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COHERENT CONFIGURATIONS

Part I: Ordinary Representation Theory

1. INTRODUCTION

Coherent configurations abstract certain features of the combinatorial structure induced in a finite set by a group acting on it. This is the first of a series of papers in which we undertake a systematic investigation of this class of combinatorial structures. Preliminary versions of some of the results have appeared in [11], [13] and [14]. The sections of the present paper are as follows:

1. Introduction.
2. Coherent configurations.
3. The Schur relations.
4. Commutative configurations.
5. Computation of multiplicities.
6. The Krein condition.
7. The centralizer algebra.
8. Common constituents.
9. Intersection matrices.
10. Coherent partitions and refinements.
11. Fusion.
12. Configurations of small rank.

In Section 2, after giving the basic definitions and some elementary consequences, we introduce two fundamental algebraic structures associated with a coherent configuration, namely, the boolean algebra of *admissable relations* and the *adjacency ring*. The action of a group on a finite set induces the structure of a coherent configuration in the set, and in this situations, which we refer to as the *group case*, the admissable relations are the invariant binary relations in the sense of Wielandt [20] and the adjacency ring is the centralizer ring of the permutation representation. Thus coherent configurations provide a combinatorial setting for centralizer ring theory of permutation representations on the one hand, and the possibility of applying methods of centralizer ring theory to combinatorics on the other. In practice, such processes as *fusion* (see Section 10) may produce coherent configurations

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which are not afforded by group actions from ones which are. For such reasons, as is to be expected, the development of a general combinatorial theory broadens the scope of applications to group theory.

One of the elementary consequences of the axioms is the decomposition into *fibers* which exhibits a coherent configuration as a collection of *homogeneous* configurations pasted together in a 'coherent' fashion. In the group case the fibers are the orbits and homogeneity is equivalent to transitivity. Homogeneous configurations with trivial pairing are equivalent to *association schemes* as defined by Bose and Mesner in [2], and pairing coincides with the standard notion in the group case.

The main focus of the present paper is on the adjacency ring, and the *intersection numbers* which turn out to be structure constants for the adjacency ring. Sections 3 through 8 are devoted to the ordinary representation theory, i.e., to the absolutely irreducible representations, of the adjacency ring. A central role is played by the *standard character*. The multiplicities of the irreducible characters in this character are determined by the intersection numbers, and possibilities for computing them are discussed in Section 4. The *Schur relations* are established and applied to obtain general versions of results of Frame [7] and Wielandt [19] in Section 3, and in Section 6 to obtain a general version of the *Krein condition* of L. L. Scott Jr. [16]. Besides being configurational versions, our results are a little more general in the group case than the original ones, dropping the assumption of transitivity in Frame's result and assumptions of transitivity and multiplicity freeness of the permutation character in the results of Wielandt and Scott. Further extensions to *weighted adjacency algebras* will be given in Part II where we study *weighted configurations.*

As an application of the present version of the Krein condition we can show that if a generalized quadrangle or octagon has $s+1$ points on each line and $t+1$ lines through each point, with $t>1$, then $s\leq t^2$. We postpone this application to Part III where we study homogeneous configurations (and where the admissable relations play a more central role). This inequality for generalized quadrangles was proved by a completely different method in [12].

The centralizer algebra of the adjacency algebra is used in Section 8 to relate the representation theory of a coherent configuration to that of the configurations based on its fibers.

The intersection numbers are arranged into matrices in Section 9, called the *intersection matrices,* which coincide in the appropriate situation with the intersection matrices introduced in [10]. Sections 10 and 11 are concerned with the questions of when the processes of *refinement* and *fusion* produce new coherent configurations.

Finally in Section 12 we take a brief look at configurations of small *rank,* indicating connections with strongly regular graphs, projective designs, partial geometries and families of linked projective designs. A survey of configurations of rank 4 is the subject of Part IV.

2. Coherent configurations

If X is a finite nonempty set and $\mathcal{O}$ is a set of nonempty binary relations on X, so that $\mathcal{O}$ is a subset of the power set $\mathscr{P}(X^2)$ of the cartesian square of X, then we call $(X, \mathcal{O})$ the *configuration based on* X with $\mathcal{O}$ as its set of *basic relations*. We call $n=|X|$ the *degree* and $r=|\mathcal{O}|$ the *rank*.

In case X is a G-space and $\mathcal{O}$ is the totality of G-orbits in X^2 (under componentwise action) we say that $(X, \mathcal{O})$ is *afforded* by G, or by the action of G on X, or by the G-space X, and refer to this situation as the *group case*.

The members of the Boolean subalgebra R of $\mathscr{P}(X^2)$ generated by $\mathcal{O}$ are the *admissable relations* of $(X, \mathcal{O})$. In the group case this terminology coincides with that of Wielandt [20].

An *isomorphism* of a configuration $(X, \mathcal{O})$ onto a configuration $(X_1, \mathcal{O}_1)$ is a bijection of X onto X_1 which induces a bijection of $\mathcal{O}$ onto $\mathcal{O}_1$. Thus the automorphism group $\mathrm{Aut}(X, \mathcal{O})$ of $(X, \mathcal{O})$ is the subgroup of the symmetric group Σ_X on X consisting of those permutations which induce permutations of $\mathcal{O}$. The action of this group on $\mathcal{O}$ gives an exact sequence

$$1 \to \mathrm{Aut}^*(X, \mathcal{O}) \to \mathrm{Aut}(X, \dot{\mathcal{O}}) \to \Sigma_{\mathcal{O}}.$$

We refer to $\mathrm{Aut}^*(X, \mathcal{O})$ as the group of *strict automorphisms* of $(X, \mathcal{O})$, namely, $\mathrm{Aut}^*(X, \mathcal{O})$ is the group of those permutations of X which act trivially on $\mathcal{O}$. If $H \leq \mathrm{Aut}^*(X, \mathcal{O})$ and H affords $(X, \mathcal{O})$, then $N(H) \leq \mathrm{Aut}(X, \mathcal{O}) = N(\mathrm{Aut}^*(X, \mathcal{O}))$ where $N(H)$ denotes the normalizer in Σ_X of H.

In the group case, G acts as a group of strict automorphisms of $(X, \mathcal{O})$.

A configuration $(X_0, \mathcal{O}_0)$ is a *subconfiguration* of $(X, \mathcal{O})$ if $X_0 \subseteq X$ and every member of $\mathcal{O}_0$ is a subset of some member of $\mathcal{O}$. The configuration afforded by a group G acting as a group of strict automorphism of $(X, \mathcal{O})$ is a subconfiguration of $(X, \mathcal{O})$ in this sense. The *full subconfiguration* $[X_0]$ of $(X, \mathcal{O})$ based on a subset $X_0 \neq \emptyset$ of X is $(X_0, \{f \cap X_0^2 \mid f \in \mathcal{O} \text{ and } f \cap X_0^2 \neq \emptyset\})$.

A $(g_1, g_2, \ldots, g_s)$-*path from* x *to* y, where $g_1, \ldots, g_s \in \mathscr{P}(X^2)$ and $x, y \in X$, is a $(s+1)$-tuple $(x_0, x_1, \ldots, x_s) \in X^{s+1}$ such that $x_0 = x$, $x_s = y$, and $(x_{i-1}, x_i) \in g_i$, $i = 1. 2, \ldots, s$. Such a path is *closed* if $x = y$. A (g)-path, $g \in \mathscr{P}(X^2)$ is therefore just an element of g, that is, an edge in the graph (X, g), and is often refered to as a *g-edge*.

If K is a commutative ring and X and Y are finite nonempty sets, we write $\mathrm{Mat}_K(X, Y)$ for the K-module of matrices with coefficients in K having rows indexed by X and columns by Y. That is, $\mathrm{Mat}_K(X, Y)$ is the totality of maps $\phi: X\times Y\to K$ with the structure of a K-module according to the pointwise operations. We write $\mathrm{Mat}_K X$ for $\mathrm{Mat}_K(X, X)$ regarded as a K-algebra with respect to matrix multiplication. For $\phi, \psi\in\mathrm{Mat}_K X$, $\phi\psi$ denotes the matrix product and $\phi\circ\psi$ denotes the pointwise (i.e., *Hadamard*) product.

For $F\subseteq X^2$, Φ_F will denote the characteristic function of F, or, what is the same thing, the adjacency matrix of the graph (X, F). Thus $\Phi_F\in\mathrm{Mat}_{\mathbf{Z}} X$ and $\Phi_F(x, y)=1$ or 0 according as $(x, y)\in F$ or X^2-F.

The class of configurations to be studied here is defined by the following four axioms.

(I) $\mathcal{O}$ is a partition of X^2.

(II) If $f\in\mathcal{O}$ and $f\cap I\neq\emptyset$, where $I=I_X=\{(x, x)\mid x\in X\}$, then $f\subseteq I$.

(III) $f\in\mathcal{O}$ implies $f^{\cup}=\{(y, x)\mid (x, y)\in f\}\in\mathcal{O}$.

(IV) For $f, g, h\in\mathcal{O}$ and $(x, y)\in h$, the number a_{fgh} of (f, g)-paths from x to y is independent of the choice of x and y.

We call a configuration satisfying axioms (I) through (IV) *coherent*, and say that $\mathcal{O}$ is a *coherent set of relations* on X.

For simplicity we will assume axiom (I) throughout the following discussion of the axioms. Then R, regarded as a vector space over $GF(2)$ (with symmetric difference as addition) has $\mathcal{O}$ as basis and so has dimension r. The submodule $\Gamma=\{\phi: X^2\to \mathbf{Z}\mid \phi|f$ is constant for all $f\in\mathcal{O}\}$ of $\mathrm{Mat}_{\mathbf{Z}} X$ is a free Abelian group of rank r with $\mathcal{B}=\{\Phi_f\mid f\in\mathcal{O}\}$ as basis. We call $\mathcal{B}$ the *standard basis* of Γ. If K is any commutative ring with identity element, we have an obvious homomorphism of Γ onto a subgroup of $\mathrm{Mat}_K X$ which spans the free K-submodule $K\Gamma=\{\phi: X^2\to K\mid \phi\mid f$ is constant for all $f\in\mathcal{O}\}$ of rank r.

For $f, g, \in\mathcal{O}$, $\Phi_f\circ\Phi_g=\delta_{fg}\Phi_f$ (where $\circ$ indicates the pointwise, or Hadamard, product) so Γ^0 is a subring of $(\mathrm{Mat}_{\mathbf{Z}} X)^0$ where the superscript indicates that the operations are the pointwise ones.

In terms of the adjacency matrices, axiom (I) is equivalent to

(2.1) $\sum_{\phi\in\mathcal{B}}\phi=\sum_{f\in\mathcal{O}}\Phi_f=\Phi$, *the 'all 1' matrix, i.e.*, $\Phi(x, y)=1$ *for all* $x, y\in X$.

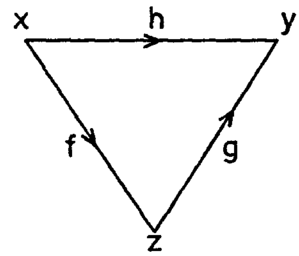
Axiom (II) is equivalent to the condition that $I\in R$, or, since Φ_I is the identity matrix, to the condition

(2.2) *The identity matrix is in* Γ.

For $F\in X^2$, $F^{\cup}=\{(y, x)\mid (x, y)\in F\}$ is called the *converse* of F. Clearly $\Phi_{F^{\cup}}=(\Phi_F)^t$, the transpose of Φ_F. Axiom (III) means that $\mathcal{O}$ is closed under the converse map $f\to f^{\cup}$, which we call the *pairing* on $(X, \mathcal{O})$, and is equivalent to

(2.3) *$\mathscr{B}$ is closed under the transpose map.*

Now assume axiom (IV) and take f, g, $h \in \mathcal{O}$ and $(x, y) \in h$. Then a_{fgh} is the number of subconfigurations



We see that if $f(x) = \{y \in X \mid (x, y) \in f\}$, then

(2.4) *For $f, g, h \in \mathcal{O}$ and $(x, y) \in h$, $a_{fgh} = |f(x) \cap g^{\cup}(y)| =$ the number of g-edges from $f(x)$ to y.*

For $E, F \in \mathscr{P}(X^2)$, the *composite EF* is defined by $(x, y) \in EF \Leftrightarrow$ there exists $z \in X$ such that $(x, z) \in E$ and $(z, y) \in F$. For $f, g \in \mathcal{O}$, axiom (IV) implies that

(2.5) *$fg = \sum_{h \in \mathcal{O}} \hat{a}_{fgh} h$ where $\hat{a}_{fgh} = 0$ or 1 according as $a_{fgh} = 0$ or $\neq 0$.*

It follows that R is closed under composition. This property of R is equivalent to the weaker axiom obtained by replacing 'number of' by 'existence of' in axiom (IV).

Axiom (IV) also implies that Γ is closed under matrix multiplication. Namely, if $f, g, h \in \mathcal{O}$ and $(x, y) \in h$, then by axiom (IV),

$$\begin{aligned}\Phi_f \Phi_g (x, y) &= \sum_{z \in X} \Phi_f (x, z) \Phi_g (z, y) \\ &= a_{fgh} = a_{fgh} \Phi_h (x, y).\end{aligned}$$

Hence

(2.6) *$\Phi_f \Phi_g = \sum_{h \in \mathcal{O}} a_{fgh} \Phi_h$ for $f, g \in \mathcal{O}$.*

Thus Γ is a subring of $\mathrm{Mat}_{\mathbf{Z}} X$ and can be referred to as the *adjacency ring* of $(X, \mathcal{O})$. In the group case Γ is the centralizer in $\mathrm{Mat}_{\mathbf{Z}} X$ of the permutation representation of G. If K is any commutative ring with identity element, then $K\Gamma$ is a K-subalgebra of $\mathrm{Mat}_K X$, so we call it the *adjacency algebra of* $(X, \mathcal{O})$ *over* K, and $\Gamma \to K\Gamma$ is a homomorphism of rings. Axiom (IV) is equivalent to

(2.7) *For some field K of characteristic* 0, $K\Gamma$ *is a ring.*

It is now clear that coherent configurations are characterized in terms of the set $\mathscr{B}$ of basic adjacency matrices as follows.

(2.8) $\mathscr{B} \subseteq \mathrm{Mat}_Z X$ *is a finite set of* (0, 1)*-matrices satisfying* (2.1), (2.2), (2.3) *and* (2.7). *Conversely given a finite set* $\mathscr{B} \subseteq \mathrm{Mat}_Z X$ *of* (0, 1)*-matrices satisfying these four conditions, then*

$$\mathcal{O} = \{\mathrm{spt}\,\phi \mid \phi \in \mathscr{B}\}$$

(*where* $\mathrm{spt}\,\phi = \{(x, y) \in X^2 \mid \phi(x, y) \neq 0\}$) *is a coherent set of relations on X.*

A main source of interest in and examples of coherent configurations comes from the almost immediate fact that

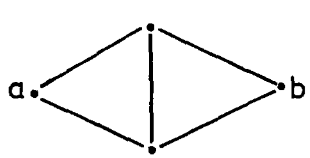
(2.9) *In the group case*, $(X, \mathcal{O})$ *is coherent.*

We pause to mention two examples, the first illustrating one way, natural from a combinational point of view, in which non-group case examples can arise from group case ones, and the second showing that Γ does not determine $(X, \mathcal{O})$.

(1) In its action on the totality X of lines of affine space over $GF(q)$, assuming that the dimension is at least 3, the affine group affords the rank 4 configuration with $\mathcal{O} = \{I, f_1, f_2, f_3\}$ where

f_1 = all pairs of distinct intersecting lines,
f_2 = all pairs of distinct parallel lines, and
f_3 = all pairs of skew lines.

It easy to verify that the rank 3 configuration $(X, \tilde{\mathcal{O}})$ obtained from $(X, \mathcal{O})$ by fusing f_2 and f_3, i.e. with $\tilde{\mathcal{O}} = \{I, f_1, f_2 + f_3\}$, is again coherent. But if we choose two nonintersecting lines a and b and count the number of subconfigurations of $(X, \tilde{\mathcal{O}})$ of the form



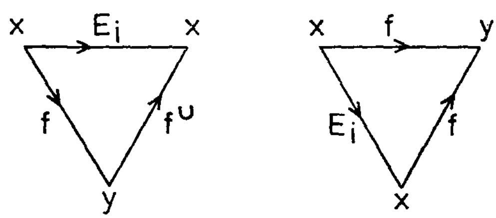
where f_1-edges are indicated by lines, we find that, if $q > 2$, this number depends on whether a and b are parallel or skew. Hence for $q > 2$, $(X, \tilde{\mathcal{O}})$ cannot be afforded by a group. For $q = 2$, lines are just pairs of points and $(X, \tilde{\mathcal{O}})$ is afforded by the action of Σ_X on the 2-element subsets.

(2) The adjacency ring Γ does not determine $(X, \mathcal{O})$. In fact we can have two nonisomorphic coherent configurations $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ with $|X|=|X'|$ and a bijection $\begin{cases}\mathcal{O}\to\mathcal{O}'\\ f\mapsto f'\end{cases}$ such that $a_{fgh}=a_{f'g'h'}$ for all $f, g, h\in\mathcal{O}$. An example of this is given by the two rank 3 configurations afforded by the action of $S_4(q)$ on (i) the set of all points and (ii) the set of all totally isotropic lines of 3-dimensional symplectic projective space over $GF(q)$.

Now we return to some basic consequences of the axioms. For the rest of this section we assume that $(X, \mathcal{O})$ *is coherent*. Then $I=E_1+\cdots+E_t$ with $E_i\in\mathcal{O}$, $1\le i\le t$. Put $X_i=$domain $E_i=$range E_i, then $\{X_i \mid 1\le i\le t\}$ is a partition of X. We call the X_i the *fibers* of $(X, \mathcal{O})$ and say that $(X, \mathcal{O})$ is *homogeneous* if there is just one fiber, or, equivalently, if $I\in\mathcal{O}$. The homogeneous configurations with trivial pairing are equivalent to *association schemes*[1] as defined in [2]. Now put $\mathcal{O}^{ij}=\{f\in\mathcal{O} \mid \text{domain}\, f=X_i \text{ and range}\, f=X_j\}$.

(2.10) *$\{\mathcal{O}^{ij} \mid 1\le i,j\le t\}$ is a partition of $\mathcal{O}$.*

Proof. Suppose that $f\in\mathcal{O}$. We have to show that $f\in\mathcal{O}^{ij}$ for some i, j. We have dom $f\cap X_i\neq\emptyset$ for some i, so we can find $x\in X_i$ and $y\in X$ such that $(x, y)\in f$. This means that $a_{ff^{\cup}E_i}\neq 0$ and $a_{E_iff}\neq 0$



Suppose $x_1\in X_i$, then there must be an $(f, f^{\cup})$-path (x_1, y_1, x_1), but then $(x_1, y_1)\in f$ so $x_1\in$domain f. Suppose $x_2\in$domain f, then there exists y_2 such that $(x_2, y_2)\in f$, and there must be an (E_i, f)-path (x_2, x_2, y_2). But then $x_2\in$domain $E_i=X_i$. This proves that domain $f=X_i$. Similarly domain $f^{\cup}=X_j$ for some j. Since range $f=$domain $f^{\cup}$, we are done.

A consequence of (2.10) is that the full subconfiguration $[Y]$ based on a union Y of fibers is coherent and has for its fibers the fibers of $(X, \mathcal{O})$ which it contains. In particular (with a slight abuse of notation) $[X_i]=(X_i, \mathcal{O}^{ii})$ is

[1] P. Delsarte, in his very interesting dissertation 'An Algebraic Approach to the Association Schemes of Coding Theory', presented to the Universite Catholique de Louvain, Faculte des Sciences Appliques (1973), uses the term association scheme in a way which is equivalent to our use of commutative configuration. Commutative configurations are necessarily homogeneous.

coherent and homogeneous. The degree of $[X_i]$ is $n_i = |X_i|$ and

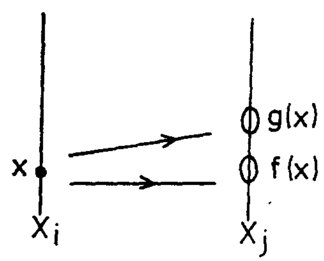
(2.11) $$n = \sum_{i=1}^{t} n_i$$

We call $r_{ij} = |\mathcal{O}^{ij}|$ the *cross rank of* X_i *and* X_j. Then $[X_i]$ has rank r_{ii} and

(2.12) $$r = \sum_{1<i,j<t} r_{ij} \text{ and } r_{ij} = r_{ji}.$$

Aut$(X, \mathcal{O})$ permutes the fibers, Aut*$(X, \mathcal{O})$ is contained in the kernel N of this action, and we have homomorphisms $N \to$ Aut$[X_i]$ and Aut*$(X, \mathcal{O})$ $\to$ Aut*$[X_i]$. In the group case the X_i are the G-orbits in X.

If $x \in X_i$, then $\{f(x) \mid f \in \mathcal{O}^{ij}\}$ is a partition of X_j.



Since $|f(x)| = a_{ff^\cup E_i}$, independent of $x \in X_i$, we denote this number by n_f and call it the *subdegree* corresponding to f. We have

(2.13) *For* $1 \leq i, j \leq t$, $n_j = \sum_{f \in \mathcal{O}^{ij}} n_f$

and

(2.14) *For* $f \in \mathcal{O}^{ij}$, $|f| = |f^\cup| = n_i n_f = n_j n_{f^\cup}$.

In particular $n_i = n_j$ implies $n_f = n_{f^\cup}$ for $f \in \mathcal{O}^{ij}$.

The strict automorphisms fixing $x \in X_i$ fix the sets $f(x)$ for $f \in \mathcal{O}^{ij}$. In the group case these are the orbits for G_x in X_j.

Put $R^{ij} = \langle \mathcal{O}^{ij} \rangle_{GF(2)}$ so that $R = \sum_{1<i,j<t} \oplus R^{ij}$. If $F \in R^{ij}$, $F \neq \emptyset$, then $F = f_1 + f_2 + \cdots + f_s$ with $f_\alpha \in \mathcal{O}^{ij}$, $i \leq \alpha \leq s$. For $(x, y) \in F$ we have corresponding partitions of $F(x)$ and $F^\cup(y)$, namely

$$\{f_\alpha(x) \mid 1 \leq \alpha \leq s\} \quad \text{and} \quad \{f_\alpha^\cup(y) \mid 1 \leq \alpha \leq s\}.$$

In the group case the fibres are just the G-orbits in X, and (2.10) is immediate in this case. For $x \in X_i$, $\{f(x) \mid f \in \mathcal{O}^{ij}\}$ is the set of G_x-orbits in X_j.

We now list some basic properties of the intersection numbers a_{fgh}.

(2.15) *For f, g, h,*

(a) $a_{fgh} \neq 0$ *implies* $f \in \mathcal{O}^{ij}$, $g \in \mathcal{O}^{jk}$ *and* $h \in \mathcal{O}^{ik}$ for some i, j, k.

(b) $a_{fgh} = a_{g^\cup f^\cup h^\cup}$.

Assertion (a) follows from (2.10) and (b) from (2.4).

(2.16) (a) *If* $f_1, \ldots, f_s \in \mathcal{O}$, $s \geq 3$, *then the sum*

$$\sum_{(\lambda) \in \mathcal{O}^{s-3}} a_{f_1 f_2 \lambda_1} a_{\lambda_1 f_3 \lambda_2} \cdots a_{\lambda_{s-3} f_{s-1} f_s^\cup} |f|$$

is independent of cyclic permutations of $f_1, \ldots, f_s$. *In particular*

(b) *If* $f, g, h \in \mathcal{O}$, *then* $a_{fgh^\cup}|h| = a_{hfg^\cup}|g| = a_{ghf^\cup}|f|$.

Proof. Put $\Phi_1 = \Phi_{f_1}$. The total number of closed $(f_1, \ldots, f_s)$-paths is clearly independent of such cyclic permutations, and this number is

$$\sum_{(x) \in X^s} \Phi_1(x_1, x_2)\, \Phi_2(x_2, x_3) \cdots \Phi_s(x_s, x_1)$$
$$= \sum_{x_1 \in X} (\Phi_1 \Phi_2 \cdots \Phi_s)(x_1, x_1).$$

By (2.4) we aee that

(2.17) *For* $f \in \mathcal{O}^{ij}$, $g \in \mathcal{O}^{jk}$ *and* $h \in \mathcal{O}^{ik}$, $a_{E_\alpha gh} = \delta_{\alpha i}\delta_{\alpha j}\delta_{gh}$, $a_{fE_\alpha h} = \delta_{\alpha j}\delta_{\alpha k}\delta_{fh}$ *and* $a_{fgE_\alpha} = \delta_{\alpha i}\delta_{\alpha k}\delta_{fg^\cup} n_f$.

(2.18) (a) *If* $g \in \mathcal{O}^{jk}$ *and* $h \in \mathcal{O}^{ik}$, *then*

$$\sum_{f \in \mathcal{O}} a_{fgh} = \sum_{f \in \mathcal{O}^{ij}} a_{fgh} = n_{g^\cup}.$$

(b) *If* $f \in \mathcal{O}^{ij}$ *and* $h \in \mathcal{O}^{ik}$, *then*

$$\sum_{g \in \mathcal{O}} a_{fgh} = \sum_{g \in \mathcal{O}^{jk}} a_{fgh} = n_f.$$

The fact that Γ is a Z-algebra with the a_{fgh} as structure constants implies

(2.19) $\sum_{k \in \mathcal{O}} a_{fgk} a_{khl} = \sum_{k \in \mathcal{O}} a_{ghk} a_{fkl}$ *for all* $f, g, h \in \mathcal{O}$.

The right regular representation of Γ provides an isomorphism $\phi \mapsto \hat{\phi}$ of Γ onto a subring $\hat{\Gamma}$ of $\mathrm{Mat}_{\mathbf{Z}}\,\mathcal{O}$ where

(2.20) $\hat{\Phi}_f(g, h) = a_{gfh}$ *for all* $f, g, h \in \mathcal{O}$, and $\hat{\phi}$, $\phi \in \Gamma$, is defined by linearity. The properties (2.15)–(2.18) of the intersection numbers can be regarded as properties of the set $\hat{\mathscr{B}} = \{\hat{\Phi}_f \mid f \in \mathcal{O}\}$ of $r \times r$ matrices. For example, (b) of

(2.15) and (b) of (2.16) imply

$$(2.21)\qquad \Delta(\hat{\Phi}_g)^t = \hat{\Phi}_{g^\cup}\Delta$$

where $\Delta \in \mathrm{Mat}_Z \mathcal{O}$ *is the diagonal matrix such that* $\Delta(g, h) = \delta_{gh} n_g$.

3. The Schur relations

We assume once more that $(X, \mathcal{O})$ is coherent of degree n and rank r, and use freely the terminology and notations introduced in Section 3. We turn now to the representation theory of the adjacency ring Γ. Since we concern ourselves here only with the absolutely irreducible representations, we replace Γ by the adjacency algebra $C = \mathbf{C}\Gamma$ over $\mathbf{C}$ and refer to this algebra simply as the *adjacency algebra.* C is a $\mathbf{C}$-subalgebra of $\mathrm{Mat}_{\mathbf{C}} X$ containing the identity matrix, and, since C is closed under the conjugate transpose map,

(3.1) *C is semisimple.*

The vector space $\mathbf{C}X$ has the structure of a (left) module over $\mathrm{Mat}_{\mathbf{C}} X$ according to

$$\phi x = \sum_{y \in X} \phi(y, x)\, y \qquad (\phi \in \mathrm{Mat}_{\mathbf{C}} X, x \in X).$$

Regarded as a module over C, $\mathbf{C}X$ will be denoted by M and called the *standard module.*

For $f \in \mathcal{O}^{ij}$ and $x \in X_\alpha$, $\Phi_f x = \delta_{\alpha j} \sum_{y \in f^\cup(x)} y$. Put $x_\alpha = \sum_{x \in X_\alpha} x$, then

(3.2) $\Phi_f \cdot x_\alpha = \delta_{j\alpha} n_f x_i$ *for* $f \in \mathcal{O}^{ij}$. Thus $\langle x_1, \ldots, x_t \rangle_{\mathbf{C}}$ is a C-submodule of M and is clearly irreducible. We call it the *principal submodule.*

Let $\varepsilon_1, \ldots, \varepsilon_m$ be the central primitive idempotents of C, so that

$$C = C_1 \oplus \cdots \oplus C_m,$$

where $C_i = \varepsilon_i C$, $i \leq i \leq m$, is the unique decomposition of C into simple two-sided components. Each C_i is isomorphic with a full matrix algebra over $\mathbf{C}$, of degree e_i, say, that is, $C_i \simeq \mathbf{C}_{e_i}$, and

$$(3.3)\qquad r = \sum_{i=1}^{m} e_i^2.$$

Let $\zeta_1, \ldots, \zeta_m$ be the irreducible characters of C so numbered that $\zeta_i(1) = e_i$ and $\zeta_i(\varepsilon_j) = \delta_{ij} e_i$, $1 \leq i \leq m$. The standard module M decomposes

$$M = M_1 \oplus \cdots \oplus M_m,$$

where $M_i = e_i M$ is a direct sum of, say, z_i isomorphic irreducible submodules affording ζ_i. Then

$$(3.4) \qquad n = \sum_{i=1}^{m} z_i e_i$$

and if ζ is the character afforded by M, so that $\zeta(\sigma) = \text{trace}\ \sigma$ for $\sigma \in C$, then

$$(3.5) \qquad \zeta = \sum_{i=1}^{m} z_i \zeta_i .$$

We call ζ the *standard character*, m the *reduced rank*, $e_1, \ldots, e_m$ the *irreducible degrees* and $z_1, \ldots, z_m$ the corresponding *multiplicities.*

We choose the numbering so that M_1 is the principal submodule. Then by (3.2)

$$e_1 = t \quad \text{and} \quad z_1 = 1 .$$

We refer to ε_1 as the *principal idempotent* and ζ_1 as the *principal character.* We have

$$(3.6) \qquad \zeta_1(\Phi_f) = \delta_{ij} n_f \quad \text{for} \quad f \in \mathcal{O}^{ij}.$$

The algebra C can be *completely reduced,* that is, there is a nonsingular matrix $U \in \text{Mat}_{\mathbf{C}} X$, and even a unitary one if we need that, such that

$$U^{-1} \phi U = \text{diag}\,(\Delta_1(\phi), \underbrace{\Delta_2(\phi), \ldots, \Delta_2(\phi)}_{z_2}, \ldots, \underbrace{\Delta_m(\phi), \ldots, \Delta_m(\phi)}_{z_m})$$

for all $\phi \in C$, where $\Delta_1, \ldots, \Delta_m$ are inequivalent irreducible representations of C with Δ_i affording ζ_i, $1 \le i \le m$. We write

$$\Delta_\alpha(\phi) = (a^\alpha_{ij}(\phi)).$$

There exist elements $\varepsilon^\alpha_{ij} \in C$, $1 \le i, j \le e_\alpha$, $1 \le \alpha \le m$, such that $\zeta_\beta(\varepsilon^\alpha_{ij}\phi) = \delta_{\alpha\beta} a^\alpha_{ij}(\phi)$ for $\phi \in C$ and $a^\alpha_{kl}(\varepsilon^\beta_{ij}) = \delta_{\alpha\beta}\delta_{il}\delta_{jk}$. Then $\{\varepsilon^\alpha_{ij} \mid 1 \le i, j \le e_\alpha, 1 \le \alpha \le m\}$ is a basis of C and $\varepsilon^{(\alpha)} = \sum_{i=1}^{e_\alpha} \varepsilon^\alpha_{ii}$ is a decomposition of $\varepsilon^{(\alpha)}$ into a sum of orthogonal primitive idempotents. We determine this new basis in terms of the standard basis. Fix i, j and α for the moment and write

$$\varepsilon^\alpha_{ij} = \varepsilon = \sum_{f \in \mathcal{O}} a_f \Phi_f \quad \text{for} \quad g \in \mathcal{O},$$

put $\tilde{\Phi}_g = (1/|g|)\Phi_{g^\cup}$. Then, if $g \in \mathcal{O}^{kl}$,

$$\varepsilon \tilde{\Phi}_g = \sum_f a_f \Phi_f \tilde{\Phi}_g = (1/|g|) \sum_f \sum_h a_f a_{fg^\cup h} \Phi_h$$

so

$$\zeta(\varepsilon\tilde{\Phi}_g) = (1/|g|)\sum_{s=1}^{t}\sum_{f\in\mathcal{O}} a_f a_{fg\cup E_s n_s = a_g}.$$

On the other hand, $\zeta=\sum_{\beta=1}^{m} z_\beta\zeta_\beta$, so

$$\zeta(\varepsilon\tilde{\Phi}_g) = \sum_{\beta=1}^{m} z_\beta\zeta_\beta(\varepsilon\tilde{\Phi}_g) = z_\alpha a_{ij}^\alpha(\tilde{\Phi}_g).$$

Hence $a_g=z_\alpha a_{ij}^\alpha(\tilde{\Phi}_g)$, and therefore

$$\varepsilon_{ij}^\alpha = z_\alpha\sum_{g\in\mathcal{O}} a_{ij}^\alpha(\tilde{\Phi}_g)\,\Phi_g,$$

and

(3.7) $$\varepsilon^{(\alpha)} = z_\alpha\sum_{g\in\mathcal{O}} \zeta_\alpha(\tilde{\Phi}_g)\,\Phi_g,$$

where $\tilde{\Phi}_g=(1/|g|)\Phi_{g^\cup}$. Applying a_{uv}^β and ζ_β respectively we get

(3.8) *(The Schur relations)*

$$\sum_{g\in\mathcal{O}} a_{ij}^\alpha(\tilde{\Phi}_g)\, a_{uv}^\beta(\Phi_g) = \delta_{\alpha\beta}\delta_{iv}\delta_{ju}(1/z_\alpha)$$

and

(3.9) *(The Orthogonality relations)*

$$\sum_{g\in\mathcal{O}} \zeta_\alpha(\tilde{\Phi}_g)\,\zeta_\beta(\Phi_g) = \delta_{\alpha\beta}(e_\alpha/z_\alpha).$$

For some applications it is convenient to rewrite the Schur relations. First, list the a_{ij}^α; $a_1, a_2, \ldots, a_r$. We can assume that $a_1, a_2, \ldots, a_{t^2}$ are the a_{ij}^1 in some order. If $a_\lambda=a_{ij}^\alpha$, put $a_{\bar\lambda}=a_{ji}^\alpha$ and $h_\lambda=z_\alpha$. In this notation, (3.8) becomes

(3.10) $$\sum_{g\in\mathcal{O}} a_\lambda(\tilde{\Phi}_g)\, a_\mu(\Phi_g) = \delta_{\lambda\bar\mu}(1/h_\lambda).$$

Now list the Φ_g:$\Phi_1, \Phi_2, \ldots, \Phi_r$. We can assume that $\Phi_i=\Phi_{E_i}$, $1<i<t$. If $\Phi_j=\Phi_g$, put $m_i=|g|$ and $\Phi_{\hat{i}}=\Phi_{g^\cup}$. Define matrices $A=(A_{\lambda s})$, $H=(H_{\lambda s})$ and $M=(M_{\lambda s})$ by

$$A_{\lambda s} = a_\lambda(\Phi_s), \qquad H_{\lambda s} = \delta_{\lambda s}h_\lambda \qquad \text{and} \qquad M_{\lambda s} = \delta_{\lambda s}m_\lambda.$$

There exist permutation matrices P and Q with $P^2=Q^2=1$, such that $(AP)_{\lambda s}=a_\lambda(\Phi_{\hat{s}})$ and $(QA)_{\lambda s}=a_{\bar\lambda}(\Phi_s)$. Now (3.8) becomes

(3.11) *(Matrix form of the Schur relations)*

$$APM^{-1}(QA)^t = H^{-1}$$

which can be rewritten as

$$(3.12) \qquad (QA)^t HAP = M.$$

The orthogonality relations (3.9) can be written in terms of the *character table*

$$Z = (\zeta_\alpha(\Phi_f))$$

as

$$(3.13) \qquad ZPM^{-1}Z^t = \operatorname{diag}(t, (e_2/z_2), \ldots, (e_m/z_m)).$$

But we do not have an analogue of (3.12) unless $r = m$ (i.e., C is commutative) in which case $Z = A$.

Consideration of determinants and, more generally, of elementary divisors in (3.12) gives the following two results which are extensions of results of Frame [7] and Wielandt ([19], (30.4)) respectively. Together with the further extensions to be given in Part II (cf. [14]) they should be compared with Curtis and Fossum [4] and Keller [15]. We use the fact that, since the structure constants a_{fgh} are rational integers, we may assume that the matrices $\Delta_\alpha(\Phi_f)$ have algebraic integer coefficients for all $f \in \mathcal{O}$, $1 \le \alpha \le m$.

(3.14) *The Frame Quotient*

$$N = \left(\prod_{k=1}^{t} n_k^{-2} \prod_{i=1}^{r} m_i\right) \Big/ \prod_{\alpha=1}^{m} z_\alpha^{e_\alpha^2}$$
$$= \left(\prod_{k=1}^{t} n_k^{r_{k1}+\cdots+r_{kt}-2} \prod_{g\in\mathcal{O}} n_g\right) \Big/ \prod_{\alpha=1}^{m} z_\alpha^{e_\alpha^2}$$

is a rational integer. If the irreducible representations Δ_i, $1 \le i \le m$, can be written in $\mathcal{Q}$, i.e. if $\mathcal{Q}$ is a splitting field for C, then N is a square.

The factor $\prod_{k=1}^{t} n_k^{-2}$ comes from the fact that, by the Schur relations,

$$\det A = \prod_{k=1}^{t} n_k \cdot \det A_0$$

where A_0 is a suitable submatrix of A.

(3.15) *If a prime power q divides z_α for l distinct values $\alpha_1, \ldots, \alpha_l$ of α, then q divides $|g|$ for $e_{\alpha_1}^2 + \cdots + e_{\alpha_l}^2$ distinct $g \in \mathcal{O}$.*

A nondegenerate bilinear form (,) on C is defined by

$$(\Phi_f, \Phi_g) = \delta_{fg^\cup} |f| \qquad (f, g \in \mathcal{O}).$$

The basis $\tilde{\mathscr{B}} = \{\tilde{\Phi}_f \mid f \in \mathcal{O}\}$ is dual to the standard basis in the sense that $(\Phi_f, \tilde{\Phi}_g) = \delta_{fg}$ for all f, $g \in \mathcal{O}$, and the form is associative in the sense that $(\alpha\beta, \gamma) = (\alpha, \beta\gamma)$ for all $\alpha, \beta, \gamma \in C$. In fact, if f, g and $h \in \mathcal{O}$, then

$$(\Phi_f \Phi_g, \Phi_h) = \sum_{k \in \mathcal{O}} a_{fgh} (\Phi_k, \Phi_h) = a_{fgh^\cup} |h| = a_{ghf^\cup} |f|$$

$$= \sum_{k \in \mathcal{O}} a_{ghk} (\Phi_f, \Phi_k) = (\Phi_f, \Phi_g \Phi_h).$$

This form will play a role in studying the integral and modular representation of Γ.

4. Commutative configurations

We shall say that a coherent configuration $(X, \mathcal{O})$ is *commutative* if the following equivalent conditions hold:

(1) Γ is commutative
(2) C is commutative
(3) $a_{fgh} = a_{gfh}$ for all $f, g, h \in \mathcal{O}$
(4) $r = m$
(5) $e_1 = e_2 = \cdots = e_m = 1$.

A commutative configuration is necessarily homogeneous. The pairing $f \mapsto f^\cup$ on $(X, \mathcal{O})$ is trivial if and only if all the basic adjacency matrices Φ_f, $f \in \mathcal{O}$, are symmetric; such a configuration is commutative.

(4.1) *A homogeneous configuration of rank* ≤ 5 *is commutative.*

(4.1) follows at once from

(4.2) *If* $r > 2$ *and* $t = 1$, *then* $m \geq 3$.

Proof. Assume $r > 2$, $t = 1$ and $m = 2$. Then $r = 1 + e_2^2$ and $\zeta = \zeta_1 + z_2\zeta_2$ with $\zeta_1(1) = 1$, $\zeta_2(1) = e_2$. We have $\sum_{f \in \mathcal{O}} \Phi_f = \Phi$, the 'all 1' matrix, and $\zeta(\Phi) = n$. Moreover, $\zeta_1(\Phi_f) = n_f$ for $f \in \mathcal{O}$. Hence $\zeta_1(\Phi) = \sum_{f \in \mathcal{O}} n_f = n$, so $\zeta_2(\Phi) = 0$. On the other hand, for $f \in \mathcal{O}$, $f \neq I$,

$$0 = \zeta(\Phi_f) = n_f + z_2\zeta_2(\Phi_f)$$

so $\zeta_2(\Phi_f) = -(n_f/z_2) \leq -1$ since $\zeta_2(\Phi_f)$ is an algebraic integer. Hence

$$0 = \zeta_2(\Phi) \leq \zeta_2(1) - (r-1) = e_2 - e_2^2 < 0$$

which is a contradiction. □

The configuration of rank 6 afforded by the action of $PSL_3(q)$ on the set of incident point-line flags in the projective plane over $GF(g)$ is homogeneous of rank 6 and is not commutative.

5. Computation of the multiplicities

The following parameters have arisen for a coherent configuration $(X, \mathcal{O})$

n	degree
r	rank
a_{fgh}	intersection numbers
t	number of fibres
n_i	degrees of the fibres
r_{ij}	cross ranks
n_f	subdegrees
m	reduced rank
ζ_α	irreducible characters
a_{ij}^α	coefficient functions of the irreducible representations
e_α	irreducible degrees
z_α	multiplicities.

Assume that the family $(a_{fgh} \mid f, g, h \in \mathcal{O})$ of intersection numbers is on hand. We claim that all the remaining entries in the list are determined (though the configuration itself is not as we have noted in Section 2). Put $\hat{\mathscr{B}} = \{\hat{\Phi}_f \mid f \in \mathcal{O}\}$ where $\hat{\Phi}_f \in \mathrm{Mat}_{\mathbf{Z}}\mathcal{O}$ is defined by (2.19). As in Section 2, the entries in the list through the subdegrees are easily obtained explicitly, starting from the fact that the diagonal matrices in $\hat{\mathscr{B}}$ are the $\hat{\Phi}_{E_i}$, $1 \leq i \leq t$, and

$$\mathcal{O}^{ij} = \{f \in \mathcal{O} \mid a_{E_i f f} \neq 0 \text{ and } a_{f E_j f} \neq 0\}.$$

Since $\Phi_f \mapsto \hat{\Phi}_f$, $f \in \mathcal{O}$, determines an isomorphism $\Phi \mapsto \hat{\Phi}$ of C onto the subalgebra $\hat{C} = \mathbf{C}\hat{\Gamma} = \langle \hat{\Phi}_f \mid f \in \mathcal{O} \rangle_{\mathbf{C}}$ of $\mathrm{Mat}_{\mathbf{C}}\mathcal{O}$, C and $\hat{C}$ have the same number m of simple two-sided components and the same character table

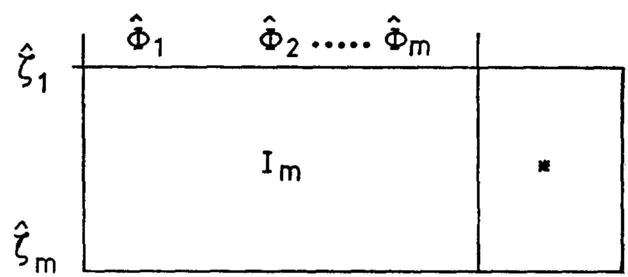
$$Z = (\zeta_\alpha(\Phi_f)) = (\hat{\zeta}_\alpha(\hat{\Phi}_f)).$$

If $\hat{\Delta}_\alpha$ is an irreducible representation of $\hat{C}$ affording $\hat{\zeta}_\alpha$ then $\Delta_\alpha(\Phi_f) = \hat{\Delta}_\alpha(\hat{\Phi}_f)$, $f \in \mathcal{O}$, determines an irreducible representation of C affording ζ_α. The character of $\hat{C}$ afforded by $\mathbf{C}\mathcal{O}$ is $\hat{\zeta} = \sum_{\alpha=1}^m e_\alpha \hat{\zeta}_\alpha$. The multiplicities z_α are obtained from the character table Z by the orthogonality relations (3.9). The integrality of the z_α is a condition on the intersection numbers which is readily seem to be independent of the previously mentioned conditions.

Note that the semisimplicity of $\hat{C}$ is not an independent condition since it is implied by (2.20).

There are other possibilities for computing multiplicities which sometimes have advantages. We make some remarks in this connection here.

(i) Reduction of the character table to the form



by elementary column operations produces elements $\hat{\Phi}_1, \ldots, \hat{\Phi}_m \in \hat{C}$ such that $\hat{\zeta}_i(\hat{\Phi}_j) = \delta_{ij}$. If $\hat{\Phi}_i = \sum_{f\in\mathcal{O}} \alpha_{if}\hat{\Phi}_f$, then $\Phi_i = \sum \alpha_{if}\Phi_f$ is the inverse image of $\hat{\Phi}_i$ in C. Since $\zeta = \sum z_j\zeta_j$.

$$(5.1) \qquad z_i = \zeta(\Phi_i) = \sum_{f\in\mathcal{O}} \alpha_{if}\zeta(\Phi_f) = \sum_{k=1}^{t} \alpha_{iE_k} n_k .$$

(ii) Let ϕ be in the center of C. Then $\Delta_\alpha(\phi) = \theta_\alpha I_\alpha$, a scalar matrix, and $\theta_\alpha = \zeta_\alpha(\phi)/e_\alpha$ is an eigenvalue of ϕ. Suppose that ϕ has the maximum number m of distinct eigenvalues (in fact there is such an element $\phi \in \Gamma$). Then the multiplicities of these eigenvalues as eigenvalues of $\hat{\phi}$ and ϕ respectively, where $\hat{\phi}$ is the image in $\hat{C}$ of ϕ, are respectively e_i^2, $1 \le i \le m$ and $e_i z_i$, $1 \le i \le m$.

(iii) If $\phi \in C$ and θ is an eigenvalue of ϕ, then the multiplicities of θ as an eigenvalue of ϕ and $\hat{\phi}$ respectively are

$$(5.2) \qquad \hat{\mu}_\theta = \frac{\text{trace } F_\theta(\hat{\phi})}{F_\theta(\theta)}$$

and

$$(5.3) \qquad \mu_\theta = \frac{\text{trace } F_\theta(\phi)}{F_\theta(\theta)},$$

where $F(T) \in \mathbf{C}[T]$ is a nonzero polynomial such that $F(\phi) = 0$ (i.e., $F(\hat{\phi}) = 0$) and $F_\theta(T) = F(T)/(T-\theta)^\mu$ where μ is the multiplicity of θ as a root of $F(T) = 0$.

To use these formulas we need trace $\phi^\alpha = \zeta(\phi^\alpha)$ and trace $\hat{\phi}^\alpha = \hat{\zeta}(\hat{\phi}^\alpha)$. If $\phi^\alpha = \sum_{h\in\mathcal{O}} \beta_{\alpha\mu}\Phi_h$, then $\hat{\phi}^\alpha = \sum_{h\in\mathcal{O}} \beta_{\alpha h}\hat{\Phi}_h$ and

$$\zeta(\phi^\alpha) = \sum_{k=1}^{t} b_{\alpha E_k} n_k$$

while

$$\hat{\zeta}(\hat{\phi}) = \sum_{h\in\mathcal{O}} b_{\alpha h}\,\hat{\zeta}(\hat{\Phi}_h) = \sum_{h\ni\mathcal{O}}\ \sum_{g\in\mathcal{O}} b_{\alpha h} a_{ghg}.$$

(iv) If the distinct eigenvalues $\theta_1, \theta_2, \ldots, \theta_s$ of $\phi\in C$ are given, then their respective multiplicities $\hat{\mu}_1, \ldots, \hat{\mu}_s$ and $\mu_1, \ldots, \mu_s$ as eigenvalues of $\hat{\phi}$ and ϕ are determined by

$$\sum_{i=1}^{s} \hat{\mu}_i\theta_i^\alpha = \hat{\zeta}(\hat{\phi}^\alpha) \qquad (0 \le \alpha \le s-1) \tag{5.4}$$

and

$$\sum \mu_i\theta_i^\alpha = \zeta(\phi^\alpha) \qquad (0 \le \alpha \le s-1). \tag{5.5}$$

6. The Krein condition

The notation of this section is that of Sections 2 and 3. Our aim is to determine the positive semidefinite Hermitian matrices in C, and to apply Schur's theorem on pointwise products of such matrices to get a condition on the a_λ. We call the resulting condition (6.4) the *Krein condition* because of its relation to what L. Scott calls the Krein condition in [16].

We use C^0 to denote C regarded as an algebra with respect to pointwise addition and multiplication, and determine the structure constants of C^0 with respect to the basis $\{\varepsilon_\lambda \mid 1 \le \lambda \le r\}$, where $\varepsilon_\lambda = \varepsilon_{ij}^\alpha$ if $a_\lambda = a_{ij}^\alpha$, so that

$$\varepsilon_\lambda = h_\lambda \sum_{f\in\mathcal{O}} a_\lambda(\tilde{\Phi}_g)\,\Phi_g$$

by (3.7). Write

$$\varepsilon_\lambda \circ \varepsilon_\mu = \sum_{\nu=1}^{r} c_{\lambda\mu\nu}\varepsilon_\nu$$

and evaluate at $(x, y)\in f^\cup$ to get

$$h_\lambda h_\mu \frac{a_\lambda(\Phi_f)\,a_\mu(\Phi_f)}{|f|^2} = \sum_\nu c_{\lambda\mu\nu} h_\nu a_\nu\,(\tilde{\Phi}_{f^\cup}). \tag{6.1}$$

Multiplying by $a_{\bar{\delta}}(\Phi_{f\cup})$ and summing over $f \in \mathcal{O}$ we obtain by (3.10) that

$$(6.2) \qquad c_{\lambda\mu\delta} = h_\lambda h_\mu \sum_{f \in \mathcal{O}} \frac{a_\lambda(\Phi_f)\, a_\mu(\Phi_f)\, a_{\bar{\delta}}(\Phi_{f\cup})}{|f|^2}.$$

Now assume that the following equivalent conditions (1) through (3) hold.

(1) $\Delta_\alpha(\varphi)^* = \Delta_\alpha(\varphi^*)$ for all $\varphi \in C$, $1 \le \alpha \le m$, where * denotes conjugate transpose.

(2) $\overline{a_\lambda(\Phi_f)} = a_{\bar{\lambda}}(\Phi_{f\cup})$ for all $f \in \mathcal{O}$, $1 \le \lambda \le r$.

(3) $\varepsilon_\lambda^* = \varepsilon_{\bar{\lambda}}$, $1 \le \lambda \le r$.

This is equivalent to assuming that the complete reduction of C has been effected by a unitary matrix. Then for $\lambda = \bar{\lambda}$, ε_λ is a projection, i.e., an Hermitian matrix with eigenvalues 0 and 1.

Fix $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$. By Schur's theorem, $\varepsilon_\lambda \circ \varepsilon_\mu$ is a positive semidefinite Hermitian matrix with all its eigenvalues in the interval [0, 1]. If $a_\delta = a_{ij}^\alpha$, put

$$(6.3) \qquad c_{ij}^\alpha = \sum_{f \in \mathcal{O}} \frac{a_\lambda(\Phi_f)\, a_\mu(\Phi_f)\, \overline{a_\delta(\Phi_f)}}{|f|^2}.$$

Then $\Delta_\alpha(\varepsilon_\lambda \circ \varepsilon_\mu) = h_\lambda h_\mu (c_{ij}^\alpha)$, so

(6.4) *If $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$, then for $1 \le \alpha \le m$ the $e_\alpha \times e_\alpha$ matrix $C_\alpha = (c_{ij}^\alpha)$ is a positive semidefinite hermitian matrix with all its eigenvalues in the interval $[0, 1/h_\lambda h_\mu]$.*

If ζ_α, ζ_β and ζ_γ are linear characters, then (6.4) becomes

$$(6.5) \qquad 0 \le \sum_{f \in \mathcal{O}} \frac{\zeta_\alpha(\Phi_f)\, \zeta_\beta(\Phi_f)\, \overline{\zeta_\gamma(\Phi_f)}}{|f|^2} \le \frac{1}{z_\alpha z_\beta}.$$

The first inequality of (6.5) is that of [16] in the case considered there.

That the Krein condition is already effective for rank 3 (as already observed by Scott) will be shown in Section 12. An application to generalized polygons, which does not depend on any of the rest of Part I or on Part II, is given in Part III.

7. The centralizer algebra

We apply the theory of centralizer algebras. Let $V = V(C)$ be the centralizer algebra of C,

$$V = \{\psi \in \mathrm{Mat}_{\mathbf{C}} X \mid \psi\varphi = \varphi\psi \text{ for all } \varphi \in C\}.$$

Then V is semisimple and the central primitive idempotents $\varepsilon_1, \ldots, \varepsilon_m$ of C coincide with those of V. If $\chi_1, \ldots, \chi_m$ are the corresponding irreducible characters of V, then $\chi_i(1) = z_i$, $1 \le i \le m$. The irreducible character of V corresponding to the principal idempotent is called the *principal character* of V. The *standard character* of V is the character $\chi = \sum_{i=1}^m e_i \chi_i$ afforded by $\mathbf{C}X$.

The centers of C and V are both equal to $C \cap V$, and

(7.1) $\chi_\alpha(\psi) = (z_\alpha / e_\alpha)\zeta_\alpha(\psi)$ *for* $\psi \in C \cap V$, $i \le \alpha \le m$. Setting $c(\varphi) = \sum_{f \in \mathcal{O}} \tilde{\Phi}_f \varphi \Phi_f$ for $\varphi \in C$ defines a linear map c of C onto its center.

(7.2) $\Delta_\alpha(c(\varphi)) = (\zeta_\alpha(\varphi)/z_\alpha) I_{e_\alpha}$ *for* $\varphi \in C$, $1 \le \alpha \le m$. Namely, the (i, l)-coefficient of $\Delta_\alpha(c(\varphi))$ is

$$\sum_{i,j,k} a_{ij}^\alpha(\tilde{\Phi}_f)\, a_{jk}^\alpha(\varphi)\, a_{kl}^\alpha(\Phi_f)$$

$$= \sum_{j,k} a_{jk}^\alpha(\varphi) \sum_f a_{ij}^\alpha(\tilde{\Phi}_f)\, a_{kl}(\Phi_f)$$

$$= \delta_{il} \sum_j a_{jj}^\alpha(\varphi) \frac{1}{z_\alpha} = \delta_{il} \frac{\zeta_\alpha(\varphi)}{z_\alpha}$$

by the Schur relations (3.8). By (7.1) and (7.2),

(7.3) $\chi_\alpha(c(\varphi)) = \zeta_\alpha(\varphi)$ *for* $\varphi \in C$, $1 \le \alpha \le m$.

In the group case, let $\Pi: G \to \mathrm{Mat}_{\mathbf{C}} X$ be the permutation representation of G affording $(X, \mathcal{O})$. Then $V = \langle \Pi(G) \rangle_{\mathbf{C}}$ and $C = V(V(C))$ is the centralizer algebra of Π. The permutation character is $\chi\Pi = \sum_{i=1}^m e_i (\chi_i \Pi)$ and $\chi_i \Pi$, $1 \le i \le m$, are its irreducible constituents. In particular, in the group case, the irreducible degrees and multiplicities of $(X, \mathcal{O})$ are the multiplicities and degrees of the irreducible constituents of the permutation character.

In the general case, $\mathrm{Aut}^*(X, \mathcal{O})$ is isomorphic with the group of all permutation matrices in V. If G is a group acting on X as a group of strict automorphisms of $(X, \mathcal{O})$ and $\Pi: G \to \mathrm{Mat}_{\mathbf{C}} X$ is the corresponding permutation representation of G, then $\chi\Pi$ is the permutation character of G. The characters $\chi_i \Pi$ are all irreducible if and only if G affords $(X, \mathcal{O})$.

Recall that an absolutely irreducible character ζ_α of Π is of the *first kind* if it is afforded by a real representation, of the *second kind* if it is afforded by a representation which is equivalent to its complex conjugate but is not of the first kind, and of the *third kind* if it is not of the first or second kind. We say that χ_α is of the same kind as ζ_α. We want to prove the following result which is well-known for the group case.

(7.4) *The number of symmetric $f \in \mathcal{O}$ is equal to the number of irreducible constituents of χ of the first kind minus the number of the second kind, counted with their multiplicities.*

We begin by proving the version of a theorem of Frobenius and Schur which holds for adjacency algebras.

(7.5) *Put* $\nu(\alpha) = \sum_{f \in \mathcal{O}} (1/|f|) \zeta_\alpha(\Phi_f^2)$. *Then*

$$\nu(\alpha) = \begin{cases} (e_\alpha/z_\alpha) & \text{if } \zeta_\alpha \text{ is of the first kind} \\ -(e_\alpha/z_\alpha) & \text{if } \zeta_\alpha \text{ is of the second kind} \\ 0 & \text{if } \zeta_\alpha \text{ is of the third kind.} \end{cases}$$

Proof. We may suppose that $\Delta_\alpha(\varphi)^* = \Delta_\alpha(\varphi^*)$ for all α and all $\varphi \in C$, i.e., that $\overline{a_{ij}^\alpha(\Phi_f)} = a_{ji}^\alpha(\Phi_{f^\cup})$ for all α, i, j and all $f \in \mathcal{O}$.

Now $\zeta_\alpha(\Phi_f^2) = \sum_{i,j} a_{ij}^\alpha(\Phi_f) a_{ji}^\alpha(\Phi_f)$, so $\nu(\alpha) = \sum_{i,j} \sum_f a_{ij}^\alpha(\Phi_f) \overline{a_{ij}^\alpha(\check{\Phi}_f)}$. By the Schur relations, if ζ_α is of the third kind, then $\nu(\alpha) = 0$. Suppose that ζ_α is of the first or second kind. Then there is a unitary matrix $U = (u_{ij})$ such that $\overline{\Delta_\alpha(\varphi)} = U^{-1} \Delta_\alpha(\varphi) U$ for all $\varphi \in C$, and such that $U^t = U$ or $-U$ according as ζ_α is of the first or second kind. Then $\overline{a_{ij}(\varphi)} = \pm \sum_{k,l} \bar{u}_{ik} a_{kl}^\alpha(\varphi) u_{lj}$, so

$$\nu(\alpha) = \pm \sum_{i,j,k} \bar{u}_{ik} u_{lj} \sum_f a_{ij}^\alpha(\Phi_f)\, a_{kl}^\alpha(\check{\Phi}_f)$$

$$= \pm \sum_{i,j} \bar{u}_{ij} u_{ij} \frac{1}{z_\alpha} = \pm \frac{e_\alpha}{z_\alpha}. \quad \square$$

(7.4) is equivalent to

(7.6) *The number of symmetric $f \in \mathcal{O}$ is $\Sigma' e_\alpha - \Sigma'' e_\alpha$, where the first sum is over the α such that ζ_α is of the first kind and the second sum is over the α such that ζ_α is of the second kind.*

Proof. We evaluate the sum $\sum_{f \in \mathcal{O}} (1/|f|) \zeta(\Phi_f^2)$, where $\zeta = \sum_{i=1}^m z_\alpha \zeta_\alpha$ is the standard character of C, in two ways. On the one hand this sum is equal to $\sum_{\alpha=1}^m z_\alpha \nu(\alpha)$, and so to $\Sigma' e_\alpha - \Sigma'' e_\alpha$ by (7.5). On the other hand, $(1/|f|)\zeta(\Phi_f^2) = \delta_{ff^\cup}$, so the sum is equal to the number of symmetric $f \in \mathcal{O}$. $\square$

The adjacency algebra of the configuration afforded by the regular action of a group G on itself is anti-isomorphic with the group algebra CG in such a way that the standard basis corresponds to the group elements and the standard character to the regular character. Thus group algebras are a special class of adjacency algebras and such results as (3.8), (3.9) and (7.6) reduce to standard results in this case (with $e_\alpha = z_\alpha$, $1 \le \alpha \le m$, and $|f| = |G|$

for all $f \in \mathcal{O}$). In particular, (7.6) reduces to a well-known result about involutions.

Further standard results can be obtained by letting $G \times G$ act on G according to

$$x^{(g,h)} = g^{-1}xh \qquad (x, g, h \in G).$$

In this case the adjacency algebra is isomorphic with the center of $\mathbf{C}G$ in such a way that the standard basis corresponds to the class sums.

8. Common constituents

We know by Section 2 that the full subconfiguration based on a union of fibers is coherent and it is natural to ask how its irreducible degrees, etc., are related to those of the configuration $(X, \mathcal{O})$.

Assume that $X = X^{(1)} + X^{(2)}$ where $X^{(i)}$ is a nonempty union of fibers of $(X, \mathcal{O})$, $i = 1, 2$, and put $\mathcal{O}_{ij} = \{f \in \mathcal{O} \mid f \subseteq X^{(i)} \times X^{(j)}\}$ and put $\varrho_{ij} = |\mathcal{O}_{ij}|$. Then $[X^{(i)}] = (X^{(i)}, \mathcal{O}_{ii})$ is coherent of rank ϱ_{ii}.

We use a superscript (i) to indicate parameters attached to $[X^{(i)}]$. Thus $C^{(i)}$ is the adjacency algebra of $[X^{(i)}]$, $\varepsilon_\alpha^{(i)}$ are its central primitive idempotents, and $e_\alpha^{(i)}$ and $z_\alpha^{(i)}$ are the corresponding irreducible degrees and multiplicities, $1 \le \alpha \le m^{(i)}$. $V^{(i)}$ is the centralizer algebra (in $\mathrm{Mat}_{\mathbf{C}} X^{(i)}$) of $C^{(i)}$ and $\chi^{(i)} = \sum_{\alpha=1}^{m^{(i)}} e_\alpha^{(i)} \chi_\alpha^{(i)}$ is the standard character of $V^{(i)}$, afforded by $\mathbf{C}X^{(i)}$, where $\chi_\alpha^{(i)}$ is the irreducible character of $V^{(i)}$ corresponding to $\varepsilon_\alpha^{(i)}$. As usual we choose the notation so that $\varepsilon_1^{(i)}$ and $\chi_1^{(i)}$ are principal.

The adjacency algebra C of $(X, \mathcal{O})$ has the vector space decomposition

$$C = C_{11} \oplus C_{12} \oplus C_{21} \oplus C_{22},$$

where $C_{ij} = \langle \Phi_f \mid f \in \mathcal{O}_{ij} \rangle_{\mathbf{C}}$, and $\dim C_{ij} = \varrho_{ij}$. We identify C_{ii} with $C^{(i)}$ under the isomorphism $\sigma \mapsto \sigma \mid X^{(i)} \times X^{(i)}$, $\sigma \in C_{ii}$.

Take $\phi \in V = V(C)$, then ϕ commutes with every $\sigma \in C^{(i)}$. Taking $\sigma = \sum_{\alpha=1}^{m^{(i)}} \varepsilon_\alpha^{(i)}$ we see that $\phi \mid X^{(i)} \times X^{(j)} = 0$ for $i \ne j$. So $\mathbf{C}X = \mathbf{C}X^{(1)} \oplus \mathbf{C}X^{(2)}$ as V-module and we have representations

$$\Pi_i: \begin{cases} V \to \mathrm{Mat}_{\mathbf{C}}\, X^{(i)} \\ \phi \mapsto \phi \mid X^{(i)} \times X^{(i)} \end{cases}$$

with $\Pi_i(V) \subseteq V^{(i)}$. For the standard character χ of V we have $\chi = \lambda_1 + \lambda_2$ where

$$\lambda_i = \chi_\alpha^{(i)} \Pi_i = \sum_{\alpha=1}^{m^{(i)}} e_\alpha^{(i)} (\chi^{(i)} \Pi_i)$$

is the character of V afforded by Π_i. We show that $\chi_\alpha^{(i)}\Pi_i$ is irreducible and $\Pi_i(V)=V^{(i)}$ for $1\le\alpha\le m^{(i)}$, $i=1,2$.

If μ and ν are characters of V and $\mu=\sum_{i=1}^m\alpha_i\chi_i$, $\nu=\sum_{i=1}^m\beta_i\chi_i$ are the decompositions into irreducible constituents, we write $\langle\mu,\nu\rangle=\sum_{i=1}^m\alpha_i\beta_i$. Now

$$\begin{aligned}\varrho_{11}+2\varrho_{12}+\varrho_{22} &= \dim_{\mathbf{C}} C\\ &= \langle\chi,\chi\rangle\\ &= \langle\lambda_1,\lambda_1\rangle+2\langle\lambda_1,\lambda_2\rangle+\langle\lambda_2,\lambda_2\rangle\end{aligned}$$

and

$$\begin{aligned}\langle\lambda_i,\lambda_i\rangle &= \sum_\alpha (e_\alpha^{(i)})^2\langle\chi_\alpha^{(i)}\Pi_i,\chi_\alpha^{(i)}\Pi_i\rangle\\ &\ge \sum_\alpha (e_\alpha^{(i)})^2=\varrho_{ii}.\end{aligned}$$

But

$$\begin{cases} C_{12}\to \mathrm{Mat}_{\mathbf{C}}(X^{(1)},X^{(2)})\\ \phi\quad\mapsto \phi\mid X^{(1)}\times X^{(2)}\end{cases}$$

maps C_{12} isomorphically onto the intertwining module $V(\Pi_1,\Pi_2)$ so $\varrho_{12}=\langle\lambda_1,\lambda_2\rangle$. The irreducibility of the $\chi_\alpha^{(i)}\Pi_i$ and the equality $\Pi_i(V)=V^{(i)}$ now follow.

We put $m^{(1,2)}$ equal to the number of distinct irreducible constituents common to λ_1 and λ_2 and choose the numbering so that

$$\chi_\alpha^{(1)}\Pi_1=\chi_\alpha^{(2)}\Pi_2 \qquad\text{for}\qquad 1\le\alpha\le m^{(1,2)}.$$

Then

$$\varrho_{12}=\langle\lambda_1,\lambda_2\rangle=\sum_{\alpha=1}^{m^{(1,2)}}e_\alpha^{(1)}e_\alpha^{(2)}. \tag{8.1}$$

Note that

(8.2) *The maximum rank of* $\phi\in C_{12}$ *is*

$$\sum_{\alpha=1}^{m^{(1,2)}}\min\{e_\alpha^{(1)},e_\alpha^{(2)}\}\cdot z_\alpha.$$

Write $\chi_{i\alpha}=\chi_\alpha^{(i)}\Pi_i$, $1\le\alpha\le m^{(i)}$, $i=1,2$, and $\chi_\alpha=\chi_{1\alpha}=\chi_{2\alpha}$, $1\le\alpha\le m^{(1,2)}$. Then

$$\chi=\sum_{\alpha=1}^{m^{(1,2)}}(e_\alpha^{(1)}+e_\alpha^{(2)})\chi_\alpha;\quad \sum_{i=1}^{2}\ \sum_{\alpha=m^{(1,2)}+1}^{m^{(i)}}e_\alpha^{(i)}\chi_{i\alpha} \tag{8.3}$$

is the decomposition of χ *into irreducible constituents.*

The reduced degree of C is

(8.4) $$m = m^{(1)} + m^{(2)} - m^{(1,2)}$$

and

(8.5) *The irreducible degrees and multiplicities for* $(X, \mathcal{O})$ *are* $e_\alpha^{(1)}+e_\alpha^{(2)}$ *with multiplicities* $z_\alpha^{(1)}=z_\alpha^{(2)}$, $1\le\alpha\le m^{(1,2)}$, *and* $e_\alpha^{(i)}$ *with multiplicity* $z_\alpha^{(i)}$, $m^{(1,2)}<\alpha\le m^{(i)}$, $i=1, 2$.

(8.6) *Assume that* $[X^{(1)}]$ *and* $[X^{(2)}]$ *are commutative (and so are fibers). Then*

(a) *the number of irreducible constituents common to* $\chi^{(1)}$ *and* $\chi^{(2)}$ *is* r_{12}, *and*

(b) *assuming further that* $r_{11}=r_{12}$ *and that* G *is a group of strict automorphisms of* $(X, \mathcal{O})$, *we have that*

(i) *the number of* G*-orbits in* $X^{(1)}\le$ *the number of* G*-orbits in* $X^{(2)}$, *and*

(ii) *for* $x\in X^{(1)}$, $y\in X^{(2)}$ *and* G *transitive on* $X^{(2)}$. *The number of* G_x*-orbits in* $X^{(1)}\le$ *the number of* G_x*-orbits in* $X^{(2)}=$ *the number of* G_y*-orbits in* $X^{(1)}$ $=$ *the number of* G_y*-orbits in* $X^{(2)}$.

Proof. Commutativity of the $X^{(i)}$ implies that all the $e_\alpha^{(i)}$ are 1, so (a) follows from (8.1). In particular, if $r_{11}=r_{12}$, then $\chi^{(2)}=\chi^{(1)}+\theta$, with θ a character of V. If G is a group of strict automorphisms of $(X, \mathcal{O})$ and $\Pi: G\to \mathrm{Mat}_{\mathbf{C}} X$ is the permutation representation, then $\chi^{(i)}\Pi$ is the permutation character afforded by the action of G on $X^{(i)}$. Now (1) and (ii) of (b) follow by standard arguments.

It is easy to see from the above considerations that the primitive central idempotents of V (= those of C) are

$$\varepsilon_\alpha = \varepsilon_\alpha^{(1)} + \varepsilon_\alpha^{(2)}, \qquad 1 \le \alpha \le m^{(1,2)}$$

and

$$\varepsilon_\alpha^{(i)}, m^{(1,2)} < \alpha \le m^{(i)}, \qquad i = 1, 2.$$

Hence the simple components of C are $C=C_\alpha^{(1)}\oplus U_\alpha\oplus U_\alpha^t\oplus C_\alpha^{(2)}$, $1\le\alpha \le m^{(1,2)}$, and $C_\alpha^{(i)}$, $m^{(1,2)}<\alpha\le m^{(i)}$, $i=1, 2$, where $C_\alpha^{(i)}=\varepsilon_\alpha^{(i)}C^{(i)}$ are the simple components of $C^{(i)}$ and $U_\alpha=\varepsilon_\alpha^{(1)}C_{12}\varepsilon_\alpha^{(2)}$. Thus $C_{12}=U_1\oplus\cdots\oplus U_{m^{(1,2)}}$ where each U_α is the sum of the members of anisomorphism class of irreducible submodules of C_{12} as a $(C^{(1)}, C^{(2)})$-bimodule. The corresponding components of M are $M_\alpha=M_\alpha^{(1)}+M_\alpha^{(2)}$, $1\le\alpha\le m^{(1,2)}$, and $M^{(i)}$, $m^{(1,2)}<\alpha \le m^{(i)}$, $i=1, 2$, where $M_\alpha^{(i)}=\varepsilon_\alpha^{(i)}\mathbf{C}X^{(i)}$.

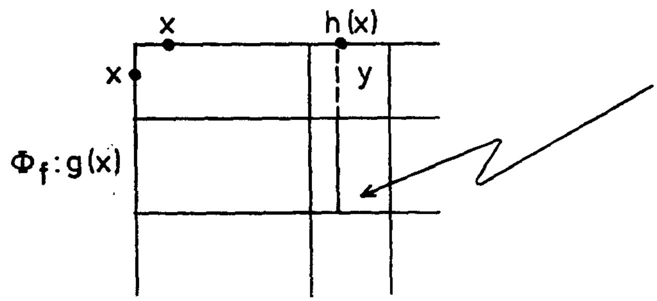
If $\varepsilon_1^{(i)}$ and $\zeta_1^{(i)}$ are principal, $i=1, 2$, then ε_1 and ζ_1 are principal.

9. Intersection matrices

The matrices $\hat{\phi}$, $\phi \in \Gamma$ introduced at the end of Section 2 can be broken up into sums of matrices $\phi^{(\alpha)}$, $1 \leq \alpha \leq t$, as follows. Choose $x \in X_\alpha$, then $\{g(x) \mid g \in \mathcal{O}, \operatorname{dom} g = X_\alpha\}$ is a partition of X. Take $f, g, h \in \mathcal{O}$ with $\operatorname{dom} g = \operatorname{dom} h = X_\alpha$. For $(x, y) \in h$,

$$\sum_{z \in g(x)} \Phi_f(z, y) = |g(x) \cap f^{\cup}(y)| = a_{gfh},$$

which is independent of the choice of $(x, y) \in h$. That is, each block of Φ_f, blocked according to this partition of X, has constant column sum. This block has column sum a_{gfh}



A matrix $\Phi_f^{(\alpha)} \in \operatorname{Mat}_Z \mathcal{O}$, $f \in \mathcal{O}$, is obtained on putting

$$\Phi_f^{(\alpha)}(g, h) = \begin{cases} a_{gfh} \text{ if } \operatorname{dom} g = \operatorname{dom} h = X_\alpha \\ 0 \text{ otherwise} \end{cases}.$$

For $\phi \in \Gamma$, $\phi = \sum_{f \in \mathcal{O}} a_f \Phi_f$, $a_f \in Z$, and we can put $\phi^{(\alpha)} = \sum a_f \Phi_f^{(\alpha)} \in \operatorname{Mat}_Z \mathcal{O}$, and $\phi^{(\alpha)}$ can be obtained from ϕ by the same blocking process. It is clear that $\hat{\phi} = \sum_{\alpha=1}^{t} \phi^{(\alpha)}$, and it follows that $\phi \mapsto \phi^{(\alpha)}$ is a ring homomorphism of Γ onto a subring $\Gamma^{(\alpha)}$ of $\operatorname{Mat}_Z \mathcal{O}$, such that $\hat{\Gamma} = \sum_{\alpha=1}^{t} \Gamma^{(\alpha)}$. Of course in the homogeneous case, $\phi^{(1)} = \hat{\phi}$ and $\Gamma^{(1)} = \hat{\Gamma}$.

In the inhomogeneous case it is often convenient to discard blocks of zeros. We have

$$\Gamma = \sum_{1 < i, j < t} \oplus \Gamma^{ij} \qquad \text{(vector space direct sum)}$$

where $\Gamma^{ij} = \langle \Phi_f \mid f \in \mathcal{O}^{ij} \rangle_Z$. For $\phi \in \Gamma^{ij}$, $\phi \mid X_\alpha \times X_\beta = 0$ unless $\alpha = i$ and $\beta = j$. Thus $\phi \mapsto \phi \mid X_i \times X_j$ is an isomorphism of Γ^{ij} onto a subgroup M^{ij} of $\operatorname{Mat}_Z(X_i, X_j)$. M^{ij} has basis $\{\mu_f \mid f \in \mathcal{O}^{ij}\}$, where

$$\mu_f = \Phi_f \mid X_i \times X_j$$

and

(9.1) *For* $f \in \mathcal{O}^{ij}$ *and* $g \in \mathcal{O}^{jk}$,

$$\mu_f \mu_g = \sum_{h \in \mathcal{O}^{ik}} a_{fgh} \mu_h .$$

In particular, $\Gamma^{ii} \to M^{ii}$ is an isomorphism of Γ^{ii} onto the adjacency ring M^{ii} of $[X_i]$ and $\{\mu_f \mid f \in \mathcal{O}^{ii}\}$ is the standard basis of M^{ii}.

We have further that

$$\Gamma^{(\alpha)} = \sum_{1<i,j<t} \oplus \Gamma_\alpha^{ij} \qquad \text{(vector space direct sum)}$$

when $\Gamma_\alpha^{ij} = \{\phi^{(\alpha)} \mid \phi \in \Gamma^{ij}\}$. If $\phi \in \Gamma^{ij}$, then $\phi^{(\alpha)}(g, h) = 0$ if $(g, h) \notin \mathcal{O}^{\alpha i} \times \mathcal{O}^{\alpha j}$. Therefore $\phi^{(\alpha)} \mapsto \phi^{(\alpha)} \mid \mathcal{O}^{\alpha i} \times \mathcal{O}^{\alpha j}$ is an isomorphism of Γ^{ij} onto a subgroup M_α^{ij} of $\mathrm{Mat}_{\mathbf{Z}}(\mathcal{O}^{\alpha i}, \mathcal{O}^{\alpha j})$. Clearly the diagram

$$(*) \qquad \begin{array}{ccc} \Gamma^{ij} & \to & M^{ij} \\ \downarrow & & \downarrow \\ \Gamma_\alpha^{ij} & \to & M_\alpha^{ij} \end{array}$$

commutes. For $f \in \mathcal{O}^{ij}$ we put $\mu_f^{(\alpha)} = \Phi_f^{(\alpha)} \mid \mathcal{O}^{\alpha i} \times \mathcal{O}^{\alpha j}$, then $\{\mu_f^{(\alpha)} \mid f \in \mathcal{O}^{ij}\}$ is a basis of M_α^{ij}.

We call $(\mu_f^{(\alpha)} \mid f \in \mathcal{O}, 1 \leq \alpha \leq t)$ the family of *basic intersection matrices* for $(X, \mathcal{O})$. Then $(\mu_f^{(\alpha)} \mid f \in \mathcal{O}^{\alpha\alpha})$ is the family of basic intersection matrices for $[X_\alpha]$. If $(X, \mathcal{O})$ is homogeneous, then $\mu_f^{(1)} = \hat{\Phi}_f$ and $\hat{\mathscr{B}} = (\hat{\Phi}_f \mid f \in \mathcal{O})$ is the family of basic intersection matrices in this case. We have

(9.2) *If* $f \in \mathcal{O}^{ij}$, *then* $\mu_f^{(\alpha)} \in \mathrm{Mat}_{\mathbf{Z}}(\mathcal{O}^{\alpha i}, \mathcal{O}^{\alpha j})$ *and* $\mu_f^{(\alpha)}(g, h) = a_{gfh}$ *for* $g \in \mathcal{O}^{\alpha i}$, $h \in \mathcal{O}^{\alpha j}$.

(9.3) *For* $f \in \mathcal{O}^{ij}$ *and* $g \in \mathcal{O}^{jk}$,

$$\mu_f^{(\alpha)} \mu_g^{(\alpha)} = \sum_{h \in \mathcal{O}^{ik}} a_{fgh} \mu_h^{(\alpha)} .$$

Knowing the family of basic intersection matrices is equivalent to knowing the family of intersection numbers. Conditions on the intersection numbers, especially (2.15) through (2.19) are readily interpreted as conditions on the basic intersection matrices. We translate some of these here which are often used in matrix form.

(9.4) $\mu_{E_i}^{(\alpha)}$ *is the identity matrix in* $\mathrm{Mat}_{\mathbf{Z}} \mathcal{O}^{\alpha i}$.

(9.5) *If* $f \in \mathcal{O}^{ij}$, *then* $\mu_f^{(\alpha)} \varrho = \sigma(\mu_{f^\cup}^{(\alpha)})^t$ *where* $\varrho \in \mathrm{Mat}_{\mathbf{Z}} \mathcal{O}^{\alpha j}$ *and* $0 \in \mathrm{Mat}_{\mathbf{Z}} \mathcal{O}^{\alpha i}$ *are diagonal matrices such that* $\varrho(h, h) = n_h$ *for* $h \in \mathcal{O}^{\alpha j}$ *and* $\sigma(g, g) = n_g$ *for* $g \in \mathcal{O}^{\alpha i}$.

(9.6) $\mu_f^{(\alpha)}$ *has column sum* $n_{f^\cup}$ *and* $\sum_{f \in \mathcal{O}^{ij}} \mu_f^{(\alpha)}(g, h) = n_g$ *for all* $g \in \mathcal{O}^{\alpha i}$, $h \in \mathcal{O}^{\alpha j}$.

10. Coherent partitions and refinements

Let $(X, \mathcal{O})$ be coherent. A partition $\mathcal{P}$ of X is *coherent* (for $(X, \mathcal{O})$) if, for all $S, T, \in \mathcal{P}$ and all $f \in \mathcal{O}$, the number of f-edges from S to a point $y \in T$ is independent of the choice of $y \in T$. We then denote this number by $\Phi_f^{\mathcal{P}}(S, T)$, thus defining $\Phi_f^{\mathcal{P}} \in \mathrm{Mat}_{\mathbf{Z}}\mathcal{P}$ such that

$$\Phi_f^{\mathcal{P}}(S, T) = |S \cap f^{\cup}(y)| = \sum_{x \in S} \Phi_f(x, y)$$

for $S, T \in \mathcal{P}$ and $y \in T$. Thus, a partition $\mathcal{P}$ of X is coherent if and only if, for all $f \in \mathcal{O}$, the blocks of Φ_f blocked according to $\mathcal{P}$ have constant column sum; and then $\Phi_f^{\mathcal{P}}$ is obtained from Φ_f be replacing each block by its column sum. For $\phi \in \Gamma$ we can write $\phi^{\mathcal{P}}$ for the matrix in $\mathrm{Mat}_{\mathbf{Z}}\mathcal{P}$ obtained from ϕ in this way. It follows that

(10.1) *If $\mathcal{P}$ is a coherent partition for $(X, \mathcal{O})$, then $\phi \mapsto \phi^{\mathcal{P}}$ is a homorphism of Γ onto a subring $\Gamma^{\mathcal{P}}$ of* $\mathrm{Mat}_{\mathbf{Z}}\mathcal{P}$.

Let $\mathcal{P}$ be a coherent partition for $(X, \mathcal{O})$ and let $S \in \mathcal{P}$. Then $S \cap X_i \neq \phi$ for some i, and if $x \in S \cap X_i$ and $y \in S$, then the number of E_i-edges from S to x, namely one, is equal to the number from S to y. Hence $y \in X_i$ and $S \subseteq X_i$. In particular

(10.2) *If $\mathcal{P}$ is a coherent partition for $(X, \mathcal{O})$, then $\{S \in \mathcal{P} \mid S \subseteq X_i\}$ is a coherent partition of $[X_i]$, $i = 1, 2, \ldots, t$.*

A first example of a coherent partition arose in Section 10 where we chose $x \in X_\alpha$ and showed that $\mathcal{P} = \{g(x) \mid g \in \mathcal{O}, \mathrm{dom}\, g = X_\alpha\}$ satisfies the coherence condition. In this case, for $\phi \in \Gamma$ and $g, h \in \mathcal{O}$ such that $\mathrm{dom}\, g = \mathrm{dom}\, h = X_\alpha$, $\phi^{\mathcal{P}}(g(x), h(x)) = \phi^{(\alpha)}(g, h)$ as defined in Section 9.

A refinement $\mathcal{O}_0$ of the partition $\mathcal{O}$ of X^2 will be called *coherent* if the subconfiguration $(X, \mathcal{O}_0)$ of $(X, \mathcal{O})$ is coherent. We also refer to $(X, \mathcal{O}_0)$ as a *coherent refinement* of $(X, \mathcal{O})$ in this case.

(10.3) *The fibers of a coherent refinement $(X, \mathcal{O}_0)$ of $(X, \mathcal{O})$ constitute a coherent partition of $(X, \mathcal{O})$.*

Proof. Let S, T be fibers for $(X, \mathcal{O}_0)$ and take $f \in \mathcal{O}$. Then $f = \sum_{g \in \mathcal{Q}} g$ for a uniquely determined subset $\mathcal{Q}$ of $\mathcal{O}_0$. The number of f-edges from S to $y \in T$ is the sum for $g \in \mathcal{Q}$ of the numbers of g-edges from S to $y \in T$, and so is equal to $\sum_{g \in \mathcal{Q}_0} n_{g^{\cup}}$, where $\mathcal{Q}_0 = \{g \in \mathcal{Q} \mid \mathrm{dom}\, g = S \text{ and range } g = T\}$. □

(10.4) *The subconfiguration afforded by a group H acting as a group of strict automorphisms of $(X, \mathcal{O})$ is a coherent refinement of $(X, \mathcal{O})$, and hence the set of H-orbits in X is a coherent partition for $(X, \mathcal{O})$.*

(10.4) is an immediate corollary to (9.3) since the orbits for H in X are the fibers of the configuration afforded by H. Of course (10.4) is also easily obtained directly.

11. Fusion

As always, $(X, \mathcal{O})$ is coherent. Suppose that a partition $\tilde{\mathcal{O}}$ of X^2 is obtained by *fusion* from $\mathcal{O}$, i.e., that $\mathcal{O}$ is a refinement of $\tilde{\mathcal{O}}$. This means that there is an equivalence relation $\sim$ on $\mathcal{O}$ such that

$$\tilde{\mathcal{O}} = \{\tilde{f} \mid f \in \mathcal{O}\},$$

where $\tilde{f} = \sum_{g \sim f} g$.

(11.1) *$(X, \tilde{\mathcal{O}})$ is coherent if and only if*

(i) *$f \sim E_i$ for some i implies $f = E_j$ for some j,*
(ii) *$f \sim g$ implies $f^{\cup} \sim g^{\cup}$, and*
(iii) *for all $f, g, h, h_1 \in \mathcal{O}$, $h \sim h_1$*

implies that $\sum_{\substack{u \sim f \\ v \sim g}} a_{uvh} = \sum_{\substack{u \sim f \\ v \sim g}} a_{uvh_1}$.

If $(X, \tilde{\mathcal{O}})$ is coherent, then its intersection numbers are $a_{\tilde{f}\tilde{g}\tilde{h}} = \sum_{\substack{u \sim f \\ v \sim g}} a_{uvh}$.

(11.2) *Let H be a group acting on $\mathcal{O}$ and suppose that $\tilde{\mathcal{O}}$ is obtained by fusion of the H-orbits, so that for $f \in \mathcal{O}$, $\tilde{f} = \cup_{\sigma \in H} f^{\sigma}$. Then each of the following conditions* (a) *and* (b) *implies coherence of $(X, \tilde{\mathcal{O}})$.*

(a) (i) *for each $f \in \mathcal{O}$ and $\sigma \in H$ there exists a $\tau \in H$ such that $(f^{\sigma})^{\cup} = (f^{\cup})^{\tau}$, and*
(ii) *each $\sigma \in H$ induces an automorphism of Γ according to $\Phi_f \mapsto \Phi_{f^\sigma}$ for $f \in \mathcal{O}$.*
(b) *the action of H on $\mathcal{O}$ is induced by an action of H as a group of automorphism of $(X, \mathcal{O})$.*

(11.2a) is a consequence of (11.1) and (11.2b) follows from (11.2a). In particular, (11.2b) implies that fusion of the orbits for $\mathrm{Aut}(X, \mathcal{O})$ in $\mathcal{O}$ yields a coherent configuration, which, in the group case will be just the configuration afforded by $\mathrm{Aut}(X, \mathcal{O})$, and in general will be a refinement of that.

(11.3) *Assume that $\tilde{\mathcal{O}}$ is obtained from $\mathcal{O}$ by fusing each $f \in \mathcal{O}$ with its converse $f^{\cup}$, so that $\tilde{f} = f \cup f^{\cup}$. Then each of the following conditions* (a) *through* (c) *implies coherence of $(X, \tilde{\mathcal{O}})$.*

(a) *$\Phi_{\tilde{f}}$ and $\Phi_{\tilde{g}}$ commute for all $f, g \in \mathcal{O}$.*
(b) *Γ is commutative.*
(c) *there is a subgroup H of* $\mathrm{Aut}(X, \mathcal{O})$ *having $\{\{f \cup f^{\cup}\} \mid f \in \mathcal{O}\}$ as its set of orbits in $\mathcal{O}$.*

(11.3a) follows from (11.1), (11.3b) from (11.2a) or (11.3a), and (11.3c) from (11.2b).

Remarks about (11.3)

In the situation of (11.3):

(i) Coherence of $(X, \mathcal{O})$ implies homogeneity of $(X, \mathcal{O})$.

(ii) We do not have examples of homogeneous configurations $(X, \mathcal{O})$ for which $(X, \tilde{\mathcal{O}})$ is incoherent, not do we have such examples for which (11.3c) fails to hold.

(iii) It is easy to prove, and presumably well-known, that if a group G acts transitively on X and has a regular elementary Abelian normal subgroup V, then there is an involution $\sigma \in N(G)$ such that $f^{\sigma} = f^{\cup}$ for all G-orbits f in X^2.

12. Configurations of Small Rank

We now take a brief look at coherent configurations $(X, \mathcal{O})$ of degree n and rank r for some small values of r.

Suppose first that $r \leq 3$. Then the Equation (2.12) implies that $t=1$, i.e., $(X, \mathcal{O})$ is homogeneous. If $r=1$, then $n=1$ and the configuration is just a loop. If $r=2$, then $\mathcal{O} = \{I, f = X^2 - I\}$ and (X, f) is a clique. This is the configuration afforded by a group acting doubly transitively on X. The first possibility for non-trivial application of our theory is $r=3$, where $\mathcal{O} = \{I, f, g\}$. In fact this case served as a model for the general development and can be used now to illustrate it.

There are two cases according as the pairing is trivial or not.

(1) *Rank 3 Configurations with Trivial Pairing*

In this case (X, f) and (X, g) are a pair of complementary strongly regular graphs, and every strongly regular graph arises in this way from a coherent rank 3 configuration with trivial pairing. A rank 3 permutation group of even order affords a configuration of this kind. We refer to [9] and [8] for discussions of rank 3 permutation groups and strongly regular graphs.

In the notation of [8], $A = \Phi_f$, $B = \Phi_g$, $k = n_f$, $l = n_g$, $\lambda = a_{fff}$ and $\mu = a_{ffg}$. The intersection matrices $\neq 1$ are

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ k & \lambda & \mu \\ 0 & k-\lambda-1 & k-\mu \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & l-\bar{\mu} & l-\bar{\lambda}-1 \\ l & \bar{\mu} & \bar{\lambda} \end{pmatrix}$$

with $\bar{\lambda}=l-k+\mu-1$ and $\bar{\mu}=l-k+\lambda+1$ and $\mu l=k(k-\lambda-1)$. These conditions, together with the assumption that the entries of $\hat{A}$ and $\hat{B}$ are non-negative integers embody all the conditions on intersection numbers mentioned in this paper, except for the integrality of the multiplicities z_2 and z_3 and the Krein condition.

The character table is

	I	A	B
ζ_1	1	k	l
ζ_2	1	r	$-(r+1)$
λ_3	1	s	$-(s+1)$

where $\{{r \atop s}\}=(\lambda-\mu\pm\sqrt{d})/2$ with $d=(\lambda-\mu)^2+4(k-\mu)$. Namely k, r and s are the eigenvalues of A and $F(A)=\mu J$ where $F(T)=T^2-(\lambda-\mu)T-(k-\mu)$ and J is the 'all 1' matrix. We have $k\geq r\geq 0>s$, with $k=r$ if and only if $\mu=0$ if and only if (X,f) is not connected, and $r=0$ if and only if $\mu=k$ if and only if (X, g) is not connected.

Various methods as in Section 5 can be used to compute z_2 and z_3. From some points of view at least, the most convenient formulas are given by solving the system (5.5), which in the present case is,

$$\begin{cases} 1 + z_2 + z_3 = n \\ k + rz_2 + sz_3 = 0 \end{cases}$$

to get

$$\begin{Bmatrix} z_2 \\ z_3 \end{Bmatrix} = \frac{1}{2}\left\{k + l \mp \frac{(k + l)(\lambda - \mu) + 2k}{\sqrt{d}}\right\}$$

as in [7]. This divides the rank 3 configurations with trivial pairing into two types, (I) and (II), according as $\Delta=0$ or $\Delta\neq 0$, where $\Delta=(k+l)(\lambda-\mu)+2k$. We have type (I) if and only if $z_2=z_3$, and then $k=l=2\mu$. For type (II) we know that d is a square and $\sqrt{d}\mid\Delta$, $2\sqrt{d}\nmid\Delta$ if n is even, while $2\sqrt{d}\mid\Delta$ if n is odd.

In the present case the Krein condition (6.5) gives the following set of inequalities.

$$0 \leq 1 + \frac{r^3}{k^2} - \frac{(r+1)^3}{l^2} \leq \frac{n^2}{z_2^2},$$

$$0 \leq 1 + \frac{s^2 r}{k^2} - \frac{(s+1)^2(r+1)}{l^2} \leq \frac{n^2}{z_2 z_3},$$

$$0 \leq 1 + \frac{r^2 s}{k^2} - \frac{(r+1)^2(s+1)}{l^2} \leq \frac{n^2}{z_2 z_3},$$

$$0 \leq 1 + \frac{s^3}{k^2} - \frac{(s+1)^3}{l^2} \leq \frac{n^2}{z_3^2}.$$

The following are some sets of rank 3 parameters which satisfy all the other conditions mentioned in this paper but fail to satisfy the Krein condition.

n	k	λ	r	s
28	9	0	1	-5
56	22	3	1	-10
63	22	1	1	-11
64	21	0	1	-11
144	65	16	1	-25
154	51	8	2	-15

(These are the first 6 on a list of such cases found by a program written by R. Scott.) The first 5 have a character value $-(r+1)=-2$ and have been eliminated by other means as part of the classification of strongly regular graphs with minimum eigenvalue -2. ([17], see also [18] and [8].)

Unfortunately the Krein condition is vacuous for type (I).

(2) *Rank 3 Configurations with Nontrivial Pairing*

A rank 3 permutation group of odd order affords such a configuration. Such a group is solvable by the Feit Thompson Theorem, and primitive. The solvable primitive rank 3 groups have been classified by Foulser [6] and, independently, by Dornhoff [5].

For a rank 3 configuration with nontrival pairing we have $k=l$ and $A^t=B$, where $k=n_f$, $l=n_g$, $g=f^{\cup}$, so $A=\Phi_f$ and $B=\Phi_g$. The intersection matrices $\neq 1$ are seen to be

$$\hat{A}=\begin{pmatrix}0 & 1 & 0\\ 0 & a & a+1\\ k & a & a\end{pmatrix} \quad\text{and}\quad \hat{B}=\begin{pmatrix}0 & 0 & 1\\ k & a & a\\ 0 & a+1 & a\end{pmatrix}$$

with $k=2a+1$. The character table is

$$\begin{pmatrix}1 & k & k\\ 1 & r & r\\ 1 & r & r\end{pmatrix},$$

where $r=(-1+i\sqrt{4a+3})/2$ and r is the complex conjugate of r. The multiplicities are $z_2=z_3=k$. The Krein condition is not an independent condition.

An analysis of homogeneous configurations of rank 4 will be made in Part V of this series of papers. We now glance briefly at inhomogeneous configurations *of small rank*. We have seen that $r\geq 4$ if $t\geq 2$ from the Equation (2.12), and the same equation implies that $r\geq 8$ if $t\geq 3$. If $t=2$ and

$4 \leq r \leq 7$, then $r_{ij}=1$ for some i, j, so $r_{12}=1$. We can always join two homogeneous configurations $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ together in a 'trivial' coherent way by forming

$$(x_1 \cup x_2, \mathcal{O}_1 \cup \mathcal{O}_2 \cup \{x_1 \times x_2, x_2 \times x_1\})$$

where we assume without essential loss in generality that X_1 and X_2 are disjoint sets. The result is a coherent configuration having X_1 and X_2 as it fibers and $r_{12}=1$. Thus the possible coherent configurations with $t=2$ and $4 \leq r \leq 7$ can be obtained by joining together trivially two configuration of ranks r_{11} and r_{22} such that $r_{11}+r_{22}=r=2$.

The first 'nontrivial' inhomogeneous case has $t=2$, $r=8$ and $r_{11}=r_{12}=r_{22}=2$. Such a configuration has two fibers X_1 and X_2, and $\mathcal{O}^{12}=\{f, g\}$, so that (X_1, X_2, f) and (X_1, X_2, g) are complementary incidence structures. By (8.6), $|X_1|=|X_2|$, so the coherence conditions imply that these incidence structures are (possibly degenerate) projective designs. Conversely, every (possibly degenerate) projective design arises from a coherent configuration in this way. The degenerate ones correspond to the cases $n_f=1$ and $n_f=|X_2|-1$.

More generally, if $(X, \mathcal{O})$ is a coherent configuration with t fibers and $r_{ij}=2$ for $1 \leq i, j \leq t$, then $(X, \mathcal{O})$ is equivalent to a family of *linked projective designs* as defined by P. J. Cameron.

In the same way we see that partial geometries (cf. [12]) are equivalent to certain coherent configurations with $t=2$ and $9 \leq r \leq 11$.

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