \sigma_{\xi\eta} \end{aligned} \tag{5}$$

The terminology and form of Eqs. (5) will be used in future work involving stress-strain relations.

It is now necessary to consider a specific kind of orthotropic body, one which is constructed by imbedding a series of parallel, straight cords lying in a plane into a sheet-like matrix of more elastic material. This is illustrated in Fig. 3. It is seen that maximum and minimum moduli of elasticity are to be found in the x and y directions.

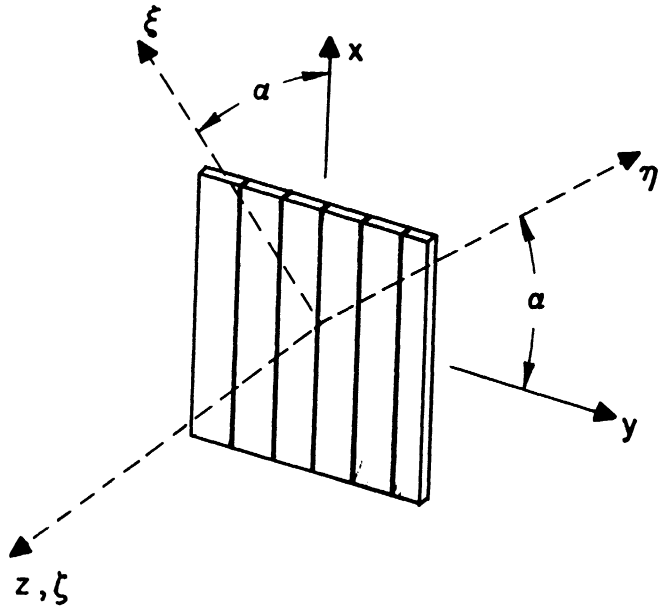

Fig. 3. Schematic view of a single ply of cord imbedded in rubber.

It will be assumed that the elastic constants E_x, E_y, F_{xy}, and G_{xy} are known. Generally, these constants could be obtained by rather straightforward elastic tests. However, a different set of constants may be applicable depending on whether or not the textile cords are in a state of tension or in a state of compression, because it was demonstrated[4] that completely different cord moduli prevailed depending upon the sign of the cord stress. Since it is anticipated that the textile cord will contribute almost all the resistance to elastic deformation of a sheet such as illustrated in Fig. 3, it is reasonable to assume that the action of the cord modulus will to a great extent govern the action of the sheet modulus. For many purposes it will be possible to treat these laminated structures as being either all in tension or all in compression. In those cases the choice of the appropriate elastic constants is clear and no difficulty should arise. Occasionally a situation might arise in which some cords in a laminated structure are in tension while others are in compression. It would then be necessary to determine ahead of time which cords were in each state and to use the appropriate elastic constants for these cords, and hence for the sheets of which they are a part. For purposes of clarity of notation, the elastic constants will be written as E_x, E_y, and so forth for those cases in which tension loads are implied, or cases where the sign of the load does not matter, as in dealing with general formula or manipulations. For those specific cases where it is necessary to utilize the compression elastic constants of the sheet of Fig. 3, these will be written as E'_x, E'_y, F'_{xy}, and G'_{xy}. For most of the development which follows, it will not be necessary to utilize these primed elastic constants and they will not really enter into the discussion again until results of the experimental tests are compared with theoretical predictions later in this report. Then they are of some importance. The use of these compression elastic properties is quite necessary to explain fully the action of laminated cord-rubber structures, and their inclusion in this discussion is not merely for academic completeness. Finally, with the addition of this idea, the sheet of Fig. 3 now has eight elastic constants, four pertaining to a state of tension in the cords and four pertaining to a state of compression.

Again referring to Fig. 3, consider a known stress state σ_x, σ_y, and σ_{xy}, with a corresponding known strain state ϵ_x, ϵ_y, and ϵ_{xy}. It is desired to determine the corresponding stress and strain state referred to the orthogonal axes ξ and η, obtained by rotating the xyz axis about the z arm through an angle α. Let $\ell_{\xi x}$, $\ell_{\xi y}$ be direction cosines of the ξ axis with respect to x and y axes, and let $\ell_{\eta x}$ and $\ell_{\eta y}$ be those of the η axis. Then it can be shown[1,3] that the stress components referred to the ξ, η axis, written in terms of the known components in the x, y axes, are

$$\begin{aligned} \sigma_\xi &= \sigma_x \ell_{\xi x}^2 + 2\sigma_{xy}\ell_{\xi x}\ell_{\xi y} + \sigma_y \ell_{\xi y}^2 \\ \sigma_\eta &= \sigma_x \ell_{\eta x}^2 + 2\sigma_{xy}\ell_{\eta x}\ell_{\eta y} + \sigma_y \ell_{\eta y}^2 \\ \sigma_{\xi\eta} &= \sigma_x \ell_{\xi x}\ell_{\eta x} + \sigma_{xy}\ell_{\xi x}\ell_{\eta y} + \sigma_{yx}\ell_{\xi y}\ell_{\eta x} + \sigma_y \ell_{\xi y}\ell_{\eta y} \end{aligned} \tag{6}$$

Similarly, the strain components in the ξ, η directions may be expressed as

$$\begin{aligned} \epsilon_\xi &= \epsilon_x \ell_{\xi x}^2 + \epsilon_y \ell_{\xi y}^2 + \epsilon_{xy}\ell_{\xi x}\ell_{\xi y} \\ \epsilon_\eta &= \epsilon_x \ell_{\eta x}^2 + \epsilon_y \ell_{\eta y}^2 + \epsilon_{xy}\ell_{\eta x}\ell_{\eta y} \\ \epsilon_{\xi\eta} &= 2\epsilon_x \ell_{\xi x}\ell_{\eta x} + 2\epsilon_y \ell_{\xi y}\ell_{\eta y} + \epsilon_{xy}(\ell_{\xi x}\ell_{\eta y} + \ell_{\xi y}\ell_{\eta x}) \end{aligned} \tag{7}$$

Eqs. (6) and (7) are perfectly general and are quite independent of any possible orthotropy possessed by the element in question. For example, if the stress state in the ξ, η directions is known then it may be used to express the stress state in the x, y directions by means of the relations analogous to Eq. (6), these being

$$\begin{aligned}
\sigma_x &= \sigma_\xi \ell_{x\xi}^2 + 2\sigma_{\xi\eta}\ell_{x\xi}\ell_{x\eta} + \sigma_\eta \ell_{x\eta}^2 \\
\sigma_y &= \sigma_\xi \ell_{y\xi}^2 + 2\sigma_{\xi\eta}\ell_{y\xi}\ell_{y\eta} + \sigma_\eta \ell_{y\eta}^2 \\
\sigma_{xy} &= \sigma_\xi \ell_{x\xi}\ell_{y\xi} + \sigma_{\xi\eta}\ell_{x\xi}\ell_{y\eta} + \sigma_{\xi\eta}\ell_{x\eta}\ell_{y\xi} + \sigma_\eta \ell_{x\eta}\ell_{y\eta}
\end{aligned} \tag{8}$$

These Eqs. (8) may be obtained by a change of subscripts from Eq. (6), letting ξ and x be interchanged as well as y and η.

The generalized form of Hooke's law may be written using the stresses as independent variables with respect to any desired coordinate axes, for example,

$$\boxed{\begin{aligned}
\epsilon_\xi &= a_{11}\sigma_\xi + a_{12}\sigma_\eta + a_{13}\sigma_{\xi\eta} \\
\epsilon_\eta &= a_{21}\sigma_\xi + a_{22}\sigma_\eta + a_{23}\sigma_{\xi\eta} \\
\epsilon_{\xi\eta} &= a_{31}\sigma_\xi + a_{32}\sigma_\eta + a_{33}\sigma_{\xi\eta}
\end{aligned}} \tag{9}$$

Equations (9) are basic to this development and will be treated in some detail.

To determine the values of the various coefficients a_{11} through a_{33} of Eqs. (9), the stress σ_ξ may be considered known while the stress σ_η and $\sigma_{\xi\eta}$ are set equal to zero. Then it follows that

$$a_{11} = \frac{\epsilon_\xi}{\sigma_\xi} \qquad a_{21} = \frac{\epsilon_\eta}{\sigma_\xi} \qquad a_{31} = \frac{\epsilon_{\xi\eta}}{\sigma_\xi} \tag{10}$$

From Eqs. (7) it is seen that ϵ_ξ, ϵ_η, and $\epsilon_{\xi\eta}$ may be expressed in terms of ϵ_x, ϵ_y, and ϵ_{xy}. Next, noting that the axes of orthotropy for the body of Fig. 3 is the xyz axis, Eqs. (5) may be written for this body as

$$\epsilon_x = \frac{\sigma_x}{E_x} - \frac{\sigma_y}{F_{xy}} \qquad \epsilon_y = -\frac{\sigma_x}{F_{xy}} + \frac{\sigma_y}{E_y} \qquad \epsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}} \tag{11}$$

Using these equations, and noting from Eqs. (18) that σ_x, σ_y, and σ_{xy} may be expressed in terms of the nonzero σ_ξ, it is seen that finally ϵ_ξ can be written in terms of σ_ξ which allows a_{11} to be found. A single example of this will be worked out. Others follow in a similar manner by letting other stress components take on nonzero values, such as by letting σ_η become nonzero while σ_ξ and $\sigma_{\xi\eta}$ vanish. The example chosen here is that of the determination of the constant a_{11}. For this case, from Eqs. (8), and recalling that σ_ξ is the only nonzero stress component,

$$\sigma_x = \sigma_\xi \ell_{x\xi}^2 \qquad \sigma_y = \sigma_\xi \ell_{y\xi}^2 \qquad \sigma_{xy} = \sigma_\xi \ell_{x\xi}\ell_{y\xi} \tag{i}$$

Substituting into Eqs. (11) gives

$$\epsilon_x = \sigma_\xi\left(\frac{\ell_{x\xi}^2}{E_x} - \frac{\ell_{y\xi}^2}{F_{xy}}\right) \qquad \epsilon_y = \sigma_\xi\left(-\frac{\ell_{x\xi}^2}{F_{xy}} + \frac{\ell_{y\xi}^2}{E_y}\right) \qquad \epsilon_{xy} = \sigma_\xi\left(\frac{\ell_{x\xi}\ell_{y\xi}}{G_{xy}}\right) \tag{k}$$

Substituting these into the first of Eqs. (7) gives

$$\epsilon_\xi = \sigma_\xi\left[\frac{\ell_{\xi x}^4}{E_x} - \frac{\ell_{\xi x}^2\ell_{y\xi}^2}{F_{xy}} - \frac{\ell_{\xi y}^2\ell_{x\xi}^2}{F_{xy}} + \frac{\ell_{\xi y}^4}{E_y} + \frac{\ell_{\xi x}^2\ell_{\xi y}^2}{G_{xy}}\right] \tag{ℓ}$$

and substituting this into the first of Eqs. (10), along with the values of the direction cosines,

$$\ell_{\xi x} = \cos\alpha \qquad \ell_{\xi y} = -\sin\alpha$$

$$\ell_{\eta y} = \cos\alpha \qquad \ell_{\eta x} = \sin\alpha$$

one obtains

$$a_{11} = \frac{\cos^4\alpha}{E_x} + \frac{\sin^4\alpha}{E_y} + \sin^2\alpha \cos^2\alpha\left(\frac{1}{G_{xy}} - \frac{2}{F_{xy}}\right)$$

By identical procedures one may also obtain

$$a_{22} = \frac{\sin^4\alpha}{E_x} + \frac{\cos^4\alpha}{E_y} + \sin^2\alpha \cos^2\alpha\left(\frac{1}{G_{xy}} - \frac{2}{F_{xy}}\right)$$

$$a_{33} = 4\left[\frac{\sin^2\alpha \cos^2\alpha}{E_x} + \frac{\sin^2\alpha \cos^2\alpha}{E_y} + \frac{2\sin^2\alpha \cos^2\alpha}{F_{xy}}\right] + \frac{(\cos^2\alpha - \sin^2\alpha)^2}{G_{xy}}$$

$$a_{21} = a_{12} = \frac{\sin^2\alpha \cos^2\alpha}{E_x} + \frac{\sin^2\alpha \cos^2\alpha}{E_y} - \left(\frac{\sin^4\alpha + \cos^4\alpha}{F_{xy}}\right) - \frac{\sin^2\alpha \cos^2\alpha}{G_{xy}}$$

$$a_{13} = a_{31} = \frac{2\cos^3\alpha \sin\alpha}{E_x} - \frac{2\sin^3\alpha \cos\alpha}{E_y}$$

$$- \cos\alpha \sin\alpha(\cos^2\alpha - \sin^2\alpha)\left(\frac{1}{G_{xy}} - \frac{2}{F_{xy}}\right)$$

$$a_{32} = a_{23} = \frac{2\sin^3\alpha \cos\alpha}{E_x} - \frac{2\cos^3\alpha \sin\alpha}{E_y}$$

$$+ \sin\alpha \cos\alpha(\cos^2\alpha - \sin^2\alpha)\left(\frac{1}{G_{xy}} - \frac{2}{F_{xy}}\right) \qquad (12)$$

Equations (9) and (12) now allow the properties of an orthotropic sheet (for example one ply of tire carcass material) to be predicted in any direction ξ, η at an angle α with the cord direction xy. This knowledge can be used to construct a two-ply structure.

Consider now a sheet of the type shown in Fig. 3 but inclined at some angle α to the orthotropic sheet axis x and y. This is illustrated in Fig. 4. Imagine that we desire to extend this sheet in the ξ and η directions by means of normal

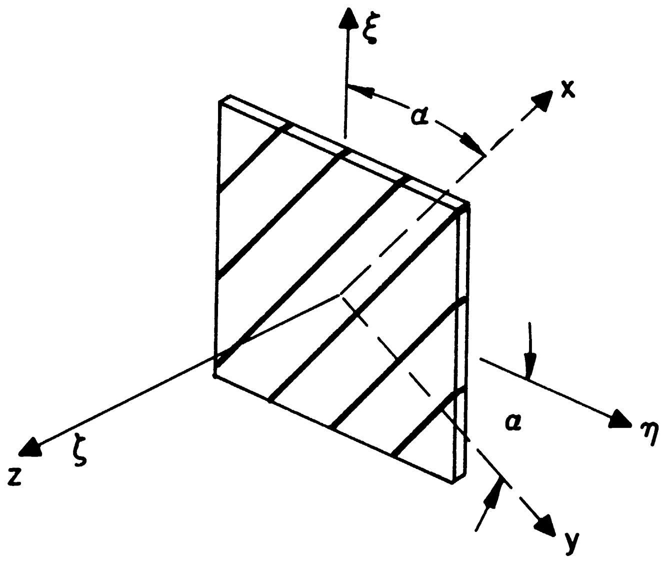

Fig. 4. Single ply of cord imbedded in rubber at angle α to the vertical ξ-axis.

stresses acting on the edges. Since the orthotropic axis x and y do not coincide with the ξ, η axis, this is not possible; distortions $\epsilon_{\xi\eta}$ will inevitably accompany the application of any set of normal stresses σ_ξ and σ_η. For that reason we must admit the existence of shearing stresses $\sigma_{\xi\eta}$ as necessary for distortionless extension of an element such as that of Fig. 4, where the reference and orthotropic axes are not the same. With this provision in mind, one may go directly to Eqs. (9) and presume no distortion under loading, i.e., the loads will be adjusted to prevent distortion. Then Eqs. (9) become

$$\epsilon_\xi = a_{11}\sigma_\xi + a_{12}\sigma_\eta + a_{13}\sigma_{\xi\eta}$$

$$\epsilon_\eta = a_{21}\sigma_\xi + a_{22}\sigma_\eta + a_{23}\sigma_{\xi\eta}$$

$$\epsilon_{\xi\eta} = 0 = a_{31}\sigma_\xi + a_{32}\sigma_\eta + a_{33}\sigma_{\xi\eta}$$

From these it may be seen that the only stress which is not independent can be chosen to be $\sigma_{\xi\eta}$, since it can be expressed in terms of σ_ξ and σ_η from the last of Eqs. (13), and this expression can then be substituted into the first two of Eqs. (13). Doing this, one obtains

$$\sigma_{\xi\eta} = -\frac{a_{31}}{a_{33}}\sigma_\xi - \frac{a_{32}}{a_{33}}\sigma_\eta$$

$$\epsilon_\xi = \left(a_{11} - \frac{a_{13}a_{31}}{a_{33}}\right)\sigma_\xi + \left(a_{12} - \frac{a_{13}a_{32}}{a_{33}}\right)\sigma_\eta$$

$$\epsilon_\eta = \left(a_{21} - \frac{a_{23}a_{31}}{a_{33}}\right)\sigma_\xi + \left(a_{22} - \frac{a_{23}a_{32}}{a_{33}}\right)\sigma_\eta \qquad (14)$$

$\sigma_{\xi\eta}$ is the stress that must be furnished from some external source to obtain a distortionless extension. In view of the previously determined symmetry $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{32} = a_{23}$, it is seen by comparison of this set of equations with Eqs. (5) that

$$\frac{1}{E_\xi} = \left(a_{11} - \frac{a_{13}^2}{a_{33}}\right) \qquad \frac{1}{E_\eta} = \left(a_{22} - \frac{a_{23}^2}{a_{33}}\right)$$

$$\frac{1}{F_{\xi\eta}} = \left(a_{12} - \frac{a_{13}a_{23}}{a_{33}}\right) \qquad (15)$$

In connection with Eqs. (15), if one were to return to Eqs. (12) and utilize the compression characteristics E'_x, E'_y, and so forth, a completely different set of constants a'_{11}, a'_{22}, etc., would exist, and these might very well be denoted as the compression constants since they are calculated using the compression characteristics of the original sheet of Fig. 3. In that case one might logically associate a prime with each of these a_{ij} terms, and their use in Eqs. (15) would result in a set of moduli E'_ξ, E'_η, and $F'_{\xi\eta}$ completely different from those using the elastic constants associated with tension. It is further implied in the construction of Eqs. (15) that all the cords in the structure will be either in tension or in compression. Equations (15) would not normally hold for those cases where some cords were in tension and some cords in compression.

It is now necessary to consider the last elastic constant, the relationship between shearing strain and shearing stress. This elastic constant will be given the symbol $G_{\xi\eta}$. Three different possible shear moduli $G_{\xi\eta}$ exist: one associated with all cords in compression; another with all cords in tension; and the third with one ply or more in tension while the others are in compression. The first two cases may be worked out in a very straightforward way and will be presented at this point. The last case involves some additional complications and appears at the end of this section of the report. The important point is that it is possible to calculate $G_{\xi\eta}$.

To obtain the $G_{\xi\eta}$ associated with either cord compression or cord tension, it is necessary to postulate an extension-free distortion, or pure distortion $\epsilon_{\xi\eta}$, which occurs as the result of the proper application of the stresses σ_ξ, σ_η, and $\sigma_{\xi\eta}$. Equations (9) become, when applied to this case,

$$\begin{aligned} \epsilon_\xi &= 0 = a_{11}\sigma_\xi + a_{12}\sigma_\eta + a_{13}\sigma_{\xi\eta} \\ \epsilon_\eta &= 0 = a_{21}\sigma_\xi + a_{22}\sigma_\eta + a_{23}\sigma_{\xi\eta} \\ \epsilon_{\xi\eta} &= a_{31}\sigma_\xi + a_{32}\sigma_\eta + a_{33}\sigma_{\xi\eta} \end{aligned} \qquad \text{(m)}$$

The first two of these may be used to express σ_ξ and σ_η in terms of $\sigma_{\xi\eta}$. This results in

$$\sigma_\eta = -\frac{\left(a_{23} - \frac{a_{12}a_{13}}{a_{11}}\right)}{\left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)}\sigma_{\xi\eta} \tag{16a}$$

$$\sigma_\xi = \left[\frac{a_{12}}{a_{11}}\frac{\left(a_{23} - \frac{a_{12}a_{13}}{a_{11}}\right)}{\left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)} - \frac{a_{13}}{a_{11}}\right]\sigma_{\xi\eta} \tag{16b}$$

and finally these may be used to obtain

$$\boxed{\frac{1}{G_{\xi\eta}} = \left\{- a_{31}\left[\left(\frac{a_{12}}{a_{11}}\right)\frac{(a_{12}a_{13} - a_{11}a_{23})}{(a_{11}a_{22} - a_{12}^2)} + \frac{a_{13}}{a_{11}}\right] + (a_{32})\frac{(a_{12}a_{13} - a_{11}a_{23})}{(a_{11}a_{22} - a_{12}^2)} + a_{33}\right\}} \tag{17}$$

Equations (15) and (17) are extremely important as they allow the elastic constants of an orthotropic structure to be calculated in any direction as a function of the terms a_{11}, a_{22}, --- a_{33}. These, in turn, are determined completely by the elastic properties of the sheet in the directions of the orthotropic axes, i.e., parallel and perpendicular to the cord directions, as well as the angular orientation of the sheet. As mentioned previously, Eqs. (17) represent two of the possible three shear moduli of a laminated structure. It can be used to represent the shear modulus for those structures where all cords are either in tension or compression, depending on whether a_{ij} or a'_{ij}'s are used in it. The third case, in which one ply is in tension and one ply is in compression, will be dealt with at a later time.

It is seen that the elastic properties given by Eqs. (15) and (17) are identical to the elastic properties of the compound orthotropic structure of Fig. 1 provided that the two plys which make up Fig. 1 are identical in all respects and provided that provisions are made for obtaining these "extra" stresses such as given by Eqs. (14) or (16a) and (16b). It will be assumed throughout this discussion that a two-ply structure is under consideration, each ply being identical in composition. Under this condition, the problem of the origin and nature of these additional or "extra" stresses will be discussed in subsequent pages.

A considerable amount of extremely interesting information can be obtain by a close inspection of the techniques used to obtain Eqs. (15) and (17). Some of this can only be extracted numerically and will be covered by a separate report. The main ideas will be discussed here in so far as possible.

First, consider Eq. (14) in which the shear stress necessary for the existence of a distortion-free extension of the element is given in terms of the normal stresses which may exist σ_ξ and σ_η. This shear stress must be supplied whenever a pure extension takes place without distortion. In the case of an orthotropic body made up of two layers such as in Fig. 1, it is known from Eqs. (5) that the application of only normal stresses along the edges will result in only normal extensions ϵ_ξ and ϵ_η, with zero distortion $\epsilon_{\xi\eta}$. Thus, the only possible origin of the shear stress of Eq. (14) is in the interply shear stresses set up due to rotation of cords with respect to one another. In fact, Eq. (14) gives the shearing component of the total interply shear stress as a function of the externally applied stresses σ_ξ and σ_η.* This means that in cases of pure extension where $\epsilon_{\xi\eta}$ vanishes, the interply stress can be determined by calculating or measuring the normal stresses σ_η and σ_ξ and then using Eq. (14). For this reason Eq. (14) is of some importance, since from it the influence of design changes on one kind of interply stress can be predicted. In particular, the effect of cord angle or elastic moduli of materials on interply stress of this kind can easily be found.

The discussion in the preceding paragraph indicates that for those cases where shear distortion $\epsilon_{\xi\eta}$ vanishes the interply stress can be determined from Eq. (14) if σ_ξ and σ_η are known. It is now necessary to consider the more general case when the shear distortion $\epsilon_{\xi\eta}$ does not vanish. To do this, it is necessary to consider the manner in which the orthotropic two-ply structure of Fig. 1 is formed from plys such as shown in Fig. 4. In all cases a film of rubber is interspersed between the cords in different layers, often called the insulating rubber. This material acts as a spring so that one expects the interply stresses generated between plies to arise due to the relative rotation of cords over one another. These stresses act positively on one ply and negatively on another since they must be equal and opposite. One could thus say that the total stress acting at any point in any ply is made up of twc components, one due to the interply stresses and one due to external stresses ap-

*A problem of terminology arises here. Any stresses transmitted between plies must arise from shearing stresses between these plies. These in turn can have distortion-causing effects or extension-causing effects on each separate ply. The total stress transmitted between plies will henceforth be referred to as the <u>interply stress</u>, and its components will be referred to as the <u>shearing</u> (distortion-causing) and <u>normal</u> (extension-causing) components.

plied by some outside agency. It is convenient now to let unprimed stress components represent those arising from external sources, while primed components represent those arising from the elastic interply connection. For example, $\sigma_{\xi\eta}$ denotes an externally applied shear stress while $\sigma'_{\xi\eta}$ denotes a shear stress component rising from interply action.

Returning to the two-ply construction of Fig. 1, it is now possible to write equations similar to Eq. (9) about each of the plies, recognizing that one ply will have an orientation angle $+\alpha$ and the other an orientation angle $-\alpha$. It should be noted that for a sign change of the angle of orientation, the only two of the terms of Eq. (12) which change sign are $a_{13} = a_{31}$ and $a_{32} = a_{23}$. The total stresses will be split into internal and external parts previously discussed. Using these ideas, one may write equations similar to Eq. (9) for each ply as follows:

Ply 1: $(+\alpha)$

$$\epsilon_\xi = a_{11}(\sigma_\xi + \sigma'_\xi) + a_{12}(\sigma_\eta + \sigma'_\eta) + a_{13}(\sigma_{\xi\eta} + \sigma'_{\xi\eta})$$

$$\epsilon_\eta = a_{21}(\sigma_\xi + \sigma'_\xi) + a_{22}(\sigma_\eta + \sigma'_\eta) + a_{23}(\sigma_{\xi\eta} + \sigma'_{\xi\eta})$$

$$\epsilon_{\xi\eta} = a_{31}(\sigma_\xi + \sigma'_\xi) + a_{32}(\sigma_\eta + \sigma'_\eta) + a_{33}(\sigma_{\xi\eta} + \sigma'_{\xi\eta}) \qquad (18)$$

Ply 2: $(-\alpha)$

$$\epsilon_\xi = a_{11}(\sigma_\xi - \sigma'_\xi) + a_{12}(\sigma_\eta - \sigma'_\eta) - a_{13}(\sigma_{\xi\eta} - \sigma'_{\xi\eta})$$

$$\epsilon_\eta = a_{21}(\sigma_\xi - \sigma'_\xi) + a_{22}(\sigma_\eta - \sigma'_\eta) - a_{23}(\sigma_{\xi\eta} - \sigma'_{\xi\eta})$$

$$\epsilon_{\xi\eta} = - a_{31}(\sigma_\xi - \sigma'_\xi) - a_{32}(\sigma_\eta - \sigma'_\eta) + a_{33}(\sigma_{\xi\eta} - \sigma'_{\xi\eta}) \qquad (19)$$

The strains of the two plies, expressed as Eqs. (18) and (19), must be identical in each of the plies since they are intimately joined by means of the insulation rubber. Hence they may be equated, provided it is understood that the strains in question here are average strains taken over several times the cord spacing. Setting the first of Eqs. (18) equal to the first of (19), the second equal to the second and the third equal to the third, one obtains three equations in the three components σ'_ξ, σ'_η, $\sigma'_{\xi\eta}$ of the total interply stress, denoted σ'. Omitting the intermediate algebra, this gives

$$\sigma'_\eta = \frac{(a_{12}a_{13} - a_{11}a_{23})}{(a_{11}a_{22} - a_{12}^2)}\,\sigma_{\xi\eta}$$

$$\sigma'_\xi = -\left[\left(\frac{a_{12}}{a_{11}}\right)\frac{(a_{12}a_{13} - a_{11}a_{23})}{(a_{11}a_{22} - a_{12}^2)} + \frac{a_{13}}{a_{11}}\right]\sigma_{\xi\eta} \qquad (20)$$

$$\sigma'_{\xi\eta} = -\left(\frac{a_{13}}{a_{33}}\right)\sigma_\xi - \left(\frac{a_{23}}{a_{33}}\right)\sigma_\eta$$

These Eqs. (20) were previously obtained as Eqs. (14), (16a), and (16b) but their significance was not brought out at that time since another line of thought was being pursued. One could thus say that Eqs. (18) and (19) form a statically determinate set from which both the interply shear stresses and the strains can be obtained, once the external stresses are specified. It should again be emphasized in connection with Eqs. (20) that they are valid only for those cases where all cords of a two-ply structure are in tension or all cords of a two-ply structure are in compression. In the latter case Eqs. (20) should be written using the elastic constants a'_{ij}, corresponding to the a_{ij} terms evaluated using the compressive elastic constants of a single sheet.

Equations (20) result in the conclusion that the application of external normal stresses σ_ξ and σ_η cause the interply stress to be composed entirely of its shearing component $\sigma'_{\xi\eta}$. Only external shear stresses $\sigma_{\xi\eta}$ can generate interply normal stress components. While Eqs. (20) are worthy of considerably more interpretation than is possible to give here, it will simply be pointed out that, if one is willing to calculate or measure experimentally the externally applied stresses σ_ξ, σ_η and $\sigma_{\xi\eta}$, the interply stress components σ'_η, σ'_ξ, and $\sigma'_{\xi\eta}$ can be determined immediately from Eqs. (20). Having done this, the total interply stress σ' can be formed from the usual two-dimensional expression

$$\sigma'_{\substack{max\\min}} = \left(\frac{\sigma'_\xi + \sigma'_\eta}{2}\right) \pm \left[\left(\frac{\sigma'_\xi - \sigma'_\eta}{2}\right)^2 + (\sigma'_{\xi\eta})^2\right]^{1/2} \qquad (21)$$

Equation (21) thus gives the total interply stress which must be carried as a shear stress across the bond between the two plies.

A much more thorough and complete illustration of the physical interpretation and implications of Eq. (21) will be made in a future report. Our objective here is merely to note the origin and nature of these interply stresses as they affect the elastic properties of the structure.

Equations (15) and (17) describe the normal elastic constants of the system. These are the constants which prevail when all the cords in a two-ply structure are in tension, or alternately, if the appropriate primed values of the a_{ij} terms are used, then Eqs. (15) and (17) give the appropriate elastic constants of the cord-rubber structure under conditions of the cord being in compression. It happens occasionally that the external forces applied to a two-ply structure are of such a nature that the cords of one ply are in tension while the other ply has its cords in compression. In this case, as has been mentioned previously, a completely different set of equations must be written and one could derive from this set a completely new series of elastic constants. It can be shown that the only type of stresses which can cause such a situation in a two-ply structure are externally applied shearing stresses. In the particular series of experiments performed as a check on the validity of this theory, and discussed in succeeding sections of this report, only a single case occurred where it was necessary to utilize a calculated value of an elastic constant under conditions of one ply having cord tension while the other ply had cord compression. This situation occurred during torsion testing of cylindrical tubes and determination of the resulting shear modulus $G_{\xi\eta}$ from the data. Such a loading condition produced shear stresses and no normal stresses. Thus, cords in alternate plies were forced into tension and compression and the analysis which follows had to be performed to explain the phenomena observed. It would have been, of course, incorrect not to take these effects into account.

It now becomes quite apparent that the elastic characteristics of a laminated cord-rubber structure cannot be calculated unless one is aware of the state of stress in the structure. One must first of all decide whether all the cords are in tension, all are in compression, or one ply is in tension and the other in compression. This determination is further complicated by the fact that interply, as well as external, stresses act on each cord-rubber element so that the complete answer to whether or not a cord is in compression can be determined only after a fairly tedious numerical calculation. A report has been prepared which endeavors to reduce the labor associated with determining the stress state in a laminated cord-rubber structure by means of presenting a series of rather general calculations which should hold for almost all materials. This information is being incorporated into Technical Report No. 4 of the Tire and Suspensions Systems Research Group, and will be issued shortly.

The study of two-ply structures of this third type is begun by writing a set of equations analogous to Eqs. (18) and (19), but now taking into account that the elastic compliance of each ply is different and hence that each ply may carry a portion of the external loads somehow proportional to its elastic characteristics. If one disassembles this two-ply structure into each of its constituent plies, the stress state is assumed to be that shown in Fig. 5.

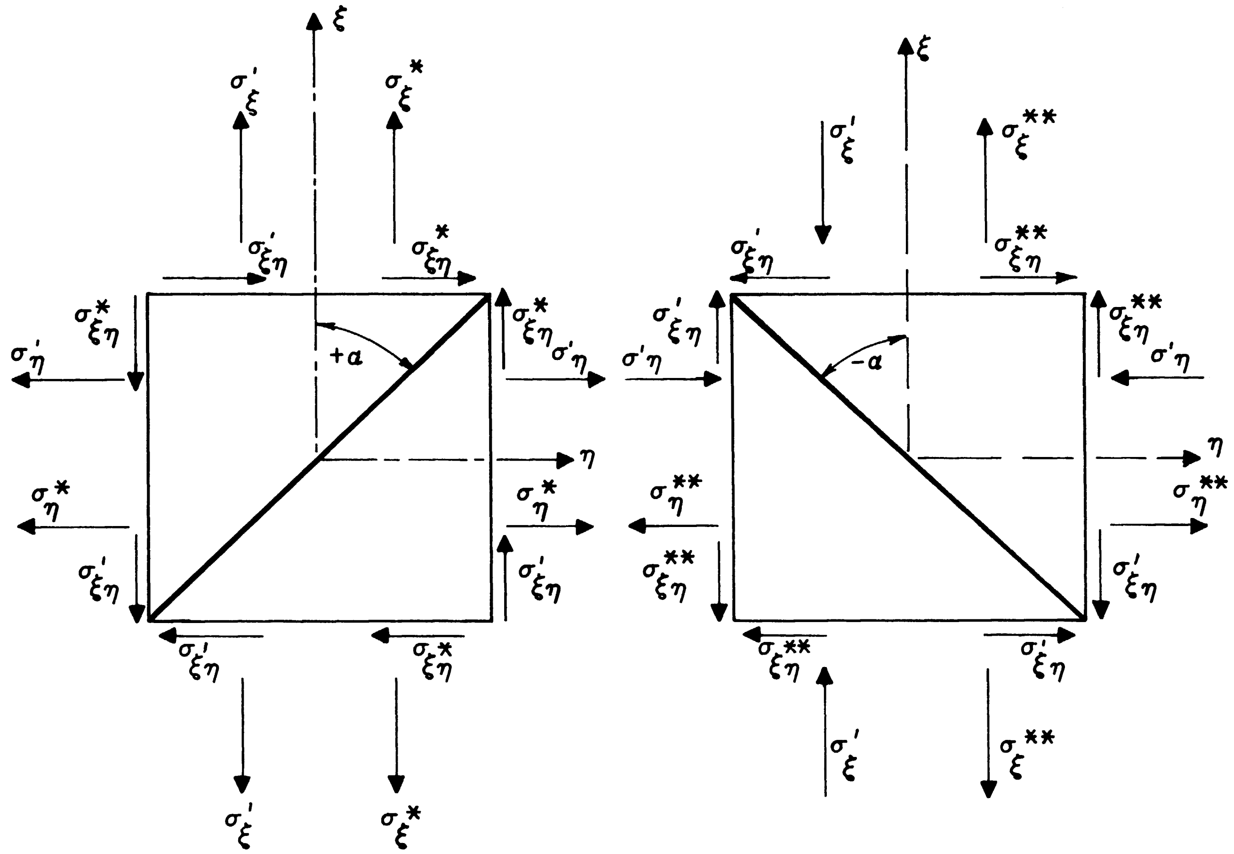

Fig. 5. General plane state of stress on each ply of a two-ply structure.

In Fig. 5, it is seen that one of the plies is assumed to carry an external shear stress of magnitude $\sigma^{*}_{\xi\eta}$ while the second ply is assumed to carry an external shear stress of magnitude $\sigma^{**}_{\xi\eta}$. Similar distributions are used for σ_{ξ} and σ_{η}. This simply reflects the fact that, due to the unequal moduli of the plies, unequal loads will also be carried by these plies. Our previous developments yielding Eqs. (15) and (17) implied equal load-carrying ability of each of the plies since in those cases both plies had cords either wholly in tension or wholly in compression. Here this is not the case and one must therefore note that the load-carrying ability of each of the plies is not necessarily the same.

Writing Eqs. (9) for each of the two plies in this structure, and noting that ply 1, which is assumed to be in tension, will be described by a set of a_{ij}'s which are calculated from tension elastic characteristics of a single sheet, one obtains the first three of Eqs. (22). Ply 2, assumed to be in compression due to the presence of the compressive interply shear stresses, may also be described elastically by Eqs. (9), now written using a'_{ij} terms based on compressive elastic characteristics of a single sheet. This results in the second set of three of Eqs. (22). The last three of Eqs. (22) are merely a statement that the sum of the stresses carried by the two plies equals the total average stress carried by the composite cord-rubber structure. In writing these equations, the angular argument of the a_{ij} is expressed in parentheses immediately following each term, while primed and unprimed a_{ij} refer to values of these constants being based on compression and tension elastic properties, respectively.

Ply 1

$$\epsilon_\xi = [a_{11}(+\alpha)](\sigma^*_\xi + \sigma'_\xi) + [a_{12}(+\alpha)](\sigma^*_\eta + \sigma'_\eta) + [a_{13}(+\alpha)](\sigma^*_{\xi\eta} + \sigma'_{\xi\eta})$$

$$\epsilon_\eta = [a_{21}(+\alpha)](\sigma^*_\xi + \sigma'_\xi) + [a_{22}(+\alpha)](\sigma^*_\eta + \sigma'_\eta) + [a_{23}(+\alpha)](\sigma^*_{\xi\eta} + \sigma'_{\xi\eta})$$

$$\epsilon_{\xi\eta} = [a_{31}(+\alpha)](\sigma^*_\xi + \sigma'_\xi) + [a_{32}(+\alpha)](\sigma^*_\eta + \sigma'_\eta) + [a_{33}(+\alpha)](\sigma^*_{\xi\eta} + \sigma'_{\xi\eta})$$

Ply 2

$$\epsilon_\xi = [a'_{11}(-\alpha)](\sigma^{**}_\xi - \sigma'_\xi) + [a'_{12}(-\alpha)](\sigma^{**}_\eta - \sigma'_\eta) + [a'_{13}(-\alpha)](\sigma^{**}_{\xi\eta} - \sigma'_{\xi\eta})$$

$$\epsilon_\eta = [a'_{21}(-\alpha)](\sigma^{**}_\xi - \sigma'_\xi) + [a'_{22}(-\alpha)](\sigma^{**}_\eta - \sigma'_\eta) + [a'_{23}(-\alpha)](\sigma^{**}_{\xi\eta} - \sigma'_{\xi\eta})$$

$$\epsilon_{\xi\eta} = [a'_{31}(-\alpha)](\sigma^{**}_\xi - \sigma'_\xi) + [a'_{32}(-\alpha)](\sigma^{**}_\eta - \sigma'_\eta) + [a'_{33}(-\alpha)](\sigma^{**}_{\xi\eta} - \sigma'_{\xi\eta})$$

Equations linking both plies (22)

$$2\sigma_\xi = \sigma^*_\xi + \sigma^{**}_\xi$$

$$2\sigma_\eta = \sigma^*_\eta + \sigma^{**}_\eta$$

$$2\sigma_{\xi\eta} = \sigma^*_{\xi\eta} + \sigma^{**}_{\xi\eta}$$

In Eqs. (22) the unknowns may be considered to be ϵ_ξ, ϵ_η, $\epsilon_{\xi\eta}$, σ^*_ξ, σ^*_η, $\sigma^*_{\xi\eta}$, σ^{**}_ξ, σ^{**}_η, $\sigma^{**}_{\xi\eta}$, σ'_ξ, σ'_η, and $\sigma'_{\xi\eta}$, assuming that the averaged external stresses σ_ξ, σ_η, and $\sigma_{\xi\eta}$ are given. All the a_{ij} and a'_{ij} are known. Thus, Eqs. (22) are indeterminate as they stand since twelve unknowns are present in nine equations. It is clear that three more equations are needed to resolve the indeterminacy. The exact form of these additional equations is not clear at this time.

If three additional equations were available to complete the set (22), one would be able to carry out the solution and nine elastic constants would result, since these would be expressible as ratios of strain ϵ_ξ, ϵ_η, and $\epsilon_{\xi\eta}$ to stress σ_ξ, σ_η, and $\sigma_{\xi\eta}$. Only six of these would be independent due to the requirement that the strain-energy function be single-valued. However, the important point here is that in this case the body is clearly not orthropic, but is just a general plane anisotropic structure. For that reason, dealing with this kind of condition in subsequent calculations will be considerably more complicated than with orthotropic laminates such as are represented by the cases discussed earlier.

Lacking the means for an exact solution of Eqs. (22), one must resort to aproximate techniques. The scheme used here is based on a physical relationship developed in Eqs. (20), in which it was shown that the presence of external normal stresses results in only the shearing component of interply stress, while the presence of external shearing stress generates only normal components of the interply stress. While Eq. (20) was developed for materials in which both plies were in either tension or compression, it will be assumed that the physical relationship just quoted still holds true for this case. This assumption means that it is possible now to couple together certain kinds of external and interply stresses, and reduce the number of unknowns in each set of equations so that each set becomes statically determinate.

As a first example of this, consider the situation resulting from the application of external shear stresses to a two-ply structure of the type in question here. Since in the previous development of Eq. (20) no interply shearing stress components would exist, it is presumed that none exist here and the equations governing this case are, when written in the same form as Eqs. (22), expressed as Eqs. (23) below:

Ply 1: $(+\alpha)$

$$\epsilon_\xi = [a_{11}(+\alpha)]\sigma'_\xi + [a_{12}(+\alpha)]\sigma'_\eta + [a_{13}(+\alpha)]\sigma^*_{\xi\eta}$$

$$\epsilon_\eta = [a_{21}(+\alpha)]\sigma'_\xi + [a_{22}(+\alpha)]\sigma'_\eta + [a_{23}(+\alpha)]\sigma^*_{\xi\eta}$$

$$\epsilon_{\xi\eta} = [a_{31}(+\alpha)]\sigma'_\xi + [a_{32}(+\alpha)]\sigma'_\eta + [a_{33}(+\alpha)]\sigma^*_{\xi\eta}$$

Ply 2: $(-\alpha)$

$$\epsilon_\xi = [a'_{11}(-\alpha)](-\sigma'_\xi) + [a'_{12}(-\alpha)](-\sigma'_\eta) + [a'_{13}(-\alpha)](\sigma^{**}_{\xi\eta})$$

$$\epsilon_\eta = [a'_{21}(-\alpha)](-\sigma'_\xi) + [a'_{22}(-\alpha)](-\sigma'_\eta) + [a'_{23}(-\alpha)](\sigma^{**}_{\xi\eta})$$

$$\epsilon_{\xi\eta} = [a'_{31}(-\alpha)](-\sigma'_\xi) + [a'_{32}(-\alpha)](-\sigma'_\eta) + [a'_{33}(-\alpha)](\sigma^{**}_{\xi\eta})$$

Plies 1 and 2: (23)

$$2\sigma_{\xi\eta} = \sigma_{\xi\eta} + \sigma^*_{\xi\eta}$$

In Eqs. (23) one may treat the unknowns as being ϵ_ξ, ϵ_η, $\epsilon_{\xi\eta}$, σ'_ξ, σ'_η, $\sigma^*_{\xi\eta}$, and $\sigma^{**}_{\xi\eta}$. This means that one may specify $\sigma_{\xi\eta}$ and expect to be able to calculate $\epsilon_{\xi\eta}$. This allows the apparent shear modulus to be calculated at once, this modulus being the ratio of applied external shearing stress to the resulting shearing strain in the absence of other external stresses. The algebraic manipulations necessary to get this are fairly lengthy and will only be reviewed. One first solves the last of Eqs. (23) for $\sigma^*_{\xi\eta}$ and substitutes this into the first six of these equations. The first and fourth of Eqs. (23) are then equated, the second and fifth are equated and the third and sixth as well. This results in a set of Eq. (24):

$$
\begin{aligned}
0 &= (a_{11} + a'_{11})\sigma'_{\xi} + (a_{12} + a'_{12})\sigma'_{\eta} - (a_{13} + a'_{13})\sigma^{**}_{\xi\eta} + 2(a_{13})\sigma_{\xi\eta} \\
0 &= (a_{21} + a'_{21})\sigma'_{\xi} + (a_{22} + a'_{22})\sigma'_{\eta} - (a_{23} + a'_{23})\sigma^{**}_{\xi\eta} + 2(a_{23})\sigma_{\xi\eta} \qquad (24) \\
0 &= (a_{31} + a'_{31})\sigma'_{\xi} + (a_{32} + a'_{32})\sigma'_{\eta} - (a_{33} + a'_{33})\sigma^{**}_{\xi\eta} + 2(a_{33})\sigma_{\xi\eta}
\end{aligned}
$$

Using the value of $\sigma^{*}_{\xi\eta}$ determined from the last of Eqs. (23), one may substitute this into the third and sixth of Eqs. (23) and may add these two. This gives

$$
2\epsilon_{\xi\eta} = (a_{31} - a'_{31})\sigma'_{\xi} + (a_{32} - a'_{32})\sigma'_{\eta} - (a_{33} - a'_{33})\sigma^{**}_{\xi\eta} + 2(a_{33})\sigma_{\xi\eta}. \qquad (n)
$$

The first of Eqs. (24) may be solved for σ'_{ξ} to give

$$
\sigma'_{\xi} = -\frac{(a_{12} + a'_{12})}{(a_{11} + a'_{11})}\sigma'_{\eta} + \frac{(a_{13} + a'_{13})}{(a_{11} + a'_{11})}\sigma^{**}_{\xi\eta} - \frac{2a_{13}}{(a_{11} + a'_{11})}\sigma_{\xi\eta}. \qquad (o)
$$

Substituting the results from Eqs. (o) into the last two of Eqs. (24) and into Eq. (n) gives the following three equations:

$$
\begin{aligned}
0 = &\left[(a_{22} + a'_{22}) - \frac{(a_{12} + a'_{12})(a_{21} + a'_{21})}{(a_{11} + a'_{11})}\right]\sigma'_{\eta} \\
&+ \left[-(a_{23} + a'_{23}) + \frac{(a_{13} + a'_{13})(a_{21} + a'_{21})}{(a_{11} + a'_{11})}\right]\sigma^{**}_{\xi\eta} \\
&+ \left[2a_{23} - \frac{2a_{13}(a_{21} + a'_{21})}{(a_{11} + a'_{11})}\right]\sigma_{\xi\eta} \qquad (25) \\
0 = &\left[(a_{32} + a'_{32}) - \frac{(a_{12} + a'_{12})(a_{31} + a'_{31})}{(a_{11} + a'_{11})}\right]\sigma'_{\eta} \\
&+ \left[-(a_{33} + a'_{33}) + \frac{(a_{13} + a'_{13})(a_{31} + a'_{31})}{(a_{11} + a'_{11})}\right]\sigma^{**}_{\xi\eta} \\
&+ \left[2a_{33} - \frac{2a_{13}(a_{31} + a'_{31})}{(a_{11} + a'_{11})}\right]\sigma_{\xi\eta}
\end{aligned}
$$

$$2\epsilon_{\xi\eta} = \left[(a_{32} - a'_{32}) - \frac{(a_{12} + a'_{12})(a_{31} - a'_{31})}{(a_{11} + a'_{11})}\right] \sigma'_{\eta}$$

$$+ \left[- (a_{33} - a'_{33}) + \frac{(a_{13} + a'_{13})(a_{31} - a'_{31})}{(a_{11} + a'_{11})}\right] \sigma^{**}_{\xi\eta} \qquad (25)$$

$$+ \left[2a_{33} - \frac{2a_{13}(a_{31} - a'_{31})}{(a_{11} + a'_{11})}\right] \sigma_{\xi\eta}$$

To simplify future algebraic manipulations, Eqs. (24) may be rewritten as Eqs. (25) by means of a number of substitutions.

$$0 = C\sigma'_{\eta} + D\sigma^{**}_{\xi\eta} + K\sigma_{\xi\eta}$$

$$0 = L\sigma'_{\eta} + M\sigma^{**}_{\xi\eta} + N\sigma_{\xi\eta} \qquad (26)$$

$$2\epsilon_{\xi\eta} = P\sigma'_{\eta} + Q\sigma^{**}_{\xi\eta} + R\sigma_{\xi\eta}$$

where the following constants have been defined

$$C = \left[(a_{22} + a'_{22}) - \frac{(a_{12} + a'_{12})(a_{21} + a'_{21})}{(a_{11} + a'_{11})}\right]$$

$$D = \left[- (a_{23} + a'_{23}) + \frac{(a_{13} + a'_{13})(a_{21} + a'_{21})}{(a_{11} + a'_{11})}\right]$$

$$K = \left[2a_{23} - \frac{2a_{13}(a_{21} + a'_{21})}{(a_{11} + a'_{11})}\right]$$

$$L = \left[(a_{32} + a'_{32}) - \frac{(a_{12} + a'_{12})(a_{31} + a'_{31})}{(a_{11} + a'_{11})}\right] \qquad (p)$$

$$M = \left[- (a_{33} + a'_{33}) + \frac{(a_{13} + a'_{13})(a_{31} + a'_{31})}{(a_{11} + a'_{11})}\right]$$

$$N = \left[2a_{33} - \frac{2a_{13}(a_{31} + a'_{31})}{(a_{11} + a'_{11})}\right]$$

$$P = \left[(a_{32} - a'_{32}) - \frac{(a_{12} + a'_{12})(a_{31} - a'_{31})}{(a_{11} + a'_{11})}\right]$$

$$Q = \left[-(a_{33} - a'_{33}) + \frac{(a_{13} + a'_{13})(a_{31} - a'_{31})}{(a_{11} + a'_{11})} \right]$$

$$R = \left[2a'_{33} - \frac{2a_{13}(a_{31} - a'_{31})}{(a_{11} + a'_{11})} \right]$$

The first of Eqs. (26) may be solved for σ'_η giving

$$\sigma'_\eta = -\frac{D}{C}\sigma^{**}_{\xi\eta} - \frac{K}{C}\sigma_{\xi\eta} \tag{q}$$

Substituting Eq. (q) into the second and third of Eqs. (26), one obtains

$$0 = \left(-\frac{LD}{C} + M\right)\sigma^{**}_{\xi\eta} + \left(-\frac{KL}{C} + N\right)\sigma_{\xi\eta}$$

$$2\epsilon_{\xi\eta} = \left(-\frac{DP}{C} + Q\right)\sigma^{**}_{\xi\eta} + \left(-\frac{KP}{C} + R\right)\sigma_{\xi\eta} \tag{r}$$

If the first of Eqs. (r) is solved for $\sigma^{**}_{\xi\eta}$, one obtains

$$\sigma^{**}_{\xi\eta} = \left(\frac{KL - CN}{CM - DL}\right)\sigma_{\xi\eta} \tag{s}$$

If Eq. (s) is substituted into the second of Eqs. (r) then one obtains

$$\frac{2\epsilon_{\xi\eta}}{\sigma_{\xi\eta}} = \left(\frac{KL - CN}{CM - DL}\right)\left(\frac{CQ - DP}{C}\right) + \left(\frac{CR - KP}{C}\right) \tag{t}$$

Hence the shear modulus $G_{\xi\eta}$ is given by the expression

$$G_{\xi\eta} = 2\left[\frac{C(CM - DL)}{(KL - CN)(CQ - DP) + (CR - KP)(CM - DL)}\right] \tag{27}$$

It is possible to plot out the calculated values of $G_{\xi\eta}$ as a function of the elastic characteristics of the materials which make up the individual lamina, and of the cord angle of the plies. These calculations are performed in succeeding sections of the report where test data are compared with theoretical predictions.

It is also possible to perform a similar analysis for the determination of the terms E_ξ, E_η, and $F_{\xi\eta}$. In general the application of external normal stresses can be shown to influence the cords in such a way that they all (that is, both plies) tend to go into tension or into compression. Therefore in constructing this analysis one must realize that we are dealing with what is probably a fairly rare case from a technical point of view, since a rather specific set of external stress conditions is required for its existence. Specfically, it would require that rather severe shear stresses act on the specimen simultaneously with normal stresses. Under those conditions, it would be possible to have extension or contraction of the element with one ply having cord tension and the other ply having cord compression. This case will now be examined in detail. Before writing the equations governing it, it should again be noted that when conditions become such that two plies do not carry identical loads, then the laminated structure no longer is orthotropic. In the case of one ply in tension and the other in compression, unequal loads are carried by the two plys since compatibility of deformation is required and one ply is much stiffer than another. Therefore, the analysis which follows is again based on an approximating assumption which reduces Eqs. (22) from a statically indeterminate to a determinate set. This assumption is again gotten from the physical result obtained from the determinate case of Eqs. (18) and (19), in which the application of external normal stresses results in the generation of only shearing components of the interply stress. It will be assumed that the same thing happens here so that one may write equations concerning the two plies in which external normal stresses and interply shearing stress components are the only ones acting.

The equations governing the case in which normal stresses are applied to a structure composed of two laminates, one whose cords are in tension and one whose cords are in compression, can then be written by referring to Fig. 6.

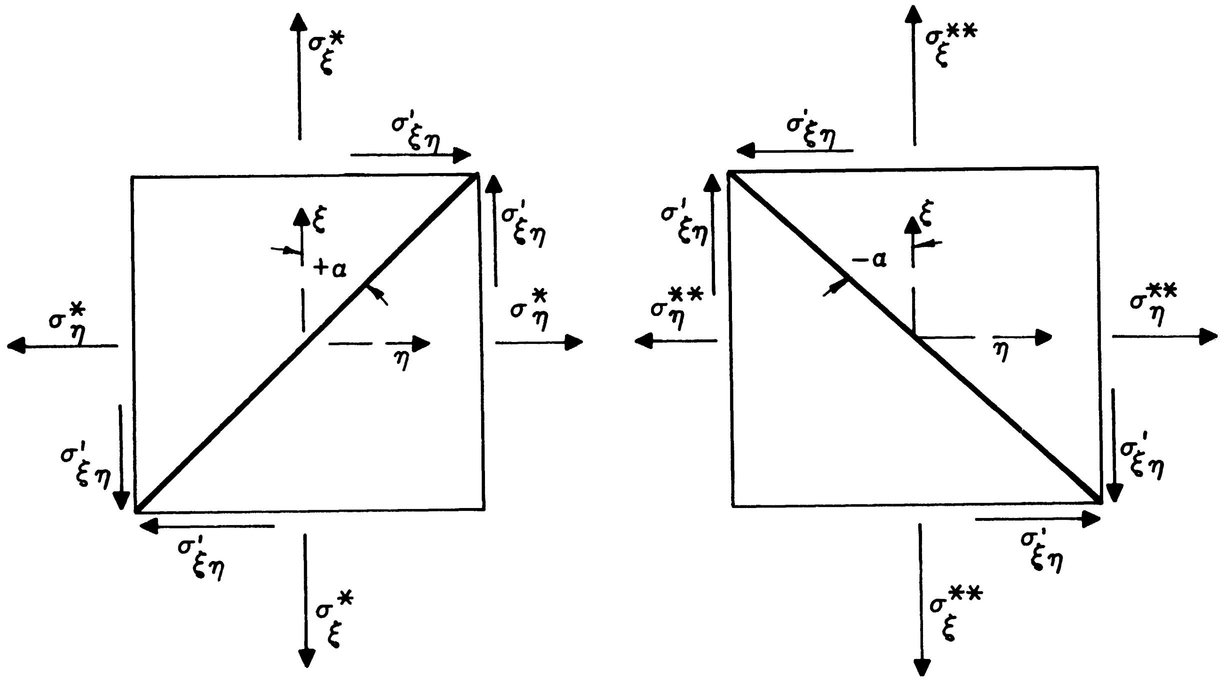

Fig. 6. Two-ply structure with only normal external stresses acting, along with the shearing component of the interply stress.

It may be seen that the only difference in the sign of the loads acting on the two plies is that equal but opposite interply shear stresses act on the tension and compression parts. The normal stresses associated with the tension ply are denoted by a super star, while the normal stresses associated with the compression ply are denoted by a super double star. The equations similar to Eqs. (9), but now written for the tension and compression plies, are as follows:

Ply 1: $(+\alpha)$ Tension

$$\epsilon_\xi = a_{11}\sigma^*_\eta + a_{12}\sigma^*_\eta + a_{13}\sigma'_{\xi\eta}$$

$$\epsilon_\eta = a_{21}\sigma^*_\xi + a_{22}\sigma^*_\eta + a_{23}\sigma'_{\xi\eta}$$

$$\epsilon_{\xi\eta} = a_{31}\sigma^*_\xi + a_{32}\sigma^*_\eta + a_{33}\sigma'_{\xi\eta}$$

Ply 2: $(-\alpha)$ Compression

$$\epsilon_\xi = a'_{11}\sigma^{**}_\xi + a'_{12}\sigma^{**}_\eta + a'_{13}\sigma'_{\xi\eta}$$

$$\epsilon_\eta = a'_{21}\sigma^{**}_\xi + a'_{22}\sigma^{**}_\eta + a'_{23}\sigma'_{\xi\eta}$$

$$\epsilon_{\xi\eta} = a'_{31}\sigma^{**}_\xi + a'_{32}\sigma^{**}_\eta - a'_{33}\sigma'_{\xi\eta} \qquad (28)$$

Both plies 1 and 2

$$2\sigma_\xi = \sigma^*_\xi + \sigma^{**}_\xi$$

$$2\sigma_\eta = \sigma^*_\eta + \sigma^{**}_\eta$$

The solution of these equations is nothing more than a long algebraic manipulation whose inclusion in this report would not serve any particular purpose. The general method employed here is to eliminate terms from the equations one at a time until at last one obtains a set of relationships between normal strains and stresses similar to those of Eq. (5). Having this, it is easy to identify the various elastic moduli concerned in this problem. Out of these equations one obtains the following expressions:

$$\epsilon_\xi = \left(\frac{\Pi_1}{\delta}\right)\sigma_\xi + \left(\frac{\Pi_2}{\delta}\right)\sigma_\eta$$

$$\epsilon_\eta = \left(\frac{\Pi_3}{\psi}\right)\sigma_\xi \quad \left(\frac{\Pi_4}{\psi}\right)\sigma_\eta \qquad (29)$$

where

$$\pi_1 = 2\left[\omega(\bar{D}\bar{E} - \bar{F}\bar{C}) + a_{13}'a_{31}(\bar{B}\bar{E} - \bar{A}\bar{F}) + a_{13}'a_{21}(\bar{A}\bar{D} - \bar{B}\bar{C})\right]$$

$$\pi_2 = 2\left[\beta(\bar{D}\bar{E} - \bar{F}\bar{C}) + a_{13}'a_{32}(\bar{B}\bar{E} - \bar{A}\bar{F}) + a_{13}'a_{22}(\bar{A}\bar{D} - \bar{B}\bar{C})\right]$$

$$\pi_3 = 2\left[\theta(\bar{H}\bar{K} - \bar{G}\bar{L}) + a_{23}'a_{31}(\bar{K}\bar{J} - \bar{I}\bar{L}) + a_{23}'a_{11}(\bar{I}\bar{H} - \bar{G}\bar{J})\right]$$

$$\pi_4 = 2\left[\gamma(\bar{H}\bar{K} - \bar{G}\bar{L}) + a_{23}'a_{32}(\bar{K}\bar{J} - \bar{I}\bar{L}) + a_{23}'a_{12}(\bar{I}\bar{H} - \bar{G}\bar{J})\right]$$

and

$$\delta = (a_{33} + a_{33}')\left[\bar{E}(\bar{D} - \bar{B}) + \bar{F}(\bar{A} - \bar{C})\right] + (a_{23} - a_{23}')(\bar{B}\bar{C} - \bar{A}\bar{D})$$

$$\psi = (a_{33} + a_{33}')\left[\bar{K}(\bar{H} - \bar{J}) + \bar{L}(\bar{J} - \bar{G})\right] + (a_{13} - a_{13}')(\bar{G}\bar{J} - \bar{I}\bar{H})$$

$$\bar{A} = a_{13}(a_{31} + a_{31}') - a_{11}(a_{33} + a_{33}')$$

$$\bar{B} = a_{13}(a_{32} + a_{32}') - a_{12}(a_{33} + a_{33}')$$

$$\bar{C} = a_{13}'(a_{31} + a_{31}') + a_{11}'(a_{33} + a_{33}')$$

$$\bar{D} = a_{13}'(a_{32} + a_{32}') + a_{12}'(a_{33} + a_{33}')$$

$$\bar{E} = a_{11}'(a_{23} - a_{23}') + a_{13}'(a_{21} + a_{21}')$$

$$\bar{F} = a_{12}'(a_{23} - a_{23}') + a_{13}'(a_{22} + a_{22}')$$

$$\bar{G} = a_{21}'(a_{33} + a_{33}') + a_{23}'(a_{31} + a_{31}')$$

$$\bar{H} = a_{22}'(a_{33} + a_{33}') + a_{23}'(a_{32} + a_{32}')$$

$$\bar{I} = a_{23}(a_{31} + a_{31}') - a_{21}(a_{33} + a_{33}')$$

$$\bar{J} = a_{23}(a_{32} + a_{32}') - a_{22}(a_{33} + a_{33}')$$

$$\bar{K} = a_{21}'(a_{13} - a_{13}') + a_{23}'(a_{11} + a_{11}')$$

$$\bar{L} = a_{22}'(a_{13} - a_{13}') + a_{23}'(a_{12} + a_{12}')$$

$$\omega = a_{11}(a_{33} + a_{33}') - a_{13}^2$$

$$\beta = a_{12}(a_{33} + a_{33}') - a_{13}a_{32}$$

$$\gamma = a_{22}(a_{33} + a_{33}') - a_{23}^2$$

$$\theta = a_{21}(a_{33} + a_{33}') - a_{23}a_{31}$$

where

$$\Pi_1 = 2\left[\omega(\bar{D}\bar{E} - \bar{F}\bar{C}) + a'_{13}a_{31}(\bar{B}\bar{E} - \bar{A}\bar{F}) + a'_{13}a_{21}(\bar{A}\bar{D} - \bar{B}\bar{C})\right]$$

$$\Pi_2 = 2\left[\beta(\bar{D}\bar{E} - \bar{F}\bar{C}) + a'_{13}a_{32}(\bar{B}\bar{E} - \bar{A}\bar{F}) + a'_{13}a_{22}(\bar{A}\bar{D} - \bar{B}\bar{C})\right]$$

$$\Pi_3 = 2\left[\theta(\bar{H}\bar{K} - \bar{G}\bar{L}) + a'_{23}a_{31}(\bar{K}\bar{J} - \bar{I}\bar{L}) + a'_{23}a_{11}(\bar{I}\bar{H} - \bar{G}\bar{J})\right]$$

$$\Pi_4 = 2\left[\gamma(\bar{H}\bar{K} - \bar{G}\bar{L}) + a'_{23}a_{32}(\bar{K}\bar{J} - \bar{I}\bar{L}) + a'_{23}a_{12}(\bar{I}\bar{H} - \bar{G}\bar{J})\right]$$

and

$$\delta = (a_{33} + a'_{33})\left[\bar{E}(\bar{D} - \bar{B}) + \bar{F}(\bar{A} - \bar{C})\right] + (a_{23} - a'_{23})(\bar{B}\bar{C} - \bar{A}\bar{D})$$

$$\psi = (a_{33} + a'_{33})\left[\bar{K}(\bar{H} - \bar{J}) + \bar{L}(\bar{J} - \bar{G})\right] + (a_{13} - a'_{13})(\bar{G}\bar{J} - \bar{I}\bar{H})$$

$$\bar{A} = a_{13}(a_{31} + a'_{31}) - a_{11}(a_{33} + a'_{33})$$

$$\bar{B} = a_{13}(a_{32} + a'_{32}) - a_{12}(a_{33} + a'_{33})$$

$$\bar{C} = a'_{13}(a_{31} + a'_{31}) + a'_{11}(a_{33} + a'_{33})$$

$$\bar{D} = a'_{13}(a_{32} + a'_{32}) + a'_{12}(a_{33} + a'_{33})$$

$$\bar{E} = a'_{11}(a_{23} - a'_{23}) + a'_{13}(a_{21} + a'_{21})$$

$$\bar{F} = a'_{12}(a_{23} - a'_{23}) + a'_{13}(a_{22} + a'_{22})$$

$$\bar{G} = a'_{21}(a_{33} + a'_{33}) + a'_{23}(a_{31} + a'_{31})$$

$$\bar{H} = a'_{22}(a_{33} + a'_{33}) + a'_{23}(a_{32} + a'_{32})$$

$$\bar{I} = a_{23}(a_{31} + a'_{31}) - a_{21}(a_{33} + a'_{33})$$

$$\bar{J} = a_{23}(a_{32} + a'_{32}) - a_{22}(a_{33} + a'_{33})$$

$$\bar{K} = a'_{21}(a_{13} - a'_{13}) + a'_{23}(a_{11} + a'_{11})$$

$$\bar{L} = a'_{22}(a_{13} - a'_{13}) + a'_{23}(a_{12} + a'_{12})$$

$$\omega = a_{11}(a_{33} + a'_{33}) - a_{13}^2$$

$$\beta = a_{12}(a_{33} + a'_{33}) - a_{13}a_{32}$$

$$\gamma = a_{22}(a_{33} + a'_{33}) - a_{23}^2$$

$$\theta = a_{21}(a_{33} + a'_{33}) - a_{23}a_{31}$$

These expressions are so long and involved that it is not possible to infer the effect of design charges on physical properties directly. This will have to await numerical calculations at some later time.

V. EXPERIMENTAL MEASUREMENT OF ELASTIC CHARACTERISTICS

Theoretical predictions of elastic characteristics are most meaningful when substantiated by laboratory tests. For that reason, plans were made very early in this program to obtain test specimens which would be used to measure most of the elastic characteristics of plane cord-rubber laminates. Some thought was given to picking a specimen geometry which would allow the greatest number and type of tests to be run easily. A cylindrical tube was chosen for this purpose since it may be subjected to the following types of stress state:

a. Pure tension in the axial direction.

b. Pure tension in the circumferential direction by means of internal pressure.

c. Torsion.

d. Circumferential compression by means of an internal vacuum.

e. Longitudinal compression.

f. Various combinations of tension and compression by means of the use of external tensile or compressive loads and internal pressure or vacuum.

These tubes were made up of four plies composed of standard cord and coat stocks, with alternate plies oppositely directed. This design was chosen before the extreme importance of compression properties became apparent, and the emphasis in choosing the geometry of the design was placed on its suitability for tension, internal pressure and torsion work. For that reason, the specimens described here are relatively thin-walled. However, future specimens used in investigations of this type should have a greater wall thickness to diameter ratio to allow more efficient compression testing to be done. The compressive load-carrying characteristics are, of course, limited by the buckling characteristics of these tubes. A greater wall-thickness-to-diameter ratio would increase this buckling load considerably.

This section of the report describes testing of the cylinders under conditions of uniaxial tension and compression and under conditions of pure torsion. No combined stress work is reported here. It is believed that the tests just mentioned can be used to explain most of the physical phenomena inherent in this problem, although it is planned to perform combined stress tests using these same specimens at a later time.

The dimensions of the test specimens are shown in Fig. 7. The cord fabric used in their construction was rayon and two sets of test specimens were ordered, each set having one specimen with α of 0, 15, 30, 45, and 60°. The same rubber was used in manufacturing all these specimens.

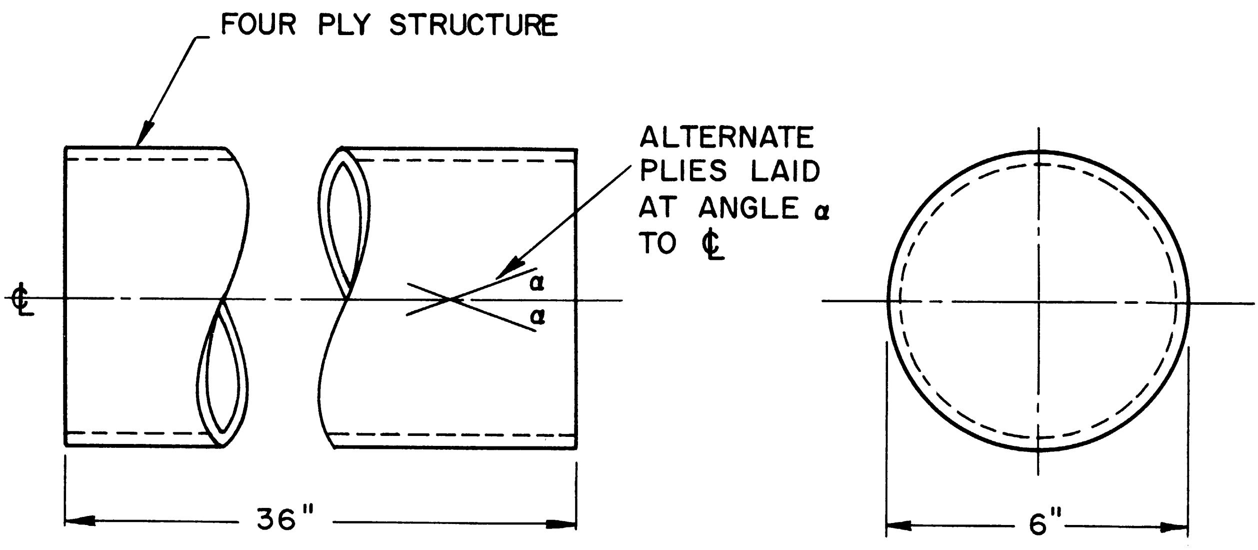

Fig. 7. Tubular test specimen showing dimensions.

This seems an appropriate place to discuss the types of information to be drawn from these experiments. Previous sections of this report have shown that elastic constants of plain cord-rubber laminates may be calculated based upon all cords in tension, all cords in compression, or one ply in tension and one ply in compression, for the case of a two-ply structure. In any one of these three cases a relationship exists between normal stress and resulting normal strain. This relationship, in the first two cases, is given the symbol E_ξ or E'_ξ and in the latter case is given the symbol a_{11}. In either event, the quantity desired from the experiment is this relationship between applied normal stress and resulting extensional strain. It may be seen from Fig. 7 that perhaps the simplest possibility here would be application of direct tensile forces to the cylindrical tube with measurement of the resulting deformation in the axial direction. With the specimens previously described, the extensional modulus E can be measured through a range of cord angles running from 0 to 60°. This range may be increased to include 75° and 90° by inflating the 15° and 0° specimens with nitrogen and then balancing the longitudinal tensile force induced by the pressure acting on the ends with an applied longitudinal compressive load. Under these conditions the circumferential strain caused by the internal pressure in the 15° specimen is equivalent to the axial strain which one would observe in a 75° specimen subjected to an axial stress equal to the hoop stress induced by the internal pressure. A similar situation exists for the 0° specimen and a 90° axial stress. Thus, a complete range of included half angles α, from 0° to 90°, could be covered by this series of experiments.

One additional complication existing here is the sign of the cord strain in each of these tests. If the cord strain is positive, the quantity to be measured will be E_ξ. If the cord strain is negative, the quantity measured is E'_ξ, and it will be necessary, for correct interpretation of these tests, to predetermine the sign of the cord strain and to use that information in calculating the appropriate value of the modulus to compare with experimental results. In a later report, the subject of the sign of the cord strain is discussed in some detail and a technique advanced for determining it. This technique was worked out for application here and was used in the calculations which appear subsequently.

Both rubber and rayon cord exhibit some nonlinearities in their stress-strain curves. For that reason, large strains applied to these specimens resulted in some curvature of the resulting stress-strain records. While this curvature is not severe enough to require a nonlinear representation of the stress-strain curve, it is such that one may obtain somewhat different values for the slope at different values of strain of the specimen. Thus to obtain systematic correlation of the data from these specimens, it appeared necessary to arrive at some common base from which to reduce the data. Various possibilities presented themselves for this. A common total load is not convenient because the load carrying capacity of the different specimens is vastly different. A common cord strain would be an excellent criterion to use as a point at which one should determine modulus, but it is not possible, under loading conditions employed here, to produce the same amount of strain in the cords of all the specimens. It was decided, therefore, that a common strain in the over-all composite body should be chosen as a criterion for deciding the points at which all data would be taken. This common strain was chosen to be

3% in all specimens except 0° and 15°, in which this strain was chosen to be 1%.

On each of the tests in which the relationship between normal stresses and extensional strains was measured, it was possible to determine simultaneously the diametral extension or contraction of the specimen. The ratio of this strain to the applied stresses has been given the symbol F in the preceding analytical work, and this factor was found by measuring the strain at right angles to the direction of the applied stresses. In most cases the strain measuring equipment necessary to do this was similar to that used to measure the strain in the direction of the applied load, and these devices will be described later in this section. The comments made earlier concerning the sign of the cord strain and the value of the strain at which readings were taken also apply to this factor F.

Experimental work done on the specimens utilized the end fixtures shown in Figs. 8 and 9. These fixtures are composed of two basic parts, shown most clearly in Fig 8. The tapered plug fits into the end of the tube and the four piece ring fits tightly over the tube. Figure 9 shows one end of the tube, partially mounted in the fixture. The maximum load obtainable with these fixtures in tension is approximately 7500 lb, at which time slip was observed to begin between the tapered plug and the outer rings.

The tests carried out for this report were performed on either a Riehle tensile testing machine or a Baldwin-Emery model FGT testing machine. These are both screw-type testing machines with controllable head velocities. Thus, the strain rate of the specimens could be maintained at a constant value throughout the duration of the test.

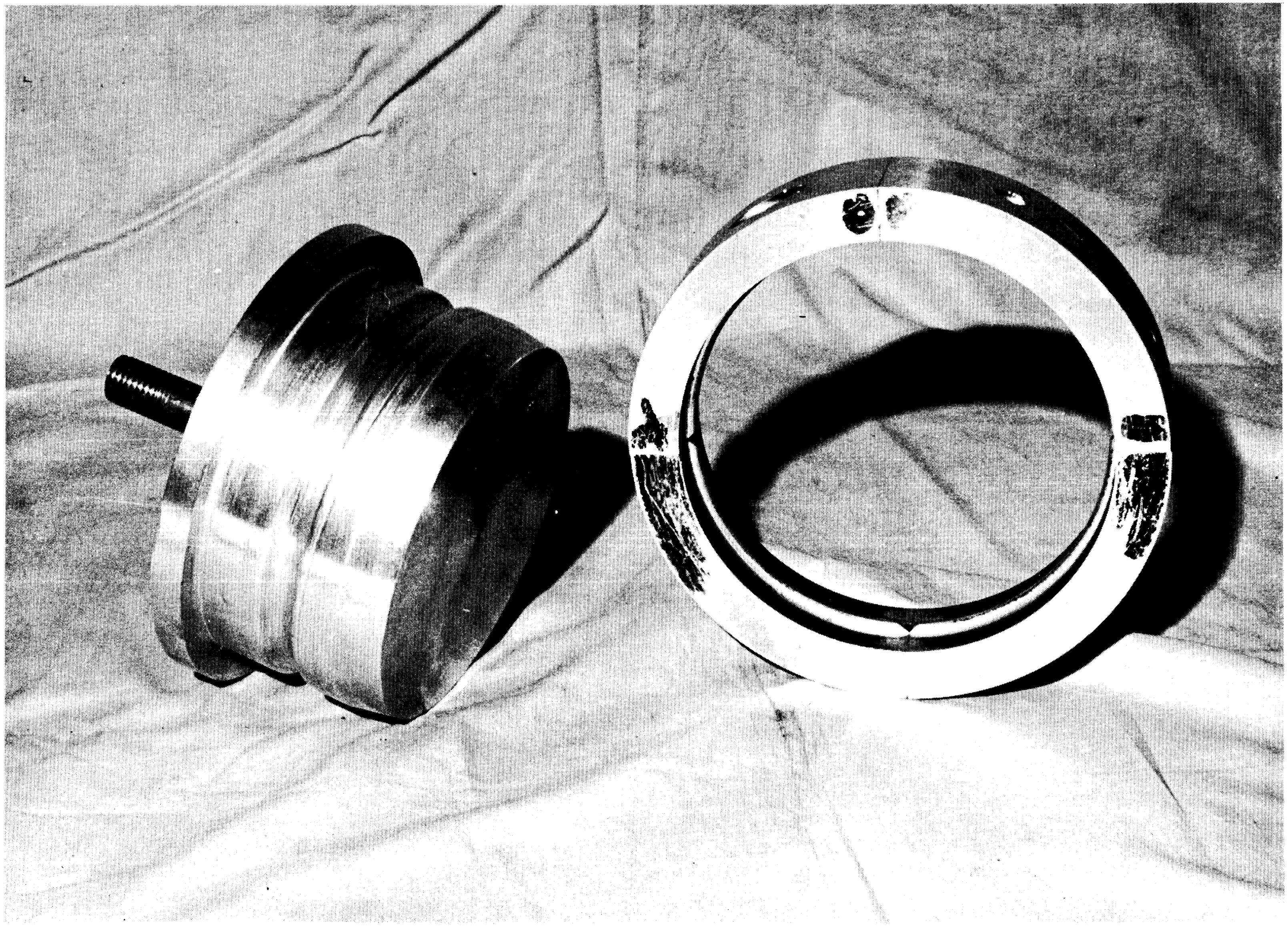

Fig. 8. End fittings used to attach tubular test specimens to testing machine.

Fig. 9. Tubular specimen with one end fitting attached.

Two types of strain measurements were utilized in these experiments. All specimens except the 0° and 15° models were elastic enough so that an ordinary machinist's scale could be used to measure the axial strain. This was accomplished by attaching a machinist's scale with a smallest division of 0.2 in. to the specimen by means of pins inserted through holes drilled near one end of the scale. These pins pierced the wall of the tubular specimen. Near the other end of the machinist's scale a small pointer was attached to the specimen, and, as the specimen was loaded, a change in length was observed. An additional scale was attached to the specimen on its opposite side, that is, 180° from the first scale, so that the two readings could be averaged and the effects of any slight bending in the tubes cancelled out. Figure 10 show the scale system described here.

The lateral or diametral strain was measured with two dial gauges placed on opposite sides near the middle of the specimen, so as to measure changes in the transverse position of the specimen. As the load was applied, the change in diamter was recorded by adding together the changes observed from the two dials. These dials are also shown in Fig. 10.

The rather crude techniques described above were quite adequate for measuring the large strains exhibited by the softer specimens. However, the 0° and 15° specimens were much more difficult to handle since even a 1% strain required fairly large loads, due to the high tension modulus of the tube. It was therefore necessary to refine the strain-measuring techniques in these cases. The axial measurements were made with a two-piece extensometer developed specifically for this purpose and illustrated in Fig. 11. A machinist's dial was attached to the long member which has a pin in its end opposite the dial.

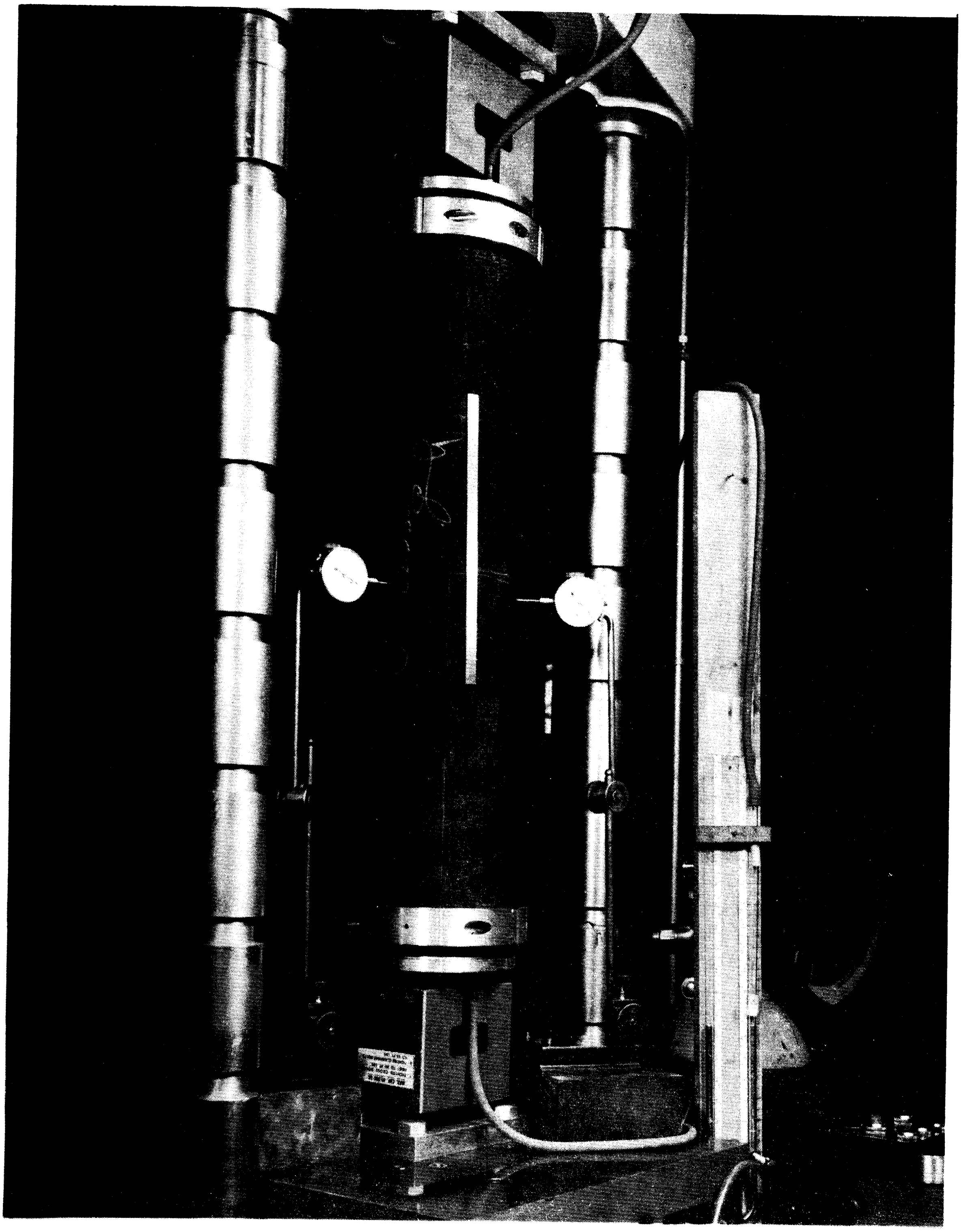

Fig. 10. Machinist's scale extensometer.

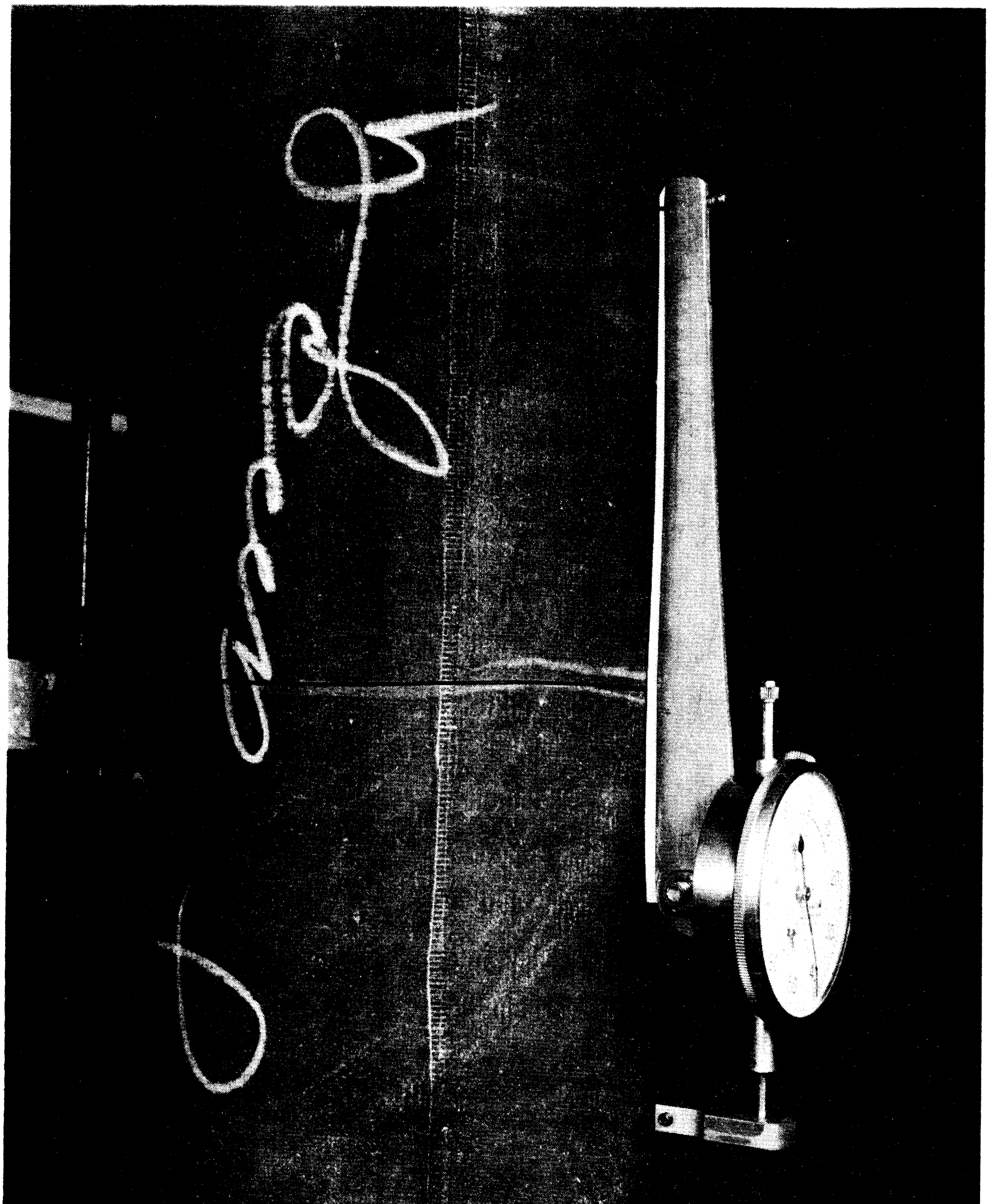

Fig. 11. A low-force extensometer developed for measurement of axial strain in tubes.

This pin is inserted through the wall of the specimen, and serves to support the weight of the specimen. The moving end of the extensometer is a small attachment which clamps to the stem of the dial gauge. This piece also has a pin which is inserted through the wall of the specimen. As the tube is loaded, the dial gauge records the change in length between the two pins. This same extensometer can, of course, be used to find both the tension and compression moduli.

The diametral strain for the 0° and 15° specimens was obtained using the device shown in Fig. 12. Two dial gauges were attached to the rigid cross bar which in turn was supported by a chemical ring stand. Each end of a lubricated plastic string was attached to the stem of one dial gauge, and the line was wrapped once completely around the specimen so that, as the specimen was loaded, the dial gauges recorded the change in the circumference of the tube. Several trails with this device indicated that with adequate lubrication and by using an inextensible plastic line, errors in this type of strain measurement could be kept negligible.

Due to the extreme restraining influences of the grips on the ends of the specimen, it was not possible to utilize over-all changes in length of the specimen as a measure of the strain. Several trials showed that these end effects were simply too severe and that accurate strain measurements could be obtained only by utilizing the central portion of the tubes. It is hoped that future test work of this general nature can be performed on specimens whose length to diameter ratio is somewhat greater than was the case here.

Some preliminarly experiments were performed to evaluate cycling effects

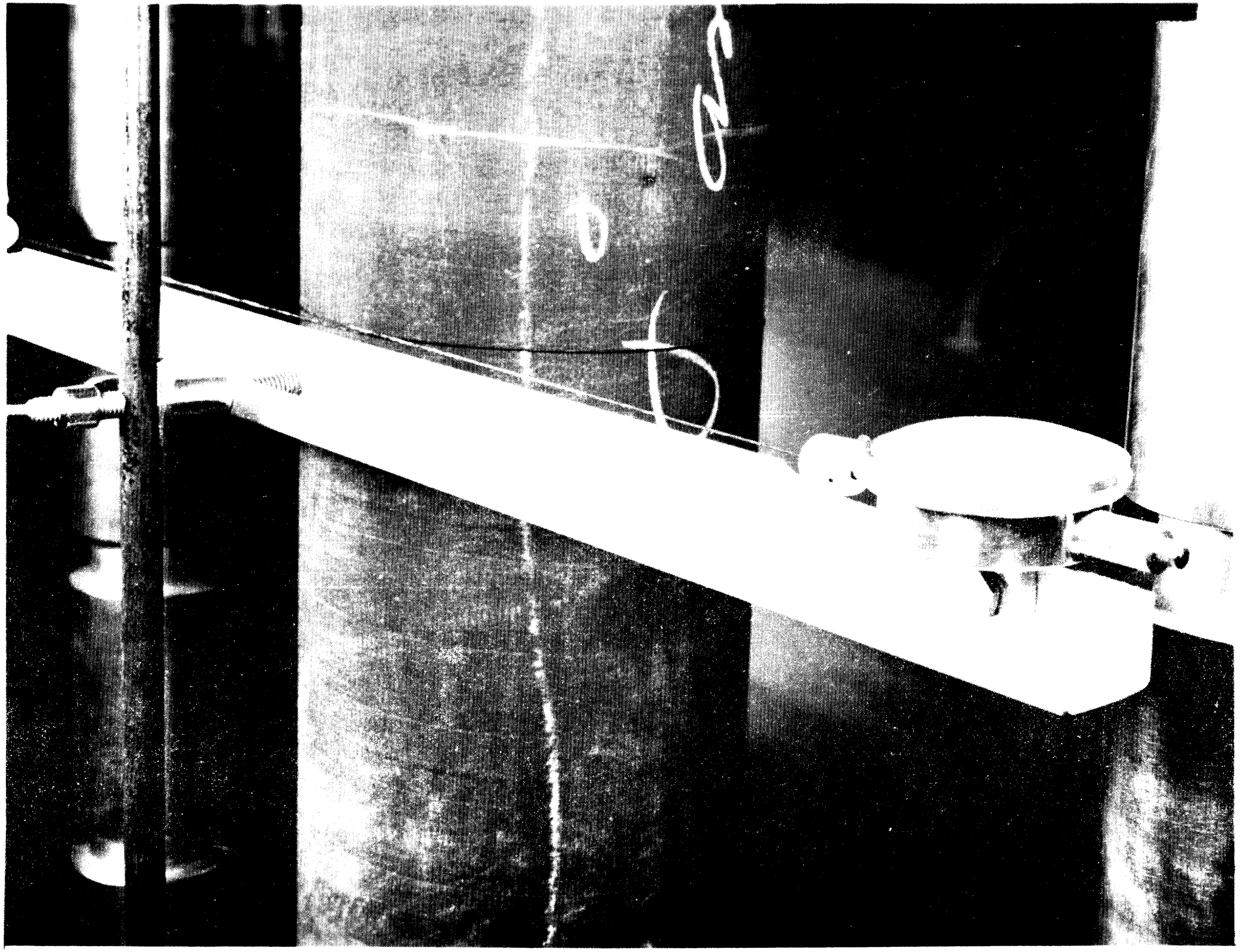

Fig. 12. An extensometer for measuring circumferential contraction in tubes.

in these specimens. It was found that cycling the load on a specimen had very little effect on the slope of the stress-strain curve. The curve shifted along the strain axis due to hysteresis and relaxation, but the slope, and hence modulus, remained essentially constant While investigating the cycling effects, it was also possible to investigate unloading characteristics of the specimens, and it was found that the stress-strain curves for unloading remained essentially parallel to the loading curve.

Tests were performed on a particular specimen by first attaching the specimen to the end fittings. These, in turn, were fixed to the weighing and moving heads of the testing machine. The strain-measuring instrumentation was attached to the specimen and the loading process started at constant strain rate. In those cases where internal pressure was to be used, a standard nitrogen bottle and regulator system were used for purposes of providing it. The pressure was monitored by an external mercury manometer so that axial loads and internal pressure could be controlled simultaneously by a single operator. In that way it was possible, wherever necessary, to provide simultaneous loadings of the specimens up to the proper stress state. The resulting strains were read on the loading portion of the curve as well as on the unloading portion. Plots of the resulting data allowed the moduli at 1% or 3% strain to be determined easily.

The torsion experiments, which form the second part of this experimental program are still incomplete. It was originally thought that a single set of these would be sufficient to demonstrate correspondence between predicted stiffnesses and those measured in the laboratory. However, the complexity of the shear modulus situation will necessitate an extensive series of experiments in the future. Most of the complexity of the calculations dealing with the shear modulus arises from the fact that the application of an external shear stress causes one ply of a two-ply structure to go into tension while the other ply is forced into compression. In the absence of normal stresses, it is thus true that the shearing modulus of a two-ply structure is governed by the indeterminate set of Eqs. (22). Since no exact solution is available for these, it is necessary to rely on the approximate solutions described in the preceding section of this report. These are somewhat more complicated algebracially than the solutions associated with both plies being in tension of in compression.

To investigate the validity of the approximate solutions described by Eqs. (23) through (28), a series of tests was performed using equipment immediately available for torsion work, and utilizing the end fixtures previously described. These tests lacked the proper equipment for control of the external stress state on the torsion specimens, and, therefore, they should be considered as approximate. This is because a two-ply structure, in which one ply is in a state of tension and the other ply in a state of compression, is not an orthotropic body. Therefore, a tubular specimen made of this kind of material will exhibit length changes when subjected to a pure twisting moment. Thus, a torsional test would not only require that the torsion specimen be free to move axially without incurring restraint, but should also be long enough so that if changes in diameter take place then these may do so freely in the center section of a long specimen. The torsion machine used for these tests has one movable head so that the condition of longitudinal freedom can be realized to some degree. However, this movable head is relatively massive as compared with the stiffness of the specimens and it is considered quite likely that loads induced by lengthening or shortening of the specimens in torsion played some part in the measured over all shear stiffness. Furthermore, there is some question as to whether these tubes possess quite as large a length to diameter ratio as would be desirable, in this case, for purposes of avoiding all traces of end effects near the central region of the specimen. An attempt was made to cancel out the effects of these two apparent imperfections in testing geometry and apparatus by recording data for both clockwise and counter-clockwise rotation of the tubular specimens in torsion. An average value was taken between these two sets. It might be noted that in all cases the shear modulus was observed to be greater in one direction than in the other. This technique should eliminate the effects of axial load but, of course, may not eliminate the effects of the proximity of end fittings on the changes in diameter.

A Riehle torsion testing machine was used to apply a twisting moment to the specimens, and to measure the value of this moment through a dead weight scale system. A troptometer was constructed of lightweight aluminum and was attached to the center portion of the tubes to measure relative angles of rotation. Figure 13 shows the testing machine and troptometer as it was used in these experiments.

During the actual conduct of the tests, loads were limited by tendencies of the specimen to buckle. In addition, the torque carrying capacity of tubes of various cord angle varied over a large range, and so provisions had to be made for choosing a common base from which to measure shear modulus. A value

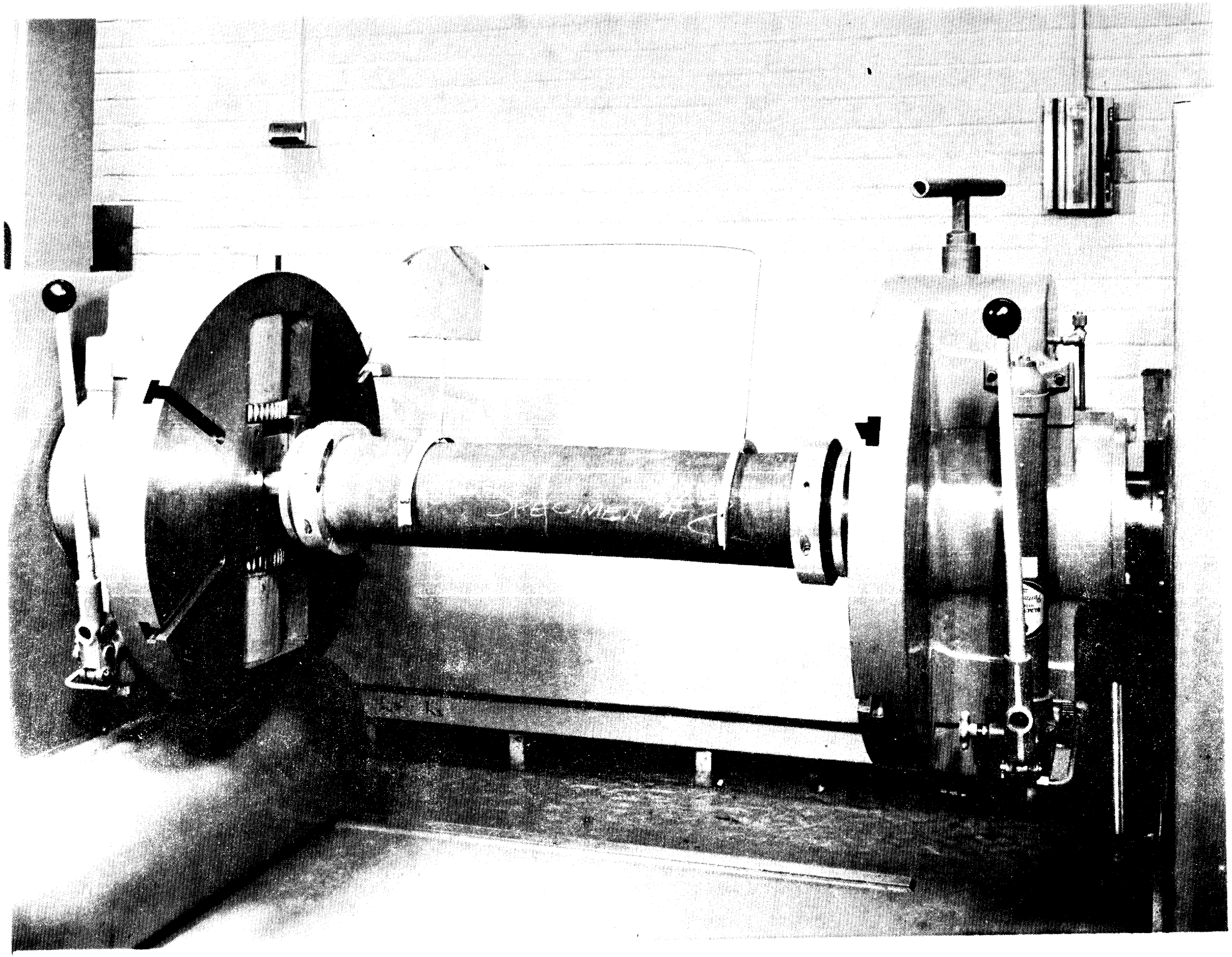

Fig. 13. Torsion test showing the troptometer for measurement of angular changes.

of shearing strain of 3% was selected. Strain rate was again maintained constant since the moving head of the machine is driven by a controllable speed motor. No tests were performed utilizing internal pressure.

It is anticipated that a future set of tests will be performed with different speciemsn and different apparatus to verify the expressions previously derived for shear modulus under conditions of all cords being in tension or compression as well as under conditions of one ply being in tension and one in compression. This will require the construction of grips and fittings to properly monitor the stress state acting on the specimens. It is hoped to report the results of that work in the future.

VI. COMPARISON OF EXPERIMENTAL RESULTS WITH THEORETICAL PREDICTIONS

The tension modulus E_ξ is, by definition, the ratio of the axial stress over the axial strain when only an axial load is applied to the body. The experimental value of E_ξ at different cord angles was obtained using specimens described in the preceding section and using that portion of the load strain curve where the axial strain was approximately 3%. The curve shown in Fig. 14 is a typical load-strain curve with the appropriate designation of that portion of the curve from whence E_ξ was obtained. Tests similar to this over the entire range of half angles from 0° to 90° enabled E_ξ to be determined completely for the tubes. The experimental data obtained from these tests is shown as circles on the graph of Fig. 15.

Measurements of E_x, E_y, F_{xy}, and G_{xy} can be made on the 0° tube of the series. These measurements were carried out in axactly the same way as those of the other tubes. Having these four quantities available, the analytical expressions developed in Section IV of this report allow the modulus of elasticity E_ξ to be calculated at any cord half-angle α. Similarly, $F_{\xi\eta}$ can also be calculated under these same conditions. Examination of the loading processes indicated that, for angles greater than approximately 55°, specimens loaded in tension actually possessed cords loaded in compression. For this reason, it was also necessary to measure the elastic characteristics E_x', E_y', G_{xy}' , and F_{xy}' by a proper set of compression tests. This information allowed the elastic constants E_ξ' and $F_{\xi\eta}'$ to be calculated for those angles where the tubular specimens exhibited cord compression. Thus, Figs. 15 and 16 can be drawn showing

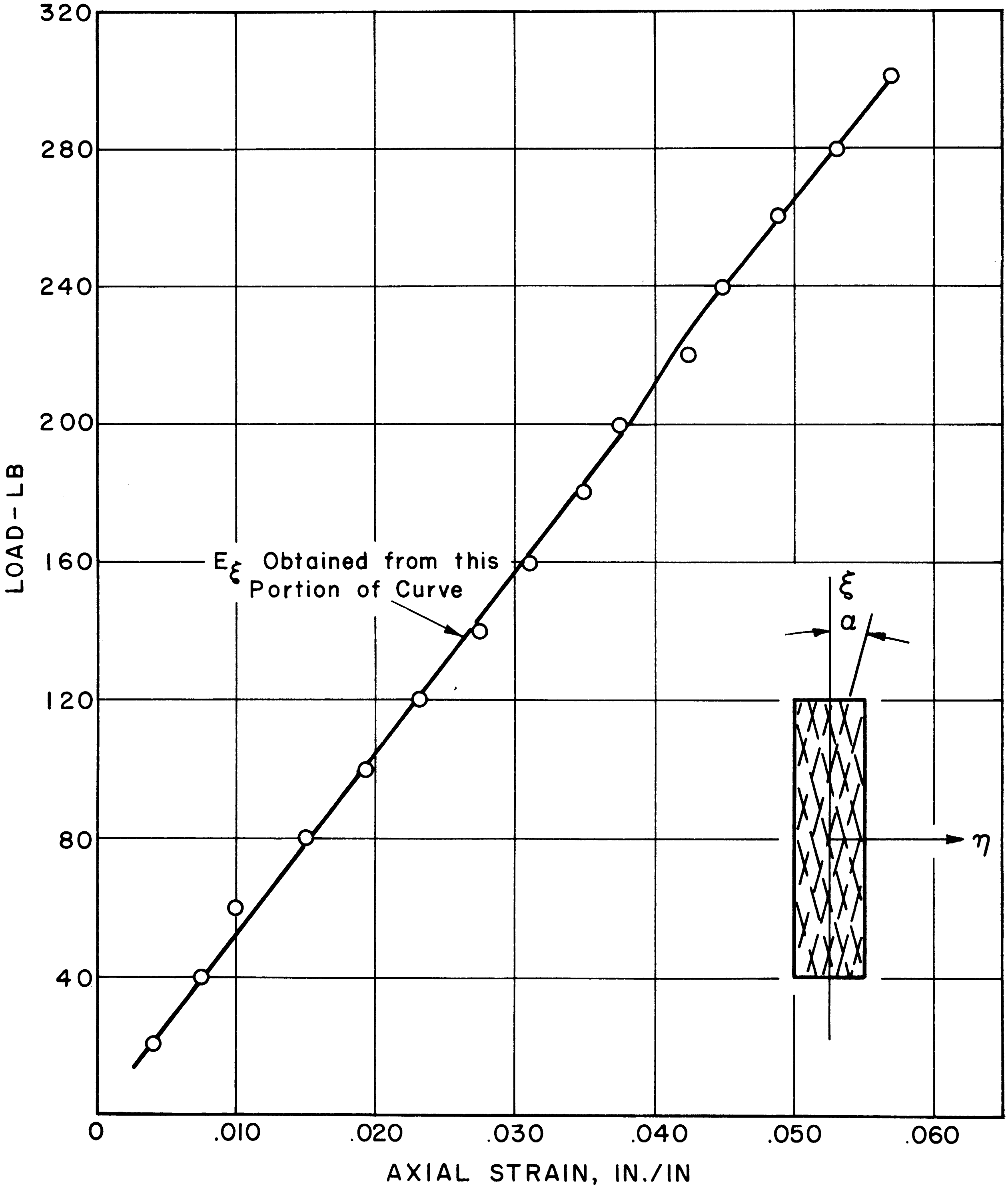

Fig. 14. Typical load-axial strain curve for cylindrical tubes subjected to axial tension.

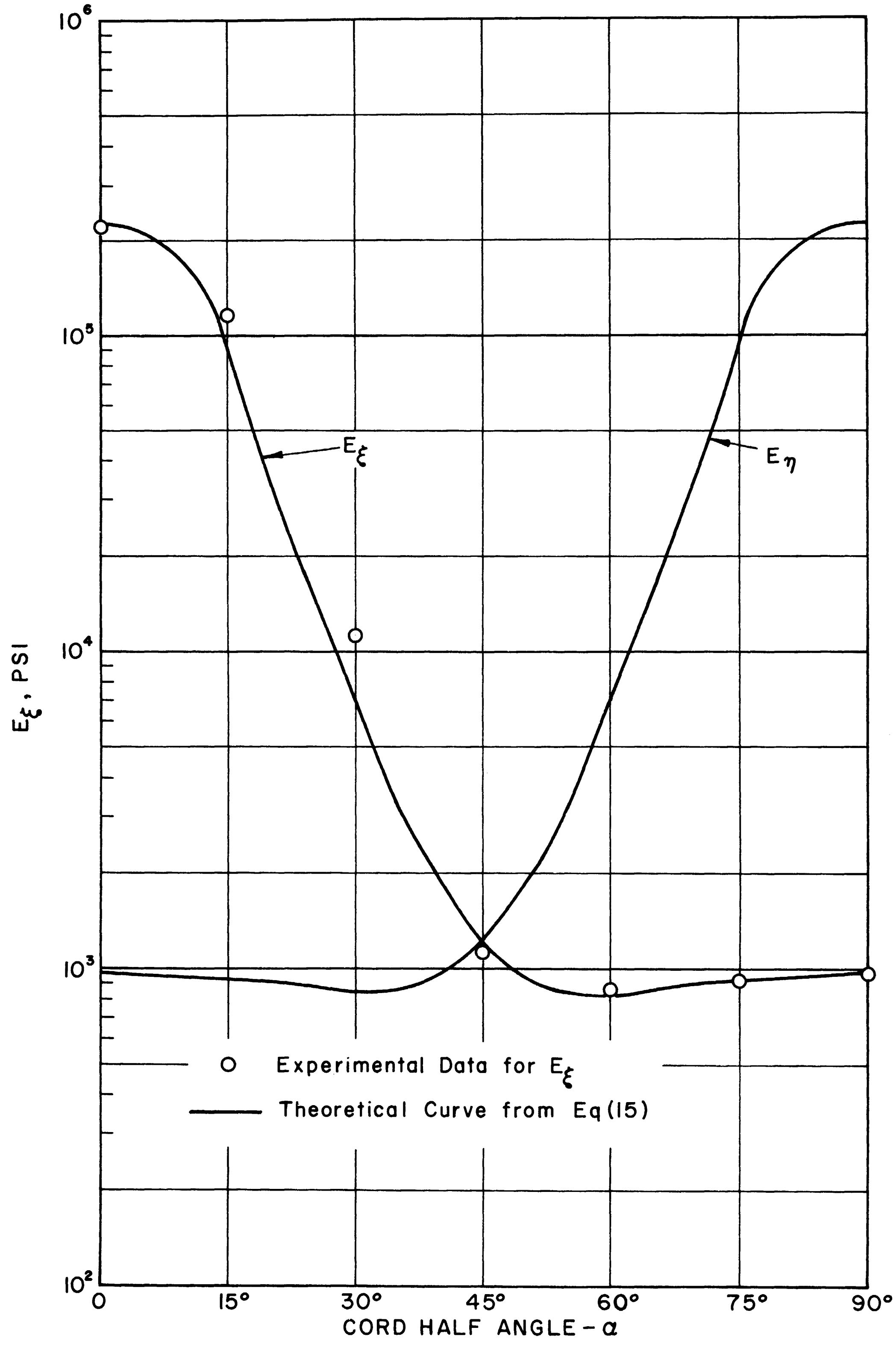

Fig. 15. Moduli E_ξ and E_η as predicted from Eq. (15) vs. cord half angle α, along with experimental values for $F_{\xi\eta}$.

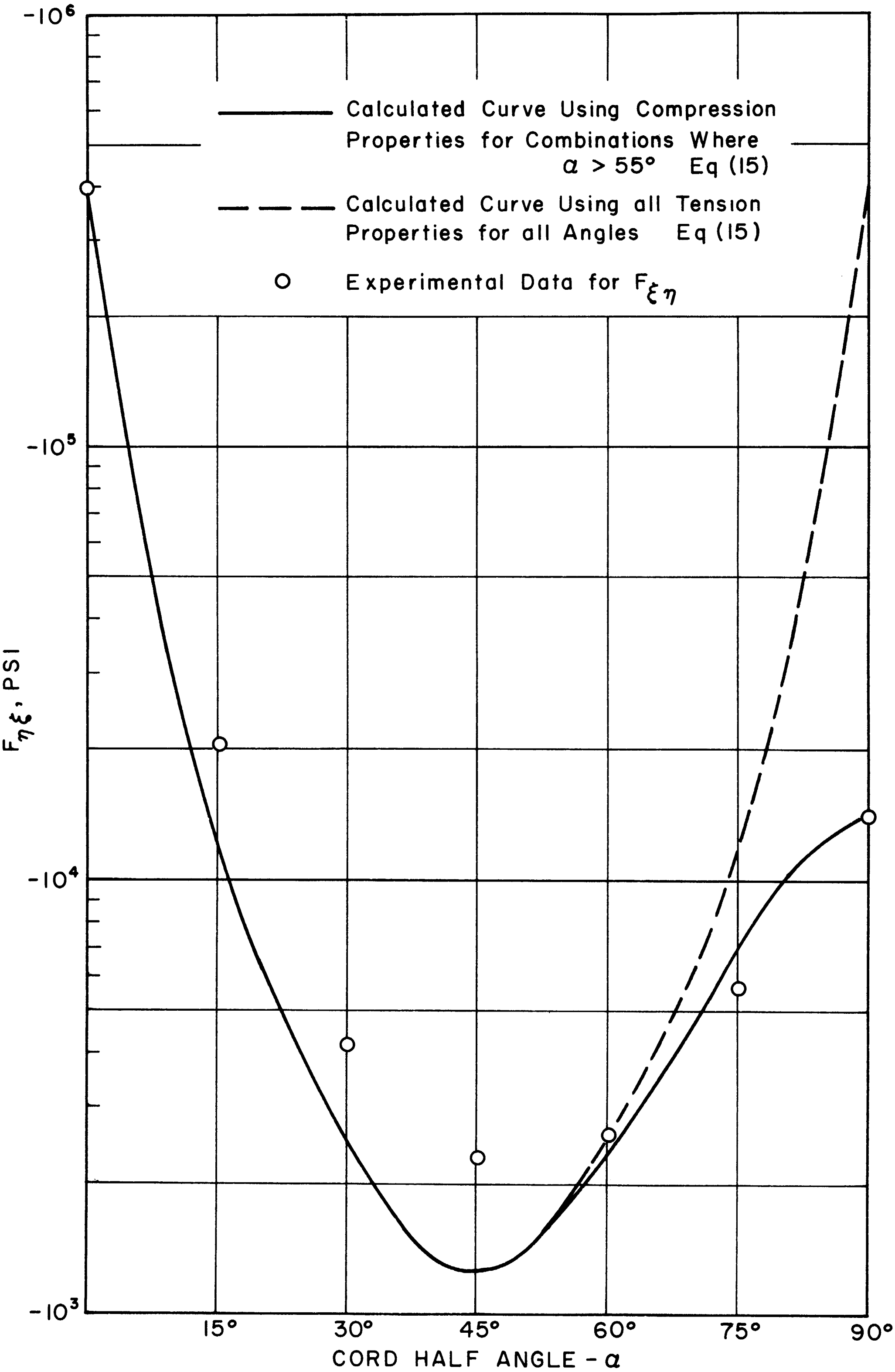

Fig. 16. Modulus $F_{\xi\eta}$ as predicted from Eq. (15) vs. cord half angle α, along with experimental values for $F_{\xi\eta}$.

the moduli E and F as prediced by Eq. (15). Notice that the dashed line represents that portion of the curve which would prevail if the compression properties did not enter into the problem, while the solid line represents which actually is predicted from the equations previously derived. Notice that in Fig. 15 there is no difference between the curve obtained using tension properties throughout the range of angles 0° to 90°, and the curve obtained using all compression properties for half-angles α greater than 55°. Figure 15 also shows the companion curve, E_η, and as might be suspected, the tension modulus in this direction is the mirror image about the 45° line of the tension modulus E_ξ. Experimental data obtained from the laboratory tests are also shown on Fig. 15 as circles, and it is seen the correlation between theory and experiment is generally quite good.

Certain comments can be made here concerning the shape of the curve shown in Fig. 15. Most important, it is seen that for cord half-angles lying between 0° add 45° the tension modulus is extremely dependent upon the cord angle. In contrast to this, it may be noted that for cord half-angles between 45° and 90°, the modulus is nearly independent of the angle. This would indicate that changes in cord angle might result in only minor changes in the elastic characteristics of cord-rubber structures for those cases where this cord half-angle exceeds 45°. Noting that this curve is drawn on a semi-logarithmic scale, it is possible to see that immense changes in stiffness can be generated by relatively small changes of angle. For example, the ratio of stiff nesses at cord half-angles of 15° and 30° is more than 10 to 1. One might conclude from a curve such as this that structures made by laminating plies composed of relatively stiff materials imbedded in a soft matrix might have other applications than the pneumatic tire, since the elastic characteristics of these structures can be varied over such a wide range.

In Fig. 16, experimental values of the cross modulus $F_{\xi\eta}$ are presented along with calculations based on Eqs. (15) and using the appropriate tension or compression elastic characteristics E_x, E_y, G_{xy}, and F_{xy}. Again, the compression properties become applicable at cord half-angles greater than approximately 55°, and it is seen in this case that the difference between the tension and compression characteristics is appreciable. If the textile cords did not go into a state of compression at angles greater than 55°, then the dashed line of Fig. 16 indicates that the quantity $F_{\xi\eta}$ would be summetrical about the 45° line of the half-angle α. Since the textile cord does indeed change its properties in compression, an unsymmetrical curve results, and this curve is seen to have some agreement with the measured values of cross modulus. The agreement is not particularly close but the general shape of the function seems to be maintained.

Again, the effective cross modulus $F_{\xi\eta}$ was obtained by measuring the strain at right angles to the direction of applied stress. This was generally done at values of specimen strain of approximately 3%. Figure 17 shows a typical load strain curve obtained from one of these tests, and the area from which the cross modulus was obtained is marked out.

One of the basic elastic constants used in the elasticity of homogeneous isotropic materials is the ratio between the lateral and the axial strain caused by an axial stress. This quantity, called Poisson's ratio, has a value ranging from approximately 0.1 to 0.5 for most ordinary materials, with rubber having a value very close to 0.5. One may physically associate the property

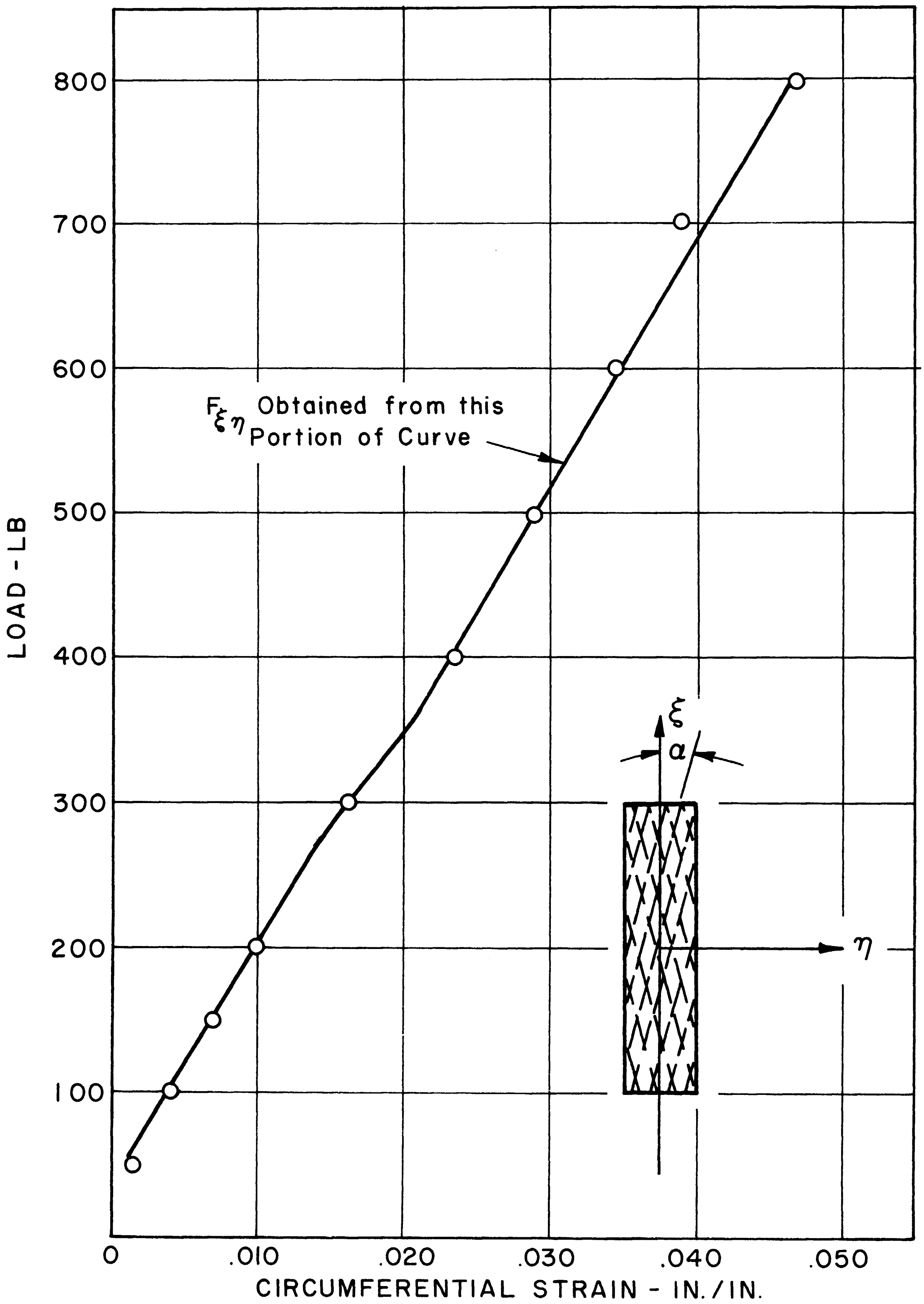

Fig. 17. Typical axial load vs. circumferential strain curve for cylindrical tubes subjected to tension loads.

of infinite bulk modulus, or incompressibility, with the value of Poisson's ratio of 0.5 for an isotropic body. As has been pointed out previously in this report, a cord-rubber combination is not homogeneous nor is it isotropic. It is not, therefore, surprising to find cord-rubber combinations exhibiting values of Poisson's ratio outside those ordinarily obtained from normal materials possessing isotropy.

Poisson's ratio may be obtained in terms of the notation used in this report by taking the ratio E_{ξ} over $F_{\xi\eta}$, which is the ratio of lateral to axial strain. This has been calculated from the experimental data and it is presented as a function of the cord half-angle α in Fig. 18. Calculations have been performed using the appropriate tension or compression elastic data and using the results of Eqs. (15). These numerical results are presented in Fig. 18 as the solid line. It may be seen that the correlation between theory and experiment, while not particularly good, at least indicates correspondence of the shapes of the curves and of the orders of magnitude of the constants involved. It is interesting to note that Poisson's ratios in excess of one are possible. This means that at the proper cord angles, a tubular cord-rubber specimen will contract more in its diameter than it will elongate when subjected to a tensile load in the direction of its elongation. This is undoubtedly the physical basis of the well-known Chinese finger puzzle.

Torsion tests of the cylindrical tube specimens were performed over a range of half-angles from 0° to 60°, under conditions previously described in which some doubt is thrown on the validity of the results. A typical torque-angle of twist curve is shown in Fig. 19, indicating the linear nature of stress-strain relation. The shear modulus was determined from the region

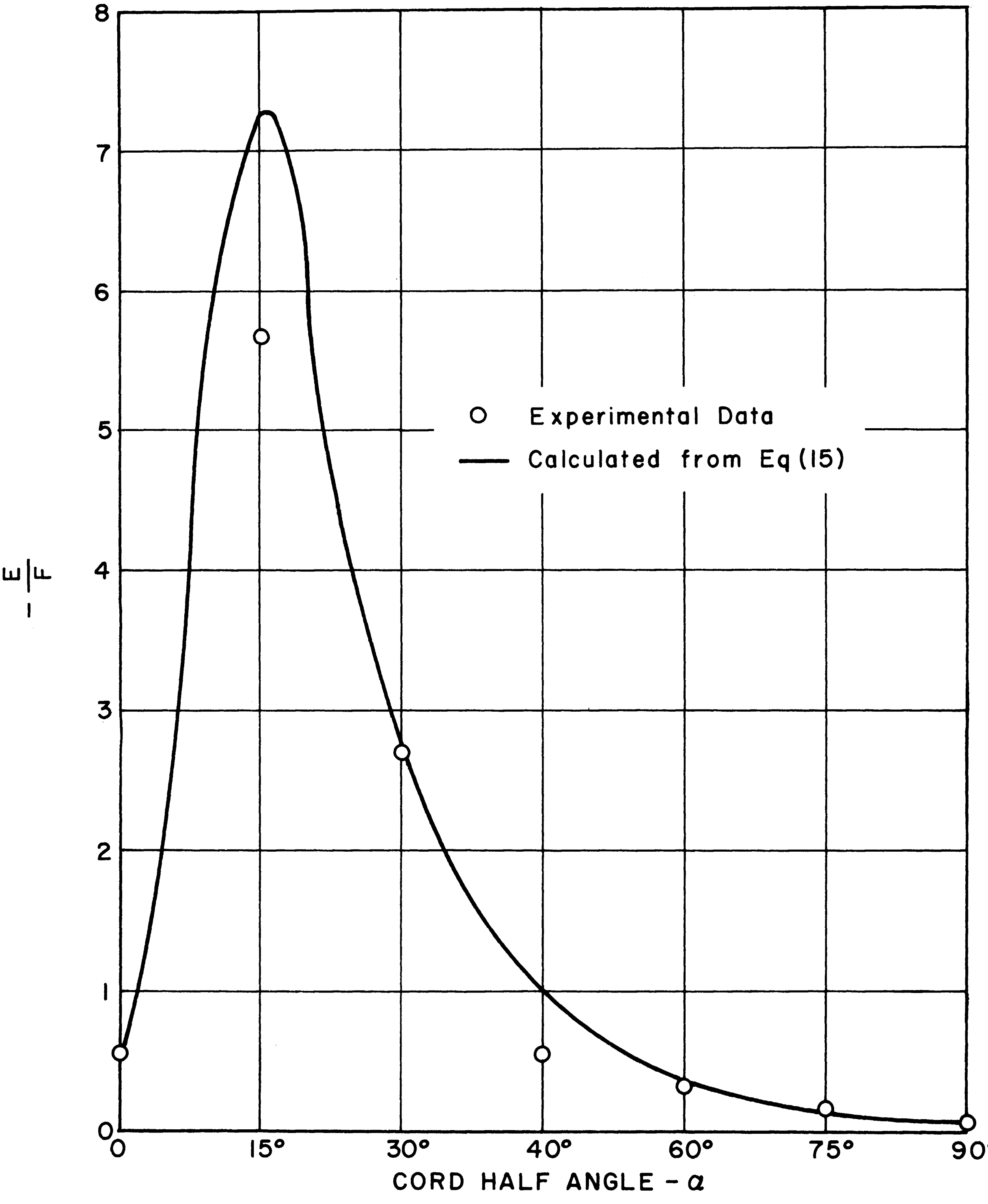

Fig. 18. Poisson's ratio as predicted from Eq. (15) vs. cord half life angle α, along with experimental values for Poisson's ratio.

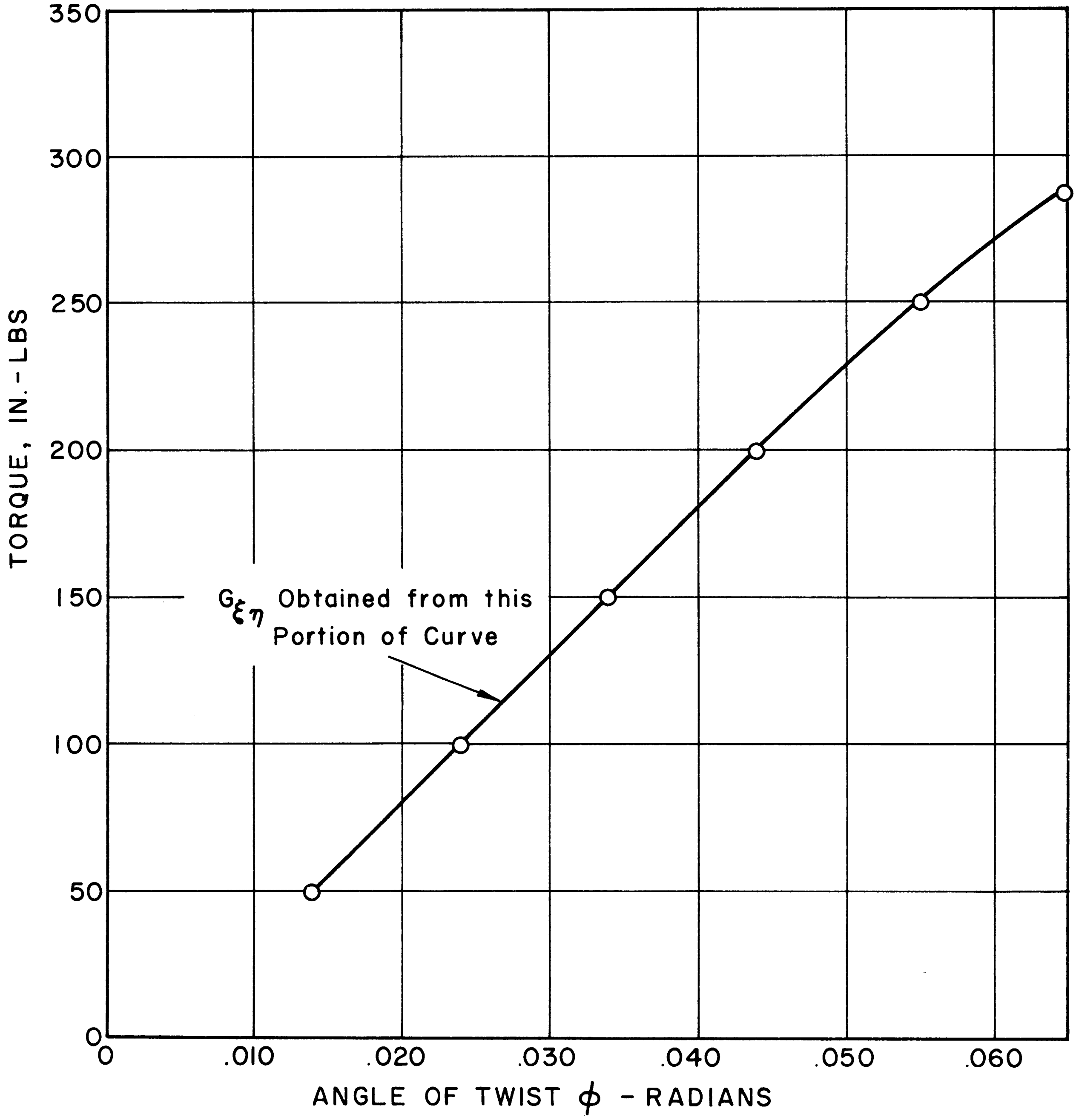

Fig. 19. Typical torque-angle of twist curve for cylindrical tubes subjected to tension loads.

surrounding 3% strain. These shear moduli are given by the experimental points in Fig. 20 as a function of half-angle. In addition, calculations of the shear modulus of Eq. (27) were performed using the elastic constants E_x, E_y, F_{xy}, G_{xy}, E'_x, E'_y, F'_{xy}, and G'_{xy} obtained from tests on the 0° tubes. The results of these calculations are presented in Fig. 20 as the solid line, and it is seen that the agreement with experiment is not very good. For comparison, the term $1/a_{33}$, which is physically the direct proportionality constant between shearing stress and shear strain, is also shown and is clearly an inadequate representation.

The lack of agreement between theory and experiment in Fig. 20 may be due to two causes. First, the basic assumptions leading to Eq. (27) may be deficient in which case the approximate solution of the indeterminate equations is simply not very good. Secondly, some trial calculations showed $G_{\xi\eta}$ is extremely dependent upon the elastic constant F'_{xy}, which is without doubt the most difficult of the eight tension-compression elastic constants to measure accurately. The measurements made on F'_{xy} in our laboratory tests are such that, at one end of their spectrum, nearly perfect agreement between the calculated and measured $G_{\xi\eta}$ is obtained, while the result of their average value is presented in Fig. 20. Clearly, resolution of this problem must await future work.

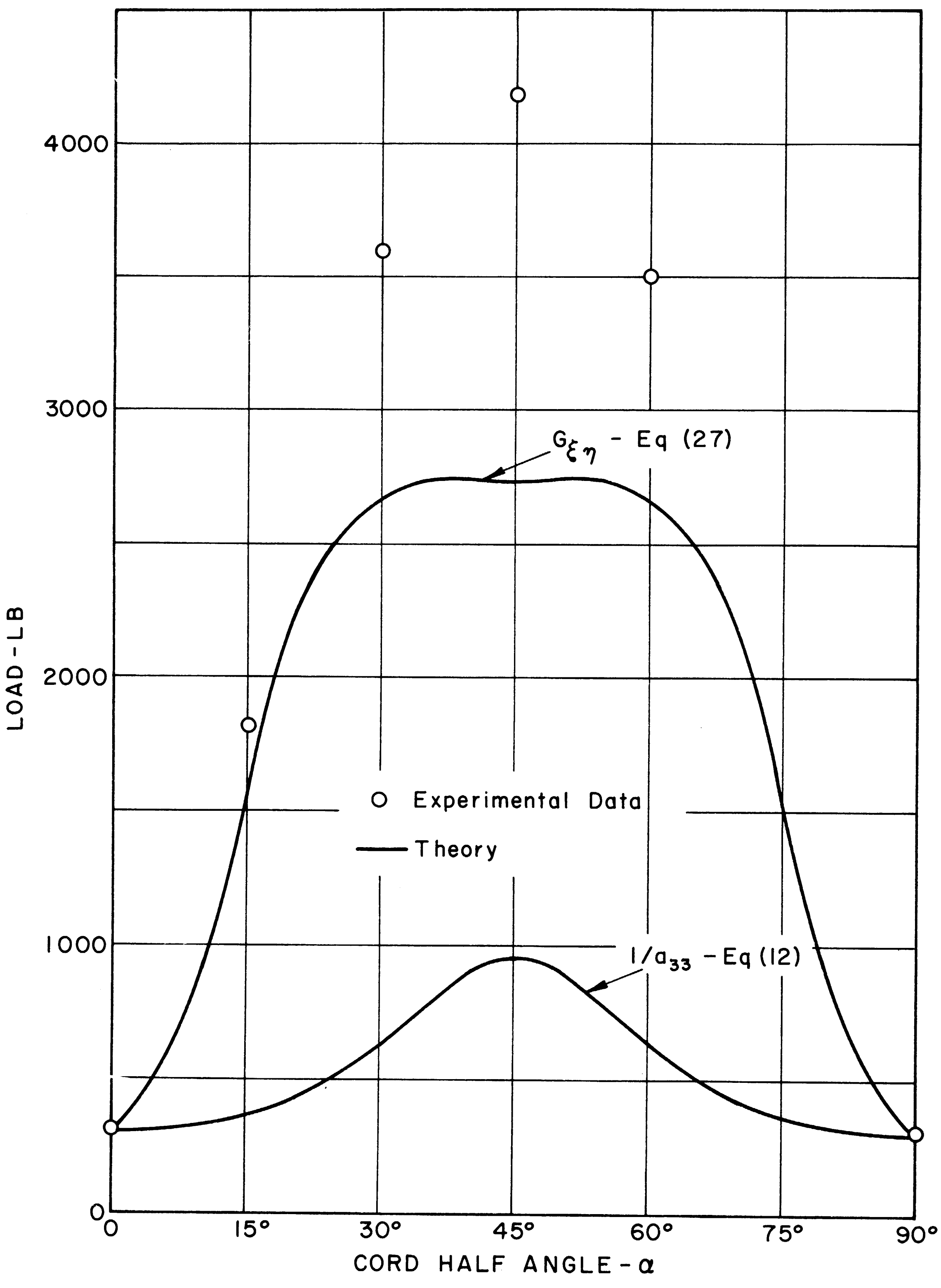

Fig. 20. Modulus $G_{\xi\eta}$ as predicted from Eq. (27) vs. cord half angle α, along with experimental values for $G_{\xi\eta}$. The term $1/a_{33}$ is shown for comparison.

VII. ACKNOWLEDGMENTS

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