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COHERENT CONFIGURATIONS

II: *Weights*

1. Introduction

We consider certain partitions $\mathcal{O}$ of X^2 where X is a finite set, and certain matrices w: $X^2 \to \mathbb{C}$ which we refer to as *weights* on $(X, \mathcal{O})$. Under a suitable coherence condition there is associated with w an algebra of matrices ϕ: $X^2 \to \mathbb{C}$ which we call the *adjacency algebra* of w. In case $(X, \mathcal{O})$ is a coherent configuration, its adjacency algebra as defined in Part I coincides with the adjacency algebra of the 'all 1' weight. With certain exceptions the ordinary theory of Part I extends to adjacency algebras of coherent weights. Our main purpose here is to indicate these extensions.

Our primary source of motivation and examples is the fact that the centralizer algebra of a monomial representation of a finite group is the adjacency algebra of a suitable weight. It should be noted, though, that there is no shortage of coherent weights which do not belong to this 'group case'. For instance, a regular 2-graph can be viewed as arising from a weight, and this weight belongs to the group case if and only if the automorphism group of the 2-graph is 2-transitive. Also, such processes as fusion can carry one out of the group case.

In this paper, [2] is referred to as Part I, and references to (I.a) or (I.a.b) are to section a or statement (a.b) of Part I. Details are frequently omitted when the extension from Part I is straight forward.

The sections of the present paper are as follows:

A detailed discussion of the group case is given in [4].

2. Coherent and regular weights

In this paper we consider *configurations* $(X, \mathcal{O})$ consisting of a finite nonempty set X and a set $\mathcal{O}$ of binary relations on X. By $R = R(X, \mathcal{O})$ we denote the

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Boolean subalgebra generated by $\mathcal{O}$ of the Boolean algebra of all subsets of X^2. We say that $(X, \mathcal{O})$ is *based* on X and refer to the members of $\mathcal{O}$ and R as the *basic* and *admissible relations* respectively. We call $(X, \mathcal{O})$ *precoherent* if the following axioms (I), (II) and (III) are satisfied.

(I) *$\mathcal{O}$ is a partition* of X^2.

(II) $i := \{(x, x) \mid x \in X\} \in R$.

(III) $f \in \mathcal{O}$ *implies* $f^{\cup} := \{(y, x) \mid (x, y) \in f\} \in \mathcal{O}$.

These are the first three of the four axioms defining coherent configurations in (I.2). The notations and terminology of (I.2) will be used freely.

Let $(X, \mathcal{O})$ be precoherent. Then any admissible relation is uniquely expressible as a union of basic relations, so in particular $i = \bigcup_{\alpha=1}^{t} E_\alpha$, $E_\alpha \in \mathcal{O}$.

A *weight* on $(X, \mathcal{O})$ is a matrix $w\colon X^2 \to \mathbb{C}$ such that

(1) w is Hermitian

(2) $|w(x, y)| = 0$ or 1 and $w(x, x) = 1$ for all $x, y \in X$, and

(3) $\operatorname{spt} w := \{(x, y) \in X^2 | w(x, y) \neq 0\} \in R$.

Let w be a weight on $(X, \mathcal{O})$. Then $\operatorname{spt} w = \bigcup_{f \in \mathcal{O}_w} f$, where $\mathcal{O}_w$ is a subset of $\mathcal{O}$ closed under the converse map $f \mapsto f^{\cup}$, and the basic components E_α of i are in $\mathcal{O}_w$. We call $r_w := |\mathcal{O}_w|$ the *rank* of w.

The 'all 1' matrix Φ and the identity matrix I are weights on $(X, \mathcal{O})$ which we refer to as the *standard* and *trivial* weights respectively. Clearly $\mathcal{O}_\Phi = \mathcal{O}$ and $\mathcal{O}_I = \{E_1, E_2, \ldots, E_t\}$.

For $S \subseteq X^2$, let $w_S\colon X^2 \to \mathbb{C}$ be the matrix which coincides with w on S and is zero off S. Thus Φ_S is the characteristic function of S, i.e., the adjacency matrix of the graph (X, S), and $w_S = w \circ \Phi_S$ where $\circ$ is the pointwise (i.e., hadamard) product. Then $w = \sum_{f \in \mathcal{O}_M} w_f$, $w_f^* = w_{f^{\cup}}$ for $f \in \mathcal{O}_w$, where * denotes conjugate transpose, and $w_{E_i} = \Phi_{E_i}$ for $1 \leqslant i \leqslant t$.

We let $\mathfrak{Q}_w$ denote the r_w-dimensional linear subspace of $\operatorname{Mat}_{\mathbb{Q}(w)} X$ spanned by $\{w_f \mid f \in \mathcal{O}_w\}$, where $\mathbb{Q}(w) := \mathbb{Q}(\{w(x, y) \mid x, y \in X\})$ and $\operatorname{Mat}_{\mathbb{Q}(w)} X$ is the $\mathbb{Q}(w)$-algebra of matrices $\phi\colon X^2 \to \mathbb{Q}(w)$. The weight w will be called *coherent* if $\mathfrak{Q}_w$ is a subalgebra of $\operatorname{Mat}_{\mathbb{Q}(w)} X$.

If $f, g \in \mathcal{O}_w$ and $w_f w_g = \sum_{h \in \mathcal{O}} b_{fgh} w_h$ with $b_{fgh} \in \mathbb{C}$, then for $h \in \mathcal{O}$ and $(x, z) \in h$,

$$(1.1) \qquad b_{fgh} = \sum_{y \in X} w_f(x, y) w_g(y, z) \overline{w_h(x, z)} = \sum_{y \in f(x) \cap g^{\cup}(z)} \delta w(x, y, z),$$

where $f(x) := \{y \in X \mid (x, y) \in f\}$ and $\delta w(x, y, z) := w(y, z)\overline{w(x, z)} w(x, y)$. If w is coherent, the structure constants for $\mathfrak{Q}_w$ with respect to the basis $\{w_f \mid f \in \mathcal{O}_w\}$ are given by (1.1), and $b_{fgh} = 0$ if not all of f, g and h are in $\mathcal{O}_w$. We call $\{b_{fgh} \mid f, g, h \in \mathcal{O}\}$ the set of *structure constants for* w. We see that

(1.2) *A weight w is coherent if and only if the linear subspace $K\mathfrak{Q}_w$ of* $\mathrm{Mat}_K X$ *spanned by $\{w_f \mid f \in \mathcal{O}_w\}$ is a subalgebra for some extension field K of $\mathbb{Q}(w)$.*

Since $\mathfrak{Q}_w$ is closed under the conjugate transpose map and $I = \sum_{i=1}^t w_{E_i}$,

(1.3) *If w is coherent, then $\mathfrak{Q}_w$ is semisimple and contains the identity matrix.*

It will be convenient to say that a triangle $(x, y, z) \in X^3$ has *type* (f, g, h) if $(x, y) \in f$, $(y, z) \in g$ and $(x, z) \in h$. The *weight* of (x, y, z) is $\delta w(x, y, z)$. In this terminology, if $(x, z) \in h$, then b_{fgh} is the sum of the weights of the triangles (x, y, z) of type (f, g, h). Thus w is coherent if, and only if, for $f, g, h \in \mathcal{O}$ and $(x, z) \in h$, the sum of the weights of the triangles (x, y, z) of type (f, g, h) is independent of the choice of $(x, z) \in h$ and is zero if $h \notin \mathcal{O}_w$.

Let $X_i = \{x \in X \mid (x, x) \in E_i\}$, then $\{X_i \mid 1 \leqslant i \leqslant t\}$ is a partition of X. We refer to the X_i as the *fibers* of $(X, \mathcal{O})$.

(1.4) *If w is coherent and $f \in \mathcal{O}_w$, then* $\operatorname{dom} f \cap X_i \neq \varnothing$ *implies* $\operatorname{dom} f = X_i$.

Proof. Here, as in (I.2), $\operatorname{dom} f = \{x \in X \mid f(x) \neq \varnothing\}$. Assume $\operatorname{dom} f \cap X_i \neq \varnothing$ and take $x \in \operatorname{dom} f \cap X_i$ and $y \in f(x)$. Then $b_{E_i f f} = \delta w(x, x, y) = 1$. If $x_1 \in \operatorname{dom} f$ and $y_1 \in f(x_1)$, then $b_{E_i f f} = \sum_{z \in E_i(x_1) \cap f^\cup(y_1)} \delta w(x_1, z, y_1) = \delta w(x_1, x_1, y_1)$ so $x_1 \in X_1$ and $\operatorname{dom} f \subseteq X_i$. On the other hand, $w_{f f^\cup E_i} = \sum_{y \in f(x)} \delta w(x, y, x) = |f(x)| \neq 0$, and if $x_1 \in X_i$, $w_{f f^\cup E_i} = \sum_{y_1 \in f(x_1)} \delta w(x_1, y_1, x_1) = |f(x_1)|$. Hence $|f(x_1)| = |f(x)|$ so $x_1 \in \operatorname{dom} f$. $\square$

Thus, if we put $\mathcal{O}_w^{ij} = \{f \in \mathcal{O}_w \mid f \subseteq X_i \times X_j\}$, then by (1.4),

(1.5) *$\{\mathcal{O}_w^{ij} \mid 1 \leqslant i, j \leqslant t\}$ is a partition of $\mathcal{O}$.*

A weight w will be called *regular* if, for all $f, g, h \in \mathcal{O}$, $(x, z) \in h$, and $\alpha \in \mathbb{C}$, the number $\beta_{fgh}(\alpha)$ of triangles (x, y, z) of type (f, g, h) and weight α is independent of the choice of $(x, z) \in h$. Clearly

(1.6) *If w is a regular weight on the precoherent configuration $(X, \mathcal{O})$, then*

(a) *$(X, \mathcal{O})$ is coherent with intersection numbers $a_{fgh} = \sum_{\alpha \in \mathbb{C}} \beta_{fgh}(\alpha)$, $f, g, h \in \mathcal{O}$, and*

(b) *w is coherent with structure constants $b_{fgh} = \sum_{\alpha \in \mathbb{C}} \alpha\beta_{fgh}(\alpha), f, g, h \in \mathcal{O}$.*

Note also that

(1.7) *For a precoherent configuration $(X, \mathcal{O})$, the following are equivalent.*

(i) *$(X, \mathcal{O})$ is coherent.*

(ii) *the standard weight Φ is coherent on $(X, \mathcal{O})$.*

(iii) *the standard weight Φ is regular on $(X, \mathcal{O})$.*

If a coherent weight w is *integral* in the sense that all the values $w(x, y)$ are algebraic integers, then $\{w_f \mid f \in \mathcal{O}_w\}$ spans on order Γ_w over the ring of integers in the number field $\mathbb{Q}(w)$. If $(X, \mathcal{O})$ is coherent. Γ_Φ coincides with the adjacency ring Γ of $(X, \mathcal{O})$ as defined in (I.2).

We list now some basic properties of the structure constants $\{b_{fgh} \mid f, g, h \in \mathcal{O}\}$ of a coherent weight w.

(1.8) *If $(X, \mathcal{O})$ is coherent, then $|b_{fgh}| \leqslant a_{fgh}$, where the a_{fgh} are the intersection numbers for $(X, \mathcal{O})$.*

Proof. For $(x, z) \in h$,

$$b_{fgh} = \left|\sum_{y} w_f(x, y)w_g(y, z)\overline{w_h(x, z)}\right| \leqslant \sum_{y} \Phi_f(x, y)\Phi_g(y, z) = a_{fgh}. \quad \square$$

(1.9) $\overline{b_{fgh}} = b_{g^\cup f^\cup h^\cup}$.

Proof. $(w_f w_g)^* = w_{g^\cup} w_{f^\cup}$. $\square$

(1.10) *If $f_1, f_2, \ldots, f_s \in \mathcal{O}$, $s \geqslant 3$, then the sum*

$$(*) \qquad \sum_{(\lambda)\in\mathcal{O}^{s-3}} b_{f_1 f_2 \lambda_1} b_{\lambda_1 f_2 \lambda_2}, \ldots, b_{\lambda_{s-3} f_{s-1} f_s} |f_s|$$

is independent of cyclic permutations of $f_1, f_2, \ldots, f_s$. In particular, $b_{fgh^\cup}|h| = b_{hfg^\cup}|g| = b_{ghf^\cup}|f|$.

Proof. Put $w_i = w_{f_i}$. The expression (*) is equal to

$$\sum_{x_1\in X} (w_1 w_2 \ldots w_s)(x_1, x_1) = \sum_{(x)\in X^s} w_1(x_1, x_2)w_2(x_2, x_3)\ldots w_s(x_s, x_1)$$

which is independent of such permutations. $\square$

(1.11) *If f and g are symmetric, then b_{ffg}, b_{fgf} and b_{gff} are real.*

Proof. Apply (1.9) and (1.10). $\square$

Clearly

(1.12) *$b_{fgh} \neq 0$ implies $f \in \mathcal{O}_w^{ij}$, $g \in \mathcal{O}_w^{jk}$ and $h \in \mathcal{O}_w^{ik}$ for some i, j, k.*

And it is easy to verify that

(1.13) *If $f \in \mathcal{O}_w^{ij}$, $g \in \mathcal{O}_w^{jk}$ and $h \in \mathcal{O}_w^{ih}$, then $b_{E_\alpha gh} = \delta_{\alpha i}\delta_{\alpha j}\delta_{gh}$, $b_{fE_\alpha h} = \delta_{\alpha j}\delta_{\alpha k}\delta_{fh}$ and $b_{fgE_\alpha} = \delta_{\alpha i}\delta_{\alpha h}\delta_{fg}n_f$, where $n_f = |f(x)|$, $x \in \operatorname{dom} f$.*

A consequence of (1.13) is

(1.14) $\qquad$ trace $w_f w_g = \delta_{fg^\cup}|f|$.

3. The Schur Relations and Theorems of Frame and Wielandt

The results of Sections 3 through 8 of Part I can be regarded as results about the standard weight Φ and admit rather straightforward extensions to coherent weights w in general. In Sections 3 through 7 of the present paper we indicate some of these extensions, omitting proofs where the extension from Part I is fairly immediate.

Let $(X, \mathcal{O})$ be precoherent, $n = |X|$, and let w be a coherent weight on $(X, \mathcal{O})$. The notations for the adjacency algebra $C_w = \mathbb{C}\Omega_w$ will be the same as those used for C in (I.2), with a subscript or superscript w attached when necessary, as in Sections 4 and 6. Thus $\Delta_1, \ldots, \Delta_m$ are the inequivalent irreducible representations of C_w and $\zeta_1, \ldots, \zeta_m$ are the corresponding characters, with $\zeta_\alpha(I) = e_\alpha$. The *standard character* of C_w, i.e., the character afforded by $\mathbb{C}X$, is $\zeta = \sum_{\alpha=1}^m z_\alpha \zeta_\alpha$, and $\zeta(\phi) = \operatorname{trace} \phi$ for $\phi \in C_w$. We have

$$(3.1) \qquad r_w = \sum_{\alpha=1}^m e_\alpha^2 \quad \textit{and} \quad n = \sum_{\alpha=1}^m z_\alpha e_\alpha .$$

We write $\Delta_\alpha(\phi) = (a_{ij}^\alpha(\phi))$ for $\phi \in \mathbb{C}_w$. There is a basis $\{\varepsilon_{ij}^\alpha \mid 1 \leqslant i, j \leqslant e_\alpha, 1 \leqslant \alpha \leqslant m\}$ of C_w such that

$$(3.2) \qquad \varepsilon_{ij}^\alpha = z_\alpha \sum_{f \in \mathcal{O}_w} a_{ij}^\alpha(\tilde{w}_f) w_f \qquad \textit{where } \tilde{w}_f = \frac{1}{|f|} w_f^* .$$

The central primitive idempotents of C_w are

$$(3.3) \qquad \varepsilon^{(\alpha)} = \sum_{i=1}^m \varepsilon_{ii}^\alpha = z_\alpha \sum \zeta_\alpha(\tilde{w}_f) w_f .$$

As in (I.3) we obtain the *Schur relations*

$$(3.4) \qquad \sum_{f \in \mathcal{O}} a_{ij}^\alpha(\tilde{w}_f) a_{kl}^\beta(w_f) = \delta_{\alpha\beta} \delta_{il} \delta_{jk} \frac{1}{z_\alpha}$$

and the *orthogonality relations*

$$(3.5) \qquad \sum_{f \in \mathcal{O}_w} \zeta_\alpha(\tilde{w}_f) \zeta_\beta(w_f) = \delta_{\alpha\beta} \frac{e_\alpha}{z_\alpha},$$

It is sometimes convenient to write the Schur relations as a matrix equation. For this purpose we list the a_{ij}^α as $a_1, a_2, \ldots, a_{r_w}$, putting $a_{\bar\lambda} = a_{ji}^\alpha$ and $h_\lambda = z_\alpha$ if $a_\lambda = a_{ij}^\alpha$. Similarly we list the w_f as $w_1, \ldots, w_{r_w}$, putting $w_{\bar\iota} = w_{f^\cup}$ and $m_\iota = |f|$ if $w_\iota = w_f$. Then (3.4) becomes

$$(3.6) \qquad APM^{-1}(\mathcal{Q}A)^t = H^{-1},$$

where $A_{\lambda,s} = a_\lambda(w_s)$, $H = \operatorname{diag}(h_1, \ldots, h_{r_w})$, $M = \operatorname{diag}(m_1, \ldots, m_{r_w})$, and P and $\mathcal{Q}$ are suitable permutation matrices with $P^2 = \mathcal{Q}^2 = I$. We can rewrite (3.6) as

$$(3.7) \qquad (\mathcal{Q}A)^t HAP = M$$

which means that

$$(3.8) \qquad \sum_{\alpha,i,j} z_\alpha a_{ij}^\alpha(w_f) a_{ji}^\alpha(w_f^*) = \delta_{fg} |f| .$$

and hence

$$(3.9) \qquad w_f = \sum_{\alpha,i,j} a_{ji}^\alpha(w_f) e_{ij}^\alpha .$$

In particular

(3.10) *If C_w is commutative, then*

$$\sum_{\alpha=1}^{m} z_\alpha \zeta_\alpha(w_f)\overline{\zeta_\alpha(w_g)} = \delta_{fg}|f|.$$

Let us now assume that w is integral. Then the structure constants b_{fgh} are algebraic integers and we may assume that the matrices $\Delta_\alpha(w_f)$ have algebraic integer coefficients for all $f \in \mathcal{O}_w$, $1 \leqslant \alpha \leqslant m$. The derivation of the version (I.3.14) of Frame's theorem can now be repeated up to the point where appeal is made to the existence of the principal irreducible character. This gives

(3.11) *If w is a coherent integral weight, then $P_w/\mathcal{Q}_w$ is a rational integer, where $P_w = \prod_{g \in \mathcal{O}_w} |g|$ and $\mathcal{Q}_w = \prod_{\alpha=1}^{m} (z_\alpha)^{e_\alpha^2}$. If, in addition, Δ_α can be written in $\mathcal{O}$, $1 \leqslant \alpha \leqslant m$, then $P_w/\mathcal{Q}_w$ is a square.*

On the other hand, the version (I.3.15) of Wielandt's theorem extends without change, namely, consideration of the elementary divisors in (3.7) gives

(3.12) *If w is a coherent integral weight and q is a prime power dividing z_α for l distinct values $\alpha_1, \ldots, \alpha_l$ of α, then q divides $|g|$ for $e_{\alpha_1}^2 + \cdots + e_{\alpha_l}^2$ distinct $g \in \mathcal{O}_w$.*

When applied to the group case (see Section 10) this result should be compared with results of Curtis and Fossum [1] and Keller [5].

4. The Krein condition

Let u and v be weights on the precoherent configuration $(X, \mathcal{O})$. Then the product $u \circ v$ is a weight on $(X, \mathcal{O})$, $\operatorname{spt} u \circ v = \operatorname{spt} u \cap \operatorname{spt} v$, and $\mathcal{O}_{u \circ v} = \mathcal{O}_u \cap \mathcal{O}_v$. We write $u \leqslant v$ to mean that $\operatorname{spt} u \subseteq \operatorname{spt} v$ and u and v coincide on $\operatorname{spt} u$. Thus $u \leqslant v$ if and only if $\mathcal{O}_u \subseteq \mathcal{O}_v$ and $u = \sum_{f \in \mathcal{O}_u} v_f$. For example, $u \circ \Phi = u$ and $u \circ \bar{u} \leqslant \Phi$, where Φ is the standard weight and $\bar{u}$ is the complex conjugate of u.

Assume that $w_1, w_2, \ldots, w_s (s \geqslant 2)$ and w are coherent weights on $(X, \mathcal{O})$ such that $w_1 \circ w_2 \circ \cdots \circ w_s \leqslant w$. Then $C_{w_1} \circ C_{w_2} \circ \cdots \circ C_{w_s} \subseteq C_w$. We have a basis $\{\varepsilon_\lambda^u \mid 1 \leqslant \lambda \leqslant r_u\}$ of C_u for $u = w_i$, $1 \leqslant i \leqslant s$, and $u = w$, such that

$$\varepsilon_\lambda^u = h_\lambda^u \sum_{g \in \mathcal{O}_u} a_\lambda^u(\tilde{u}_g) u_g,$$

where the notation is that of Section 3.

Fix λ_i, $1 \leqslant \lambda_i \leqslant r_{w_i}$, for $1 \leqslant i \leqslant s$, and put $h_i = h_{\lambda_i}^{w_i}$ and $a_{if} = a_{\lambda_i}^{w_i}(w_{if})$. Then

$$\varepsilon_{\lambda_1}^{w_1} \circ \varepsilon_{\lambda_2}^{w_2} \circ \cdots \circ \varepsilon_{\lambda_s}^{w_s} = h_1 h_2 \ldots h_s \sum_{v=1}^{r_w} c_v \varepsilon_w^v,$$

where

$$c_\nu = \sum \frac{a_{1f}a_{2f}\dots a_{sf}}{|f|^s} a_\nu^w(w_f)$$

the sum being over all $f \in \mathcal{O}_{w_1} \cap \mathcal{O}_{w_2} \cap \cdots \cap \mathcal{O}_{w_s}$. Assume that for $u = u_i$, $1 \leqslant i \leqslant s$, and $u = w$, the complete reduction of C_u has been effected by a unitary matrix, i.e., that $\Delta_\alpha(\phi)^* = \Delta_\alpha(\phi^*)$ for $1 \leqslant \alpha \leqslant m_w$, $\phi \in C_w$, and assume that $\bar{\lambda}_i = \lambda_i$, $1 \leqslant i \leqslant s$. Then $\varepsilon_{\lambda_i}^{w_i}$ is a projection, so $\varepsilon_{\lambda_1}^{w_1} \circ \varepsilon_{\lambda_2}^{w_2} \circ \cdots \circ \varepsilon_{\lambda_s}^{w_s}$ is a positive semidefinite hermitian matrix with all its eigenvalues in the interval $[0, 1]$. Thus, writing $c_\nu = c_{ij}^\alpha$ if $a_\nu = a_{ij}^\alpha$, we have

(4.1) *For* $1 \leqslant \alpha \leqslant m_w$, $C_\alpha = (c_{ij}^\alpha)$ *is a positive semidefinite hermitian matrix with all its eigenvalues in the interval* $[0, 1/h_1h_2\dots h_s]$.

In particular

(4.2) *Let* $w_1, w_2, \dots, w_s$ *and* w *be coherent weights on a precoherent configuration* $(X, \mathcal{O})$, *such that* $w_1 \circ w_2 \circ \cdots \circ w_s \leqslant w$, *and let* $\rho_1, \rho_2, \dots, \rho_s$ *and* ρ *be linear characters of* $C_{w_1}, C_{w_2}, \dots, C_{w_s}$ *and* C_w *respectively. Then*

$$0 \leqslant \sum \frac{\rho_1(w_{1f})\rho_2(w_{2f})\dots\rho_s(w_{sf})\overline{\rho(w_f)}}{|f|^s} \leqslant \frac{1}{z_1z_2\dots z_s}$$

where the sum is over all $f \in \mathcal{O}_{w_1} \cap \mathcal{O}_{w_2} \cap \cdots \cap \mathcal{O}_{w_s}$ *and* z_i *is the multiplicity of* ρ_i *in the standard character of* C_w.

Concerning the applicability of (4.1) and (4.2) to the group case see the end of Section 10. The obvious extension of the argument of Part I from 2 factors to $s \geqslant 2$ factors was pointed out by Norman Briggs.

5. The centralizer algebra

Let w be a coherent weight on the precoherent configuration $(X, \mathcal{O})$. The centralizer algebra of C_w in $\mathrm{Mat}_{\mathbb{C}}\, X$ will be denoted by $V(C_w)$ and its standard character, i.e., the character afforded by $\mathbb{C}X$, will be denoted by $\chi = \chi_w$. Then χ has the decomposition

$$\chi = \sum_{\alpha=1}^{m_w} e_\alpha \chi_\alpha,$$

where $\chi_\alpha = \chi_\alpha^w$, $1 \leqslant 2 \leqslant m_w$, are the irreducible characters of $V(C_w)$ and χ_α has degree z_α. Moreover

(5.1) $\qquad \chi_\alpha(z)e_\alpha = \zeta_\alpha(z)z_\alpha$ *for all* $z \in C_w \cap V(C_w)$.

Assume that $b_{fgh} \in \mathbb{R}$ for all $f, g, h \in \mathcal{O}_w$. Then the irreducible characters of C_w and $V(C_w)$ are classified into three kinds in the usual way. For $1 \leqslant \alpha \leqslant m_w$ put

$$v_w(\alpha) = \frac{e_\alpha}{z_\alpha} \sum_{f \in \mathcal{O}_w} \frac{1}{|f|} \zeta_\alpha(w_f).$$

Then

(5.2) *ζ_α is of the first, second or third kind according as $v_w(\alpha) = 1, -1$ or 0.*

(5.3) *The number of symmetric $f \in \mathcal{O}_w$ is the number of irreducible constituents of χ_w of the first kind minus the number of the second kind, counting multiplicities.*

6. Common constituents

Let w be coherent on the precoherent configuration $(X, \mathcal{O})$ and assume that $X = X^{(1)} \dot\cup X^{(2)}$ where $X^{(1)}$ and $X^{(2)}$ are nonempty unions of fibers. Assume in addition that for $f \in \mathcal{O}$, $f \subseteq X^{(i)} \times X^{(j)}$ for some $i, j \in \{1, 2\}$, so that $\{\mathcal{O}_{ij} \mid 1 \leqslant i, j \leqslant 2\}$ is a partition of $\mathcal{O}$, where $\mathcal{O}_{ij} = \{f \in \mathcal{O} \mid f \subseteq X^{(i)} \times X^{(j)}\}$. Put $\mathcal{O}_{ij}^w = \mathcal{O}_w \cap \mathcal{O}_{ij}$. Then $[X^{(i)}] = (X^{(i)}, \mathcal{O}_{ii})$ is precoherent and $w_i = w \mid X^{(i)} \times X^{(i)}$ is a coherent weight on $[X^{(i)}]$, $i = 1, 2$. Moreover $\mathcal{O}_{w_i} = \mathcal{O}_{ii}^w$, so $[X^{(i)}]$ has rank ρ_{ii}^w where $\rho_{ij}^w = |\mathcal{O}_{ij}|$, $1 \leqslant i, j \leqslant 2$.

The adjacency algebra C_w has the vector space decomposition

$$C_w = C_{11}^w \oplus C_{12}^w \oplus C_{21}^w \oplus C_{22}^w,$$

where $C_{ij}^w = \langle w_f \mid f \in \mathcal{O}_{ij}^w \rangle_{\mathbb{C}}$. We identify C_{ii}^w with the adjacency algebra C_{w_i} under the map $\sigma \mapsto \sigma \mid X^{(i)} \times X^{(i)}$, $\sigma \in C_{ii}^w$, $i = 1, 2$.

We have $\mathbb{C}X = \mathbb{C}X^{(1)} \oplus \mathbb{C}X^{(2)}$ as a module over $V(C_w)$ so we have representations

$$\Pi_i : \begin{cases} V(C_w) \to \mathrm{Mat}_{\mathbb{C}}\, X^{(i)} \\ \phi \mapsto \phi \mid X^{(i)} \times X^{(i)}, \end{cases}$$

$i = 1, 2$, with $\Pi_i(V(C_w)) \subseteq V(C_{w_i})$. The standard character χ_w of $V(C_w)$ decomposes accordingly into a sum $\chi_w = \lambda_1 + \lambda_2$, where $\lambda_i = \chi_{w_i}\Pi_i$, χ_{w_i} being the standard character of $V(C_{w_i})$.

Put $m^{(1,2)}$ equal to the number of distinct irreducible constituents common to λ_1 and λ_2 and number the irreducible characters $\chi_\alpha^{w_i}$ of $V(C_{w_i})$ so that $\chi_\alpha^{w_1}\Pi_i = \chi_\alpha^{w_2}\Pi_2$, $1 \leqslant \alpha \leqslant m^{(1,2)}$. Then

$$(6.1) \qquad \rho_{12}^w = \sum_{\alpha=1}^{m^{(1,2)}} e_\alpha^{w_1} e_\alpha^{w_2}$$

and

$$(6.2) \qquad \max\{\mathrm{rank}\, \phi \mid \phi \in C_{12}^w\} = \sum_{\alpha=1}^{m^{(1,2)}} \min(e_\alpha^{w_1}, e_\alpha^{w_2}) z_\alpha^w.$$

If we write $\chi_{i\alpha} = \chi_\alpha^{w_i}\Pi_i$, $i \leqslant \alpha \leqslant m_{w_i}$, $i = 1, 2$, and $\chi_\alpha = \chi_{1\alpha} = \chi_{2\alpha}$, $1 \leqslant \alpha \leqslant m^{(1,2)}$, then

$$\chi_w = \sum_{\alpha=1}^{m^{(1,2)}} (e_\alpha^{w_1} + e_\alpha^{w_2})\chi_\alpha + \sum_{i=1}^{2} \sum_{\alpha=m^{(1,2)}+1}^{m_{w_i}} e_\alpha^{w_i}\chi_{i\alpha} \tag{6.3}$$

is the composition of χ_w *into irreducible constituents.*

In particular, therefore

(6.4) *The irreducible degrees of* C_w *are* $e_\alpha^{w_1} + e_\alpha^{w_2}$ *with multiplicity* $z_\alpha^{w_1} = z_\alpha^{w_2}$ *for* $1 \leqslant \alpha \leqslant m^{(1,2)}$, *and* $e_\alpha^{w_i}$ *with multiplicity* $z_\alpha^{w_i}$ *for* $m^{(1,2)} < \alpha \leqslant m_{w_i}$, $i = 1, 2$.

7. Fusion

Let w be a coherent weight on a precoherent configuration $(X, \mathcal{O})$ and assume given an equivalence relation $\sim$ on $\mathcal{O}$. For $f \in \mathcal{O}$ write $\tilde{f} = \bigcup_{g \sim f} g$ and $\tilde{\mathcal{O}} = \{\tilde{f} \mid f \in \mathcal{O}\}$. Then

(7.1) $(X, \tilde{\mathcal{O}})$ *is precoherent and* w *is coherent on* $(X, \tilde{\mathcal{O}})$ *if and only if*

(i) $f \sim E_i$ *for some* i *implies* $f = E_j$ *for some* j,

(ii) $f \sim g$ *implies* $f^\cup \sim g^\cup$, *and*

(iii) *if* $f, g, h, h_1 \in \mathcal{O}_w$ *and* $h \sim h_1$, *then* $\sum_{u \sim f; v \sim g} b_{uvh} = \sum_{u \sim f; v \sim g} b_{uvh_1}$.

In this case, the structure constants for w *on* $(X, \tilde{\mathcal{O}})$ *are* $b_{\tilde{f}\tilde{g}\tilde{h}} = \sum_{u \sim f; v \sim g} b_{uvh}$.

For example

(7.2) *If* C_w *is commutative and* $\tilde{f} = f \cup f^\cup$ *for* $f \in \mathcal{O}$, *then* $(X, \tilde{\mathcal{O}})$ *is precoherent and* w *is coherent on* $(X, \tilde{\mathcal{O}})$.

8. The casimir operator

The trace form

$$(\phi, \psi) = \operatorname{trace} \phi\psi \qquad (\phi, \psi \in C_w)$$

is a nondegenerate symmetric associative bilinear form on C_w, and

$$(w_f, \tilde{w}_g) = \delta_{fg},$$

so $\{w_f \mid f \in \mathcal{O}_w\}$ and $\{\tilde{w}_f \mid f \in \mathcal{O}_w\}$ are dual bases. Hence

$$c: \begin{cases} C_w \to C_w \\ \phi \mapsto \sum_{f \in \mathcal{O}_M} \tilde{w}_f \phi w_f \end{cases}$$

is a linear map of C_w into its center (independent of the particular choice of dual bases). By the Schur relations, the (i, j)-entry of

$$\Delta_\beta(c(e_{st}^\alpha)) = \sum_{f \in \mathcal{O}_M} \Delta_\beta(\tilde{w}_f) E_{st}^\alpha \Delta_\beta(w_f)$$

is $\delta_{\alpha\beta}\delta_{ij}\delta_{st}(1/z_\alpha)$, so

$$c(\varepsilon_{st}^{\alpha}) = \delta_{st}\frac{1}{z_\alpha}\varepsilon^{(\alpha)}.$$

In particular, therefore

$$c\left(\sum_{\alpha=1}^{m} z_\alpha \varepsilon_{11}^{\alpha}\right) = I. \tag{8.1}$$

Now assume that w is integral and let $\mathscr{o}$ be the ring of integers in $\mathbb{Q}(w)$, so that Γ_w is the $\mathscr{o}$-subalgebra of C_w spanned by $\{w_f \mid f \in \mathcal{O}_w\}$. The ideal $H = H(\Gamma_w)$ of all $\lambda \in \mu$ such that $\lambda I = c(\phi)$ for some $\phi \in \mathscr{Q}(w)\mathbb{Q}_w$ such that $\phi\tilde{w}_f \in \Gamma_w$ for all $f \in \mathcal{O}_w$ is of interest ([3], [6; V11]). In principle we can calculate H by the method described in Section 4 of [4]. By (8.1)

$$\prod_{f \in \mathcal{O}_w} |f| \in H. \tag{8.2}$$

9. Regular 2-graphs

These were defined originally by G. Higman and were studied extensively by D. Taylor [8]. A *2-graph* on a nonempty set X is a set Δ of 3-element subsets of X, called *coherent triangles*, such that the number of coherent triangles amongst the 3-element subsets of each 4-element subset of X is even. A 2-graph is *regular* if the number of coherent triangles containing two given points is constant.

Given a 2-graph (X, Δ) define $F: X^3 \to \{1, -1\}$ by $F(x, y, z) = -1$ or 1 according as $\{x, y, z\}$ is in Δ or not. Then (i) F is symmetric in the sense that the value $F(x, y, z)$ is unchanged by any permutation of x, y, and z, (ii) $F(x, x, y) = 1$ for all $x, y \in X$, and (iii) $\delta F(x, y, z, w) = 1$ for all $x, y, z, w \in X$, where

$$\delta F(x, y, z, w) = F(y, z, w)F(x, y, w)F(x, z, w)F(x, y, z).$$

Conversely, a map $F: x^3 \to \{1, -1\}$ satisfying (i), (ii), and (iii) defines a 2-graph (X, Δ). But these are precisely the maps $F = \delta w$, where $w: X^2 \to \{1, -1\}$ is a weight of rank 2 on the rank 2 configuration based on X. Regularity of w in the sense of Section 2 is equivalent to regularity of the 2-graph defined by δw. There is no distinction between coherence and regularity of w in this situation.

In addition to the fact that a substantial number of the known 2-transitive permutation groups are automorphism groups of regular 2-graphs, interest attaches to 2-graphs because of their connection with the problem of equiangular lines in Euclidean space and the related Seidel classes of strong graphs (cf. [7]).

10. The Group Case

Let $M: G \to \mathrm{Mat}_{\mathbb{C}}\, X$ be a monomial representation of a finite group G. Then there is an action $X \times G \to X$, $(x, \sigma) \mapsto x^{\sigma}$, of G on X and a map $u: G \times X \to \mathbb{C}$, $(\sigma, x) \mapsto u_{\sigma}(x)$, such that for all $\sigma, \tau \in G$, $x, y \in X$,

$$M(\sigma)(x, y) = \delta_{x^{\sigma}y} u_{\sigma}(x),$$
$$u_1(x) = 1, \quad \text{and} \quad u_{\sigma\tau}(x) = u_{\sigma}(x) u_{\tau}(x^{\sigma}).$$

We assume, as we may without loss in generality, that the value group $U_M = \langle u_{\sigma}(x) \mid \sigma \in G, x \in X \rangle$ is a group of $|G|$-th roots of unity.

Let $V(M)$ be the centralizer algebra of M, $V(M) = \{\phi \in \mathrm{Mat}_{\mathbb{C}}\, X \mid \phi M(\sigma) = M(\sigma)\phi$ for all $\sigma \in G\}$, and let $\mathcal{O}$ be the totality of G-orbits in X^2. In [4] we show that

(10.1) *There is a regular integral weight w on $(X, \mathcal{O})$ whose values are $2|G|$-th roots of unity such that $C_w = V(M)$.*

We say that w as in (10.1) is *afforded by M*, and refer to this situation as the *group case*. Because of (10.1), the results of Sections 2 through 9 can be applied in the group case. For this we must make the following two observations.

(1) The matrices $M(\sigma)$, $\sigma \in G$, span the centralizer algebra of $V(M)$. Thus, if η is the monomial character of G afforded by M, then

$$\eta = \sum_{\alpha=1}^{m} z_{\alpha}\eta_{\alpha}; \; \eta_{\alpha}(1) = e_{\alpha},$$

where $\eta_1, \ldots, \eta_m$ are the distinct irreducible constituents of η, $m = m_w$, and $\eta_{\alpha}(\sigma) = \chi_{\alpha}(M(\sigma))$, $\sigma \in G$, in the notation of Section 5.

(2) If $M_i: G \to \mathrm{Mat}_{\mathbb{C}}\, X$, $1 \leqslant i \leqslant s$, are monomial representations of G corresponding to a given action of G on X, then so is $M = M_1 \circ M_2 \circ \cdots \circ M_s$, where $M(\sigma) := M_1(\sigma) \circ M_2(\sigma) \circ \cdots \circ M_s(\sigma)$ for $\sigma \in G$. Let w_i be a weight afforded by M_i, $1 \leqslant i \leqslant s$. Since $V(M_1) \circ V(M_2) \circ \cdots \circ V(M_s) \subseteq V(M)$, there is a weight w afforded by M such that $w_1 \circ w_2 \circ \cdots \circ w_s \leqslant w$. This means that (4.1) and (4.2) can be applied.

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