COHERENT CONFIGURATIONS

II: Weights

1. Introduction

We consider certain partitions $\mathcal{O}$ of $X^2$ where $X$ is a finite set, and certain matrices $w: X^2 \rightarrow \mathbb{C}$ which we refer to as weights on $(X, \mathcal{O})$. Under a suitable coherence condition there is associated with $w$ an algebra of matrices $\phi: X^2 \rightarrow \mathbb{C}$ which we call the adjacency algebra of $w$. In case $(X, \mathcal{O})$ is a coherent configuration, its adjacency algebra as defined in Part I coincides with the adjacency algebra of the 'all 1' weight. With certain exceptions the ordinary theory of Part I extends to adjacency algebras of coherent weights. Our main purpose here is to indicate these extensions.

Our primary source of motivation and examples is the fact that the centralizer algebra of a monomial representation of a finite group is the adjacency algebra of a suitable weight. It should be noted, though, that there is no shortage of coherent weights which do not belong to this 'group case'. For instance, a regular 2-graph can be viewed as arising from a weight, and this weight belongs to the group case if and only if the automorphism group of the 2-graph is 2-transitive. Also, such processes as fusion can carry one out of the group case.

In this paper, [2] is referred to as Part I, and references to (I.a) or (I.a.b) are to section a or statement (a.b) of Part I. Details are frequently omitted when the extension from Part I is straightforward.

The sections of the present paper are as follows:

1. Introduction
2. Coherent and regular weights
3. The Schur relations and theorems of Frame and Wielandt
4. The Krein condition
5. The centralizer algebra
6. Common constituents
7. Fusion
8. The Casimir operator
9. Regular 2-graphs
10. The group case

A detailed discussion of the group case is given in [4].

2. Coherent and regular weights

In this paper we consider configurations $(X, \mathcal{O})$ consisting of a finite nonempty set $X$ and a set $\mathcal{O}$ of binary relations on $X$. By $R = R(X, \mathcal{O})$ we denote the

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The boolean subalgebra generated by $\mathcal{O}$ of the boolean algebra of all subsets of $X^2$. We say that $(X, \mathcal{O})$ is based on $X$ and refer to the members of $\mathcal{O}$ and $R$ as the basic and admissible relations respectively. We call $(X, \mathcal{O})$ precohrent if the following axioms (I), (II) and (III) are satisfied.

(I) $\mathcal{O}$ is a partition of $X^2$.

(II) $i := \{(x, x) : x \in X\} \in R$.

(III) $f \in \mathcal{O}$ implies $f^{\backarrow} := \{(y, x) : (x, y) \in f\} \in \mathcal{O}$.

These are the first three of the four axioms defining coherent configurations in (I.2). The notations and terminology of (I.2) will be used freely.

Let $(X, \mathcal{O})$ be precohrent. Then any admissible relation is uniquely expressible as a union of basic relations, so in particular $i = \bigcup_{i=1}^t E_i$, $E_i \in \mathcal{O}$.

A weight on $(X, \mathcal{O})$ is a matrix $w : X^2 \to \mathbb{C}$ such that

1. $w$ is Hermitian
2. $|w(x, y)| = 0$ or 1 and $w(x, x) = 1$ for all $x, y \in X$, and
3. $\text{spt } w := \{(x, y) \in X^2 : w(x, y) \neq 0\} \in R$.

Let $w$ be a weight on $(X, \mathcal{O})$. Then $\text{spt } w = \bigcup_{f \in \mathcal{O}_w} f$, where $\mathcal{O}_w$ is a subset of $\mathcal{O}$ closed under the converse map $f \mapsto f^{\backarrow}$, and the basic components $E_i$ of $i$ are in $\mathcal{O}_w$. We call $r_w := |\mathcal{O}_w|$ the rank of $w$.

The "all 1" matrix $\Phi$ and the identity matrix $I$ are weights on $(X, \mathcal{O})$ which we refer to as the standard and trivial weights respectively. Clearly $\mathcal{O}_\Phi = \emptyset$ and $\mathcal{O}_I = \{E_1, E_2, \ldots, E_t\}$.

For $S \subseteq X^2$, let $w_S : X^2 \to \mathbb{C}$ be the matrix which coincides with $w$ on $S$ and is zero off $S$. Thus $\Phi_S$ is the characteristic function of $S$, i.e., the adjacency matrix of the graph $(X, S)$, and $w_S = w \circ \Phi_S$ where $\circ$ is the pointwise (i.e., hadamard) product. Then $w = \sum_{f \in \mathcal{O}_w} w_f$, $w_f^* = w_{f^{\backarrow}}$ for $f \in \mathcal{O}_w$, where * denotes conjugate transpose, and $w_{E_i} = \Phi_{E_i}$ for $1 \leq i \leq t$.

We let $\mathcal{Q}_w$ denote the $r_w$-dimensional linear subspace of $\text{Mat}_{\mathcal{Q}(w)}X$ spanned by $\{w_f \mid f \in \mathcal{O}_w\}$, where $\mathcal{Q}(w) := \mathcal{Q}(\{w(x, y) \mid x, y \in X\})$ and $\text{Mat}_{\mathcal{Q}(w)}X$ is the $\mathcal{Q}(w)$-algebra of matrices $\Phi : X^2 \to \mathcal{Q}(w)$. The weight $w$ will be called coherent if $\mathcal{Q}_w$ is a subalgebra of $\text{Mat}_{\mathcal{Q}(w)}X$.

If $f, g \in \mathcal{O}_w$ and $w_f w_g = \sum_{h \in \mathcal{O}} b_{fgh} w_h$ with $b_{fgh} \in \mathbb{C}$, then for $h \in \mathcal{O}$ and $(x, z) \in h$,

\begin{equation}
    b_{fgh} = \sum_{y \in X} w_f(x, y)w_g(y, z)\overline{w_h(x, z)} = \sum_{y \in f(x) \cap g(y)} \delta w(x, y, z),
\end{equation}

where $f(x) := \{y \in X \mid (x, y) \in f\}$ and $\delta w(x, y, z) := w(y, z)\overline{w(x, z)}w(x, y)$.

If $w$ is coherent, the structure constants for $\mathcal{Q}_w$ with respect to the basis $\{w_f \mid f \in \mathcal{O}_w\}$ are given by (1.1), and $b_{fgh} = 0$ if not all of $f, g$ and $h$ are in $\mathcal{O}_w$.

We call $\{b_{fgh} \mid f, g, h \in \mathcal{O}\}$ the set of structure constants for $w$. We see that
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(1.2) A weight \( w \) is coherent if and only if the linear subspace \( K\Omega_w \) of \( \text{Mat}_K X \) spanned by \( \{ w_f \mid f \in \mathcal{O}_w \} \) is a subalgebra for some extension field \( K \) of \( \mathbb{Q}(w) \).

Since \( \mathcal{O}_w \) is closed under the conjugate transpose map and \( f = \sum_{i=1}^t w_{E_i} \),

(1.3) If \( w \) is coherent, then \( \mathcal{O}_w \) is semisimple and contains the identity matrix.

It will be convenient to say that a triangle \( (x, y, z) \in X^3 \) has type \((f, g, h)\) if \((x, y) \in f, (y, z) \in g\) and \((x, z) \in h\). The weight of \((x, y, z)\) is \( \delta w(x, y, z) \). In this terminology, if \((x, z) \in h\), then \( b_{fgh} \) is the sum of the weights of the triangles \((x, y, z)\) of type \((f, g, h)\). Thus \( w \) is coherent if, and only if, for \( f, g, h \in \mathcal{O} \) and \((x, z) \in h\), the sum of the weights of the triangles \((x, y, z)\) of type \((f, g, h)\) is independent of the choice of \((x, z) \in h\) and is zero if \( h \notin \mathcal{O}_w \).

Let \( X_i = \{ x \in X \mid (x, x) \in E_i \} \), then \( \{ X_i \mid 1 \leq i \leq t \} \) is a partition of \( X \). We refer to the \( X_i \) as the fibers of \((X, \mathcal{O})\).

(1.4) If \( w \) is coherent and \( f \in \mathcal{O}_w \), then \( \text{dom} f \cap X_i \neq \emptyset \) implies \( \text{dom} f = X_i \).

Proof. Here, as in (I.2), \( \text{dom} f = \{ x \in X \mid f(x) \neq \emptyset \} \). Assume \( \text{dom} f \cap X_i \neq \emptyset \) and take \( x \in \text{dom} f \cap X_i \) and \( y \in f(x) \). Then \( b_{E_i f} = \delta w(x, x, y) = 1 \).

If \( x_1 \in \text{dom} f \) and \( y_1 \in f(x_1) \), then \( b_{E_i f} = \sum_{(x_1) \in f(x)} \delta w(x_1, z, y_1) = \delta w(x_1, x_1, y_1) \) so \( x_1 \in X_1 \) and \( \text{dom} f \subseteq X_i \). On the other hand, \( w_{fuv_{E_i}} = \sum_{(x_1) \in f(x)} \delta w(x, y, x) = |f(x)| \neq 0 \), and if \( x_1 \in X_i \), \( w_{fuv_{E_i}} = \sum_{y_1 \in f(x_1)} \delta w(x_1, y_1, y_1) = |f(x_1)| \). Hence \( |f(x_1)| = |f(x)| \) so \( x_1 \in \text{dom} f \).

Thus, if we put \( \mathcal{O}_w^{i,j} = \{ f \in \mathcal{O}_w \mid f \subseteq X_i \times X_j \} \), then by (1.4),

(1.5) \( \{ \mathcal{O}_w^{i,j} \mid 1 \leq i, j \leq t \} \) is a partition of \( \mathcal{O} \).

A weight \( w \) will be called regular if, for all \( f, g, h \in \mathcal{O} \), \((x, z) \in h\), and \( \alpha \in \mathbb{C} \), the number \( \beta_{fgh}(\alpha) \) of triangles \((x, y, z)\) of type \((f, g, h)\) and weight \( \alpha \) is independent of the choice of \((x, z) \in h\). Clearly

(1.6) If \( w \) is a regular weight on the precoherent configuration \((X, \mathcal{O})\), then

(a) \((X, \mathcal{O})\) is coherent with intersection numbers \( a_{fgh} = \sum_{\alpha \in \mathbb{C}} \beta_{fgh}(\alpha) \), \( f, g, h \in \mathcal{O} \), and

(b) \( w \) is coherent with structure constants \( b_{fgh} = \sum_{\alpha \in \mathbb{C}} \alpha \beta_{fgh}(\alpha) \), \( f, g, h \in \mathcal{O} \).

Note also that

(1.7) For a precoherent configuration \((X, \mathcal{O})\), the following are equivalent.

(i) \((X, \mathcal{O})\) is coherent.

(ii) the standard weight \( \Phi \) is coherent on \((X, \mathcal{O})\).

(iii) the standard weight \( \Phi \) is regular on \((X, \mathcal{O})\).

If a coherent weight \( w \) is integral in the sense that all the values \( w(x, y) \) are algebraic integers, then \( \{ w_f \mid f \in \mathcal{O}_w \} \) spans on order \( \Gamma_w \) over the ring of integers in the number field \( \mathbb{Q}(w) \). If \((X, \mathcal{O})\) is coherent, \( \Gamma_w \) coincides with the adjacency ring \( \Gamma \) of \((X, \mathcal{O})\) as defined in (I.2).

We list now some basic properties of the structure constants \( b_{fgh} \), \( f, g, h \in \mathcal{O} \) of a coherent weight \( w \).
(1.8) If \((X, \mathcal{O})\) is coherent, then \(|b_{fgh}| \leq a_{fgh}\), where the \(a_{fgh}\) are the intersection numbers for \((X, \mathcal{O})\).

Proof. For \((x, z) \in h,\)

\[
b_{fgh} = \left| \sum_y w_f(x, y)w_g(y, z)\overline{w_h(x, z)} \right|
\]

\[\leq \sum_y \Phi_f(x, y)\Phi_g(y, z) = a_{fgh}.\]

\((1.9)\)

Proof. \((w_tw_o)^* = w_ow_w.\)

(1.10) If \(f_1, f_2, \ldots, f_s \in \mathcal{O},\ s \geq 3,\) then the sum

\[\sum_{(\lambda_\mu ) \in \Lambda^s} b_{f_1f_2\lambda_1}b_{f_2f_3\lambda_2} \cdots b_{f_{s-1}s-1f_{s-1}f_1}f_s\]

is independent of cyclic permutations of \(f_1, f_2, \ldots, f_s.\) In particular, \(b_{fgh}|h| = b_{fgh}, |g| = b_{fgh}.|f|\)

Proof. Put \(w_i = w_{f_i}.\) The expression (*) is equal to

\[\sum_{x_1x_2 \cdots x_s} w_1w_2 \cdots w_s(x_1, x_1) = \sum_{x_1x_2 \cdots x_s} w_1(x_1, x_2)w_2(x_2, x_3) \cdots w_s(x_s, x_1)\]

which is independent of such permutations.

(1.11) If \(f\) and \(g\) are symmetric, then \(b_{f_0f}, b_{f_0f}\) and \(b_{f_0f}\) are real.

Proof. Apply (1.9) and (1.10). Clearly

(1.12) \(b_{fgh} \neq 0\) implies \(f \in \mathcal{O}_w^i, g \in \mathcal{O}_w^i,\) and \(h \in \mathcal{O}_w^i\) for some \(i, j, k.\)

And it is easy to verify that

(1.13) If \(f \in \mathcal{O}_w^i, g \in \mathcal{O}_w^j,\) and \(h \in \mathcal{O}_w^k,\) then \(b_{fgh} = \delta_{ai}\delta_{aj}\delta_{ah},\ b_{fgh} = \delta_{aj}\delta_{ah}\delta_{fh}\)

and \(b_{fgh} = \delta_{ai}\delta_{ah}\delta_{fh},\) where \(n = |f(x)|, x \in \text{dom} f.\)

A consequence of (1.13) is

\[
(1.14) \quad \text{trace } w_f w_g = \delta_{fg}|f|.\]

3. The Schur relations and theorems of Frame and Wielandt

The results of Sections 3 through 8 of Part I can be regarded as results about the standard weight \(\Phi\) and admit rather straightforward extensions to coherent weights \(w\) in general. In Sections 3 through 7 of the present paper we indicate some of these extensions, omitting proofs where the extension from Part I is fairly immediate.

Let \((X, \mathcal{O})\) be precoherent, \(n = |X|,\) and let \(w\) be a coherent weight on \((X, \mathcal{O}).\) The notations for the adjacency algebra \(C_w = \mathbb{C}Q_w\) will be the same.
as those used for $C$ in (1.2), with a subscript or superscript $w$ attached when necessary, as in Sections 4 and 6. Thus $\Delta_1, \ldots, \Delta_m$ are the inequivalent irreducible representations of $C_w$ and $\zeta_1, \ldots, \zeta_m$ are the corresponding characters, with $\zeta_\alpha(I) = e_\alpha$. The standard character of $C_w$, i.e., the character afforded by $\mathbb{C}X$, is $\zeta = \sum_{\alpha=1}^n z_\alpha \zeta_\alpha$, and $\zeta(\phi) = \text{trace } \phi$ for $\phi \in C_w$. We have

\begin{equation}
(3.1) \quad r_w = \sum_{\alpha=1}^m e_\alpha^2 \quad \text{and} \quad n = \sum_{\alpha=1}^m z_\alpha e_\alpha.
\end{equation}

We write $\Delta_\alpha(\phi) = (a_\alpha^\phi(\phi))$ for $\phi \in C_w$. There is a basis $\{e_{ij} \mid 1 \leq i, j \leq e_a, 1 \leq \alpha \leq m\}$ of $C_w$ such that

\begin{equation}
(3.2) \quad e_{ij} = z_\alpha \sum_{f \in \mathcal{G}_w} a_{ij}^f(\tilde{w}_f)w_f \quad \text{where} \quad \tilde{w}_f = \frac{1}{|f|} w_f^*.
\end{equation}

The central primitive idempotents of $C_w$ are

\begin{equation}
(3.3) \quad \epsilon^\alpha = \sum_{t=1}^m e_{tt} = z_\alpha \sum_{f \in \mathcal{G}_w} \zeta_\alpha(\tilde{w}_f)w_f.
\end{equation}

As in (I.3) we obtain the Schur relations

\begin{equation}
(3.4) \quad \sum_{f, g \in \mathcal{G}_w} a_{ij}^f(\tilde{w}_f)a_{jk}^g(\tilde{w}_g) = \delta_{ij} \delta_{ik} \frac{1}{z_\alpha}
\end{equation}

and the orthogonality relations

\begin{equation}
(3.5) \quad \sum_{f \in \mathcal{G}_w} \zeta_\alpha(\tilde{w}_f)\zeta_\beta(\tilde{w}_f) = \delta_{\alpha \beta} \frac{\epsilon_\alpha}{z_\alpha}.
\end{equation}

It is sometimes convenient to write the Schur relations as a matrix equation. For this purpose we list the $a_{ij}$ as $a_1, a_2, \ldots, a_{e_{aw}}$, putting $a_\lambda = a^\lambda_w$ and $h_\lambda = z_\lambda$ if $\lambda_n = a^\lambda_w$. Similarly we list the $w_f$ as $w_1, \ldots, w_{r_w}$, putting $w_1 = w_f^0$ and $m_1 = |f|$ if $w_1 = w_f$. Then (3.4) becomes

\begin{equation}
(3.6) \quad APM^{-1}(\mathcal{A})^t = H^{-1},
\end{equation}

where $A_{\lambda r} = a_\lambda(w_r)$, $H = \text{diag}(h_1, \ldots, h_{r_w})$, $M = \text{diag}(m_1, \ldots, m_{r_w})$, and $P$ and $\mathcal{A}$ are suitable permutation matrices with $P^2 = \mathcal{A}^2 = I$. We can rewrite (3.6) as

\begin{equation}
(3.7) \quad (\mathcal{A})^t HAP = M
\end{equation}

which means that

\begin{equation}
(3.8) \quad \sum_{\alpha, i, j} z_\alpha a_{ij}^\alpha(w_f)a_{ji}^\alpha(w_f^*) = \delta_{i,0}|f|.
\end{equation}

and hence

\begin{equation}
(3.9) \quad w_f = \sum_{\alpha, i, j} a_{ij}^\alpha(w_f)e_{ij}^f.
\end{equation}
In particular

(3.10) If $C_w$ is commutative, then
\[ \sum_{a=1}^{m} z_a \delta_a(w_f) \overline{\delta_a(w_f)} = \delta_{/f}[f]. \]

Let us now assume that $w$ is integral. Then the structure constants $b_{fgh}$ are algebraic integers and we may assume that the matrices $\Delta_a(w_f)$ have algebraic integer coefficients for all $f \in \Theta_w$, $1 \leq \alpha \leq m$. The derivation of the version (1.3.14) of Frame's theorem can now be repeated up to the point where appeal is made to the existence of the principal irreducible character. This gives

(3.11) If $w$ is a coherent integral weight, then $P_w/\Theta_w$ is a rational integer, where $P_w = \prod_{g \in \Theta_w} |g|$ and $\Theta_w = \prod_{a=1}^{m} (z_a)^2$. If, in addition, $\Delta_a$ can be written in $\Theta$, $1 \leq \alpha \leq m$, then $P_w/\Theta_w$ is a square.

On the other hand, the version (1.3.15) of Wielandt's theorem extends without change, namely, consideration of the elementary divisors in (3.7) gives

(3.12) If $w$ is a coherent integral weight and $q$ is a prime power dividing $z_a$ for $l$ distinct values $a_1, \ldots, a_l$ of $a$, then $q$ divides $|g|$ for $e_{a_1}^2 + \cdots + e_{a_l}^2$ distinct $g \in \Theta_w$.

When applied to the group case (see Section 10) this result should be compared with results of Curtis and Fossum [1] and Keller [5].

4. THE KREIN CONDITION

Let $u$ and $v$ be weights on the precoherent configuration $(X, \Theta)$. Then the product $u \circ v$ is a weight on $(X, \Theta)$, $\text{spt} u \circ v = \text{spt} u \cap \text{spt} v$, and $\Theta_{u \circ v} = \Theta_u \cap \Theta_v$. We write $u \leq v$ to mean that $\text{spt} u \subseteq \text{spt} v$ and $u$ and $v$ coincide on $\text{spt} u$. Thus $u \leq v$ if and only if $\Theta_u \subseteq \Theta_v$ and $u = \sum_{g \in \Theta_u} v_f$. For example, $u \circ \Phi = u$ and $u \circ \bar{u} \leq \Phi$, where $\Phi$ is the standard weight and $\bar{u}$ is the complex conjugate of $u$.

Assume that $w_1, w_2, \ldots, w_s (s \geq 2)$ and $w$ are coherent weights on $(X, \Theta)$ such that $w_1 \circ w_2 \circ \cdots \circ w_s \leq w$. Then $C_{w_1} \circ C_{w_2} \circ \cdots \circ C_{w_s} \subseteq C_w$. We have a basis $\{e_{t}^\lambda | 1 \leq \lambda \leq r_u\}$ of $C_u$ for $u = w_i$, $1 \leq i \leq s$, and $u = w$, such that
\[ e_{t}^\lambda = h_{t}^\lambda \sum_{g \in \Theta_u} a_{t}^g (\overline{\delta_t(g)}) u_g, \]
where the notation is that of Section 3.

Fix $\lambda_i$, $1 \leq \lambda_i \leq r_{w_i}$, for $1 \leq i \leq s$, and put $h_i = h_{\lambda_i}^i$ and $a_{ij} = a_{\lambda_i}^{\lambda_j}(w_{ij})$. Then
\[ e_{t}^{\lambda_1} \circ e_{t}^{\lambda_2} \circ \cdots \circ e_{t}^{\lambda_s} = h_1 h_2 \cdots h_s \sum_{v=1}^{r} c_v a_{t}^v, \]
where
\[ c_v = \sum \frac{a_{1r}a_{2r} \cdots a_{sr}}{|f|} a_v(w_f) \]
the sum being over all \( f \in \Theta_{w_1} \cap \Theta_{w_2} \cap \cdots \cap \Theta_{w_s} \). Assume that for \( u = u_1, 1 \leq i \leq s, \) and \( u = w \), the complete reduction of \( C_u \) has been effected by a unitary matrix, i.e., that \( \Delta_a(\phi)* = \Delta_a(\phi) \) for \( 1 \leq a \leq m_w, \phi \in C_w \), and assume that \( \lambda_i = \lambda_i \), \( 1 \leq i \leq s \). Then \( e_w^a \) is a projection, so \( e_w^{a_1} \circ e_w^{a_2} \circ \cdots \circ e_w^{a_s} \) is a positive semidefinite hermitian matrix with all its eigenvalues in the interval \([0, 1]\). Thus, writing \( c_v = c_v^a \) if \( a_v = a^a \), we have

(4.1) For \( 1 \leq a \leq m_w \), \( C_a = (c_v^a) \) is a positive semidefinite hermitian matrix with all its eigenvalues in the interval \([0, 1/h_1h_2 \cdots h_s]\).

In particular

(4.2) Let \( w_1, w_2, \ldots, w_s \) and \( w \) be coherent weights on a precoherent configuration \((X, \Theta)\), such that \( w_1 \circ w_2 \circ \cdots \circ w_s \leq w \), and let \( \rho_1, \rho_2, \ldots, \rho_s \) and \( \rho \) be linear characters of \( C_{w_1}, C_{w_2}, \ldots, C_{w_s} \) and \( C_w \) respectively. Then
\[ 0 \leq \sum \frac{\rho_1(w_{1r})\rho_2(w_{2r}) \cdots \rho_s(w_{sr})\rho(w_f)}{|f|^s} \leq \frac{1}{z_1z_2 \cdots z_s} \]
where the sum is over all \( f \in \Theta_{w_1} \cap \Theta_{w_2} \cap \cdots \cap \Theta_{w_s} \) and \( z_i \) is the multiplicity of \( \rho_i \) in the standard character of \( C_{w_i} \).

Concerning the applicability of (4.1) and (4.2) to the group case see the end of Section 10. The obvious extension of the argument of Part I from 2 factors to \( s \geq 2 \) factors was pointed out by Norman Briggs.

5. THE CENTRALIZER ALGEBRA

Let \( w \) be a coherent weight on the precoherent configuration \((X, \Theta)\). The centralizer algebra of \( C_w \) in \( \text{Mat}_c \) \( X \) will be denoted by \( V(C_w) \) and its standard character, i.e., the character afforded by \( \mathcal{C} X \), will be denoted by \( \chi = \chi_w \).

Then \( \chi \) has the decomposition
\[ \chi = \sum_{a=1}^{m_w} e_a \chi_a, \]
where \( \chi_a = \chi_a^w \), \( 1 \leq a \leq m_w \), are the irreducible characters of \( V(C_w) \) and \( \chi_a \) has degree \( z_a \). Moreover

(5.1) \[ \chi_a(z)e_a = \zeta_a(z)z_a \text{ for all } z \in C_w \cap V(C_w). \]

Assume that \( b_{fgh} \in \mathbb{R} \) for all \( f, g, h \in \Theta_w \). Then the irreducible characters of \( C_w \) and \( V(C_w) \) are classified into three kinds in the usual way. For \( 1 \leq a \leq m_w \) put

\[ v_a(w) = \frac{e_a}{z_a} \sum_{f \in \Theta_w} \frac{1}{|f|} \zeta_a(w_f). \]
(5.2) $\zeta_\alpha$ is of the first, second or third kind according as $v_\omega(\alpha) = 1$, $-1$ or $0$.

(5.3) The number of symmetric $f \in \mathcal{O}_w$ is the number of irreducible constituents of $\chi_w$ of the first kind minus the number of the second kind, counting multiplicities.

6. Common Constituents

Let $w$ be coherent on the precoherent configuration $(X, \mathcal{O})$ and assume that $X = X^{(1)} \cup X^{(2)}$ where $X^{(1)}$ and $X^{(2)}$ are nonempty unions of fibers. Assume in addition that for $f \in \mathcal{O}$, $f \subseteq X^{(i)} \times X^{(j)}$ for some $i, j \in \{1, 2\}$, so that $\{\mathcal{O}_{ij} \mid 1 \leq i, j \leq 2\}$ is a partition of $\mathcal{O}$, where $\mathcal{O}_{ij} = \{f \in \mathcal{O} \mid f \subseteq X^{(i)} \times X^{(j)}\}$. Put $\mathcal{O}_{ii} = \mathcal{O}_w \cap \mathcal{O}_{ij}$. Then $[X^{(i)}] = (X^{(i)}, \mathcal{O}_w)$ is precoherent and $w_i = w \mid X^{(i)} \times X^{(i)}$ is a coherent weight on $[X^{(i)}]$, $i = 1, 2$. Moreover $\mathcal{O}_{wi} = \mathcal{O}_{ii}$, so $[X^{(i)}]$ has rank $\rho_w^i$ where $\rho_w^i = |\mathcal{O}_{ii}|, 1 \leq i, j \leq 2$.

The adjacency algebra $C_w$ has the vector space decomposition

$$C_w = C_{11} \oplus C_{12} \oplus C_{21} \oplus C_{22},$$

where $C_{ij}^w = \langle w \mid f \in \mathcal{O}_{ij}\rangle_C$. We identify $C_{ii}^w$ with the adjacency algebra $C_{wi}$ under the map $\sigma \mapsto \sigma \mid X^{(i)} \times X^{(i)} \in C_{ii}^w$, $i = 1, 2$.

We have $\mathbb{C}X = \mathbb{C}X^{(1)} \oplus \mathbb{C}X^{(2)}$ as a module over $V(C_w)$ so we have representations

$$\Pi_i : \left\{ V(C_w) \rightarrow \text{Mat}_{\mathbb{C}} X^{(i)}, \right.$$

$$\phi \mapsto \phi \mid X^{(i)} \times X^{(i)}, \left. \right\}$$

$i = 1, 2$, with $\Pi_i(V(C_w)) \subseteq V(C_{wi})$. The standard character $\chi_w$ of $V(C_w)$ decomposes accordingly into a sum $\chi_w = \lambda_1 + \lambda_2$, where $\lambda_i = \chi_{wi} \Pi_i$, $\chi_{wi}$ being the standard character of $V(C_{wi})$.

Put $m^{(1, 2)}$ equal to the number of distinct irreducible constituents common to $\lambda_1$ and $\lambda_2$ and number the irreducible characters $\chi_w^{\alpha_i}$ of $V(C_{wi})$ so that $\chi_w^{\alpha_i} \Pi_i = \chi_w^{\alpha_2} \Pi_2, 1 \leq \alpha \leq m^{(1, 2)}$. Then

$$\rho_{12}^{w_1} = \sum_{\alpha = 1}^{m^{(1, 2)}} e_{\alpha}^{w_1} e_{\alpha}^{w_2}$$

and

$$\max\{\text{rank } \phi \mid \phi \in C_{12}^w\} = \sum_{\alpha = 1}^{m^{(1, 2)}} \min(e_{\alpha}^{w_1}, e_{\alpha}^{w_2}) \chi_{w_\alpha}.$$
If we write $\chi_{1\alpha} = x_{\alpha}^{11}\Pi_i$, $i \leq \alpha \leq m_{w_1}$, $i = 1, 2$, and $\chi_{2\alpha} = x_{\alpha}^{22}$, then

$$
\chi_{\alpha} = \sum_{\alpha = 1}^{m_{(1,2)}} (e_{\alpha}^{21} + e_{\alpha}^{22})X_{\alpha}^{11} + \sum_{\alpha = 1}^{m_{w_1}} \sum_{\alpha = m_{(1,2)} + 1}^{m_{w_1}} e_{\alpha}^{22}X_{\alpha}^{a}
$$

is the composition of $\chi_{\alpha}$ into irreducible constituents.

In particular, therefore

$$
\chi_{\alpha} = \sum_{\alpha = 1}^{m_{(1,2)}} (e_{\alpha}^{21} + e_{\alpha}^{22})X_{\alpha}^{11} + \sum_{\alpha = 1}^{m_{w_1}} \sum_{\alpha = m_{(1,2)} + 1}^{m_{w_1}} e_{\alpha}^{22}X_{\alpha}^{a}
$$

(6.3)

The irreducible degrees of $C_w$ are $e_{\alpha}^{21} + e_{\alpha}^{22}$ with multiplicity $z_{\alpha}^{21} = z_{\alpha}^{22}$ for $1 \leq \alpha \leq m_{(1,2)}$, and $e_{\alpha}^{22}$ with multiplicity $z_{\alpha}^{22}$ for $m_{(1,2)} < \alpha \leq m_{w_1}$, $i = 1, 2$.

7. Fusion

Let $w$ be a coherent weight on a precoherent configuration $(X, \emptyset)$ and assume given an equivalence relation $\sim$ on $\emptyset$. For $f \in \emptyset$ write $f = \bigcup_{g \sim f} g$ and $\emptyset = \{f^\circ | f \in \emptyset\}$. Then

(7.1) $(X, \emptyset)$ is precoherent and $w$ is coherent on $(X, \emptyset)$ if and only if

(i) $f \sim E$ for some $i$ implies $f = E_i$ for some $j$,

(ii) $f \sim g$ implies $f^\circ \sim g^\circ$, and

(iii) if $f, g, h, h_1 \in \emptyset$ and $h \sim h_1$, then $\sum_{u \sim f; v \sim g} b_{uh} = \sum_{u \sim f; v \sim g} b_{uh_1}$.

In this case, the structure constants for $w$ on $(X, \emptyset)$ are $b_{f^\circ h} = \sum_{u \sim f; v \sim g} b_{uh}$.

For example

(7.2) If $C_w$ is commutative and $f = f \cup f^\circ$ for $f \in \emptyset$, then $(X, \emptyset)$ is precoherent and $w$ is coherent on $(X, \emptyset)$.

8. The Casimir operator

The trace form

$$(\phi, \psi) = \text{trace } \phi \psi \quad (\phi, \psi \in C_w)$$

is a nondegenerate symmetric associative bilinear form on $C_w$, and

$$(w_f, \tilde{w}_g) = \delta_{fg},$$

so $\{w_f | f \in \emptyset\}$ and $\{\tilde{w}_f | f \in \emptyset\}$ are dual bases. Hence

$$C: \left\{ \begin{array}{ccc}
\phi & \rightarrow & C_w \\
\phi_f & \mapsto & \sum_{f \in \emptyset} \tilde{w}_f \phi w_f
\end{array} \right.$$

is a linear map of $C_w$ into its center (independent of the particular choice of dual bases). By the Schur relations, the $(i, j)$-entry of

$$\Delta_{\phi}(c_{\alpha \beta}^{22}) = \sum_{f \in \emptyset} \Delta_{\phi}(\tilde{w}_{\alpha}) E_{\beta}^{22} \Delta_{\phi}(w_f)$$
is $\delta_{zt} \delta_{st}(1/z_t)$, so

$$c(e_{zt}^e) = \delta_{zt} \frac{1}{z_t} e^{(t)}.$$  

In particular, therefore

$$(8.1) \quad c\left( \sum_{z_{zt}=1} z_{zt} e_{zt}^e \right) = I.$$  

Now assume that $w$ is integral and let $\sigma$ be the ring of integers in $Q(w)$, so that $\Gamma_w$ is the $e$-subalgebra of $C_w$ spanned by $\{w_f \mid f \in \mathcal{O}_w\}$. The ideal $H = H(\Gamma_w)$ of all $\lambda \in \mu$ such that $\lambda I = c(\phi)$ for some $\phi \in \mathcal{L}(w)Q_w$ such that $\phi \tilde{w} \in \Gamma_w$ for all $f \in \mathcal{O}_w$ is of interest ([3], [6; VI 1]). In principle we can calculate $H$ by the method described in Section 4 of [4]. By (8.1)

$$\prod_{f \in \mathcal{O}_w} |f| \in H.$$  

9. Regular 2-graphs

These were defined originally by G. Higman and were studied extensively by D. Taylor [8]. A 2-graph on a nonempty set $X$ is a set $\Delta$ of 3-element subsets of $X$, called coherent triangles, such that the number of coherent triangles amongst the 3-element subsets of each 4-element subset of $X$ is even. A 2-graph is regular if the number of coherent triangles containing two given points is constant.

Given a 2-graph $(X, \Delta)$ define $F: X^3 \rightarrow \{1, -1\}$ by $F(x, y, z) = -1$ or 1 according as $\{x, y, z\}$ is in $\Delta$ or not. Then (i) $F$ is symmetric in the sense that the value $F(x, y, z)$ is unchanged by any permutation of $x, y,$ and $z$, (ii) $F(x, x, y) = 1$ for all $x, y \in X$, and (iii) $\delta F(x, y, z, w) = 1$ for all $x, y, z, w \in X$, where

$$\delta F(x, y, z, w) = F(y, z, w)F(x, y, w)F(x, z, w)F(x, y, z).$$

Conversely, a map $F: x^3 \rightarrow \{1, -1\}$ satisfying (i), (ii), and (iii) defines a 2-graph $(X, \Delta)$. But these are precisely the maps $F = \delta w$, where $w: X^2 \rightarrow \{1, -1\}$ is a weight of rank 2 on the rank 2 configuration based on $X$. Regularity of $w$ in the sense of Section 2 is equivalent to regularity of the 2-graph defined by $\delta w$. There is no distinction between coherence and regularity of $w$ in this situation.

In addition to the fact that a substantial number of the known 2-transitive permutation groups are automorphism groups of regular 2-graphs, interest attaches to 2-graphs because of their connection with the problem of equiangular lines in Euclidean space and the related Seidel classes of strong graphs (cf. [7]).
10. The group case

Let $M : G \to \text{Mat}_\mathbb{C} X$ be a monomial representation of a finite group $G$. Then there is an action $X \times G \to X$, $(x, \sigma) \mapsto x^\sigma$, of $G$ on $X$ and a map $u : G \times X \to \mathbb{C}, (\sigma, x) \mapsto u_\sigma(x)$, such that for all $\sigma, \tau \in G$, $x, y \in X$,

$$M(\sigma)(x, y) = 3_{\sigma}^y u_\sigma(x),$$

$$u_1(x) = 1,$$

and

$$u_\sigma(x) = u_\sigma(x)u_\tau(x^\sigma).$$

We assume, as we may without loss in generality, that the value group $U_M = \langle u_\sigma(x) \mid \sigma \in G, x \in X \rangle$ is a group of $|G|$-th roots of unity.

Let $V(M)$ be the centralizer algebra of $M$, $V(M) = \{ \phi \in \text{Mat}_\mathbb{C} X \mid \phi M(\sigma) = M(\sigma)\phi \text{ for all } \sigma \in G \}$, and let $\mathcal{O}$ be the totality of $G$-orbits in $X^2$. In [4] we show that

\[(10.1) \text{ There is a regular integral weight } w \text{ on } (X, \mathcal{O}) \text{ whose values are } 2|G|\text{-th roots of unity such that } C_w = V(M).\]

We say that $w$ as in (10.1) is afforded by $M$, and refer to this situation as the group case. Because of (10.1), the results of Sections 2 through 9 can be applied in the group case. For this we must make the following two observations.

(1) The matrices $M(\sigma), \sigma \in G$, span the centralizer algebra of $V(M)$. Thus, if $\eta$ is the monomial character of $G$ afforded by $M$, then

$$\eta = \sum_{\alpha=1}^{m} z_\alpha \eta_\alpha; \eta_\alpha(1) = e_\alpha,$$

where $\eta_1, \ldots, \eta_m$ are the distinct irreducible constituents of $\eta$, $m = m_\omega$, and $\eta_\alpha(\sigma) = \chi_\alpha(M(\sigma)), \sigma \in G$, in the notation of Section 5.

(2) If $M_i : G \to \text{Mat}_\mathbb{C} X$, $1 \leq i \leq s$, are monomial representations of $G$ corresponding to a given action of $G$ on $X$, then so is $M = M_1 \circ M_2 \circ \cdots M_s$, where $M(\sigma) = M_1(\sigma) \circ M_2(\sigma) \circ \cdots M_s(\sigma)$ for $\sigma \in G$. Let $w_i$ be a weight afforded by $M_i$, $1 \leq i \leq s$. Since $V(M_1) \circ V(M_2) \circ \cdots V(M_s) \subseteq V(M)$, there is a weight $w$ afforded by $M$ such that $w_1 \circ w_2 \circ \cdots w_s \leq w$. This means that (4.1) and (4.2) can be applied.

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