STRESS CONCENTRATIONS IN LAP JOINTS

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NOMENCLATURE

**English Letters:**

A,B,C,D \( \text{Constants of integration.} \)

E \( \text{Young's modulus.} \)

G \( \text{Shear modulus.} \)

I \( \text{Cross-sectional moment of inertia.} \)

K \( \text{Spring constant of bond in tension.} \)

L \( \text{Length of lap joint.} \)

M \( \text{Bending moment.} \)

P \( \text{Membrane load.} \)

q,r,s \( \text{Variables associated with bending.} \)

t \( \text{Thickness of a ply.} \)

u \( \text{Displacement in direction of load P.} \)

x,y,z \( \text{Cartesian coordinates.} \)

**Greek Letters:**

\( \beta \) \( \text{A constant of the lap joint.} \)

\( \xi \) \( \text{Vertical deflection in bending.} \)

\( \eta \) \( \text{Bond thickness.} \)

\( \tau \) \( \text{Shear stress in bond.} \)

**Subscripts:**

u \( \text{Upper.} \)

l \( \text{Lower.} \)

R,r \( \text{Rubber.} \)
I. FOREWORD

Lap joints are of interest to the tire industry since they are difficult to analyze satisfactorily, and since the ends of lap joints can initiate failure unless the joints are properly made. They are of interest to the Tire and Suspension Systems Group since lap joints are necessary in a real tire and yet cannot be conveniently represented in an idealized shell solution. Hence, they must be analyzed separately, and their effects must be considered both as local disturbances and as secondary elements of the stress and deformation state of the whole tire.

This analysis considers a lap joint in an idealized form, in which two isotropic layers are held together by an elastic glue. Due to this idealization and simplification, the results obtained are probably more important as indicators of trends and effects than as quantitative and accurate descriptions.
II. SUMMARY

A study of lap joint stresses in both tension and bending indicates that under both loading conditions the maximum value of the lap joint stress occurs at the edge of the joint. The most practical means of reducing this peak stress is to soften the bond used to secure the lap joint, either by decreasing the modulus of the rubber or by increasing the bond thickness.
III. PHYSICAL CONSIDERATIONS

In this study, lap joints are assumed to occur in two places in the pneumatic tire:

(a) The joint formed when one or more breaker plies are attached, and the lengths of members are so long compared with the thickness of the plies that end effects are negligible.

(b) Turn-ups around beads where the length of lap joint is limited and may, other considerations being disregarded, become very short provided that load-carrying ability is maintained.

In both of these cases it will be assumed that the joint is made by lapping together two plies of material which are stiff compared with the bonding agent, which is usually rubber. The ply angle will only be taken in account indirectly, in terms of a ply stiffness which can be evaluated from previous technical work done by this group (see Reference 1).

The lap joint must function in carrying both membrane and bending loads, and its stresses under both conditions are to be analyzed.

A more complete analysis of this problem for metal joints was published in Reference 2, which served as a background for the present study.
IV. LAP JOINTS UNDER MEMBRANE LOADS

In this section anisotropy is neglected and only membrane loads are considered. It will be assumed that two ply-structures are being joined together as shown in Fig. 1, where symbols and dimensions are given.

![Fig. 1. Representative lap joint.](image)

The cross-hatched portion, which represents rubber or other bonding material of thickness \( \eta \) and shear modulus \( G_r \), is assumed to carry no tensile load in the direction of the external force \( P \). It should be assumed that the thicknesses, \( t_f \) and \( t_u \), of the plies are very nearly equal to the diameters of the cords imbedded therein so that \( \eta \) represents the entire rubber bond between the two layers of cords. If this is not assumed, additional problems such as the resulting shear deformation of the upper and lower plies must be considered. It might be safe to assume that \( t_f \) and \( t_u \) are cord diameters. Any small additional layer of rubber on either ply will be considered only in \( \eta \), which will thus represent the thickness of the bond plus twice the thickness.
of this additional layer. Diagrams of individual sections of length $dx$ are shown at sections $m-n$ in Fig. 2.

![Diagram](image)

**Fig. 2. Forces acting on joint elements.**

The equations of equilibrium of the sections shown in Fig. 2. are:

$$\tau dx + \frac{dP_u}{dx} dx = 0$$  \hspace{1cm} (1)

$$-\tau dx + \frac{dP_l}{dx} dx = 0.$$  \hspace{1cm} (2)

A relation between shear stress and displacement differential is needed. Assume that

$$\tau = \frac{G_R}{\eta} (u_l - u_u)$$  \hspace{1cm} (3)

where $G_R$ is the rubber or bonding material modulus, and $u_l$ and $u_u$ are displacements in the $x$ direction of the lower and upper layers respectively.
Using Eq. (3) in Eqs. (1) and (2), one obtains

\[
\frac{GR}{\eta} (u_{Ix} - u_u) + \frac{dP_I}{dx} = 0
\]

\[
- \frac{GR}{\eta} (u_{Ix} - u_u) + \frac{dP_u}{dx} = 0.
\]  (4)

Assuming that tensile forces cause the displacements, then

\[
P_I = t_I E_I \frac{du_I}{dx}
\]

\[
P_u = t_u E_u \frac{du_u}{dx}
\]  (5)

where the unit width of the strips in Fig. 1 is assumed. Inserting Eqs. (5) into Eqs. (4) gives

\[
\frac{GR}{\eta} (u_{Ix} - u_u) + t_u E_u \frac{d^2 u_u}{dx^2} = 0
\]

\[
- \frac{GR}{\eta} (u_{Ix} - u_u) + t_I E_I \frac{d^2 u_I}{dx^2} = 0
\]  (6)

Solving the first of Eqs. (6) for \(u_I\) and substituting it into the second gives

\[
\frac{d^4 u_u}{dx^4} - \frac{(1 + \frac{t_u E_u}{t_I E_I})}{[t_u E_u/(GR/\eta)]} \frac{d^2 u_u}{dx^2} = 0.
\]  (7)

The solution to Eq. (7) is

\[
u_u = A e^{\beta x} + B e^{-\beta x} + Cx + D
\]  (8)

where

\[
\beta^2 = (1 + \frac{t_u E_u}{t_I E_I})/(t_u E_u \eta/G_R)
\]
and A, B, C, and D are constants of integration. From Eqs. (6), \( u_f \) becomes

\[
\begin{align*}
\left( \frac{1}{\beta^2} - \frac{t_u E_u}{G_R} \right) e^{\beta x} + \left( \frac{1}{\beta^2} - \frac{t_u E_u}{G_R} \right) e^{-\beta x} + C x + D.
\end{align*}
\]

(9)

So far this solution has been general, without reference to specific end conditions. It is now necessary to treat two different applications separately, since one results in a much simpler solution than the other.

**Case 1:**

Here it will be assumed that only the origin of the lap joint is of interest. The lengths of the upper and lower pieces are presumed to be so long that their ends are very far away, as in the case of a breaker ply which is attached to a carcass body. Under these assumptions, the boundary conditions require that the strain \( du_u/dx \) or \( du_f/dx \) remain bounded as \( x \to \infty \). Hence \( A = 0 \). In addition, the following boundary conditions are valid:

(a) \( (u_f)_{x=0} = 0 \)

(b) \( t_u E_u \left( \frac{\partial u_u}{\partial x} \right)_{x=\infty} = t_u E_u \frac{u_u}{E_u t_u + E_f t_f} \cdot P \)

(c) \( t_u E_u \left( \frac{\partial u_u}{\partial x} \right)_{x=0} = 0 \).

Using these and omitting the intermediate algebra, one obtains:

\[
\begin{align*}
u_u &= B \left( \frac{e^{-\beta x}}{\beta^2} - \frac{1}{\beta^2} + \frac{t_u E_u}{G_R/\eta} \right) + \frac{P x}{E_u t_u + E_f t_f} \\
u_f &= B \left( \frac{1}{\beta^2} - \frac{t_u E_u}{G_R/\eta} \right) (e^{-\beta x} - 1) + \frac{P x}{E_u t_u + E_f t_f}
\end{align*}
\]

(10)
where

\[ B = \frac{P\beta}{E_{tu} + E_{tf} \ell} \cdot \]

Finally, since displacements and hence strains have been obtained, the shear stress \( \tau \) can be obtained from Eq. (3) to give

\[ \tau = -\frac{P \cdot \beta t_u E_u}{E_{tu} + E_{tf} \ell} e^{-\beta x} \quad (11) \]

with the intermediate algebra again omitted. The maximum value of \( \tau \) occurs at \( x = 0 \).

The quantity \( \beta^2 \) can be written

\[ \beta^2 = \left( 1 + \frac{t_u E_u}{t_f E_f} \right) \left( \frac{G_R}{\eta} \right). \]

Using this at \( x = 0 \) gives, from Eq. (11),

\[ \tau_{\text{max}} = -P \left[ \frac{G_R}{\eta} \frac{E_{tu} = E_{tf} \ell + E_{tu} \ell} \right]^{1/2} \quad (12) \]

This maximum lap joint stress occurs at the edge of the joint. It is seen that for a given load \( P \) the only ways in which this stress can be reduced are to:

1. Decrease the rubber modulus \( G_R \) between the main carcass \( f \) and the extra ply \( u \).

2. Increase the bond thickness \( \eta \) by adding additional rubber, e.g., by using a gum strip.

3. Decrease the extra ply thickness \( t_u \). (Perhaps tapering the end would be of use, as shown in Fig. 3.)
4. Decrease the extra ply modulus \( E_u \).

5. Decrease the stress level \( P/(E_{ft} + E_{tu}) \) by increasing \( E_{ft} \).

Equation (11) indicates that lap-joint shear stresses decay exponentially away from the end \( x = 0 \) of the joint. Hence most lap-joint failures in breaker plies can be expected to occur at the end of the breaker ply. It should also be noted, as shown by Eq. (12), that the stress concentration effects become more important as the number of plies in a real tire decreases, this is because, compared with \( E_{ft} \), the ratio

\[
\frac{E_{tu}}{E_{ft}(E_{ft} + E_{tu})}
\]

becomes larger as the quantity \( E_{tu} \) increases.

**Case 2:**

In turning a ply up around a bead, a lap joint of fixed, and sometimes short, length is formed. In this case it is probably not valid to assume that one end does not influence another; therefore a solution for Eqs. (6) and (9) must be obtained for a finite length. The solutions can then be used to
determine the form of the shear stress in this joint.

Consider for simplicity the physical situation illustrated by Fig. 4. Two plies are joined by a lap of length \( L \), in the assumed absence of bending moments.

![Diagram](image)

**Fig. 4.** Two-ply structure joined by lap of length \( L \).

The notation used for Fig. 4 is the same as that used for Fig. 1. Here, however, the boundary conditions for Eqs. (8) and (9) are

\[
\left( u_f \right)_{x=0} = 0 \quad \text{(13a)} \\
\frac{t_u E_u}{t_u E_u + t_f E_f} \left( \frac{\partial u_f}{\partial x} \right)_{x=L} = \frac{P t_u E_u}{t_u E_u + t_f E_f} \quad \text{(13b)} \\
\left( \frac{\partial u_u}{\partial x} \right)_{x=0} = 0 \quad \text{(13c)} \\
\frac{t_f E_f}{t_u E_u + t_f E_f} \left( \frac{\partial u_f}{\partial x} \right)_{x=0} = P. \quad \text{(13d)}
\]

Using these conditions, and omitting the intermediate algebra, one can determine the constants of integration as follows:
\[
A = B - \left(\frac{G_R}{\eta}\right) \cdot \left(\frac{P}{(t_uE_u)(t_fE_f)}\right) \beta
\]

\[
B = \frac{P}{(e^{\beta L} - e^{-\beta L})} \left[ \frac{\beta}{2t_uE_u} \left( t_fE_f \cdot t_uE_u \cdot \beta \right) + \frac{e^{\beta L} - 1}{t_fE_f \cdot t_uE_u} \cdot \left( \frac{G_R}{\eta} \right) \right]
\]

C = \frac{(B-A)}{\beta} \quad \text{(The algebraic form of D is not required here.)}

The shear stress in the joint is given by

\[
\tau = \frac{G_R}{2} \left( u_f - u_u \right) - t_uE_u \left( \kappa \frac{e^{\beta x}}{L} + Be^{-\beta x} \right)
\]  \hspace{1cm} (14)

Substituting the values of A and B into this gives

\[
\tau = P \beta \left[ \frac{e^{\beta x}tu_Eu}{t_fE_f + tu_Eu} - \frac{2 \sinh \beta x}{e^{\beta L} - e^{-\beta L}} \left\{ \frac{1}{2} + \frac{(e^{\beta L} - 1)}{t_fE_f + tu_Eu} \right\} \right] \quad (15a)
\]

For the case where \( t_uE_u \) and \( t_fE_f \) are the same, as in a one-ply turn-up, then

\[
\tau = P \beta \left[ \frac{e^{\beta x}}{2} - \left( \frac{e^{\beta L}}{e^{\beta L} - e^{-\beta L}} \right) \sinh \beta x \right] \quad (15b)
\]

From Eq. (15b), the simpler of the two forms, one may note the following:

(a) \( \tau = 0 \) at \( x = L \).

(b) \( \tau = P \beta / 2 \) at \( x = 0 \), for any value of \( L \), and this is the maximum \( \tau \).

This is the same general result as one obtained in Case 1, in that the methods of reducing \( \tau_{\text{max}} \) depend completely on reducing the value of the parameters denoted here by \( \beta \). The possibilities for doing this were discussed on pages 11 and 12 of this report.
V. LAP JOINTS UNDER BENDING DEFORMATION

The bending problem may be approached in a fashion in which it is assumed that the two layers, each made up of a ply or a series of plies, are joined by an elastic bond of spring constant \( K \) per unit length, and that they are formed in a shape which is either curved or is straight.

![Diagram of two-layer structure joined by elastic bond](image)

**Fig. 5.** Two-layer structure joined by elastic bond.

In Fig. 5, \( \xi_0 \) represents the initial configuration (some known function of \( x \)); \( y \) stands for the location of the top ply after bending; and \( w \) is the location of the bottom ply after bending.

The equilibrium equations of the two plies are

\[
E_l I_u \frac{d^4(\xi_0 - y)}{dx^4} = K(y-w); \quad E_l I_l \frac{d^4(\xi_0 - w)}{dx^4} = -K(y-w). \tag{15}
\]

These equations imply \( \frac{d^4}{dx^4} \ll 1 \). It might be noted that for \( \frac{d^4}{dx^4} = 10^\circ \), the above equations are in error by only \( 1\% \). Therefore it would seem safe to say that in the region of the end of a lap joint, the expressions are sufficiently accurate. If the changes of variable

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are introduced, the following boundary value problems result.

\[ E_u I_u \frac{d^4 r}{dx^4} = -K(r - s); \quad \frac{d^3 r(o)}{dx^3} = \frac{d^2 r(o)}{dx^2} = \frac{dr(\infty)}{dx} = r(\infty) = 0 \]

\[ E_f I_f \frac{d^4 s}{dx^4} = K(r - s); \quad \frac{d^3 s(o)}{dx^3} = \frac{ds(\infty)}{dx} = s(\infty) = 0 \quad (17) \]

\[ E_f I_f \frac{d^2 s(o)}{dx^2} = M \]

In other words a moment, \( M \), is applied only to the lower ply at the origin.

The above system can be solved by classical methods, in which case a damped exponential results. For the present discussion it is easier to introduce the variable

\[ q = r - s = -(y - w) \]

and then subtract the two boundary value problems:

\[ \frac{d^4 q}{dx^4} = -K \left( \frac{E_u I_u + E_f I_f}{E_u I_u E_f I_f} \right) q \]

\[ \frac{d^3 q(o)}{dx^3} = \frac{dq(\infty)}{dx} = q(\infty) = 0; \quad \frac{d^2 q(o)}{dx^2} = -\frac{M}{E_f I_f}. \quad (18) \]

The general solution is

\[ q(x) = e^{\beta x}(A \cos \beta x + B \sin \beta x) + e^{-\beta x}(C \cos \beta x + D \sin \beta x) \quad (19) \]

where

\[ \beta^2 = \sqrt{\frac{K(E_u I_u + E_f I_f)}{4E_u I_u E_f I_f}}. \]
Given the boundary conditions at \( \infty \), \( A = B = 0 \). Algebraic manipulation gives

\[
D = \frac{M}{2\beta^2 E I I_f} ; \quad C = -\frac{M}{2\beta^2 E I I_f} .
\]

The complete solution is

\[
q(x) = \frac{M}{2\beta^2 E I I_f} e^{-\beta x} [\sin \beta x - \cos \beta x]. \quad (20)
\]

The tension stress in the bond is given by

\[
Kq(x) = \frac{KM}{2\beta^2 E I I_f} e^{-\beta x} [\sin \beta x - \cos \beta x] \quad (21)
\]

The maximum value is at \( x = 0 \) and is damped out as the distance from the joint edge increases.

\[
(Kq)_{\text{max}} = -\sqrt{\frac{KM^2}{E I I_f (1 + \frac{E I I_f}{E u I_u})}}. \quad (22)
\]

The spring constant of the bond, \( K \), is given by

\[
(Kq)_{\text{max}} = q_{\text{max}} = E r = E_r \frac{q(0)}{\eta} \implies K = \frac{E_r}{\eta}. \quad (23)
\]

Thus the maximum stress tending to pull the plies apart is

\[
(Kq)_{\text{max}} = -\sqrt{\frac{E r M^2}{\eta E I (1 + \frac{E I I_f}{E u I_u})}}. \quad (24)
\]

Reduction of the tensile stresses in a lap joint caused by bending is thus governed by the same principles as reduction of the shear stress. Thus we once again see that the possibilities for reduction are to:
1. Decrease the bonding rubber modulus, $E_r$.

2. Increase the bond thickness, $\eta$.

3. Increase the ply stiffness, $E_l I_l$, or decrease $E_u I_u$. This is, in effect, decreasing the ratio $t_u/t_l$. The stress concentration at the joint edge would be greater for a two-ply structure than for a four-ply structure, which enhances the chance for failure. Tapering the end of the joint might again be employed to reduce the stress concentration.
VI. REFERENCES


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