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Asymptotic Flatness of the Weil–Petersson Metric on Teichmüller Space

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Abstract. In this paper, we show that there is no negative upper bound for the sectional curvature of Teichmüller space of Riemann surfaces with the Weil–Petersson metric.

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Key words. Riemann surfaces, Teichmüller space, Weil–Petersson metric, sectional curvature, harmonic maps.

1. Introduction

Let Σ be a smooth, closed Riemann surface of genus g, with n punctures and $3g-3+n>1$. Let M_{-1} denote the space of metrics of constant curvature -1 on Σ. The group, Diff_0, of diffeomorphisms homotopic to the identity acts by pull back on M_{-1}, and we can define Teichmüller space $\mathcal{T}_{g,n}$, to be the quotient space $M_{-1}/\mathrm{Diff}_0(\Sigma)$. Note that the constant $3g-3+n$ is fundamental in Teichmüller theory: it is the complex dimension of the Teichmüller space $\mathcal{T}_{g,n}$.

The Weil–Petersson metric on Teichmüller space has been heavily studied. It is defined by the L^2 inner product on the space of holomorphic quadratic differentials, the cotangent space of Teichmüller space. This metric has many curious properties; for example, it is a Riemannian metric with negative sectional curvature [23, 29], but it is incomplete [7,28]. Also this metric is Kähler [2], and there is an upper bound $-1/2\pi(g-1)$ for the holomorphic sectional curvature and Ricci curvature [29], a result conjectured by Royden [19]. However, there is no known negative upper bound for the sectional curvatures.

In this paper, we investigate the asymptotic behavior of the sectional curvatures of the Weil–Petersson metric on Teichmüller space. Our method is to investigate harmonic maps from a nearly noded surface to nearby hyperbolic structures. One of the difficulties in estimating Weil–Petersson curvatures is working with the operator $D=-2(\Delta-2)^{-1}$, which appears in Tromba–Wolpert's curvature formula. Our approach is to study the Hopf differentials associated to harmonic maps and the analytic formulas resulting from the harmonicity of the maps. There is a natural connection between the operator D and the local variations of the energy of a harmonic map between surfaces (see Lemma 1). From this connection, we estimate the solutions to some ordinary differential equations to derive our curvature estimates. Our estimates imply that even though the sectional curvatures are negative, they are not staying away from zero. More specifically, we prove

MAIN THEOREM. *If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, then there is no negative upper bound for the sectional curvature of the Weil–Petersson metric.*

THEOREM B. *If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, and let l be the length of shortest geodesic along a path to a boundary point in the Teichmüller space, then there exists a sequence of tangent planes with Weil–Petersson sectional curvature of the order* $O(l)$.

Recently Brock and Farb [5] showed that the Weil–Petersson metric on $\mathcal{T}$ is Gromov hyperbolic if and only if $\dim_C(\mathcal{T}) \leqslant 2$. They pointed out that if the Weil–Petersson metric on $\mathcal{T}$ had curvature pinched from above by a negative constant, its Gromov hyperbolicity in the case of the surface being doubly-punctured torus or 5-punctured sphere would be an immediate consequence of the comparison theorems. In the end of their paper, they asked the following question 6.3: if $\mathrm{int}(\Sigma)$ is homeomorphic to a doubly punctured torus or 5-punctured sphere, are the sectional curvatures of the Weil–Petersson metric bounded away from zero [5]? Our paper gives a negative answer to this question.

Naturally associated to a harmonic map $w: (\Sigma, \sigma|dz|^2) \to (\Sigma, \rho|dw|^2)$ is a quadratic differential $\Phi(\sigma, \rho)dz^2$, which is holomorphic with respect to the conformal structure of σ. This association of a quadratic differential to a conformal structure then defines a map $\Phi: \mathcal{T} \to QD(\Sigma)$ from Teichmüller space $\mathcal{T}$ to the space of holomorphic quadratic differentials $QD(\Sigma)$. This map is in fact a homeomorphism [25].

As an important computational tool in geometry of Teichmüller theory, the method of harmonic maps has been studied by many people. In particular, the second variation of the energy of the harmonic map $w = w(\sigma, \rho)$ with respect to the domain structure σ (or image structure ρ) at $\sigma = \rho$ yields the Weil–Petersson metric on $\mathcal{T}$ [23, 25], and one can also establish Tromba–Wolpert's curvature formula of the Weil–Petersson metric from this method [13], (see also [25]).

The moduli space of Riemann surfaces admits a compactification, known as the Deligne–Mumford compactification [16], and any element of the compactification divisor can be thought of as a Riemann surface with nodes, a connected complex space where points have neighborhoods complex isomorphic to either $\{|z| < \varepsilon\}$ (regular points) or $\{zw = 0; |z|, |w| < \varepsilon\}$ (nodes). We can think of noded surfaces arising as elements of the compactification divisor through a pinching process: fix a family of simple closed curves on Σ such that each component of the complement of the curves has negative Euler characteristic. The noded surface is topologically the result of identifying each curve to a point, the node [4].

While writing this paper, the author learned that Scott Wolpert [30] has obtained results related to ours.

The organization of this paper is as follows. In Section 2 we give the necessary background, define our terms and introduce the notations. Section 3 is devoted to proving our main theorem. The discussion of this purpose is broken into subsections: in Section 3.1, we collect the local variational formulas associated with the harmonic maps, hence connect the local variations of the energy of the harmonic map to the operator $D=-2(\Delta-2)^{-1}$; in Section 3.2 we describe the so called 'model case' of the problem; namely, we pinch two core geodesics of two cylinders into two points, and study the asymptotic behavior of the harmonic maps between cylinders. In the model case of this problem, the surface is a pair of cylinders; in Section 3.3, we describe a sequence of two-dimensional subspaces which have asymptotically flat Weil–Petersson sectional curvatures and we will establish two related ordinary differential equations, and solve these equations explicitly, thus we are able to relate the solutions of the equations to the operator $D=-2(\Delta-2)^{-1}$, which leads us to estimate the curvature. We will establish the estimates of terms in the curvature formula when the surface is a pair of cylinders; in Section 3.4 we construct families of maps, which have small tension, and are close to the harmonic maps resulting from pinching process in Section 3.3; finally in Section 3.5 we prove our main theorem based on the estimates in Section 3.3 and the construction in Section 3.4.

2. Notations and Background

Recall that Σ is a fixed, oriented, C^∞ surface of genus $g\geqslant 1$, and $n\geqslant 0$ punctures, where $3g-3+n>1$. We denote hyperbolic metrics on Σ by $\sigma|\mathrm{d}z|^2$ and $\rho|\mathrm{d}w|^2$, where z and w are conformal coordinates on Σ. By the uniformization theorem, the set of all similarly oriented hyperbolic structures M_{-1} can be identified with the set of all conformal structures on Σ with the given orientation. Equivalently, this is the same as the set of all complex structures on Σ with the given orientation. And Teichmüller space $\mathcal{T}$ is defined to be

$$\mathcal{T}=M_{-1}/\mathrm{Dif}\,f_0(\Sigma).$$

It is well known that Teichmüller space $\mathcal{T}$ has a complex structure [2] and the cotangent space at a point $\Sigma\in\mathcal{T}$ is the space of holomorphic quadratic differentials $\Phi\,\mathrm{d}z^2$ on Σ. On Σ, there is a natural pairing of quadratic differentials and Beltrami differentials $(\mu(z)(\mathrm{d}\bar{z}/\mathrm{d}z)$ given by

$$\langle\mu,\Phi\rangle=\mathrm{Re}\int_\Sigma \mu(z)\Phi(z)\mathrm{d}z\,\mathrm{d}\bar{z}.$$

The tangent space at Σ is the space of Beltrami differentials modulo the trivial ones which are ones such that $\langle\mu,\Phi\rangle=0$. The Weil–Petersson metric on $\mathcal{T}$ is obtained by duality from the L^2-inner product on $QD(\Sigma)$

$$\langle\phi,\psi\rangle=\int_\Sigma\frac{\phi\bar{\psi}}{\sigma}\mathrm{d}z\,\mathrm{d}\bar{z},$$

where $\sigma\,\mathrm{d}z\,\mathrm{d}\bar{z}$ is the hyperbolic metric on Σ.

For a Lipschitz map $w:(\Sigma,\sigma|dz|^2)\to(\Sigma,\rho|dw|^2)$, we define the energy density of w at a point to be

$$e(w;\sigma,\rho)=\frac{\rho(w(z))}{\sigma(z)}|w_z|^2+\frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2$$

and the total energy $E(w;\sigma,\rho)=\int_\Sigma e(w;\sigma,\rho)\sigma\,\mathrm{d}z\,\mathrm{d}\bar{z}$.

A critical point of $E(w;\sigma,\rho)$ is called a harmonic map; it satisfies the Euler–Lagrange equation, namely,

$$w_{z\bar{z}}+\frac{\rho_w}{\rho}w_zw_{\bar{z}}=0.$$

The Euler–Lagrange equation for the energy is the condition for the vanishing of the *tension* which is, in local coordinates,

$$\tau(w)=\Delta w^\gamma+{}^N\Gamma^\gamma_{\alpha\beta}w^\alpha_iw^\beta_j=0.$$

It is well known [3,9,11,20,22] that given σ,ρ, there exists a unique harmonic map $w:(\Sigma,\sigma)\to(\Sigma,\rho)$ homotopic to the identity of Σ, and this map is in fact a diffeomorphism.

Thus we have a holomorphic quadratic differential $\Phi\,dz^2=\rho w_z\bar{w}_z\,\mathrm{d}z^2$, and evidently

$$\Phi=0\quad\Leftrightarrow\quad w\quad\text{is conformal}\quad\Leftrightarrow\quad\sigma=\rho,$$

where $\sigma=\rho$ means that (Σ,σ) and (Σ,ρ) are the same point in Teichmüller space $\mathcal{T}$. This describes Φ as a well-defined map $\Phi:\mathcal{T}\to QD(\Sigma)$ from Teichmüller space $\mathcal{T}$ to the space of Hopf differentials $QD(\Sigma)$. In fact, this map is a homeomorphism [25]. Also note that the map w can be extended to surfaces with finitely many punctures.

We define two auxiliary functions as follows:

$$\mathcal{H}=\mathcal{H}(\sigma,\rho)=\frac{\rho(w(z))}{\sigma(z)}|w_z|^2,$$
$$\mathcal{L}=\mathcal{L}(\sigma,\rho)=\frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2.$$

Hence, the energy density is $e=\mathcal{H}+\mathcal{L}$. Many things connected with a harmonic map between surfaces can be written in terms of $\mathcal{H}$ and $\mathcal{L}$ (and Φ). We denote on $(\Sigma,\sigma|dz|^2)$

$$\Delta=\frac{4}{\sigma}\frac{\partial^2}{\partial z\,\partial\bar{z}},\qquad K(\rho)=-\frac{2}{\rho}\frac{\partial^2}{\partial w\,\partial\bar{w}}\log\rho,\qquad K(\sigma)=-\frac{2}{\sigma}\frac{\partial^2}{\partial z\,\partial\bar{z}}\log\sigma,$$

where Δ is the Laplacian, and $K(\rho)$, $K(\sigma)$ are curvatures of the metrics ρ and σ, respectively.

The Euler–Lagrange equation gives

$$\Delta\log\mathcal{H}=-2K(\rho)\mathcal{H}+2K(\rho)\mathcal{L}+2K(\sigma).$$

When we restrict ourselves to the situation $K(\sigma)=K(\rho)=-1$, we will have the following facts [25]:

(2.1) $\mathcal{H}>0$,

(2.2) The Beltrami differential $\mu=w_{\bar{z}}/w_z=\bar{\Phi}/\sigma\mathcal{H}$,

(2.3) $\Delta\log\mathcal{H}=2\mathcal{H}-2\mathcal{L}-2$.

3. Sectional Curvature of the Weil–Petersson Metric

In this section, we will prove the main theorem of this paper, namely,

MAIN THEOREM. *If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, then there is no negative upper bound for the sectional curvature of the Weil–Petersson metric.*

Combined with the $\hbar$ fact that there is no lower bound for the sectional curvature [21], We have

COROLLARY. *There are no negative bounds for the sectional curvature of the Weil–Petersson metric when* $\dim_C(\mathcal{T})>1$.

The rest of this paper will be devoted to proving our main theorem. The discussion will be broken into five Sections. In Section 3.1, we collect some variational formulas we will need in the rest of discussion; in Section 3.2, we mostly describe the 'model case', i.e. we consider a family of pertubations of the identity map between a pair of cylinders; in Section 3.3, we describe a family of subplanes of the tangent space of Teichmüller space $\mathcal{T}_{g,2}$, and estimate terms in the curvature formula when the surface is a pair of cylinders; in Section 3.4, we construct $C^{2,\alpha}$ maps between surfaces and show that the constructed maps have small tension and are close to the harmonic maps we get from the pinching process; in Section 3.5, we prove the main theorem.

3.1. LOCAL VARIATION

We consider a family of harmonic maps $w(t)$ for t small, where $w(0)=\mathrm{id}$, the identity map. Denote by $\Phi(t)$ the family of Hopf differentials determined by $w(t)$. We can rewrite (2.3) as

$$\Delta \log \mathcal{H}(t) = 2\mathcal{H}(t) - \frac{2|\Phi(t)|^2}{\sigma^2 \mathcal{H}(t)} - 2.$$

For this equation, we see that the maximum principle will force all the odd order t-derivatives of $\mathcal{H}(t)$ to vanish, since the above equation only depends on the modulus of $\Phi(t)$ and not on its argument [25], and $\mathcal{H}(t)$ is real-analytic in t [26].

Wolf computed the t-derivative of various geometric quantities associated with the harmonic maps, and we collect these local variational formulas into our next lemma.

LEMMA 1 [25]. *For the above notations, we have*

(3.1) $\mathcal{H}(t) \geqslant 1$, *and* $\mathcal{H}(t) \equiv 1 \;\Leftrightarrow\; t=0$,

(3.2) $\dot{\mathcal{H}}(t) = \partial/\partial t^\alpha|_0\ \mathcal{H}(t) = 0$,

(3.3) $\dot{\mu} = \partial/\partial t^\alpha|_0 \mu(t) = \bar{\Phi}_\alpha/\sigma$,

(3.4) $\ddot{\mathcal{H}}(t) = \dfrac{\partial^2}{\partial t^\alpha \partial t^{\bar{\beta}}}|_0 \mathcal{H}(t) = D\dfrac{\Phi_\alpha \Phi_{\bar{\beta}}}{\sigma^2}$,
where $D = -2(\Delta-2)^{-1}$. *Evidently* D *is a self-adjoint compact integral operator with a positive kernel, and it is the identity on constant functions.*

Hence with (3.4), we obtain a partial differential equation about $\ddot{\mathcal{H}}(t)$:

$$(\Delta - 2)(\ddot{\mathcal{H}}(t)) = -2\frac{\Phi_\alpha \Phi_{\bar{\beta}}}{\sigma^2}.$$

3.2. HARMONIC MAPS BETWEEN CYLINDERS

For the sake of simplicity of exposition, we assume that our surface has no punctures. We will comment on punctured case at the end of the arguement.

Before we jump into the proof of the main theorem, in this subsection, we consider the asymptotics of harmonic maps between cylinders, where our surface is a pair of cylinders. We consider the asymptotics of a family of pertubations of the identity map between two cylinders, and we call this is the ‘Model Case’. In the model case, the surface is a pair of cylinders M_0 and M_1. In particular, consider the boundary value problem of harmonically mapping the cylinder

$$M = [l^{-1} \sin^{-1}(l), \pi l^{-1} - l^{-1} \sin^{-1}(l)] \times [0, 1]$$

with boundary identification

$$\left[\frac{\sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{\sin^{-1}(l)}{l}\right] \times \{0\} = \left[\frac{\sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{\sin^{-1}(l)}{l}\right] \times \{1\},$$

where the hyperbolic length element on M is $l\csc(lx)|dz|$, to the cylinder

$$N = [L^{-1}\sin^{-1}(L), \pi L^{-1} - L^{-1}\sin^{-1}(L)] \times [0, 1]$$

with boundary identification

$$\left[\frac{\sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{\sin^{-1}(L)}{L}\right] \times \{0\} = \left[\frac{\sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{\sin^{-1}(L)}{L}\right] \times \{1\},$$

where the hyperbolic length element on N is $L\csc(Lu)|dw|$. Here l and L are the lengths of the simple closed core geodesics in the corresponding cylinders.
Say $w = u + iv$ is this harmonic map between cylinders M and N, where

$$u(l, L; x, y) = u(l, L; x), v(x, y) = y.$$

The Euler–Lagrange equation becomes

$$u'' = L\cot(Lu)(u'^2 - 1)$$

with boundary conditions $u(\sin^{-1}(l)/l) = \sin^{-1}(L)/L$ and $u(\pi/(2l)) = \pi/(2L)$. Note that both M and N admit an anti-isometric reflection about the curves $\{\pi/2l\} \times [0, 1]$ and $\{\pi/2L\} \times [0, 1]$.

Since the quadratic differential $\Phi = \rho w_z \bar{w}_z = \frac{1}{4}L^2\csc^2(Lu)(u'^2 - 1)$ is holomorphic in M_0, we have

$$0 = \frac{\partial}{\partial \bar{z}}(\rho^2 w_z \bar{w}_z) = \frac{\partial}{\partial x}\left(\tfrac{1}{8}L^2\csc^2(Lu)(u'^2 - 1)\right).$$

Therefore $L^2\csc^2(Lu)(u'^2 - 1) = c_0(l, L)$, where $c_0(l, L)$ is independent of x, and $c_0(l, l) = 0$ since $u(x)$ is the identity map when $L = l$.

So we have

$$u' = \sqrt{1 + c_0(l, L)L^{-2}\sin^2(Lu)}$$

with boundary conditions $u(\sin^{-1}(l)/l) = \sin^{-1}(L)/L$ and $u(\pi/(2l)) = \pi/(2L)$. So the solution to the Euler–Lagrange equation can be derived from the equation

$$\int_{L^{-1}\sin^{-1}L}^{u} \frac{dv}{\sqrt{1 + c_0(l, L)L^2\sin^2(Lv)}} = x - l^{-1}\sin^{-1}(l)$$

with $c_0(l, L)$ chosen such that

$$\int_{L^{-1}\sin^{-1}L}^{\frac{\pi}{2L}} \frac{dv}{\sqrt{1 + c_0(l, L)L^2\sin^2(Lv)}} = \frac{\pi}{2l} - l^{-1}\sin^{-1} l.$$

It is not hard to show that when $l \to 0$, the solution $u(l, L; x)$ converges to a solution $u(L; x)$ to the 'noded' problem, i.e. $M = [1, +\infty)$, where we require $u(L; 1) = L^{-1} \sin^{-1}(L)$ and $\lim_{x \to +\infty} u(L; x) = \pi/2L$.

This 'noded' problem has the explicit solution as following [26]:

$$u(L; x) = L^{-1} \sin^{-1} \left\{ \frac{1 - \frac{(1-L)}{(1+L)} e^{2L(1-x)}}{1 + \frac{(1-L)}{(1+L)} e^{2L(1-x)}} \right\}$$

with the holomorphic energy

$$H_0(L; x) = \frac{L^2 x^2}{4} \left[\frac{1 + \sqrt{\frac{(1-L)}{(1+L)}} e^{L(1-x)}}{1 - \sqrt{\frac{(1-L)}{(1+L)}} e^{L(1-x)}} \right]^2.$$

3.3. ASYMPTOTICALLY FLAT SUBPLANES

Consider the surface Σ which is developing two nodes, i.e. we are pinching two nonhomotopic closed geodesics γ_0 and γ_1 on Σ to two points, say p_0 and p_1. We denote M_0 and M_1 their pinching neighborhoods, i.e. two cylinders described as M above centered at γ_0 and γ_1, respectively.

We define $M(l_0, l_1)$ be the surface with two of the Fenchel–Nielsen coordinates, namely, the hyperbolic lengths of γ_0 and γ_1, are l_0 and l_1, respectively. When we set the length of these two geodesics equal to l simultaneously, we will have a point $M(l) = M(l, l)$ in the Teichmüller space $\mathcal{T}_g$. Note that as l tends to zero, the surface is developing two nodes. At this point $M(l)$, we look at two directions. First, we fix γ_1 in M_1 having length l, and pinch γ_0 in M_0 into length $L = L(t)$, where $L(0) = l$. So the t-derivative of $\mu_0(t)$, the Beltrami differential of the resulting harmonic map, at $t = 0$ represents a tangent vector, $\dot{\mu}_0$, of Teichmüller space $\mathcal{T}_g$ at $M(l)$; we denote the resulting harmonic map by $W_0(t): M(l, l) \to M(L(t), l)$. Then we fix γ_0 in M_0 having length l, and pinch γ_1 in M_1 into length $L = L(t)$, so the t-derivative of $\mu_1(t)$ at $t = 0$ represents another tangent vector, $\dot{\mu}_1$, at $M(l)$; we denote the resulting harmonic map by $W_1(t): M(l, l) \to M(l; L(t))$. These two tangent vectors $\dot{\mu}_0$ and $\dot{\mu}_1$ will span a two-dimensional subspace of the tangent space $T_{M(l)}\mathcal{T}_g$ to $\mathcal{T}_g$, hence we obtain a family, Ω_l, of two-dimensional subspaces of the tangent space of $\mathcal{T}_g$.

PROPOSITION. *The Weil–Petersson sectional curvatures of Ω_l tend to 0 as $l \to 0$.*

It is immediate that this proposition implies our main theorem.

To begin the proof of the proposition, we look at the curvature tensor of the Weil–Petersson metric, which is given by [29]

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \left(\int_\Sigma D(\dot{\mu}_\alpha \dot{\bar{\mu}}_\beta) \dot{\mu}_\gamma \dot{\bar{\mu}}_\delta \, dA \right) + \left(\int_\Sigma D(\dot{\mu}_\alpha \dot{\bar{\mu}}_\delta) \dot{\mu}_\gamma \dot{\bar{\mu}}_\beta \, dA \right),$$

where $\mathrm{d}A$ is the area element. Then the curvature of Ω is R/g, where [29]

$$R = R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}}$$

and

$$\begin{aligned} g &= 4\langle \dot\mu_0, \dot\mu_0\rangle\langle \dot\mu_1, \dot\mu_1\rangle - 2|\langle \dot\mu_0, \dot\mu_1\rangle|^2 - 2\mathrm{Re}(\langle \dot\mu_0, \dot\mu_1\rangle)^2 \\ &= 4\langle \dot\mu_0, \dot\mu_0\rangle\langle \dot\mu_1, \dot\mu_1\rangle - 4|\langle \dot\mu_0, \dot\mu_1\rangle|^2. \end{aligned}$$

In the rest of this section, we will estimate terms in the above curvature formula in the model case, i.e. the surface is a pair of cylinders M_0 and M_1.

We denote $\phi_0(t)$ as the Hopf differential corresponding to the cylinder map $w_0(t)$ in M_0, and $\phi_1(t)$ as the Hopf differential corresponding to the cylinder map $w_1(t)$ in M_1. Here $w_0(t)\colon M_0(l) \to M_0(L(t))$ and $w_1(t)\colon M_1(l) \to M_1(L(t))$ are harmonic maps described as $w = u(x) + iy$ in Section 3.2. In M_1, we still use ϕ_0 to denote the Hopf differential corresponding to harmonic map $W_0(t)$, while in M_0, we still use ϕ_1 as the Hopf differential corresponding to harmonic map $W_1(t)$. We also denote μ_0 and μ_1 as the corresponding Beltrami differentials to ϕ_0 and ϕ_1.

We denote $a = a(l) = l^{-1}\sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1}\sin^{-1}(l)$. And in this paper, $A \sim B$ means $A/C < B < CA$ for some constant $C > 0$.

So in M_0, as in Section 3.2, we choose $L(t)$ so that $(\mathrm{d}/\mathrm{d}t)|_{t=0}c_0(t) = \mathrm{d}/(\mathrm{d}L)|_{L=l}c_0(l, L) = 4$. Here we recall that $c_0(l, L(t)) = L^2\csc^2(Lu)(u'2 - 1)$. We notice that $\dot c_0$ is never zero for all $l > 0$, otherwise we would have $\dot{\bar{w}}_z = 0$ as $\dot\phi = \sigma\dot{\bar{w}}_z$. Hence, w is a constant map by rotational invariance of the map. Thus, we have $\dot\phi_0 = (\mathrm{d}/\mathrm{d}t)|_{t=0}((1/4)c_0(t)) = 1$ in M_0.

Also, we see that $|\dot\phi_1||_{M_0} = \zeta(x, l)$ for $x \in [a, b]$, where $\zeta(x, l)$ satisfies that $\zeta(x, l) \leqslant C_1 x^{-4}$ for $x \in [a, \pi/2l]$, and $\zeta(x, l) \leqslant C_1(\pi/l - x)^{-4}$ for $x \in [\pi/2l, b]$, and $\zeta(x, 0)$ decays exponentially in $[1, +\infty]$. Here $C_1 = C_1(l)$ is positive and bounded as $l \to 0$. To see this, notice that $\dot\phi_1$ is holomorphic and $|\dot\phi_1|$ is positive, so $\log|\dot\phi_1|$ is harmonic in the cylinder M_0. Hence, we can express $\log|\dot\phi_1|$ in a Fourier series $\Sigma a_n(x)\exp(-iny)$, and we compute $0 = \Delta\log|\dot\phi_1| = \Sigma(a_n'' - n^2 a_n)\exp(-iny)$. We will see, in Section 3.4, that $\dot\phi_1$ is close to 0 in M_0. Hence, we conclude the properties ζ has. Similarly, we have $\dot\phi_1|_{M_1} = 1$ and $|\dot\phi_0||_{M_1} = \zeta(x, l)$ for $x \in [a, b]$.

Note that these estimates imply, informally, that most of the mass of $|\phi_0|$ resides in the thin part associated to γ_0, and most of the mass of $|\phi_1|$ resides in the thin part associated to γ_1.

Now in M_0, the corresponding Beltrami differential is

$$\dot\mu_0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\frac{w_{\bar z}}{w_z}\right) = \dot{\bar\phi}_0/\sigma$$

and $|\dot\mu_0|^2|_{M_0} = |\dot{\bar\phi}_0/\sigma|^2|_{M_0} = l^{-4}\sin^4(lx)$.

Also, in that same neighborhood M_0, $\dot\mu_1 = \dot{\bar\phi}_1/\sigma$. Hence

$$|\dot\mu_1|^2|_{M_0} = |\dot{\bar\phi}_1/\sigma|^2|_{M_0} = l^{-4}\sin^4(lx)\zeta^2(x, l).$$

Similarly, in M_1, we have

$$|\dot{\mu}_0|^2|_{M_1} = l^{-4}\sin^4(lx)\zeta^2(x,l),$$
$$|\dot{\mu}_1|^2|_{M_1} = l^{-4}\sin^4(lx).$$

LEMMA 2. $1/g = O(l^3)$.

We recall that $g = 4\langle\dot{\mu}_0,\dot{\mu}_0\rangle\langle\dot{\mu}_1,\dot{\mu}_1\rangle - 4|\langle\dot{\mu}_0,\dot{\mu}_1\rangle|^2$, and compute the asymptotics in l of each term. Using $|\dot{\mu}_0|^2|_{M_0} = l^{-4}\sin^4(lx)$, and noting that $a = a(l) = l^{-1}\sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1}\sin^{-1}(l)$, we have that

$$\begin{aligned}\langle\dot{\mu}_0,\dot{\mu}_0\rangle|_{M_0} &= \int_{M_0}|\dot{\mu}_0|^2\sigma\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\int_a^b|\dot{\mu}_0|^2\sigma\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\int_a^b l^{-2}\sin^2 lx\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{\pi}{2}l^{-3} - l^{-3}\sin^{-1}(l)\\ &\sim l^{-3}.\end{aligned}$$

And using $|\dot{\mu}_1|^2|_{M_0} = l^{-4}\sin^4(lx)\zeta^2(x,l)$, and $\zeta(x,l)\leqslant C_1x^{-4}$ for $x\in[a,\pi/2l]$, we have

$$\begin{aligned}\langle\dot{\mu}_1,\dot{\mu}_1\rangle|_{M_0} &= \int_{M_0}|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y\\ &= 2\int_0^1\int_a^{\pi/2l} l^{-4}\sin^4(lx)\zeta^2(x,l)\sigma\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant 2C_1^2\int_0^1\int_a^{\pi/2l} l^{-2}\sin^2(lx)x^{-8}\,\mathrm{d}x\,\mathrm{d}y\\ &= O(1).\end{aligned}$$

Also $\langle\dot{\mu}_0,\dot{\mu}_1\rangle = \int_\Sigma \dot{\mu}_0\bar{\dot{\mu}}_1\,dA$, hence,

$$\begin{aligned}\langle\dot{\mu}_0,\dot{\mu}_1\rangle|_{M_0} &= \int_{M_0}\dot{\mu}_0\dot{\mu}_1\sigma\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant C_1\int_{M_0} l^{-2}\sin^2(lx)x^{-4}\,\mathrm{d}x\,\mathrm{d}y\\ &= O(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\dot{\mu}_0,\dot{\mu}_0\rangle|_{M_1} &= O(1),\\ \langle\dot{\mu}_1,\dot{\mu}_1\rangle|_{M_1} &\sim l^{-3},\\ \langle\dot{\mu}_0,\dot{\mu}_1\rangle|_{M_1} &= O(1).\end{aligned}$$

Note that $g \geqslant (4\langle\dot{\mu}_0, \dot{\mu}_0\rangle\langle\dot{\mu}_1, \dot{\mu}_1\rangle - 4(\langle\dot{\mu}_0, \dot{\mu}_1\rangle))^2|_{M_0} \sim l^{-3}$, which completes the proof of Lemma 2.

From Lemma 2, we have

$$|R|/g = \mathrm{O}(|R|/(l^{-3})) = \mathrm{O}(|R|l^3). \tag{1}$$

Now we are left to estimate $|R|$. We have the following lemma.

LEMMA 3. $|R| \leqslant 4\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y$.

Note that $D = -2(\Delta - 2)^{-1}$ is self-adjoint, hence we have

$$\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \int_\Sigma D(|\dot{\mu}_1|^2)|\dot{\mu}_0|^2\sigma\,\mathrm{d}x\,\mathrm{d}y.$$

Therefore,

$$\begin{aligned}
R &= R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}} \\
&= 2\int_\Sigma D(\dot{\mu}_0\dot{\bar{\mu}}_1)\dot{\mu}_0\dot{\bar{\mu}}_1\sigma\,\mathrm{d}x\,\mathrm{d}y + 2\int_\Sigma D(\dot{\mu}_1\dot{\bar{\mu}}_0)\dot{\mu}_1\dot{\bar{\mu}}_0\sigma\,\mathrm{d}x\,\mathrm{d}y - \\
&\quad - \int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y - \int_\Sigma D(|\dot{\mu}_1|^2)|\dot{\mu}_0|^2\sigma\,\mathrm{d}x\,\mathrm{d}y - \\
&\quad - \int_\Sigma D(\dot{\mu}_0\dot{\bar{\mu}}_1)\dot{\mu}_1\dot{\bar{\mu}}_0\sigma\,\mathrm{d}x\,\mathrm{d}y - \int_\Sigma D(\dot{\mu}_1\dot{\bar{\mu}}_0)\dot{\mu}_0\dot{\bar{\mu}}_1\sigma\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\int_\Sigma D(\dot{\mu}_0\dot{\mu}_1)\dot{\mu}_1\dot{\mu}_0\sigma\,\mathrm{d}x\,\mathrm{d}y - 2\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The last equality follows from that here $\dot{\mu}_0$ and $\dot{\mu}_1$ are real functions. Now from Lemma 4.3 of [29], we have $|D(\dot{\mu}_0\dot{\mu}_1)| \leqslant |D(|\dot{\mu}_0|^2)|^{1/2}|D(|\dot{\mu}_1|^2)|^{1/2}$. Then an application of Hölder inequality shows that

$$\begin{aligned}
\left|\int_\Sigma D(\dot{\mu}_0\dot{\mu}_1)\dot{\mu}_1\dot{\mu}_0\,dA\right| &\leqslant \int_\Sigma |D(|\dot{\mu}_0|^2)|^{1/2}|D(|\dot{\mu}_1|^2)|^{1/2}\dot{\mu}_1\dot{\mu}_0\,\mathrm{d}A \\
&\leqslant \left(\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\,\mathrm{d}A\right)^{1/2}\left(\int_\Sigma D(|\dot{\mu}_1|^2)|\dot{\mu}_0|^2\,\mathrm{d}A\right)^{1/2} \\
&= \int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

So we will have

$$|R| \leqslant 4\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y, \tag{2}$$

which completes the proof of Lemma 3.

Therefore together with (1) and (2), to show the proposition, it is enough to show that $\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \mathrm{o}(l^{-3})$. In fact, LEMMA 4.

$$\int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma \, \mathrm{d}x \, \mathrm{d}y = \mathrm{O}(l^{-2})$$

First, from Lemma 4, we have

$$D(|\dot{\mu}_0|^2) = -2(\Delta - 2)^{-1}\frac{|\dot{\phi}_0|^2}{\sigma^2}.$$

We recall in M_0, the Hopf differential ϕ_0 is corresponding to the cylinder map $w_0(t) \colon (M_0, \sigma) \to (M_0, \rho(t))$, and $|\dot{\phi}_0| = 1$. As in Sections 2 and 3.1, we write $\mathcal{H} = (\rho(w(z))/\sigma(z))|w_z|^2$, therefore we can write $D(|\dot{\mu}_0|^2) = \ddot{\mathcal{H}}$. Then in the cylinder M_0, we have

$$(\Delta - 2)\ddot{\mathcal{H}} = -2\frac{|\dot{\phi}_0|^2}{\sigma^2} = -2l^{-4}\sin^4(lx). \tag{3}$$

A maximum principle argument implies that $\ddot{\mathcal{H}}$ is positive. This $\ddot{\mathcal{H}}$ converges to the holomorphic energy $\ddot{H}_0$ in the 'noded' problem (of Section 3.2) when x is fixed but sufficiently large in $[a, \pi/2l]$.

Therefore $\ddot{\mathcal{H}}$ is bounded on the compacta in [a,b]. Thus we can assume that $A_1(l) = \ddot{\mathcal{H}}(a) = \ddot{\mathcal{H}}(l^{-1}\sin^{-1} l) = \mathrm{O}(1) > 0$. Then $\ddot{\mathcal{H}}(x)$ solves the following differential equation:

$$(l^{-2}\sin^2(lx))\ddot{\mathcal{H}}'' - 2\ddot{\mathcal{H}} = -2l^{-4}\sin^4(lx) \tag{4}$$

with the conditions

$$\ddot{\mathcal{H}}(l^{-1}\sin^{-1} l) = A_1(l), \qquad \ddot{\mathcal{H}}'(\pi/2l) = 0.$$

Recall from Section 3.1 that all the odd order t-derivatives of $\mathcal{H}(t)$ vanish. Also notice that

$$J(x) = \frac{\sin^2(lx)}{2l^4}$$

is a particular solution to Equation (4). Hence, we can check, by the method of reduction of the solutions, the general solution to Equation (4) with the assigned conditons has the form

$$\ddot{\mathcal{H}}(l; x) = J(x) + A_2\cot(lx) + A_3(1 - lx\cot(lx)),$$

where coefficients $A_2 = A_2(l)$ and $A_3 = A_3(l)$ are constants independent of x and we can check, by substituting the solution into the assigned conditions, that they satisfy that

$$A_2 = \frac{\pi}{2}A_3 = \mathrm{O}(l^{-1}).$$

Noticing that $\zeta(x,l)\leqslant C_1x^{-4}$ in $[a,\pi/2l]$, we compute the following:

$$\begin{aligned}\int_{M_0}\ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1\int_a^b \ddot{\mathcal{H}}(x)(l^{-2}\sin^2(lx))\zeta^2(x,l)\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant 2C_1^2\Bigg(\int_0^1\int_a^{\pi/2l} J(x)l^{-2}\sin^2(lx)x^{-8}\,\mathrm{d}x\,\mathrm{d}y\,+\\ &\qquad +\int_0^1\int_a^{\pi/2l} A_2\cot(lx)l^{-2}\sin^2(lx)x^{-8}\,\mathrm{d}x\,\mathrm{d}y+\\ &\qquad +\int_0^1\int_a^{\pi/2l} A_3(1-lx\cot(lx))l^{-2}\sin^2(lx)x^{-8}\,\mathrm{d}x\,\mathrm{d}y\Bigg)\\ &= \mathrm{O}(l^{-2})+\mathrm{O}(l^{-2})+\mathrm{O}(l^{-2})\\ &= \mathrm{O}(l^{-2}).\end{aligned}$$

Therefore

$$\begin{aligned}\int_{M_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \ddot{\mathcal{H}}|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\\ &= \mathrm{O}(l^{-2}).\end{aligned}\tag{5}$$

Now let us look at the term $\int_{M_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y$. Note that in M_1, which we identify with $[a,b]\times[0,1]$, we have $D(|\dot{\mu}_0|^2)=\ddot{\mathcal{H}}$, here $\mu_0(t)$ and $\mathcal{H}$ come from the harmonic map $W_0(t)\colon M(l,l)\to M(L(t),l)$, where $|\dot{\mu}_0||_{M_1}=l^{-2}\sin^2(lx)\zeta(x,l)$, and $\mathcal{H}(t)$ solves

$$(l^{-2}\sin^2(lx))\ddot{\mathcal{H}}''-2\ddot{\mathcal{H}}=-2l^{-4}\sin^4(lx)\zeta^2(x,l)\tag{6}$$

with the conditions

$$\ddot{\mathcal{H}}(l^{-1}\sin^{-1}l)=B_1(l),\ddot{\mathcal{H}}(\pi/l-l^{-1}\sin^{-1}l)=B_2(l).$$

Here $B_1(l)$ and $B_2(l)$ are positive and bounded as $l\to 0$, since $\ddot{\mathcal{H}}$ converges to the holomorphic energy in the 'noded' problem (of Section 3.2, when $M_1=[1,\infty)$). We recall that $|\dot{\phi}_1||_{M_0}=\zeta(x,l)$, where $\zeta(x,l)\leqslant C_1x^{-4}$ for $x\in[a,\pi/2l]$ and $\zeta(x,l)\leqslant C_1(\pi/l-x)^{-4}$ for $x\in[\pi/2l,b]$.

Consider the equation

$$(l^{-2}\sin^2(lx))Y''-2Y=0\tag{7}$$

with the boundary conditions that satisfy

$$Y(l^{-1}\sin^{-1}l)=\mathrm{O}(1)\quad\text{and}\quad Y(\pi/l-l^{-1}\sin^{-1}l)=\mathrm{O}(1),\quad\text{as}\quad l\to 0.$$

We claim that there exists some $h=\mathrm{O}(l)$ such that $\ddot{\mathcal{H}}-h$ is a supersolution to (7) for $x\in[l^{-1/4},b-l^{-1/4}]$. To see this, we notice that $2|\dot{\phi}_0|^2|_{M_1}l^{-4}\sin^4(lx)$ decays rapidly as $x \to \pi/2l$ for small l. So there is a positive number $h = O(l)$ such that $2|\dot{\phi}_0|^2|_{M_1} l^{-4} \sin^4(lx) < 2h$ for $x \in [l^{-1/4}, b - l^{-1/4}]$. Now for $x \in [l^{-1/4}, b]$ we have

$$(l^{-2} \sin^2(lx))(\ddot{\mathcal{H}} - h)'' - 2(\ddot{\mathcal{H}} - h) = 2h - 2|\dot{\phi}_0|^2|_{M_1} l^{-4} \sin^4(lx) > 0. \tag{8}$$

Notice that for any constant λ, we have that if $Y(x)$ solves the equation (7), then so does $\lambda Y(x)$. So up to multiplying by a bounded constant, we have $Y|_{\partial M_1} > (\ddot{\mathcal{H}} - h)|_{\partial M_1}$. Hence $\ddot{\mathcal{H}} - h$ is a subsolution to (7) for $x \in [l^{-1/4}, b - l^{-1/4}]$. We can check the solutions to (7) have the form of

$$Y(l; x) = B_3 \cot(lx) + B_4(1 - lx \cot(lx)),$$

where constants $B_3 = B_3(l)$ and $B_4 = B_4(l)$ satisfy, from the boundary conditions for Equation (7), that

$$B_3 = \mathrm{O}(l) \quad \text{and} \quad B_4 = \mathrm{O}(l).$$

Therefore in $[l^{-1/4}, b - l^{-1/4}]$, We have $\ddot{\mathcal{H}} \leqslant h + Y(x)$. Now,

$$\begin{aligned}
\int_{M_1} Y(x)|\dot{\mu}_1|^2 \sigma \,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_a^b Y(x)(l^{-2} \sin^2(lx))\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \int_a^b B_3 \cot(lx)(l^{-2} \sin^2(lx))\,\mathrm{d}x\,\mathrm{d}y + \\
&\quad + \int_0^1 \int_a^b B_4(1 - lx \cot(lx))(l^{-2} \sin^2(lx))\,\mathrm{d}x\,\mathrm{d}y \\
&= \mathrm{O}(l^{-2}) + \mathrm{O}(l^{-2}) \\
&= \mathrm{O}(l^{-2}).
\end{aligned}$$

Also for $x \in [a, l^{-1/4}] \cup [b - l^{-1/4}, b]$, we apply maximum principle to Equation (6) and find that

$$\ddot{\mathcal{H}} \leqslant \max(l^{-4} \sin^4(lx_0)\zeta^2(x_0, l), \sup(\ddot{\mathcal{H}}|_{\partial(([a,l^{-1/4}] \cup [b - l^{-1/4}, b]) \times [0,1])}))$$

for some $x_0 \in [a, l^{-1/4}]$. Hence using the properties of $\zeta(x, l)$ in $[a, b]$, a direct computation shows that

$$\begin{aligned}
&\int_0^1 \int_a^{l^{-1/4}} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 \mathrm{d}A = \mathrm{O}(l^{-3/4}) = \mathrm{o}(l^{-1}), \\
&\int_0^1 \int_{b - l^{-1/4}}^b \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 \mathrm{d}A = \mathrm{O}(l^{-3/4}) = \mathrm{o}(l^{-1}).
\end{aligned}$$

With this and using $\ddot{\mathcal{H}} \leqslant h + Y(x)$ for $x \in [l^{-1/4}, b - l^{-1/4}]$, we have

$$
\begin{aligned}
\int_{M_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \mathrm{d}A &= \int_0^1 \int_a^b \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 \mathrm{d}A \\
&= \int_0^1 \int_{l^{-1/4}}^{b-l^{(-1/4)}} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 \mathrm{d}A + \int_0^1 \int_a^{l^{(-1/4)}} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 \mathrm{d}A + \\
&\quad + \int_0^1 \int_{b-l^{-1/4}}^{b} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 \mathrm{d}A \\
&\leqslant \int_0^1 \int_a^b (Y(x)+h)|\dot{\mu}_1|^2 \sigma \,\mathrm{d}x \,\mathrm{d}y + \mathrm{o}(l^{-1}) \\
&= \mathrm{O}(l^{-2}) + \mathrm{O}(l^{-2}) + \mathrm{o}(l^{-1}) = \mathrm{O}(l^{-2}). \qquad (9)
\end{aligned}
$$

Now combine the estimates of Lemma 3, (9) and (10), we will have

$$
\begin{aligned}
|R| &\leqslant 4 \int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma \,\mathrm{d}x \,\mathrm{d}y \\
&= 4 \left(\int_{M_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \mathrm{d}A + \int_{M_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \mathrm{d}A \right) \\
&= \mathrm{O}(l^{-2}). \qquad (10)
\end{aligned}
$$

Now with Lemma 2 we have

$$|R|/g = \mathrm{O}(l^{-2})l^3 = \mathrm{O}(l),$$

namely, $|R|/g$ tends to zero as we pinch the geodesics.

3.4. CONSTRUCTED MAPS

In Section 3.5, we estimated the terms in the curvature formula in the model case, i.e. the surface is a pair of cylinders. Essentially, in each pinching neighborhood M_i, we used the cylinder map w_i instead of the harmonic map W_i during the computation, for $i = 0, 1$. Now we are in the general setting, i.e. the surface Σ develops two nodes. In this section, we will construct a family of maps G_i to approximate the harmonic map W_i, and the essential parts of this family are the identity map of the surface restricted in the noncylindral part and the cylinder map in each pinching neighborhood. We will also show that this constructed family G_i is reasonably close to the harmonic maps W_i, for $i = 0, 1$; hence we can use the estimates we obtained in the previous subsection to the general situation.

We recall some of the notations from previous subsections. We still set M_0 and M_1 to be the pinching neighborhoods of the nodes p_0 and p_1, respectively. Also $W_0(t)$ is the harmonic map corresponding to fixing γ_1 in M_1 having length l, and pinching γ_0 in M_0 into length $L = L(t)$, where $L(0) = l$. Similarly $W_1(t)$ is the harmonic map corresponding to fixing γ_0 in M_0 into length l, and pinching γ_1 in M_1 having length $L = L(t)$. Let $w_0(t)$ and $w_1(t)$ be cylinder maps in the model case, we want to show that $W_0(t)$ is close to $w_0(t)$ in M_0 and is close to identity map outside of M_0.

We denote subsets

$$\Sigma_0=\{p\in\Sigma:\mathrm{dist}(p,\partial M_0)>1\}\quad\text{and}\quad \Sigma_1=\{p\in\Sigma:\mathrm{dist}(p,\partial M_1)>1\}.$$

Define the 1-tube of ∂M_0 as

$$B(\partial M_0,1)=\{p\in\Sigma:\mathrm{dist}(p,\partial M_0)\leqslant 1\}$$

and the 1-tube of ∂M_1 as

$$B(\partial M_1,1)=\{p\in\Sigma:\mathrm{dist}(p,\partial M_1)\leqslant 1\}.$$

We can construct a $C^{2,\alpha}$ map $G_0:\Sigma\to\Sigma$ such that

$$G_0(p)=\begin{cases} w_0(t)(p), & p\in M_0\cap\Sigma_0,\\ p, & p\in(\Sigma_0\backslash M_0),\\ g_t(p), & p\in B(\partial M_0,1),\end{cases}$$

where $g_t(p)$ in $B(\partial M_0,1)$ is constructed so that it satisfies

(1) $g_t(p)=p$ for $p\in\partial(\Sigma_0\backslash(M_0\cup B(\partial M_0,1)))$, and $g_t(p)=w_0(t)(p)$ for $p\in\partial(M_0\cap\Sigma_0)$,
(2) g_t is the identity map when $t=0$,
(3) g_t is smooth and the tension of g_t is of the order $\mathrm{O}(t)$.

We note that G_0 consists of three parts. It is the cylinder map of M_0 in $M_0\cap\Sigma_0$, the identity map in $\Sigma_0\backslash M_0$, and a smooth map in $B(\partial M_0,1)$. Among three parts of the constructed map $G_0(t)$, two of them, the identity map and the cylinder map, are harmonic hence have zero tension; thus the tension of $G_0(t)$ is concentrated in $B(\partial M_0,1)$. From Section 3.2, for the cylinder map $w_0=u(x)+iy$, we have $u'=\sqrt{1+c_0(t)L^{-2}\sin^2 Lu}$, where $c_0(0)=0,\dot{c}_0(0)=4$,. Hence, for $x\in[l^{-1}\sin^{-1}(l),l^{-1}\sin^{-1}(l)+1]$,

$$\begin{aligned}&w_{0,z}(x,y)=\tfrac{1}{2}(u'(x)+1)=\tfrac{1}{2}((1+\mathrm{O}(1)c_0(t))^{1/2}+1)=1+\mathrm{O}(1)t+\mathrm{O}(t^2),\\ &|w_{0,z}(x,y)-1|=\mathrm{O}(t)\to 0,\ (t\to 0),\\ &|w_{0,z\bar z}(x,y)|=|\tfrac{1}{4}u''(x)|=\mathrm{O}(|L\cot(Lu)(u'^2-1)|)=\mathrm{O}(t).\end{aligned}$$

Thus we can require that $|g_{t,z}|\leqslant C_2t$ and $|g_{t,z\bar z}|\leqslant C_2t$, and the constant $C_2=C_2(t,l)$ is bounded in both t and l since the coefficient of t for $c_0(t)$ is bounded for small t and small l. With the local formula of the tension in Section 2, we have $\tau(G_0(t))$, the tension of $G_0(t)$ is of the order $\mathrm{O}(t)$.

Similarly, we have a $C^{2,\alpha}$ map $G_1:\Sigma\to\Sigma$. Note that these constructed maps G_0 and G_1 are not necessarily harmonic.

Now we are about to compare the constructed family $G_0(t)$ and the family of the harmonic maps $W_0(t)$. To do this, we consider the function $Q_0=\cosh(\mathrm{dist}(W_0,G_0))-1$.

LEMMA 5. $\mathrm{dist}(W_0, G_0) \leqslant C_3 t$ *in* $B(\partial M_0, 1)$, *where the constant* $C_3 = C_3(t, l)$ *is bounded for small* t *and* l.

First, we want to show that Q_0 is a C^2 function. Notice that both the harmonic map $W_0(t)$ and the constructed map $G_0(t)$ are the identity map when $t = 0$, and both families vary smoothly without changing homotopy type in t for sufficiently small $|t|$ [8]. For all $l > 0$, and for any $\varepsilon > 0$, there exists a δ such that for $|t| < \delta$, we have $|W_0(t) - W_0(0)| < (\varepsilon/2)$ and $|G_0(t) - G_0(0)| < (\varepsilon/2)$. Therefore, the triangular inequality implies that $|W_0(t) - G_0(t)| < \varepsilon$. Since l is positive, the Collar theorem [6] implies that the surface has positive injectivity radius r bounded below, and we choose our $\varepsilon \ll r$, so Q_0 is well defined and smooth.

We follow an argument in [12]. For any unit $v \in T^1(B(\partial M_0, 1))$, the map G_0 satifies the inequality $|\|\mathrm{d}G_0(v)\| - 1| = \mathrm{O}(t)$, hence $|\mathrm{d}G_0(v)|^2 > 1 - \varepsilon_0$ where $\varepsilon_0 = \mathrm{O}(t)$, then we find that for any $x \in \Sigma$,

$$\begin{aligned}\Delta Q_0 &\geqslant \min\{|dG_0(v)|^2 : dG_0(v) \perp \gamma_x\} Q - \\ &\quad - \langle \tau(G_0), \nabla d(\bullet, W_0)|_{G_0(x)} \rangle \sinh(\mathrm{dist}(W_0, G_0)),\end{aligned} \tag{11}$$

where γ_x is the geodesic joining $G_0(x)$ to $W_0(x)$ with initial tangent vector $-\nabla d(\bullet, W_0)|_{G_0(x)}$ and terminal tangent vector $\nabla d(G_0(x), \bullet)|_{W_0(x)}$. If $G_0(t)$ does not coincide with $W_0(t)$ on $B(\partial M_0, 1)$, we must have all maxima of $Q_0(t)$ on the interior of $B(\partial M_0, 1)$, at any such maximum, we apply the inequality $|dG_0(v)|^2 > 1 - \varepsilon_0$ to (12) to find

$$0 \geqslant \Delta Q_0 \geqslant (1 - \varepsilon_0) Q_0 - \tau(G_0)(\sinh(\mathrm{dist}(W_0, G_0))$$

so that at a maximum of Q_0, we have

$$Q_0 \leqslant \frac{\tau(G_0) \sinh \mathrm{dist}(W_0, G_0)}{(1 - \varepsilon_0)}.$$

We notice that Q_0 is of the order $\mathrm{dist}^2(W_0, G_0)$ and $\sinh \mathrm{dist}(W_0, G_0)$ is of the order $\mathrm{dist}(W_0, G_0)$, this implies that $\mathrm{dist}(W_0, G_0)$ is of the order $O(t)$ in $B(\partial M_0, 1)$, which completes the proof of Lemma 5.

Remark. Lemma 5 implies that $Q_0(t)$ is of the order $\mathrm{O}(t^2)$ in $B(\partial M_0, 1)$. Note that $B(\partial M_0, 1)$ contains the boundary of the cylinder $M_0 \cap \Sigma_0$, which we identify with $[a + 1, b - 1] \times [0, 1]$, where, again, $a = a(l) = l^{-1} \sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1} \sin^{-1}(l)$. While in the cylinder $M_0 \cap \Sigma_0$, we have the inequality

$$\begin{aligned}\Delta Q_0 &\geqslant (1 - \varepsilon_0) Q_0 - \tau(G_0)(\sinh(\mathrm{dist}(W_0, G_0)) \\ &= (1 - \varepsilon_0) Q_0 - \tau(G_0)(\tanh(\mathrm{dist}(W_0, G_0))(1 + Q_0) \\ &= (1 - \varepsilon_0 - \tau(G_0)) Q_0 - \tau(G_0)(\tanh(\mathrm{dist}(W_0, G_0)) \\ &\geqslant 1/2 Q_0 - C_4 t^2,\end{aligned}$$

where the constant C_4 is bounded for small t and l. Therefore, we find that $Q_0(z,t)$ decays rapidly in $z=(x,y)$ for x close enough to $\pi/2l$. Hence we can assume that $\mathrm{dist}(W_0, G_0)$ is at most of order $C't$ in $[a+1, b-1]$, here $C'=C'(x,l)$ is no greater that $C_5 x^{-2}$ for $x\in[a+1,\pi/2l]$, and no greater than $C_5(\pi/l-x)^{-2}$ for $x\in[\pi/2l, b-1]$, where C_5 is bounded for small t and l. Both maps W_0 and G_0 are harmonic in $M_0\cap\Sigma_0$, so they are also C^1 close [8], i.e. we have $|W_{0,\bar z}-G_{0,\bar z}|\leqslant C_5 x^{-2} t$ for small t and l, when $x\in[a+1,\pi/2l]$. Thus we see that $|\dot W_{0,\bar z}-\dot G_{0,\bar z}|=C_5x^{-2}$, for $x\in[a+1,\pi/2l]$. Also, $|\dot W_{0,\bar z}-\dot G_{0,\bar z}|=C_5(\pi/l-x)^{-2}$, for $x\in[\pi/2l, b-1]$.

As before we denote ϕ_0 and ϕ_1 as Hopf differentials corresponding to harmonic maps $W_0(t)$ and $W_1(t)$, respectively. We also denote μ_0 and μ_1 as the corresponding Beltrami differentials in $M_0\cap\Sigma_0$ and $M_1\cap\Sigma_1$, respectively. Write $\phi_{G_0}=\rho G_{0,z}(t)\bar G_{0,z}(t)$ and $\phi_{G_1}=\rho G_{1,z}(t)\bar G_{1,z}(t)$. Notice that in $M_0\cap\Sigma_0$, map G_0 is the cylinder map hence harmonic, so ϕ_{G_0} is the Hopf differential corresponding to G_0. When $t=0$ we have $W_0=G_0=$ identity and $\rho=\sigma$, hence we can differentiate $\phi_{G_0}=\rho G_{0,z}(t)\bar G_{0,z}(t)$ in t at $t=0$, and find that

$$|\dot\phi_{G_0}-\dot\phi_{W_0}|=\sigma|\dot W_{0,\bar z}-\dot G_{0,\bar z}|\leqslant C_5 l^2 x^{-2}\csc^2(lx)=\mathrm{O}(1)$$

for $x\in[a+1,\pi/2l]$, and

$$|\dot\phi_{G_0}-\dot\phi_{W_0}|\leqslant C_5 l^2(\pi/l-x)^{-2}\csc^2(lx)=\mathrm{O}(1)$$

for $x\in[\pi/2l, b-1]$.
Therefore we have proved

LEMMA 6. $|\dot\phi_{G_0}-\dot\phi_{W_0}|=\mathrm{O}(1)$ *for* $x\in[a+1,b-1]$.

Similarly we have $|\dot\phi_0-\dot\phi_{G_0}||_{M_1\cap\Sigma_1}=\mathrm{O}(1)$, and $|\dot\phi_1-\dot\phi_{G_1}||_{M_0\cap\Sigma_0}=\mathrm{O}(1)$, also $|\dot\phi_1-\dot\phi_{G_1}||_{M_1\cap\Sigma_1}=\mathrm{O}(1)$.

3.5. PROOF OF THE MAIN THEOREM

In Section 3.4, we constructed families of maps and found that they are reasonably close to the corresponding families of harmonic maps between surfaces. Now we are ready to adapt the estimates in the model case to the general setting, and prove the proposition in this situation, which will imply our main theorem.

Recall from Section 3.3 that the sectional curvature is represented by R/g. With Lemma 2 in the model case, the inequality $g\geqslant g_{M_0\cap\Sigma_0}\sim l^{-3}$ still holds. Hence $|R|/g=\mathrm{O}(l^3)|R|$, so it will be sufficient to show that $|R|=\mathrm{o}(l^{-3})$. From Lemma 3, we have $|R|\leqslant 4\int_\Sigma D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y$, so we will show that Lemma 4 still holds, which implies desired curvature estimate immediately.

Now we are about to estimate $\int_\Sigma D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y$, which breaks into three integrals as follows:

$$\int_{\Sigma} D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \int_{M_0\cap\Sigma_0} D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y + \\ + \int_{M_1\cap\Sigma_1} D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y + \\ + \int_{K} D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y, \quad (12)$$

where K is the compact set disjoint from $(M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1)$.

For the third integral, from the previous discussion, because of the convergence of the harmonic maps to the harmonic maps of 'noded' problem, we have both $|\dot\mu_0|$ and $|\dot\mu_1|$ are bounded. The maximum principle implies that $D(|\dot\mu_0|^2)=\ddot{\mathcal{H}}\leqslant \sup\{|\dot\mu_0|^2\}$. Note that K is compact, hence we have the third integral is the order of O(1).

From Lemma 1, we have $(\Delta-2)\ddot{\mathcal{H}}=-2(|\dot\phi_0|^2/\sigma^2)$, so we can rewirte (12) as

$$\int_{\Sigma} D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\mathrm{d}x\mathrm{d}y \\ = \int_{M_0\cap\Sigma_0} \ddot{\mathcal{H}}|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y + \int_{M_1\cap\Sigma_1} \ddot{\mathcal{H}}|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}(1). \quad (13)$$

Now we will look at the first and the second integrals in (13). Recall that $(\Delta-2)\ddot{\mathcal{H}}^G=-2(|\dot\phi_0^G|^2/\sigma^2)$, where $\mathcal{H}^G$ is the holomorphic energy corresponding to the model case when the harmonic map is the cylinder map, and ϕ_0^G is the quadratic differential corresponding to the constructed map G_0. We also denote μ_0^G to be the Beltrami differential corresponding to ϕ_0^G. We assign similar meanings for ϕ_1^G and μ_1^G. Also recall that $|\dot\phi_0-\dot\phi_0^G|=\mathrm{O}(1)$ and $|\dot\phi_1-\dot\phi_1^G|=\mathrm{O}(1)$ in $(M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1)$, here O(1) is bounded in l for small l. So we can set some $\lambda=\mathrm{O}(1)$ (bounded in l for small l) such that $|\dot\phi_0|^2<\lambda^2|\dot\phi_0^G|^2$ and at the boundary of $(M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1)$ satisfies $\ddot{\mathcal{H}}<\lambda\ddot{\mathcal{H}}^G$. For example, we can take $\lambda=1+\max_{\partial K}((\ddot{\mathcal{H}}/\ddot{\mathcal{H}}^G), |\dot\phi_0|/|\dot\phi_0^G|)$, this $\lambda=\mathrm{O}(1)$ because at

$$\partial K=\partial((M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1)),$$

both $\ddot{\mathcal{H}}$, $\ddot{\mathcal{H}}^G$, and $|\dot\phi_0|/|\dot\phi_0^G|$ are bounded. Therefore,

$$(\Delta-2)(\ddot{\mathcal{H}}-\lambda\ddot{\mathcal{H}}^G)=2\frac{\lambda^2|\dot\phi_0^G|^2-|\dot\phi_0|^2}{\sigma^2}>0.$$

So $(\ddot{\mathcal{H}}-\lambda\ddot{\mathcal{H}}^G)$ is a subsolution to the equation $(\Delta-2)Y=0$ whose solutions have the form of $Y(l;x)=B_3\cot(lx)+B_4(1-lx\cot(lx))$, where constants B_3 and B_4 satisfy that $B_3=\mathrm{O}(l)$ and $B_4=\mathrm{O}(l)$. Hence, in $(M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1)$, we have

$\ddot{\mathcal{H}} \leqslant \lambda\ddot{\mathcal{H}}^M + B_3 \cot(lx) + B_4(1 - lx\cot(lx))$. We apply this into (13) and find that,

$$\begin{aligned}\int_\Sigma D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y &= \int_{(M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1)} \ddot{\mathcal{H}}|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}(1)\\ &\leqslant \int_{(M_0\cap\Sigma_0)} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(|\dot\mu_1|^2)\sigma\,\mathrm{d}x\,\mathrm{d}y + \\ &\quad + \int_{(M_1\cap\Sigma_1)} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(|\dot\mu_1|^2)\sigma\,\mathrm{d}x\,\mathrm{d}y + \mathrm{O}(1)\\ &\leqslant \int_{M_0\cap\Sigma_0} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(2|\dot\mu_1|^2 + 2|\dot\mu_1 - \dot\mu_1^G|^2)\mathrm{d}A + \\ &\quad + \int_{M_1\cap\Sigma_1} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(2|\dot\mu_1|^2 + 2|\dot\mu_1 - \dot\mu_1^G|^2)\mathrm{d}A + \\ &\quad + \mathrm{O}(1).\end{aligned}$$

Recalling the computation in Section 3.3, and $|\dot\mu_1 - \dot\mu_1^G| = |\dot\phi_1 - \dot\phi_1^G|/\sigma$, where $|\dot\phi_1 - \dot\phi_1^G| \leqslant C_5 l^2 x^{-2}\csc^2(lx)$ for $x \in [a+1, \pi/2l]$, we have the following:

$$\begin{aligned}&\int_{M_0} (\lambda\ddot{\mathcal{H}}^G)|\dot\mu_1^G|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(l^{-2}),\\ &\int_{M_1} (\lambda\ddot{\mathcal{H}}^G)|\dot\mu_1^G|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(l^{-2}),\\ &\int_{M_0} Y(l,x)|\dot\mu_1^G|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(1),\\ &\int_{M_1} Y(l,x)|\dot\mu_1^G|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(l^{-2}),\\ &\int_{M_0\cap\Sigma_0} (\ddot{\mathcal{H}}^G + Y(l;x))(|\dot\mu_1 - \dot\mu_1^G|^2)\mathrm{d}A = \mathrm{o}(l^{-2}),\\ &\int_{M_1\cap\Sigma_1} (\ddot{\mathcal{H}}^G + Y(l;x))(|\dot\mu_1 - \dot\mu_1^G|^2)\mathrm{d}A = \mathrm{o}(l^{-2}).\end{aligned}$$

Therefore $\int_\Sigma D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(l^{-2})$, with Lemma 3, we have $|R| \leqslant 4\int_\Sigma D(|\dot\mu_0|^2)|\dot\mu_1|^2\sigma\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(l^{-2})$. Apply this into the curvature formula, we have $|R|/g = \mathrm{O}(l) \to 0$ as $l \to 0$, which completes the proof of Proposition.

For the case of punctured surfaces, the existence of a harmonic diffeomorphism between punctured surfaces has been investigated by Wolf [26] and Lohkamp [14]. In particular, Lohkamp [14] showed that a homeomorphism between punctured surfaces is homotopic to a unique harmonic diffeomorphism with finite energy, and the holomorphic quadratic differential corresponding to the harmonic map in the homotopy class of the identity is a bijection between Teichmüller space of punctured surfaces and the space of holomorphic quadratic differentials.

In this case, the set $\Sigma\backslash((M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1))$ is no longer compact. Let K_0 be a compact surface with finitely many punctures, and $\{K_m\}$ be a compact exhaustion of K_0. We now estimate $\int_{K_0} D(|\dot\mu_0|^2)|\dot\mu_1|^2\mathrm{d}A$. Let $H(t)$ be the holomorphic energy corresponding to the harmonic map $w(t)\colon (K_0, \sigma) \to (K_0, \rho(t))$, then $H(t)$ is bounded above and below, and has nodal limit 1 near the punctures [26], hence both $|\dot{\mu}_0|^2$ and $|\dot{\mu}_1|^2$ have the order o(1) near the punctures. To see this, we consider K_0 as the union of K_m and disjoint union of finitely many punctured disks, each equipped hyperbolic metric $|dz|^2/(z^2 \log^2 z)$. Then $|\dot{\mu}_0| = \mathrm{O}(|z|\log^2 z) \to 0$ as z tends to the puncture, since the quadratic differential has a pole of at most the first order. A similar result holds for $|\dot{\mu}_1|$. We notice that ∂K_0 is the boundary of the cylinders, where the harmonic maps converge to a solution to the 'noded' problem as $l \to 0$, hence $D(|\dot{\mu}_0|^2) = \ddot{H}(t)$ is bounded on ∂K_0. Therefore, we apply Omori–Yau maximum principle [18,31] to $(\Delta - 2)\ddot{H} = -2|\dot{\mu}_0|^2$ on K_0 and obtain that $\sup(D(|\dot{\mu}_0|^2)) \leqslant \max(\sup(|\dot{\mu}_0|^2), \max(D(|\dot{\mu}_0|^2))|_{\partial K_0}) = O(1)$. Hence, we have

$$\begin{aligned}\int_{K_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA &\leqslant \int_{K_0} \sup(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA\\ &\leqslant \mathrm{O}(1)\mathrm{O}(1)\mathrm{Vol}(K_0)\\ &= \mathrm{O}(1).\end{aligned}$$

In other words, our proof carries over to the punctured case, which completes the proof of our Main theorem.

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