MINIMIZATION OF CLOSURE TIME FOR A FAST ACTING SWITCH

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Abstract

A mechanisms optimization problem involving minimization of closure time is solved in this paper using non-iterative global optimization techniques, whereas most mechanisms problems in the literature have focused on path optimization and use iterative methods. Monotonicity analysis is used successfully to identify active constaints which reduce the problem to one degree of freedom. The solution method may be applicable to a variety of dynamics problem in which algebraic relations can be constructed from the governing differential equations.

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Introduction

Previous optimization work on mechanisms has focused on path optimization problems. The problem presented in this paper represents a different class of mechanisms optimization problems; the objective is to minimize closure time. A simple design for a fast acting electrical switch, [1], is shown in Figure 1. Although this design is perhaps artificially simple, the solution does suggest an optimization method which may be applicable to more complicated designs.

The simple slider crank mechanism functions as a fast acting switch in the following way: The helical spring is compressed initially by the piston, which is held in place by a restraining pin. When the pin is released, link 2 swings shut to complete the circuit. The mechanism is assumed to be constructed of rigid parts with uniform cross section. Parts are connected with rotational contacts. Minimization of closure time implies obtaining the maximum possible acceleration over the shortest possible distance. This results in tradeoffs between the initial spring load (which depends on the spring constant and deformed spring length) and the spring shear strength, and also between the length of link 2 necessary to achieve linear motion of part 2 and the maximum width of the mechanism.

The optimization problem is originally formulated with thirteen degrees of freedom and twenty-one constraints. Global optimization is achieved using the methods presented in [2], [3], and [4]. The problem is reduced to seven degrees of freedom by direct elmination, which is used due to the large number of equality constraints. Monotonicity analysis [4] and implicit elimination are then used to identify constraint activity, and the problem is successfully reduced to one degree of freedom. The optimum is located by mapping the objective against the remaining degree of freedom on the feasible domain.

Derivation of the Mathematical Model

The objective for optimization of the fast acting switch shown in Figure 1 is minimization of the closure time, t_f. Initially time is defined only implicity, the motion of the mechanism being time dependent. Thus, the objective function is originally undefined because it must be derived from the differential equations of motion describing the slider-crank mechanism.

The slider crank mechanism has only a single degree of freedom and consequently the objective function should be rather straightforward. Typically the important variables are the system mass, damping and stiffness and the initial and final configuration of the mechanism. The optimization problem is originally stated in the form

min t_f
subject to: constraints,

where the constraints are grouped in the following categories: (1) differential equations of motion, (2) loop equations, (3) design space and geometry, (4) initial - final configuration, (5) helical spring and (6) strength of joints.

The differential equations of motion are derived using the constrained D'Alembert method [5],

$$\sum_{j=1}^{m} \vec{F}_j \cdot \frac{\partial \vec{\rho}_j}{\partial q_i} - \sum_{k=1}^{n} \vec{\lambda}_k \cdot \frac{\partial \vec{\phi}_k}{\partial q_i} = 0 \; . \qquad (1)$$

In the analysis several forces were treated as negligible. Gravitational forces and mass inertial forces on links 2 and 3 were assumed small compared to the rotational inertial forces, and frictional forces were considered negligible. The model could be further refined by including these forces, although the added complexity would make solution for the objective function very difficult. Details of the derivation of the equations of motion are given in Appendix A. The result is, in matrix form,

$$\begin{bmatrix} I_2 & 0 & 0 & -r\sin\theta_2 & r_2\cos\theta_2 \\ 0 & I_3 & 0 & -r\sin\theta_3 & r_3\cos\theta_3 \\ 0 & 0 & M_4 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{s} \\ \lambda_i \\ \lambda_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -k(s-s_{ud}) \end{Bmatrix} \qquad (2)$$

The loop equations and their derivatives express geometric constraints and result in six equalities:

$$
\begin{aligned}
& r_2\sin\theta_2 + r_3\sin\theta_3 = 0 \ , \\
& r_2\cos\theta_2 + r_3\cos\theta_3 + p_4 + s - g = 0 \ , \\
& r_2\dot{\theta}_2\sin(\theta_2 - \theta_3) - \dot{s}\cos\theta_3 = 0, \\
& r_3\dot{\theta}_3\sin(\theta_3 - \theta_2) - \dot{s}\cos\theta_2 = 0, \\
& -r_2\dot{\theta}_2^2 - r_3\ddot{\theta}_3\sin(\theta_3 - \theta_2) - r_3\dot{\theta}_3^2\cos(\theta_3 - \theta_2) + \ddot{s}\cos\theta_2 = 0, \\
& -r_2\ddot{\theta}_2\sin(\theta_2 - \theta_3) - r_2\dot{\theta}_2^2\cos(\theta_2 - \theta_3) - r_3\dot{\theta}_3^2 + \ddot{s}\cos\theta_3 = 0.
\end{aligned}
\qquad (3)
$$

Again, the details are in Appendix A.

The design space is limited, in a two-dimensional sense, to a rectangular area with height L and width 2L. The total length of link 2, L, is fixed, but the position of the joint with link 3 is variable. Thus,

$$R1: \quad r_2 \le L \ . \qquad (4)$$

The total width of the mechanism cannot exceed 2L so

$$R2: \quad g \le 2L \ . \qquad (5)$$

The angles θ_2 and θ_3, measured as shown in Figure 1, are limited by the design space such that

$$
\begin{aligned}
& 0 \le \theta_2 \le \pi/2 \ , \\
& 3\pi/2 \le \theta_3 \le 2\pi \ .
\end{aligned}
\qquad (6)
$$

With respect to this design, the path of the mechanism can be left quite arbitrary, so that the initial and final conditions result in the most important constraints on the problem. In order for any motion to take place, the spring must be compressed initially, so that

$$R3: \quad s(t = 0) = s_o < s_{ud} \ . \qquad (7)$$

The piston also has zero initial velocity

$$\dot{s}(t = 0) = \dot{s}_o = 0 \ . \qquad (8)$$

The distance between the end of link 2 and the contact must be greater than some distance B to prevent arcing of the electrical signal when the switch is open

$$R4: \theta_{2o} \le W \ . \qquad (9)$$

Further constraints are placed on θ_{2o} (although the above will likely be the least upper bound) and θ_{3o} by applying the design space criteria (6) at time t = 0

$$R5: \theta_{2o} \leq \pi/2 \ , \tag{10}$$

$$R6: \theta_{3o} \geq \frac{3\pi}{2} \ . \tag{11}$$

$$R7: \theta_{3o} \leq 2\pi \ . \tag{12}$$

Note that satisfactory closure of the switch implies the final angular position of link 2, θ_{2f}, is

$$\theta_{2f} = \pi/2 \ . \tag{13}$$

Using equation (13), application of the first two loop equations (3) at the initial and final time leads to

$$R8: \quad r_2 \sin\theta_{2o} + r_3 \sin\theta_{3o} = 0 \tag{14}$$

$$R9: \quad r_2 \cos\theta_{2o} + r_3 \cos\theta_{3o} + p_4 + s_o - g = 0, \tag{15}$$

$$R10: \quad r_2 + r_3 \sin\theta_{3f} = 0 \tag{16}$$

$$R11: \quad r_3 \cos\theta_{3f} + p_4 + s_f - g = 0. \tag{17}$$

Finally note that

$$s_f \leq s_{max} \ . \tag{18}$$

For an undamped system the spring will oscillate such that

$$s_{max} - s_{ud} = s_{ud} - s_o . \tag{19}$$

Rearranging (19) and substitution into (18) yields the approximate relation

$$R12: \quad s_f \leq 2s_{ud} - s_o \ . \tag{20}$$

The only variables with respect to the helical spring are the spring constant, k, the number of coils, N, and the undeformed spring length s_{ud}; all other quantities which affect the spring are treated as parameters. The spring constant and the number of coils are related by

$$R13: \quad k = \frac{d^4 G}{8D^3 N} \ . \tag{21}$$

The allowable deformation of the spring is limited by the spring wire shear strength [6] such that

$$\text{R14:} \quad \frac{K_s Dk(s_{ud} - s_o)}{0.17Ad^{3-m}} \leq 1 \;, \tag{22}$$

and

$$\text{R15:} \quad \frac{K_s Dk(s_f - s_{ud})}{0.17Ad^{3-m}} \leq 1 \;, \tag{23}$$

Geometrical considerations lead to

$$\text{R16:} \quad s_o \geq Nd \tag{24}$$

which is the final constraint derived from consideration of the helical spring.

Typically in the design of simple mechanisms such as considered here the joints between links will be the weakest part. The reaction forces can be calculated using the method of virtual work and equilibrium ideas; they must not exceed the joint strength. The result is

$$[M_4\ddot{s} + k(s - s_{ud})]^2 + \left[\frac{I_3\ddot{\theta}_3}{r_3\cos\theta_3} - (M_4\ddot{s} + k(s - s_{ud})\tan\theta_3\right]^2\Bigg\}^{1/2} \leq F_{rated}. \tag{25}$$

The details of the determination of reaction force is shown in Appendix B.

At this point the objective function must be derived, and the constraints should be put in a form such that the problem is well posed. This requires elimination of all the differential quantities in constraints such as equation (25), and further solving the differential equations of motion (2) for the objective function t_f. Consider the third of equations (2)

$$M_4\ddot{s} + k(s - s_{ud}) + \lambda_i = 0 \;, \tag{26}$$

and note that λ_i can be solved for in terms of θ_2 and θ_3 by writing a virtual work expression (see Appendix B)

$$\lambda_i = -\left[\frac{I_2\ddot{\theta}_2\cos\theta_3}{r_2\sin(\theta_2-\theta_3)} + \frac{I_3\ddot{\theta}_3\cos\theta_2}{r_3\sin(\theta_3-\theta_2)}\right]. \tag{27}$$

Then using the last four of equations (3) to eliminate $\ddot{\theta}_2$ and $\ddot{\theta}_3$ in (27), and substituting the result into (26)

$$m'\ddot{s} + c\dot{s}^2 + k(s - s_{ud}) = 0 \tag{28}$$

where

$$m' = M_4 + \frac{I_2 \cos^2\theta_3}{r_2^2 \sin^2(\theta_2 - \theta_3)} + \frac{I_3 \cos^2\theta_2}{r_3^2 \sin^2(\theta_3 - \theta_2)} \tag{29}$$

$$c = \left[\frac{I_2 \cos^3\theta_3}{r_2^3} + \frac{I_3 \cos^3\theta_2}{r_3^3}\right] \frac{\cos(\theta_3 - \theta_2)}{\sin^4(\theta_3 - \theta_2)} + \left[\frac{I_2 \cos\theta_2}{r_2^2\, r_3} + \frac{I_3 \cos\,\theta_3}{r_2 r_3^2}\right] \frac{\cos\theta_2 \cos\theta_3}{\sin^4(\theta_2 - \theta_3)} \; . \tag{30}$$

Since θ_2 and θ_3 are functions of time (theoretically they can be solved in terms of s), and since the $\dot{s}$ term is squared, the differential equation (28) is highly nonlinear. In order to simplify the solution of the problem, the equation is approximated by

$$m'\ddot{s} + c\dot{s} + k(s - s_{ud}) = 0 \; , \tag{31}$$

where m' and c are treated as constants with respect to time (see Appendix C):

$$m' = M_4 + \frac{.083 M_2 L^2}{r_2^2} + .021 M_3 \; , \tag{32}$$

$$c = \frac{.0083 M_2 L^2}{r_2^3} + \frac{.0042 M_3}{r_3} + \frac{.033 M_2 L^2}{r_2^{\,2} r_3} + \frac{.067 M_3}{r_2} \; . \tag{33}$$

Equation (31) together with the initial conditions (7) and (8) has a straightforward solution

$$s - s_{ud} = \frac{s_o - s_{ud}}{\sqrt{1 - \xi^2}} \, e^{-\xi\omega_n t} \sin\left(\sqrt{1 - \xi^2}\; \omega_n t + \tan^{-1}\frac{\sqrt{1 - \xi^2}}{\xi}\right) \tag{34}$$

where

R17: $$\xi = \frac{\omega_n c}{2k} = \frac{\omega_n}{2k}\left[\frac{.0083 M_2 L^2}{r_2^3} + \frac{.0042 M_3}{r_3} + \frac{.033 M_2 L^2}{r_2^{\,2} r_3} + \frac{.067 M_3}{r_2}\right] \tag{35}$$

and

$$\text{R18:} \quad \omega_n = \sqrt{\frac{k}{m'}} = k^{1/2}\left[M_4 + \frac{.083 M_2 L^2}{r_2^2} + .021 M_3\right]^{-1/2} \tag{36}$$

Note that we are concerned with the motion during, at most, one-half cycle and thus the decay should be relatively small, so that

$$e^{-\xi\omega_n t} \cong 1, \quad 0 \le t \le t_f \ . \tag{37}$$

using the above result and solving for s_f in (34)

$$s_f - s_{ud} = \frac{s_o - s_{ud}}{\sqrt{1 - \xi^2}} \sin\left(\sqrt{1 - \xi^2}\,\omega_n t_f + \tan^{-1}\frac{\sqrt{1 - \xi^2}}{\xi}\right) \tag{38}$$

Finally, the objective function is obtained by solving the above for t_f:

$$\min t_f = \frac{1}{\sqrt{1 - \xi^2}\,\omega_n}\left\{\sin^{-1}\left[\frac{s_f - s_{ud}}{s_o - s_{ud}}\sqrt{1 - \xi^2}\right] - \tan^{-1}\frac{\sqrt{1 - \xi^2}}{\xi}\right\}. \tag{39}$$

In the analysis a sinusoidal response has been assumed which implies

$$\text{R19:} \quad \xi < 1 \ . \tag{40}$$

Only equation (25) contains differentials; these can be eliminated by differentiating (38) twice to obtain s and solving for θ_3 in terms of s. The approximate result is

$$\text{R20:} \quad 1.2\,\frac{s_{ud} - s_o}{\sqrt{1 - \xi^2}}\,(M_4\omega_n^{\,2} - k) \le F_{rated} \tag{41}$$

<u>Mathematical Model</u>

Substituting symbols x_i, i = 1,..., 13 for the corresponding physical variables, the problem can now be stated in normalized form as:

$$\min t_F = \left\{\sin^{-1}\left[\frac{(x_3 - x_4)(1 - x_1^2)^{1/2}}{(x_5 - x_4)}\right]\right.$$

$$\left. - \tan^{-1}\frac{(1 - x_1^{\,2})^{1/2}}{x_1}\right\}\frac{1}{(1 - x_1^{\,2})^{1/2} x_2}$$

subject to:

: $x_7 L^{-1} \leq 1$

: $0.5\, x_{10}\, L^{-1} \leq 1$

: $x_5 x_4^{-1} \leq 1$

: $x_{13}\, W^{-1} \leq 1$

: $x_{13} \frac{2}{\pi} \leq 1$

5: $\frac{3\pi}{2} x_{12}^{-1} \leq 1$

7: $\frac{1}{2\pi} x_{12} \leq 1$

8: $-x_7 (\sin x_{13}) x_6^{-1} (\sin x_{12})^{-1} = 1$

9: $x_{10}^{-1} [x_7 \cos x_{13} + x_6 \cos x_{12} + p_4 + x_5] = 1$

R10: $-x_7^{-1} x_6 \sin x_{11} = 1$

R11: $x_{10}^{-1} [x_6 \cos x_{11} + p_4 + x_3] = 1$

R12: $x_3 [2\, x_4 - x_5]^{-1} \leq 1$

R13: $8D^3 x_8 x_9 d^{-4} G^{-1} = 1$

R14: $5.88 A^{-1} d^{m-3} K_s D\, x_8 (x_4 - x_5) \leq 1$

R15: $5.88 A^{-1} d^{m-3} K_s D x_8 (x_3 - x_4) \leq 1$

R16: $x_9 x_5^{-1} d \leq 1$

R17: $2x_1 x_2^{-1} x_8\, [.0083 M_2 x_7^{-3} L^2 + .0042 M_3 x_6^{-1} + .033 M_2 x_6^{-1} x_7^{-2} L^2 + .067 M_3 x_7^{-1}] = 1$

R18: $x_2 x_8^{-1/2} [M_4 + .083 M_2 x_7^{-2} L^2 + .021 M_3]^{1/2} = 1$

R19: $x_1 < 1$

R20: $1.2 (x_4 - x_5)(1 - x_1^2)^{-1/2} (M_4 x_2^2 - x_8)\, F_{rated}^{-1} \leq 1$.

Direct Elimination

Many of the constraints in the mathematical model are equalities and can be used to eliminate appropriate variables. In many cases the elimination leads to the derivation of additional inequality constraints from the equalities in order to preserve the original feasible domain.

Constraint R10 can be used to eliminate x_{11}, the final angle of part 3

$$x_{11} = \sin^{-1}(-x_7 x_6^{-1}) \ , \tag{42}$$

which also implies

$$\text{R10a:} \quad x_7 x_6^{-1} \leq 1 \ . \tag{43}$$

Similarly R8 and R9 are used to solve for X_{12} and X_{13}, the initial angle of parts 3 and 2, respecively. Solving for these variables exactly is fairly difficult, however, adequate bounds can be imposed. Consider

$$\text{R8:} \quad -x_7(\sin x_{13}) x_6^{-1} (\sin x_{12})^{-1} = 1 \ . \tag{44}$$

Since $x_7 x_6^{-1} \leq 1$, $\sin x_{13} \leq -\sin x_{12}$ so that

$$\text{R8a:} \quad x_{13}(2\pi - x_{12})^{-1} \leq 1 \ . \tag{45}$$

Similarly, R9 implies

$$\text{R9a:} \quad x_{10}[x_7 + x_6 + p_4 + x_5]^{-1} \leq 1. \tag{46}$$

In order to bound x_{12} and x_{13}. R8 and R9 must be solved simultaneously. Use R8 to solve for $\cos x_{13}$,

$$\cos x_{13} = [1 - (x_6 x_7^{-1})^2 \sin^2 x_{12})^{1/2} \ , \tag{47}$$

and substitute into R9

$$x_{10}^{-1}\{x_7[(x_6 x_7^{-1})^2 \sin^2 x_{12}]^{1/2} + x_6(1 - \sin^2 x_{12})^{1/2} + p_4 + x_5\} = 1 \tag{48}$$

or, rearranging,

$$x_7(1-(x_6 x_7^{-1})^2 \sin^2 x_{12})^{1/2} + x_6(1-\sin^2 x_{12})^{1/2} = x_{10} - p_4 - x_5 \tag{49}$$

Since $x_7 \leq x_6$, it follows that

$$(x_6^2 - x_6^2 \sin^2 x_{12})^{1/2} \geq (x_7^2 - x_6^2 \sin^2 x_{12})^{1/2} \ , \tag{50}$$

and

$$2x_7[1 - (x_6 x_7^{-1})^2 \sin^2 x_{12}]^{1/2} \leq x_{10} - p_4 - x_5 \ . \tag{51}$$

Finally, the following results are obtained:

$$x_{13} \geq \cos^{-1}[(x_{10} - p_4 - x_5)(2x_7)^{-1}] , \tag{52}$$

$$x_{12} \leq \sin^{-1} - \{(x_7 x_6^{-1})^2 - [(x_{10} - p_4 - x_5)(2x_6)^{-1}]^2\}^{1/2} , \tag{53}$$

$$x_{12} \geq \sin^{-1} - \{1 - [(x_{10} - p_4 - x_5)(2x_6)^{-1}]^2\}^{1/2} , \tag{54}$$

which can be substituted into R6, R8 and R9, respectively, to effectively eliminate x_{12} and x_{13}.

Variables x_9 and x_{10} can be eliminated easily using the equalities R13 and R11. From R13

$$x_9 = .125D^{-3}x_8^{-1}d^4G \tag{55}$$

and from R11, using equation (42),

$$x_{10} = (x_6^2 - x_7^2)^{1/2} + p_4 + x_3 \tag{56}$$

It is advantageous to eliminate x_2 from the problem using equality constraint R18:

$$x_2 = x_8^{1/2}[M_4 + .083M_2x_7^{-2}L^2 + .021M_3]^{-1/2} \tag{57}$$

Although this makes the objective function more complicated R18 is a difficult equality to direct, and so using R18 to eliminate x_2 is the best approach.

The problem can now be reformulated as a seven degreee of freedom problem in a form much more suitable for monotonicity analysis. Equality constraints have been used to eliminate six degrees of freedom. In the reformulation R5 is omitted since it is apparent that R4 is the least upper bound. Constraint R15 is redundant (R15 is active if R12 and R14 are both active) and it is also dropped. Using equations (54) - (57) to eliminate x_2, and x_9 to x_{13}, the problem can now be stated as:

$$\min t_F = \left\{\sin^{-1}\left[\frac{(x_3-x_4)(1-x_1^2)^{1/2}}{(x_5-x_4)}\right] - \tan^{-1}\frac{(1-x_1^2)^{1/2}}{x_1}\right\} \frac{(M_4+.083M_2x_7^{-2}L^2+.021M_3)^{1/2}}{(1-x_1^2)^{1/2}\, x_8^{1/2}}$$

subject to:

R1: $x_7 L^{-1} \leq 1$

R2: $0.5\left[\left(x_6^2 - x_7^2\right)^{1/2} + p_4 + x_3\right] L^{-1} \leq 1$

R3: $x_5 x_4^{-1} \leq 1$

R4: $2x_7 \cos W\left[\left(x_6^2 - x_7^2\right)^{1/2} + x_3 - x_5\right]^{-1} \leq 1$

R6: $1.5\pi/\sin^{-1} - \{(x_7 x_6^{-1})^2 - [(x_{10} - p_4 - x_5)(2x_6)^{-1}]^2\}^{1/2} \leq 1$

R7: $\frac{1}{2\pi} \sin^{-1} - \left|1 - [(x_{10} - p_4 - x_5)\ (2x_6)^{-1}]^2\right|^{1/2} \leq 1$

R8a: $\cos^{-1} [(x_{10} - p_4 - x_5)(2x_7)^{-1}][2\pi -$

$\sin^{-1} - \left|1 - [(x_{10} - p_4 - x_5)(2x_6)^{-7}]^2\right|^{1/2}]^{-1} \leq 1$

R9a: $x_{10}(x_7 + x_6 + p_4 + x_5)^{-1} \leq 1$

R10a: $x_7 x_6^{-1} \leq 1$

R12: $x_3(2x_4 - x_5)^{-1} \leq 1$

R14: $5.88A^{-1}\ d^{m-3}\ K_s D x_8 (x_4 - x_5) \leq 1$

R16: $.125D^{-3}\ x_8^{-1} x_5^{-1}\ d^5 G \leq 1$

R17: $$x_1 = \frac{[.0083M_2 x_7^{-3} L^2 + .0042M_3 x_6^{-1} + .033M_2 x_6^{-1} x_7^{-2} L^2 + .067M_3 x_7^{-1}]}{2x_8^{1/2}\ [M_4 + .083M_2 x_7^{-2} L^2 + .021M_3]^{1/2}}$$

R19: $x_1 < 1$

R20: $1.2(x_4 - x_5)(1 - x_1^2)^{-1/2} x_8 [M_4 (M_4 + .083M_2 x_7^{-2} L^2 + .021M_3)^{-1} - 1] F_{rated}^{-1} \leq 1.$

Monotonicity Analysis

The monotonicities of the variables in the constraints can be determined by inspection. However, the objective function depends on the sine and tangent, which are double valued functions. Thus a branch and bound technique must be used to determine the monotonicities.

With respect to the $\tan^{-1}$ term no branching need be done since the feasible domain is limited to the first quadrant and $\tan^{-1}$ is a strictly increasing function in this range. However, for the $\sin^{-1}$ term the feasible domain includes the second and third quadrants, and the monotonicities of the inverse sine are opposite each other in these quadrants. Due to constraint R3 $(x_5 - x_4)$ is always negative, so that the sign of $(x_3 - x_4)$ will determine the monotonicity. Thus, two cases must be considered: Case I with $x_3 \geq x_4$ and case II with $x_3 \leq x_4$.

Consider case I first: An additional constraint is implied by the branching,

$$\text{R21a:} \quad x_3^{-1} x_4 \leq 1 \tag{59}$$

For this case the inverse sine is increasing.

To check the monotonicity of x_4 in the objective it is sufficient to show the monotonicity of x_4 with respect to

$$g = \sin^{-1}(x_3 - x_4)(1 - x_1^2)^{1/2} (x_5 - x_4)^{-1} = \sin^{-1} v \tag{60}$$

Note that

$$\frac{\partial g}{\partial x_4} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial x_4} = \left[\frac{1}{(1 - v^2)^{1/2}}\right]\left[\frac{(1 - x_1^2)^{1/2} (x_3 - x_5)}{(x_5 - x_4)^2}\right] \tag{61}$$

Since $x_3 - x_5 \geq 0$, it follows that

$$\frac{\partial g}{\partial x_4} > 0 \tag{62}$$

and the objective is increasing with respect to x_4, everywhere.

The monotonicity of x_1 in the objective is more difficult to show. However, note that if

$$w = \sin^{-1} \frac{(x_3-x_4)(1-x_1^2)^{1/2}}{(x_5-x_4)} - \tan^{-1} \frac{(1-x_1^2)^{1/2}}{x_1} \tag{63}$$

is increasing or stationary with respect to x_1 then the objective is increasing with respect to x_1. The details are in Appendix D; the result is

$$\frac{\partial w}{\partial x_1} = \frac{1}{(1-x_1^2)^{1/2}} - \frac{A}{(1-x_1^2)^{1/2}} \geq 0 \tag{64}$$

Thus, x_5 is the only variable in the objective whose monotonicity depends on the sign of $(x_3 - x_4)$. Recalling equation (60) note that the monotonicity of x_5 in the objective depends on the sign of

$$\frac{\partial g}{\partial x_5} = \frac{\partial g}{\partial v} \frac{\partial v}{\partial x_5} = \frac{1}{(1-v^2)^{1/2}} \left[\frac{-(x_3-x_4)(1-x_1^2)^{1/2}}{(x_5-x_4)^2} \right] \tag{65}$$

For case I $x_3 - x_4 \geq 0$ and thus $\frac{\partial g}{\partial x_5} < 0$ and x_5 is decreasing The monotonicities of x_3, x_7 and x_8 in the objective can be determined by inspection.

Before the monotonicities in R17 can be determined, it is necessary to direct the equality. Since x_1, is increasing in the objective it must be decreasing in at least one constraint. Note that x_1 is not decreasing in any of the inequality constraints, and thus the equality can be directed as follows

$$\text{R17:} \quad \frac{[0083M_2x_7^{-2}L^2 + .0042M_3x_6^{-1} + .033M_2x_6^{-1}x_7^{-2}L^2 + .067M_3x_7^{-1}]}{2x_2x_8^{1/2}(M_4+.083M_2x_7^{-2}L^2 + .021M_3)} \leq 1$$

Note further that R7, R8a, and R9a are assumed inactive since they are dominated by R10a.

For case I the problem can now be expressed in the form:

$$\min\ t_f(x_1^+,\ x_3^+,\ x_4^+,\ x_5^-,\ x_7^-,\ x_8^-),$$

subject to:

R1: $(x_7+) \leq 1$

R2: $(x_3^+, x_6^+, x_7^-) \leq 1$

R3: $(x_4^-, x_5^+) < 1$

R4: $(x_3^-, x_5^+, x_6^-, x_7^+) \leq 1$

R6: $(x_5^+, x_6^-, x_7^+, x_{10}^-) < 1$

R7: $(x_5^-, x_6^-, x_{10}^+) < 1$

R8a: $(x_5^+, x_6^-, x_7^+, x_{10}^-) < 1$

R9a: $(x_5^-, x_6^-, x_7^-, x_{10}^+) < 1$

R10a: $(x_6^-, x_7^+) \leq 1$

R12: $(x_3^+, x_4^-, x_5^+) \leq 1$

R14: $(x_4^+, x_5^-, x_8^+) \leq 1$

R16: $(x_5^-, x_8^-) \leq 1$

R17: $(x_1^-, x_6^-, x_7^?, x_8^-) \leqq 1$

R19: $(x_1^+) < 1$

R20: $(x_1^+, x_4^+, x_5^-, x_7^+, x_8^+) \leq 1$

R21a: $(x_3^-, x_4^+) \leq 1$

Inactive constraints will not be written in remaining formulations.

Constraint Activity

Rules of monotonicity analysis state that for a monotonic variable in the objective there must be at least one active constraint with opposite monotonicity in that variable (for minimization) for a well posed problem. If a variable does not appear in the objective then at least two constraints with opposite monotonicities in that variable are active, or none of the constraints containing that variable are active.

Thus, constraint R2 must be active since R17 is decreasing wrt x_6 and only R2 is increasing wrt x_6. Since t_f is increasing wrt x_3, R4 and/or R21a are active. R12 must be active because of x_4, and R14 or R20 must be active because of x_8 in the objective.

The problem appears completely bounded by these constraints, and thus the activity of the remaining inequalities cannot be determined directly. Implicit elimination is the next step in attempting to reduce the problem further.

Implicit Elimination

An appropriate branching must be considered here so that all possible solutions are covered. It has already been shown that either R4 or R21a is active; now assume R4 is active. R4 is then used to eliminate x_3

$$x_3 = \phi_3(x_5^+, x_6^-, x_7^+) = 2x_7\cos W-(x_6^2-x_7^2)^{1/2} + x_5 \qquad (67)$$

In order to find the monotonicities in the objective and in R2, x_3 must be solved for explicitly and substituted into these constraints. Substituting (67) into the objective

$$t_f = \left\{\sin^{-1} \frac{2x_7\cos W-(x_6^2-x_7^2)^{1/2}+x_5-x_4}{x_5-x_4} - \tan^{-1}\frac{(1-x_1^2)}{x_1^2}\right\} f(x_1,x_7,x_8) \ . \qquad (68)$$

Only the monotonicity of x_5 in the objective cannot be determined implicity; x_5 depends only on the $\sin^{-1}$ term, which can be rearranged as

$$g = \sin^{-1}v = \sin^{-1}\left[\frac{2x_7\cos W-(x_6^2-x_7^2)^{1/2}}{x_5-x_4} + 1\right] \ . \qquad (69)$$

Then

$$\frac{\partial g}{\partial x_5} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial x_5} = \left[\frac{1}{(1-v^2)^{1/2}}\right]\left[\frac{-2x_7\cos W+(x_6^2-x_7^2)^{1/2}}{(x_5-x_4)^2}\right] \ , \qquad (70)$$

and since

$$2x_7\cos W-(x_6^2-x_7^2)^{1/2} \geq 0 \ , \qquad (71)$$

$$\frac{\partial g}{\partial x_5} < 0 \qquad (72)$$

and x_5 is still decreasing in the objective. Substitution of x_3 from (67) into R2 yields

$$0.5[2x_7\cos W + P_4 + x_5]L^{-1} \leq 1 \qquad (73)$$

which is increasing for x_5 and x_7 as expected, but independent of x_6. The remaining constraints yield to implicit elimination.

The resulting problem is

$$\text{Min } t_f(x_1^+, x_4^+, x_5^-, x_7^?, x_8^-),$$

subject to:

$$\begin{array}{ll} R1\ (x_7^+) \le 1 & R16\ (x_5^-,\ x_8^-) \le 1 \\ R2\ (x_5^+,\ x_6^o,\ x_7^+) \le 1 & R17\ (x_1^-,\ x_6^-,\ x_7^?,\ x_8^-) \le 1 \\ R10a\ (x_6^-,\ x_7^+) \le 1 & R20\ (x_1^+,\ x_4^+,\ x_5^-,\ x_7^+,\ x_8^+) \le 1 \\ R12\ (x_4^-,\ x_5^+,\ x_6^-,\ x_7^+) \le 1 & R21a\ (x_4^+,\ x_5^-,\ x_6^+,\ x_7^-) \le 1 \\ R14\ (x_4^+,\ x_5^-,\ x_8^+) \le 1 & \end{array} \qquad (74)$$

A significant result arises from the elimination: R21a must be active due to x_6. Thus R21a is always active, since either R4 or R21a is active and when R4 is assumed active R21a must also be active. The activity of R21a has a major impact on the problem (case I); without assuming R4 active, use R21a to eliminate x_3

$$x_3 = x_4 = \phi(x_4^+), \qquad (75)$$

and substitution into the objective yields

$$t_f = \left[\pi - \tan^{-1}\frac{(1-x_1^2)}{x_1^2}\right]\frac{(M_4+.083M_2x_7^{-2}L^2+.021M_3)^{1/2}}{(1-x_1^2)^{1/2}x_8^{1/2}} \qquad (76)$$

The problem (case I) is now formulated as:

$$\min\ t_f(x_1^+, x_7^-, x_8^-),$$

subject to:

$$\begin{array}{ll} R1:\ (x_7^+) \le 1 & R14:\ (x_4^+,\ x_5^-,\ x_8^+) \le 1 \\ R2:\ (x_4^+, x_6^+, x_7^-) \le 1 & R16:\ (x_5^-,\ x_8^-) \le 1 \\ R4:\ (x_4^-,\ x_5^+,\ x_6^-,\ x_7^+) \le 1 & R17:\ (x_1^-,\ x_6^-,\ x_7^?,\ x_8^-) \le 1 \\ R10a:\ (x_6^-,\ x_7^+) \le 1 & R20:\ (x_1^+,\ x_4^+,\ x_5^-,\ x_7^+,\ x_8^+) \le 1 \\ R12:\ (x_4^-,\ x_5^+) \le 1 & \end{array} \qquad (77)$$

Thus, R2 must be active because of x_6 in R17. Since R2 is active, either R4 or R12 must be active due to x_4. Either R14 or R20 is active due to x_8.

Following the same approach as before, assume R12 is active and eliminate x_4. This yields a physically absurd result, i.e.

$$x_4 = x_5 = \phi(x_5^+) \tag{78}$$

The problem becomes:

$$\min t_f(x_1^+, x_7^-, x_8^-) \quad ,$$

subject to:

$$\begin{array}{llll} \text{R1:} & (x_7^+) \leq 1 & \text{R14:} & (x_8^o) = 0 \\ \text{R2:} & (x_5^+, x_6^+, x_7^-) \leq 1 & \text{R16:} & (x_5^-, x_8^-) \leq 1 \\ \text{R4:} & (x_6^-, x_7^+) \leq 1 & \text{R17:} & (x_1^-, x_6^-, x_7^?, x_8^-) \leq 1 \\ \text{R10a:} & (x_6^-, x_7^+) \leq 1 & \text{R20:} & (x_1^o, x_7^o, x_8^o) = 0 \end{array} \tag{79}$$

If R12 is active, x_8 cannot be bounded from below as required by the objective. Thus, the assumption that R12 is active must be false and thus R4 must be active.

There is one feasible branch for case I left to consider: either R14 or R20 must be active. Again, we assume the activity of one of the constraints, in this case R20, so that

$$x_1 = \phi(x_4^-, x_5^+, x_7^-, x_8^-) \tag{80}$$

and the problem becomes:

$$\min t_f(x_4^-, x_5^+, x_7^-, x_8^-),$$

subject to:

$$\begin{array}{llll} \text{R1:} & (x_7^+) \leq 1 & \text{R14:} & (x_4^+, x_5^-, x_8^+) \leq 1 \\ \text{R2:} & (x_4^+, x_6^+, x_7^-) \leq 1 & \text{R16:} & (x_5^-, x_8^-) \leq 1 \\ \text{R4:} & (x_4^-, x_5^+, x_6^-, x_7^+) \leq 1 & \text{R17:} & (x_4^+, x_5^-, x_6^-, x_7^?, x_8^-) \leq 1 \\ \text{R10a:} & (x_6^-, x_7^+) \leq 1 & & \end{array}$$

The result is similar to the previous result; R14 must be active because of x_8, so that R14 is always active.

Without assuming R20 active, use R14 to eliminate x_4:

$$x_4 = 5.88A^{-1}d^{m-3}K_s D x_8^{-1} + x_5 = \phi_4(x_5^+, x_8^-). \quad (82)$$

The problem is stated as:

$$\min t_f(x_1^+, x_7^-, x_8^-),$$

subject to:

$$\begin{aligned} &R1: (x_7^+) \le 1 && R16: (x_5^-, x_8^-) \le 1 \\ &R2: (x_5^+, x_6^+, x_7^-, x_8^-) \le 1 && R17: (x_1^-, x_6^-, x_7^?, x_8^-) \le 1 \\ &R4: (x_5^o, x_6^-, x_7^+, x_8^+) \le 1 && R20: (x_1^+, x_7^+) \le 1 \\ &R10a: (x_6^-, x_7^+) \le 1 && \end{aligned} \quad (83)$$

Since R2 is active, x_5 must also be bounded from below. Thus R16 must be active.

Case II, that is, the range for $x_3 \le x_4$; can be reduced to exactly the same set of active constraints. For case II, the problem is initially stated as:

$$\min t_f(x_1^+, x_3^+, x_4^+, x_5^+, x_7^-, x_8^-),$$

subject to:

$$\begin{aligned} &R1: (x_7^+) \le 1 && R14: (x_4^+, x_5^-, x_8^+) \le 1 \\ &R2: (x_3^+, x_6^+, x_7^-) \le 1 && R16: (x_5^-, x_8^-) \le 1 \\ &R4: (x_3^-, x_5^+, x_6^-, x_7^+) \le 1 && R17: (x_1^-, x_6^-, x_7^?, x_8^-) \le 1 \\ &R10a: (x_6^-, x_7^+) \le 1 && R20: (x_1^+, x_4^+, x_5^-, x_7^+, x_8^+) \le 1 \\ &R12: (x_3^+, x_4^-, x_5^+) \le 1 && R21b: (x_3^+, x_4^-) \end{aligned}$$

R12 cannot be active since case II implies $x_3 \le x_4$, and thus R21b is active because of x_4 in the objective. Using R21b to eliminate x_3:

$$x_3 = x_4 = \phi_3(x_4^+) \quad (84)$$

and the problem becomes:

$$\min t_f (x_1^+, x_7^-, x_8^-)$$

subject to:

R1: $(x_7^+) \leq 1$

R2: $(x_4^+, x_6^+, x_7^-) \leq 1$

R4: $(x_4^-, x_5^+, x_6^+, x_7^+) \leq 1$

R10a: $(x_6^-, x_7^+) \leq 1$

R12: $(x_4^-, x_5^+) \leq 1$

R14: $(x_4^+, x_5^-, x_8^+) \leq 1$

R16: $(x_5^-, x_8^-) \leq 1$ (85)

R17: $(x_1^-, x_6^-, x_7^?, x_8^+) \leq 1$

R20: $(x_1^+, x_4^+, x_5^-, x_7^+, x_8^+) \leq 1$

This is exactly the form in which case I was reduced after it was determined that R21a was active, and so the two cases must reduce to the same optimum. Monotonicity analysis has successfully identified five active constraints: R2, R4, R14, R16, and R21, with the result that the problem has now one degree of freedom, which can be easily mapped to locate the optimum design point.

Generating an Optimum

The problem has been reduced to a single case with one degree of freedom. Thus, the optimal solution can be obtained by plotting the objective against the independent variable. Note, however, that x_9 (the number of spring coils) must be an integer value, and in addition the number of coils on a spring generally should be greater than 3 (practical constraint) so that $x_9 \geq 3$. Treating x_5 as the independent variable; and substituting values for the parameters gives

R16: $x_8 = .125D^{-3}d^5G\ x_5^{-1} = 103.5\ x_5^{-1}$

R14: $x_4 = [588A^{-1}d^{m-3}K_sDx_8]^{-1} + x_5 = 4.739\ x_5$

R21a: $x_3 = x_4 = 4.739x_5$ (86)

R4: $$x_7 = \frac{4L(x_3-x_5) + 2p_4(x_5-x_3) + x_5^2 - x_3^2}{(2L - p_4 - x_5)(2\cos W)}$$

R2: $x_6 = [(2L-p_4-x_3)^2 + x_7^2]^{1/2}$

The feasible range can be reduced by noting that $x_7 < L$ so that from R4

$$x_5 > 1.732(.202-L) = .00346\text{m} \quad , \qquad (87)$$

and also that $x_3 < 2L$ so that from R21a

$$x_5 < \frac{2L}{4.739} = .0844\text{m}. \qquad (88)$$

Finally, recall the original form of R16:

$$x_9 x_5^{-1} d \leq 1 \ . \tag{89}$$

Thus, substituting (87) and (88) into (89)

$$1.73 < x_9 < 42.2; \tag{90}$$

however, x_9 must be greater than 3 due to the practical constraint, so the feasible domain for x_9 consists of all integer values in the range

$$3 \leq x_9 \leq 42 \tag{91}$$

This list can be exhaustively searched; the calculations were coded due to the large number of feasible designs. The results are tabulated in Table 1. The minimum closure time is t_f = 18.9 msec at the design point:

$$\begin{array}{lll} x_1 = .0147 & x_5 = 14\text{mm}, & x_9 = 7 \text{ coils}, \\ x_2 = 84.1 \text{ rad/scc}, & x_6 = 289\text{mm}, & x_{10} = 400\text{mm}, \\ x_3 = 66\text{mm}, & x_7 = 56\text{mm}, & x_{11} = 348.9° \\ x_4 = 66\text{mm}, & x_8 = 7{,}400 \text{ N/m}, & x_{12} = 354.4° \\ & & x_{13} = 30° \ . \end{array} \tag{92}$$

The optimum design for the fast acting switch is shown, in scale, for the open and closed positions in Figures 2 and 3, respectively.

Conclusions

Monotonicity analysis identifies enough active constraints so that the global optimum is easily determined. The problem is successfully reduced to one degree of freedom with a feasible domain that consists of a discrete number of points. The global optimum was found by exhaustively checking the feasible domain - a simple task for one degree of freedom.

The active constraints lead to an interesting set of design rules. Not surprisingly, link 2 initially is positioned at as large an angle as possible without the risk of arcing (R4). The active constraints R14, R18 and R21 give information about the spring design: The spring should be designed to its ultimate strength and it should be compressed so that the wires are flush. The final spring length is equal to the undeformed length so that the spring force always acts in a direction which tends to shut the switch. The interesting aspect of the design is that part 3 should be a very long part such that it is nearly horizontal and its motion is nearly linear; this is done at the expense of building a longer spring with the capability for greater deformation and thus greater spring force (R2). Similarly the joint between parts 2 and 3 is located very near the pivot point of link 2, again so that the length of link 3 is maximized. By designing part 3 so that its motion is nearly linear and in the horizontal plane, the spring force is not "wasted" in driving part 3 vertically.

Typically mechanisms optimization problems involve path optimization and must be solved numerically. In this paper a mechanisms problem involving minimization of time has been successfully solved in closed-form using monotonicity analysis. Other classes of problems in dynamics could be solved using the approach in this paper, specifically problems in which the governing differential equations can be used to derive algebraic equations for the objective function and constraints. Possible examples of such problems include those in which the frequencies and/or mode shapes are of principal concern, and those for which an approximate solution to the differential equations can be constructed.

Nomenclature

Parameters:

L = Length of link 2 = 200 mm

p_4 = One-half piston length = 50 mm

M_2 = Mass of part 2 = 0.5 kg

M_3 = Mass of part 3 = 0.75 kg

M_4 = Mass of part 4 = 0.5 kg

D = Spring diameter = 20 mm

d = Spring were diameter = 2 mm

A = Spring wire strength coefficient = 1880 MPa

m = Spring wire strength exponent = 0.186

G = Spring wire shear modulus = 207.0 GPa

W = Maximum initial angle on part 2 = 30°(to prevent arcing)

K = Wahl correction factor = $1 + 0.5 \frac{d}{D} = 1.05$

F_{rated} = Maximum joint load = 1500N

Variables:

Mathematical model	Physical notation	Definition
x_1	ξ	Damping coefficient
x_2	ω_n	Natural frequency
x_3	s_f	Final spring length
x_4	s_{ud}	Undeformed spring length
x_5	s_o	Initial spring length
x_6	r_3	Length of part 3
x_7	r_2	Length to joint on part 2
x_8	k	Spring constant

Variables (cont.)

Mathematical model	Physical notation	Definition
x_9	N	Number of spring coils
x_{10}	g	Width of mechanism
x_{11}	θ_{3f}	Final angle of part 3
x_{12}	θ_{3o}	Initial angle of part 3
x_{13}	θ_{2o}	Initial angle of part 2
	$\theta_2 = \theta_2(t)$	Angular orientation of part 2
	$\dot{\theta}_2$	Angular velocity of part 2
	$\ddot{\theta}_2$	Angular velocity of part 2
	$\theta_3 = \theta_3(t)$	Angular orientation of part 3
	$\dot{\theta}_3$	Angular velocity of part 3
	$\ddot{\theta}_3$	Angular acceleration of part 3
	$s = s(t)$	Length of spring at time t
	$\dot{s}$	Velocity of piston
	$\ddot{s}$	Acceleration of piston

APPENDIX A

Equations of Motion

The constrained D'Alembert Method [5] is

$$\sum_{j=1}^{m} \vec{F}_j \cdot \frac{\partial \rho_j}{\partial q_i} - \sum_{k=1}^{n} \vec{\lambda}_k \cdot \frac{\partial \phi_k}{\partial q_i} = 0 \qquad (93)$$

where F_j are the m applied forces, $\vec{\rho}_j$ are vectors from ground to the point of application of F_j, q_i are the generalized coordinates θ_2, θ_3 and s, λ_k are the chord forces necessary to maintain superposition and ϕ_k are the k independent closed loops. The only forces considered are the spring force and the D'Alembert forces on each part. Three equations are derived as follows:

$$q_i = \theta_2: \quad I_2\ddot{\theta}_2 + \lambda_{1j}r_2\cos\theta_2 - \lambda_{1i}r_2\sin\theta_2 = 0 \qquad (94)$$

$$q_i = \theta_3: \quad I_3\ddot{\theta}_3 + \lambda_{1j}r_3\cos\theta_3 - \lambda_{1i}r_3\sin\theta_3 = 0 \qquad (95)$$

$$q_i = s: \quad M_4\ddot{s} + k(s - s_{ud}) + \lambda_{1i} = 0 \qquad (96)$$

Equations (94 - 96) are easily assembled into the matrix equations (2).

Loop Equations

The loop equations are dervied from a vector sum around the closed loop in the mechanism. This vector sum can be differentrated twice to yield additional equalities. Geometrical considerations lead to

$$\vec{r}_2 + \vec{r}_3 + \vec{p}_4 + \vec{s} + \vec{g} = 0 \qquad (97)$$

Taking the dot product of (97) with $\hat{i}$ and $\hat{j}$ yields the first two of equations (3). Differentiation of (97) gives

$$\dot{\theta}_2(\hat{k} \times \vec{r}_2) + \dot{\theta}_3(\hat{k} \times \vec{r}_3) + \dot{s}\hat{s} = 0 \qquad (98)$$

Dotting (98) with r_2 and r_3 results in the third and fourth of equations (3), respectively. Differentiating (97) a second time

$$\ddot{\theta}_2(\hat{k} \times \vec{r}_2) - \dot{\theta}_2^2\vec{r}_2 + \ddot{\theta}_3(\hat{k} \times \vec{r}_3) - \dot{\theta}_3^2 r_3 + \ddot{s}\hat{s} = 0 \qquad (99)$$

The last two of equations (3) are derived by taking the dot product of (99) with r_2 and r_3, respectively.

APPENDIX B

From equilibrium considerations it is easily shown that the magnitude of the reaction force is the same for every joint. To determine the reaction force between parts 3 and 4, hold part 4 fixed and consider a virtual displacement of part 3, $\delta\beta\hat{i}$. To compute the horizontal component of the reaction force, the virtual work done is summed to zero

$$-I_2\ddot{\theta}_2\delta\theta_2 - I_3\ddot{\theta}_3\delta\theta_3 + H\delta\beta = 0 \tag{100}$$

where $\delta\theta_2$ and $\delta\theta_3$ are related to $\delta\beta$ by

$$\delta\theta_2 = \frac{-\delta\beta\cos\theta_3}{r_2\sin(\theta_2-\theta_3)} \tag{101}$$

$$\delta\theta_3 = \frac{-\delta\beta\cos\theta_2}{r_3\sin(\theta_3-\theta_2)} \quad . \tag{102}$$

Substituting (101) and (102) into (100) and, cancelling $\delta\beta$ and solving for H the result is

$$H = -\frac{I_2\ddot{\theta}_2\cos\theta_3}{r_2\sin(\theta_2-\theta_3)} - \frac{I_3\ddot{\theta}_3\cos\theta_2}{r_3\sin(\theta_3-\theta_2)} \quad . \tag{103}$$

A similar analysis to determine the normal component of the reaction force yields

$$N = \frac{-I_2\ddot{\theta}_2\sin\theta_3}{r_2\sin(\theta_2-\theta_3)} - \frac{I_3\ddot{\theta}_3}{r_3}\frac{\sin\theta_2}{\sin(\theta_3-\theta_2)} \tag{104}$$

Note further that H and N are just equal to λ_i and λ_j, respectively. The magnitude of the reaction force is simply

$$F_R = (H^2 + N^2)^{1/2} \quad . \tag{105}$$

APPENDIX C

The differential equation (31) can be simplified by treating the coefficients m′ and c as constants with respect to time. These coefficients are in fact complicated functions of time as given by equations (29) and (30). By considering the expected variations in θ_2 and θ_3, m′ and c can be parameterized by making an average approximation over the time range of interest. The angles θ_2 and θ_3 can be expected to vary from 30° to 90° and 350° to 315° respectively. Also note that

$$I_2 = \frac{1}{12} M_2 L^2 \qquad I_3 = \frac{1}{12} M_3 r_3^2 \tag{106}$$

Substituting equations (106) into equations (29) and (30) and replacing the θ dependent terms with a qualitative approximation for the average value, the result for m′ and c is

$$m' \cong M_4 + \frac{1}{12}\frac{M_2 L^2}{r_2^3}(1) + \frac{1}{12} M_3 (.25) \tag{107}$$

$$c \cong \frac{1}{12}\frac{M_2 L^2}{r_2^3}(.1) + \frac{1}{12}\frac{M_3}{r_3}(.05) + \frac{1}{12} + \frac{1}{12}\frac{M_2 L^2}{r_2^2 r_3}(.4) + \frac{1}{12}\frac{M_3}{r_2}(.8) \quad . \tag{108}$$

It is important to note that the accuracy of the average values used to eliminate the θ dependence does not greatly affect the solution since it does not affect the monotonicity of the variables, and consequently the active constraints are not changed.

APPENDIX D

The monotonicity of x_1 in the objective depends on

$$\omega = \sin^{-1} \frac{(x_3-x_4)(1-x_1^2)^{1/2}}{(x_5-x_4)} - \tan^{-1} \frac{(1-x_1^2)^{1/2}}{x_1}$$

$$= \sin^{-1} v - \tan^{-1} y = k - \ell \qquad (109)$$

Partial differentiation yields

$$\frac{\partial \omega}{\partial x_1} = \frac{\partial k}{\partial v}\frac{\partial v}{\partial x_1} - \frac{\partial \ell}{\partial y}\frac{\partial y}{\partial x_1} \qquad (110)$$

$$= \frac{1}{\left[1-\left(\frac{x_3-x_4}{x_5-x_4}\right)^2(1-x_1^2)\right]^{1/2}} \frac{-x_1(x_3-x_4)}{(x_5-x_4)(1-x_1^2)^{1/2}} + \frac{1}{(1-x_1^2)^{1/2}} \qquad (111)$$

Therefore $\frac{\partial \omega}{\partial x_1} \geq 0$ if

$$\frac{x_1}{\left[\left(\frac{x_5-x_4}{x_3-x_4}\right)^2 - (1-x_1^2)\right]^{1/2}} = A \leq 1 \qquad (112)$$

Constraint R14 can be manipulated to give

$$\left(\frac{x_5-x_4}{x_3-x_4}\right)^2 \geq 1$$

so that

$$A \leq \frac{x_1}{[1-(1-x_1^2)]^{1/2}} = 1 \qquad (113)$$

Therefore $\frac{\partial \omega}{\partial x_1} \geq 0$ and the objective is increasing wrt x_1 everywhere

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References

[1] Suh, C. and Radcliffe, C., Kinematics and Machine Design, Wiley and Sons, N.Y. (1978), p 430.

[2] Papalambros, P., ME566 Global Optimization in Design, Lectures, University of Michigan, Winter 1980.

[3] Wilde, D. J., Globally Optimal Design, Wiley Interscience, N. Y. (1978)

[4] Papalambros, P. and Wilde, D. J., "Global Non-Iterative Design Optimization Using Monotonicity Analysis," Trans. ASME, J. of Mech. Design, Vol. 101, No. 4, (1979) pp 645-649.

[5] Chace, M. A., ME542 CAD-Mechanisms Lectures, University of Michigan, Winter 1980.

[6] Shigley, J. E., Mechanical Engineering Design, 3rd edition, McGraw Hill, N.Y. (1977), pp 295-308.

TABLE 1

NO. COILS X9	INITIAL SPRING LENGTH X5	UNDEFORMED & FINAL SPRING LENGTH X4,X3	SPRING CONSTANT X8	LENGTH PART 3 X6	LENGTH PART 2 TO JOINT X7	CLOSURE TIME TF
3	.00600	.02842	17250.0	.32255	.02505	.02174
4	.00800	.03790	12937.5	.31384	.03301	.02004
5	.01000	.04737	10350.0	.30536	.04078	.01924
6	.01200	.05685	8625.0	.29711	.04835	.01893
7	.01400	.06632	7392.9	.28910	.05571	.01890
8	.01600	.07580	6468.7	.28132	.06286	.01905
9	.01800	.08527	5750.0	.27378	.06981	.01932
10	.02000	.09474	5175.0	.26648	.07653	.01966
11	.02200	.10422	4704.5	.25943	.08304	.02006
12	.02400	.11369	4312.5	.25262	.08932	.02048
13	.02600	.12317	3980.8	.24607	.09537	.02093
14	.02800	.13264	3696.4	.23976	.10120	.02140
15	.03000	.14212	3450.0	.23371	.10678	.02187
16	.03200	.15159	3234.4	.22790	.11213	.02236
17	.03400	.16106	3044.1	.22235	.11722	.02284
18	.03600	.17054	2875.0	.21704	.12207	.02333
19	.03800	.18001	2723.7	.21199	.12666	.02381
20	.04000	.18949	2587.5	.20718	.13099	.02430
21	.04200	.19896	2464.3	.20262	.13506	.02478
22	.04400	.20844	2352.3	.19830	.13886	.02526
23	.04600	.21791	2250.0	.19421	.14238	.02573
24	.04800	.22739	2156.3	.19037	.14562	.02621
25	.05000	.23686	2070.0	.18675	.14857	.02668
26	.05200	.24633	1990.4	.18335	.15123	.02715
27	.05400	.25581	1916.7	.18017	.15359	.02761
28	.05600	.26528	1848.2	.17721	.15565	.02808
29	.05800	.27476	1784.5	.17445	.15739	.02854
30	.06000	.28423	1725.0	.17190	.15882	.02900
31	.06200	.29371	1669.4	.16954	.15992	.02945
32	.06400	.30318	1617.2	.16738	.16070	.02991
33	.06600	.31265	1568.2	.16540	.16113	.03037
34	.06800	.32213	1522.1	.16361	.16122	.03083
35	.07000	.33160	1478.6	.16201	.16096	.03129
36	.07200	.34108	1437.5	.16059	.16034	.03175
37	.07400	.35055	1398.6	.15935	.15935	.03221
38	.07600	.36003	1361.8	.15830	.15798	.03268
39	.07800	.36950	1326.9	.15744	.15623	.03316
40	.08000	.37898	1293.8	.15679	.15409	.03364
41	.08200	.38845	1262.2	.15635	.15155	.03414
42	.08400	.39792	1232.1	.15613	.14859	.03465

Figure 1. Fast Acting Switch - Notation

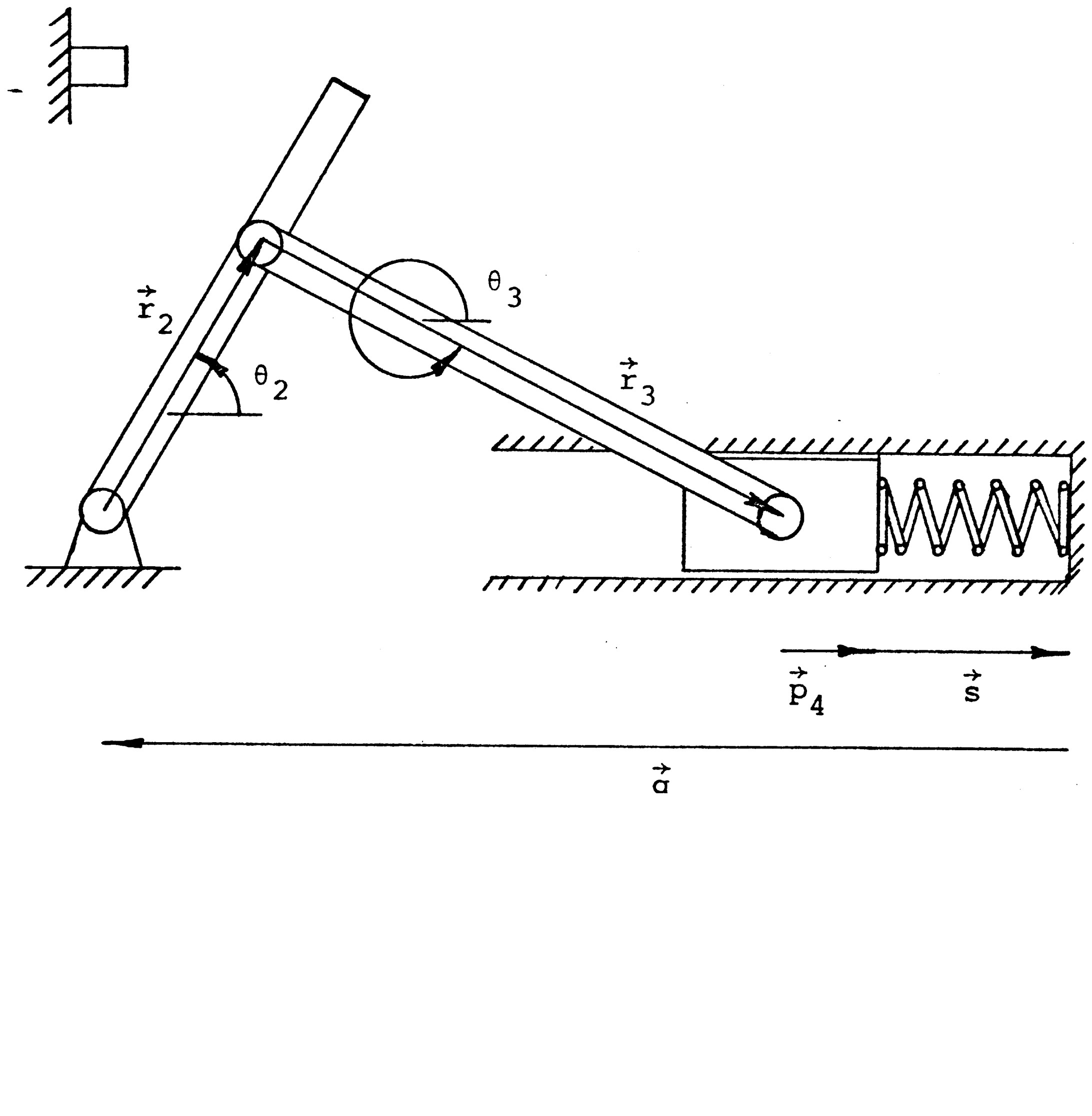

Figure 1
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FIGURE 2. OPTIMUM DESIGN - OPEN POSITION

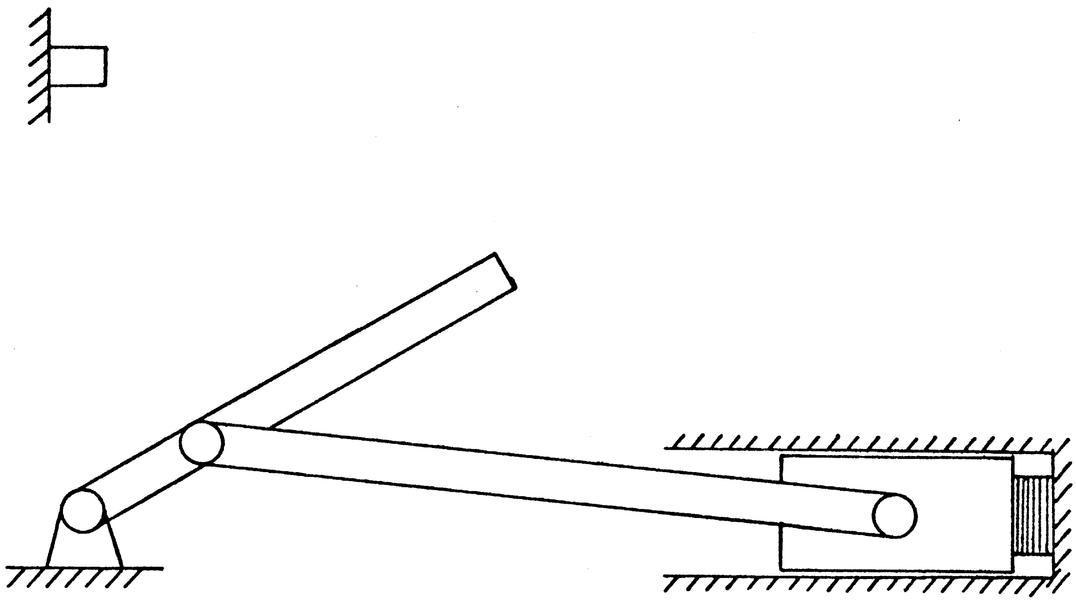

Figure 2
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FIGURE 3. OPTIMUM DESIGN - CLOSED POSITION

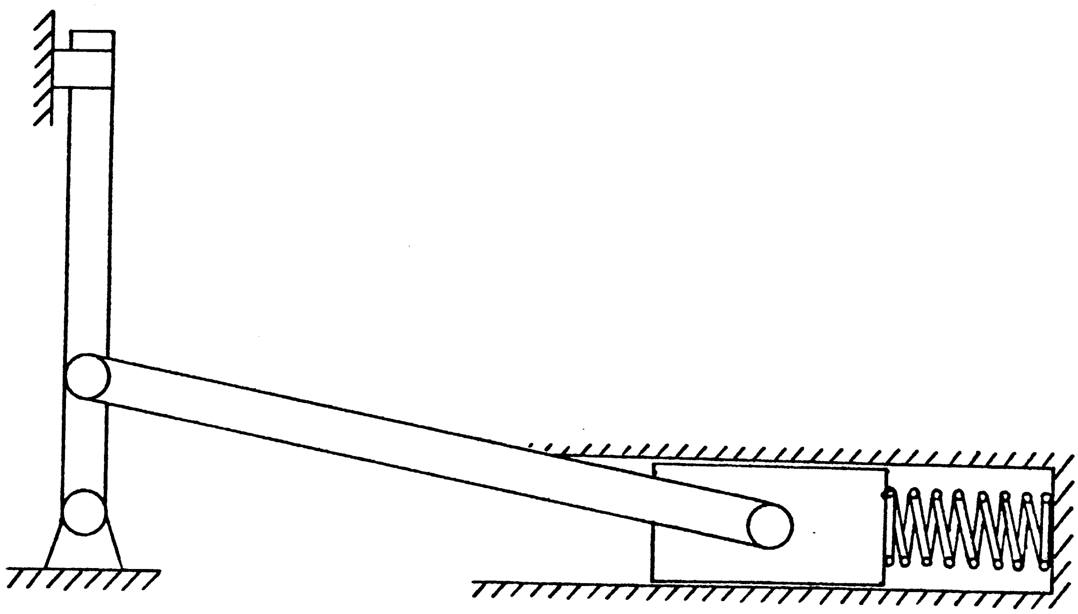

Figure 3
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