THE UNIVERSITY OF MICHIGAN

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Tire and Suspension Systems Research Group

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STRESS CONCENTRATIONS IN LAP JOINTS

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NOMENCLATURE

English Letters:

A,B,C,D	Constants of integration.
E	Young's modulus.
G	Shear modulus.
I	Cross-sectional moment of inertia.
K	Spring constant of bond in tension.
L	Length of lap joint.
M	Bending moment.
P	Membrane load.
q,r,s	Variables associated with bending.
t	Thickness of a ply.
u	Displacement in direction of load P.
x,y,z	Cartesian coordinates.

Greek Letters:

β	A constant of the lap joint.
ξ	Vertical deflection in bending.
η	Bond thickness.
τ	Shear stress in bond.

Subscripts:

u	Upper.
ℓ	Lower.
R,r	Rubber.

I. FOREWORD

Lap joints are of interest to the tire industry since they are difficult to analyze satisfactorily, and since the ends of lap joints can initiate failure unless the joints are properly made. They are of interest to the Tire and Suspension Systems Group since lap joints are necessary in a real tire and yet cannot be conveniently represented in an idealized shell solution. Hence, they must be analyzed separately, and their effects must be considered both as local disturbances and as secondary elements of the stress and deformation state of the whole tire.

This analysis considers a lap joint in an idealized form, in which two isotropic layers are held together by an elastic glue. Due to this idealization and simplification, the results obtained are probably more important as indicators of trends and effects than as quantitative and accurate descriptions.

II. SUMMARY

A study of lap joint stresses in both tension and bending indicates that under both loading conditions the maximum value of the lap joint stress occurs at the edge of the joint. The most practical means of reducing this peak stress is to soften the bond used to secure the lap joint, either by decreasing the modulus of the rubber or by increasing the bond thickness.

III. PHYSICAL CONSIDERATIONS

In this study, lap joints are assumed to occur in two places in the pneumatic tire:

(a) The joint formed when one or more breaker plies are attached, and the lengths of members are so long compared with the thickness of the plies that end effects are negligible.

(b) Turn-ups around beads where the length of lap joint is limited and may, other considerations being disregarded, become very short provided that load-carrying ability is maintained.

In both of these cases it will be assumed that the joint is made by lapping together two plies of material which are stiff compared with the bonding agent, which is usually rubber. The ply angle will only be taken in account indirectly, in terms of a ply stiffness which can be evaluated from previous technical work done by this group (see Reference 1).

The lap joint must function in carrying both membrane and bending loads, and its stresses under both conditions are to be analyzed.

A more complete analysis of this problem for metal joints was published in Reference 2, which served as a background for the present study.

IV. LAP JOINTS UNDER MEMBRANE LOADS

In this section anisotropy is neglected and only membrane loads are considered. It will be assumed that two ply-structures are being joined together as shown in Fig. 1, where symbols and dimensions are given.

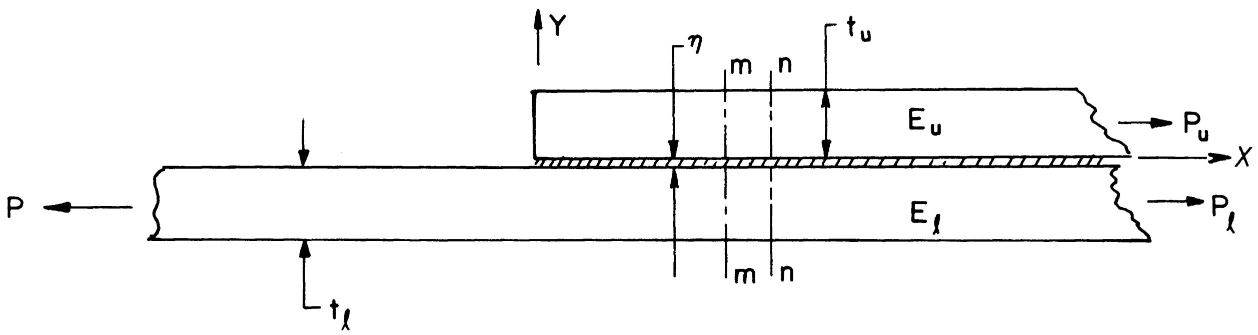

Fig. 1. Representative lap joint.

The cross-hatched portion, which represents rubber or other bonding material of thickness η and shear modulus G_r, is assumed to carry no tensile load in the direction of the external force P. It should be assumed that the thicknesses, t_ℓ and t_u, of the plies are very nearly equal to the diameters of the cords imbedded therein so that η represents the entire rubber bond between the two layers of cords. If this is not assumed, additional problems such as the resulting shear deformation of the upper and lower plies must be considered. It might be safe to assume that t_ℓ and t_u are cord diameters. Any small additional layer of rubber on either ply will be considered only in η, which will thus represent the thickness of the bond plus twice the thickness of this additional layer. Diagrams of individual sections of length dx are shown at sections m-n in Fig. 2.

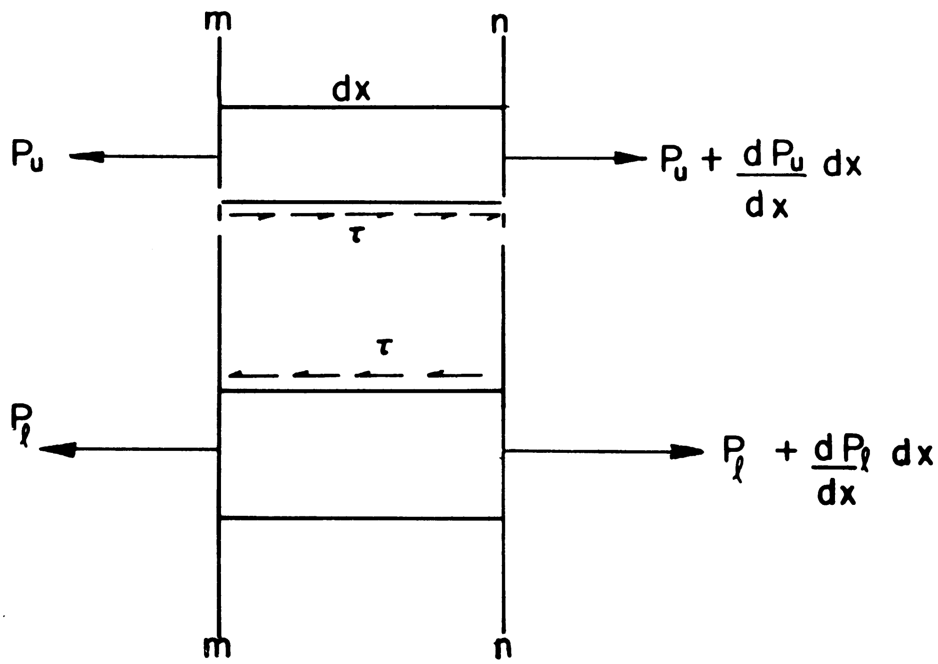

Fig. 2. Forces acting on joint elements.

The equations of equilibrium of the sections shown in Fig. 2. are:

$$\tau dx + \frac{dP_u}{dx} dx = 0 \qquad (1)$$

$$-\tau dx + \frac{dP_\ell}{dx} dx = 0. \qquad (2)$$

A relation between shear stress and displacement differential is needed. Assume that

$$\tau = \frac{G_R}{\eta} (u_\ell - u_u) \qquad (3)$$

where G_R is the rubber or bonding material modulus, and u_ℓ and u_u are displacements in the x direction of the lower and upper layers respectively.

Using Eq. (3) in Eqs. (1) and (2), one obtains

$$\frac{G_R}{\eta}(u_\ell - u_u) + \frac{dP_u}{dx} = 0$$
$$- \frac{G_R}{\eta}(u_\ell - u_u) + \frac{dP_\ell}{dx} = 0. \tag{4}$$

Assuming that tensile forces cause the displacements, then

$$P_\ell = t_\ell E_\ell \frac{du_\ell}{dx}$$
$$P_u = t_u E_u \frac{du_u}{dx} \tag{5}$$

where the unit width of the strips in Fig. 1 is assumed. Inserting Eqs. (5) into Eqs. (4) gives

$$\frac{G_R}{\eta}(u_\ell - u_u) + t_u E_u \frac{d^2 u_u}{dx^2} = 0$$
$$- \frac{G_R}{\eta}(u_\ell - u_u) + t_\ell E_\ell \frac{d^2 u_\ell}{dx^2} = 0 \tag{6}$$

Solving the first of Eqs. (6) for u_ℓ and substituting it into the second gives

$$\frac{d^4 u_u}{dx^4} - \frac{\left(1 + \frac{t_u E_u}{t_\ell E_\ell}\right)}{[t_u E_u/(G_R/\eta)]} \frac{d^2 u_u}{dx^2} = 0 . \tag{7}$$

The solution to Eq. (7) is

$$u_u = \frac{A}{\beta^2} e^{\beta x} + \frac{B}{\beta^2} e^{-\beta x} + Cx + D \tag{8}$$

where

$$\beta^2 = \left(1 + \frac{t_u E_u}{t_\ell E_\ell}\right)/(t_u E_u \eta / G_R)$$

and A,B,C, and D are constants of integration. From Eqs. (6), u_ℓ becomes

$$u_\ell = A\left(\frac{1}{\beta^2} - \frac{t_u E_u}{\frac{G_R}{\eta}}\right) e^{\beta x} + B\left(\frac{1}{\beta^2} - \frac{t_u E_u}{\frac{G_R}{\eta}}\right) e^{-\beta x} + Cx + D. \qquad (9)$$

So far this solution has been general, without reference to specific end conditions. It is now necessary to treat two different applications separately, since one results in a much simpler solution than the other.

Case 1:

Here it will be assumed that only the origin of the lap joint is of interest. The lengths of the upper and lower pieces are presumed to be so long that their ends are very far away, as in the case of a breaker ply which is attached to a carcass body. Under these assumptions, the boundary conditions require that the strain du_u/dx or du_ℓ/dx remain bounded as $x \to \infty$. Hence $A = 0$. In addition, the following boundary conditions are valid:

(a) $(u_\ell)_{x=o} = 0$

(b) $t_u E_u \left(\frac{\partial u_u}{\partial x}\right)_{x\to\infty} = \frac{t_u E_u}{E_u t_u + E_\ell t_\ell} \cdot P$

(c) $t_u E_u \left(\frac{\partial u_u}{\partial x}\right)_{x=o} = 0.$

Using these and omitting the intermediate algebra, one obtains:

$$u_u = B\left(\frac{e^{-\beta x}}{\beta^2} - \frac{1}{\beta^2} + \frac{t_u E_u}{G_R/\eta}\right) + \frac{Px}{E_u t_u + E_\ell t_\ell}$$

$$u_\ell = B\left(\frac{1}{\beta^2} - \frac{t_u E_u}{G_R/\eta}\right)\left(e^{-\beta x} - 1\right) + \frac{Px}{E_u t_u + E_\ell t_\ell} \qquad (10)$$

where

$$B = \frac{P\beta}{E_u t_u + E_\ell t_\ell} .$$

Finally, since displacements and hence strains have been obtained, the shear stress τ can be obtained from Eq. (3) to give

$$\tau = - \frac{P \cdot \beta t_u E_u}{E_u t_u + E_\ell t_\ell} e^{-\beta x} \tag{11}$$

with the intermediate algebra again omitted. The maximum value of τ occurs at $x = 0$.

The quantity β^2 can be written

$$\beta^2 = \left(\frac{1 + \frac{t_u E_u}{t_\ell E_\ell}}{t_u E_u} \right) \left(\frac{G_R}{\eta} \right) .$$

Using this at $x = 0$ gives, from Eq. (11),

$$\tau_{max} = - P \left[\left(\frac{G_R}{\eta} \right) \frac{E_u t_u}{E_\ell t_\ell (E_\ell t_\ell + E_u t_u)} \right]^{1/2} . \tag{12}$$

This maximum lap joint stress occurs at the edge of the joint. It is seen that for a given load P the only ways in which this stress can be reduced are to:

1. Decrease the rubber modulus G_R between the main carcass ℓ and the extra ply u.

2. Increase the bond thickness η by adding additional rubber, e.g., by using a gum strip.

3. Decrease the extra ply thickness t_u. (Perhaps tapering the end would be of use, as shown in Fig. 3.)

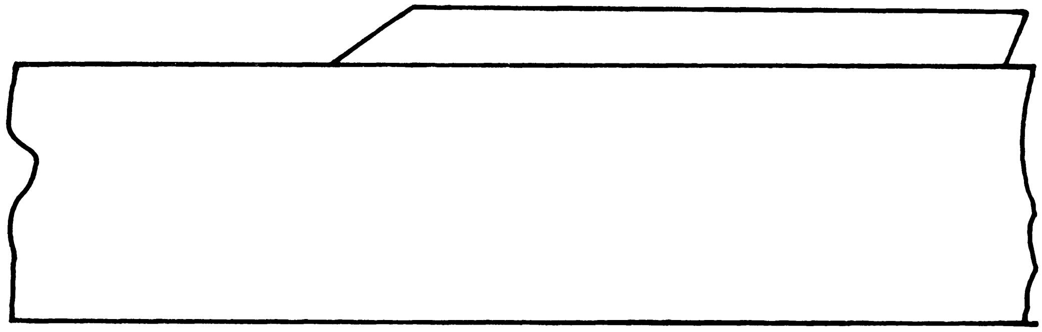

Fig. 3. Tapering to decrease bond stresses.

4. Decrease the extra ply modulus E_u.

5. Decrease the stress level $P/E_\ell t_\ell + E_u t_u$ by increasing $E_\ell t_\ell$.

Equation (11) indicates that lap-joint shear stresses decay exponentially away from the end x = 0 of the joint. Hence most lap-joint failures in breaker plies can be expected to occur at the end of the breaker ply. It should also be noted, as shown by Eq. (12), that the stress concentration effects become more important as the number of plies in a real tire decreases, this is because, compared with $E_\ell t_\ell$, the ratio

$$\frac{E_u t_u}{E_\ell t_\ell (E_\ell t_\ell + E_u t_u)}$$

becomes larger as the quantity $E_u t_u$ increases.

Case 2:

In turning a ply up around a bead, a lap joint of fixed, and sometimes short, length is formed. In this case it is probably not valid to assume that one end does not influence another; therefore a solution for Eqs. (8) and (9) must be obtained for a finite length. The solutions can then be used to determine the form of the shear stress in this joint.

Consider for simplicity the physical situation illustrated by Fig. 4. Two plies are joined by a lap of length L, in the assumed absence of bending moments.

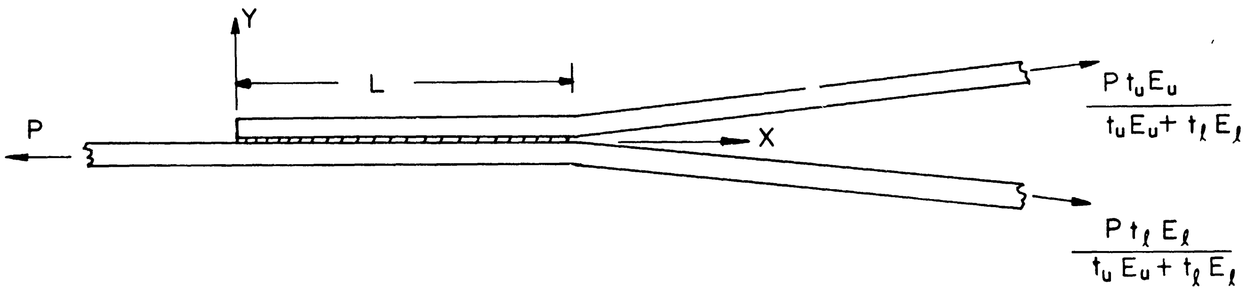

Fig. 4. Two-ply structure joined by lap of length L.

The notation used for Fig. 4 is the same as that used for Fig. 1. Here, however, the boundary conditions for Eqs. (8) and (9) are

$$(u_\ell)_{x=o} = 0 \tag{13a}$$

$$t_u E_u \left(\frac{\partial u_u}{\partial x}\right)_{x=L} = \frac{P t_u E_u}{t_u E_u + t_\ell E_\ell} \tag{13b}$$

$$t_u E_u \left(\frac{\partial u_u}{\partial x}\right)_{x=o} = 0 \tag{13c}$$

$$t_\ell E_\ell \left(\frac{\partial u_\ell}{\partial x}\right)_{x=o} = P. \tag{13d}$$

Using these conditions, and omitting the intermediate algebra, one can determine the constants of integration as follows:

$$A = B - (G_R/\eta) \cdot \frac{P}{(t_u E_u)(t_\ell E_\ell)\beta}$$

$$B = \frac{P}{(e^{\beta L} - e^{-\beta L})} \left[\frac{\beta}{2 t_u E_u} + \frac{(e^{\beta L} - 1)}{t_\ell E_\ell \cdot t_u E_u \cdot \beta} \cdot (G_R/\eta) \right]$$

$C = (B-A)/\beta$ (The algebraic form of D is not required here.)

The shear stress in the joint is given by

$$\tau = \frac{G_R}{\eta}(u_\ell - u_u) = - t_u E_u (A e^{\beta x} + B e^{-\beta x}) \qquad (14)$$

Substituting the values of A and B into this gives

$$\tau = P\beta \left[\frac{e^{\beta x} t_u E_u}{t_\ell E_\ell + t_u E_u} - \frac{2 \sinh \beta x}{e^{\beta L} - e^{-\beta L}} \left\{ \frac{1}{2} + (e^{\beta L} - 1) \frac{t_u E_u}{t_\ell E_\ell + t_u E_u} \right\} \right] \qquad (15a)$$

For the case where $t_u E_u$ and $t_\ell E_\ell$ are the same, as in a one-ply turn-up, then

$$\tau = P\beta \left[\frac{e^{\beta x}}{2} - \left(\frac{e^{\beta L}}{e^{\beta L} - e^{-\beta L}} \right) \sinh \beta x \right] \qquad (15b)$$

From Eq. (15b), the simpler of the two forms, one may note the following:

(a) $\tau = 0$ at $x = L$.

(b) $\tau = P\beta/2$ at $x = 0$, for any value of L, and this is the maximum τ.

This is the same general result as one obtained in Case 1, in that the methods of reducing τ_{max} depend completely on reducing the value of the parameters denoted here by β. The possibilities for doing this were discussed on pages 11 and 12 of this report.

V. LAP JOINTS UNDER BENDING DEFORMATION

The bending problem may be approached in a fashion in which it is assumed that the two layers, each made up of a ply or a series of plies, are joined by an elastic bond of spring constant K per unit length, and that they are formed in a shape which is either curved or is straight.

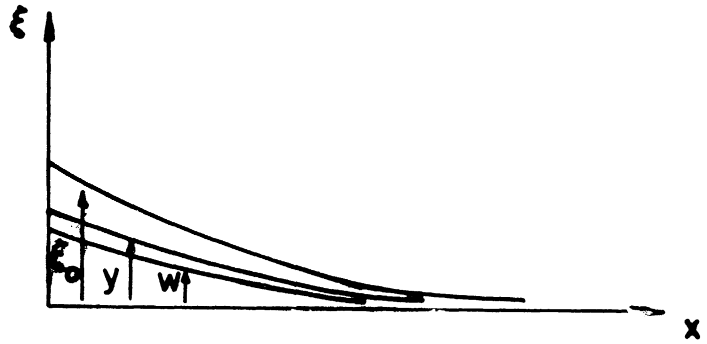

Fig. 5. Two-layer structure joined by elastic bond.

In Fig. 5, ξ_0 represents the initial configuration (some known function of x); y stands for the location of the top ply after bending; and w is the location of the bottom ply after bending.

The equilibrium equations of the two plies are

$$E_u I_u \frac{d^4(\xi_0 - y)}{dx^4} = K(y-w); \qquad E_\ell I_\ell \frac{d^4(\xi_0 - w)}{dx^4} = -K(y-w). \qquad (16)$$

These equations imply $\frac{d\xi}{dx} << 1$. It might be noted that for $\frac{d\xi}{dx} = 10°$, the above equations are in error by only 4%. Therefore it would seem safe to say that in the region of the end of a lap joint, the expressions are sufficiently accurate. If the changes of variable

$$r = \xi_o - y; \qquad s = \xi_o - w$$

are introduced, the following boundary value problems result.

$$E_u I_u \frac{d^4 r}{dx^4} = -K(r-s); \qquad \frac{d^3 r(o)}{dx^3} = \frac{d^2 r(o)}{dx^2} = \frac{dr(\infty)}{dx} = r(\infty) = 0$$

$$E_\ell I_\ell \frac{d^4 s}{dx^4} = K(r-s); \qquad \frac{d^3 s(o)}{dx^3} = \frac{ds(\infty)}{dx} = s(\infty) = 0 \tag{17}$$

$$E_\ell I_\ell \frac{d^2 s(o)}{dx^2} = M$$

In other words a moment, M, is applied only to the lower ply at the origin. The above system can be solved by classical methods, in which case a damped exponential results. For the present discussion it is easier to introduce the variable

$$q = r - s = -(y-w)$$

and then subtract the two boundary value problems:

$$\frac{d^4 q}{dx^4} = -K\left(\frac{E_u I_u + E_\ell I_\ell}{E_u I_u E_\ell I_\ell}\right) q$$

$$\frac{d^3 q(o)}{dx^3} = \frac{dq(\infty)}{dx} = q(\infty) = 0; \qquad \frac{d^2 q(o)}{dx^2} = -\frac{M}{E_\ell I_\ell}. \tag{18}$$

The general solution is

$$q(x) = e^{\beta x}(A\cos \beta x + B\sin \beta x) + e^{-\beta x}(C\cos \beta x + D\sin \beta x) \tag{19}$$

where

$$\beta^2 = \sqrt{\frac{K(E_u I_u + E_\ell I_\ell)}{4 E_u I_u E_\ell I_\ell}}.$$

Given the boundary conditions at ∞, A = B = 0. Algebraic manipulation gives

$$D = \frac{M}{2\beta^2 E_\ell I_\ell}; \qquad C = -\frac{M}{2\beta^2 E_\ell I_\ell} .$$

The complete solution is

$$q(x) = \frac{M}{2\beta^2 E_\ell I_\ell} e^{-\beta x} [\sin \beta x - \cos \beta x]. \tag{20}$$

The tension stress in the bond is given by

$$Kq(x) = \frac{KM}{2\beta^2 E_\ell I_\ell} e^{-\beta x} [\sin \beta x - \cos \beta x] \tag{21}$$

The maximum value is at x = 0 and is damped out as the distance from the joint edge increases.

$$(Kq)_{max} = -\sqrt{\frac{KM^2}{E_\ell I_\ell (1 + \frac{E_\ell I_\ell}{E_u I_u})}} . \tag{22}$$

The spring constant of the bond, K, is given by

$$(Kq)_{max} = \sigma_{max} = E_r \epsilon = E_r \frac{q(o)}{\eta} \Rightarrow K = \frac{E_r}{\eta} . \tag{23}$$

Thus the maximum stress tending to pull the plies apart is

$$(Kq)_{max} = -\sqrt{\frac{E_r M^2}{\eta E\ I\ (1 + \frac{E_\ell I_\ell}{E_u I_u})}} . \tag{24}$$

Reduction of the tensile stresses in a lap joint caused by bending is thus governed by the same principles as reduction of the shear stress. Thus we once again see that the possibilities for reduction are to:

1. Decrease the bonding rubber modulus, E_r.

2. Increase the bond thickness, η.

3. Increase the ply stiffness, $E_\ell I_\ell$, or decrease $E_u I_u$. This is, in effect, decreasing the ratio t_u/t_ℓ. The stress concentration at the joint edge would be greater for a two-ply structure than for a four-ply structure, which enhances the chance for failure. Tapering the end of the joint might again be employed to reduce the stress concentration.

VI. REFERENCES

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2. Goland, M. and Reissner, E. J., "The Stresses in Cemented Joints," *Trans. ASME*, *66*, 1944, p. A-17.

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