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STUDIES IN RADAR CROSS SECTIONS XXXIV - AN INFINITE LEGENDRE INTEGRAL TRANSFORM AND ITS INVERSE

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PREFACE

This is the thirty-fourth in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross sections of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers, and (d) low and high density ionization phenomena.

K. M. Siegel

SUMMARY

An integral transform is defined in which the kernel is a solution of Legendre's equation and the integration is over an infinite range of the angular variable θ. The inversion formula is stated. It is then confirmed; rigorously, for the particular case when the original function is a rectangular pulse, and heuristically for the general case by an appeal to the delta function concept. The standard series expansion in terms of Legendre polynomials is recovered when the original function is even and of period 2π in θ.

A methodical, though still formal, derivation of the transform relations is given in an Appendix.

SECTION I

INTRODUCTION

The object of this paper is to present an infinite Legendre integral transform and its inverse. The search for such a transform was stimulated by an investigation into a boundary value problem of mathematical physics, and the background to this is now outlined in order to explain how the requirement for the transform arose.

In the problems of the diffraction of monochromatic waves by circular cylinders and spheres the "classical" series forms of solution become intractable as the radii of the diffracting obstacles increase much beyond a wavelength. However, solutions useful when the bodies are large can be obtained from contour integral representations, in which the variable of integration is the separation "constant", ν say. The dominant part of the solution is then not itself periodic in the angular variable θ, although, of course, the complete solution is, with period 2π. It seems that a logical way of deriving the dominant part is to express by means of an integral over ν, in which the integrand is the appropriate separable solution of the wave equation, a function of θ which is identical with the incident field at the surface of the diffracting body over a 2π range of θ and which is zero outside this range. This is easily done in the case of the circular cylinder, because the angular part of the separable solution of the wave equation in cylindrical coordinates is $\exp(i\gamma\,\theta)$, and the theory of the infinite Fourier integral transform immediately gives the required representation. In the case of the sphere, however, the corresponding part of the separable solution of the wave equation in spherical polar coordinates is a function satisfying Legendre's equation; the procedure thus suggests the desirability of seeking the inverse of an infinite integral transform in which the kernel is a Legendre function whose behavior for all real values of θ is in some sense analogous to $\exp(i\gamma\theta)$.

A treatment of Fourier integral theory which the mathematical physicist finds useful and acceptable consists first in establishing rigorously, by contour integration, the existence of the transform and its inverse when the original function is a step function; this leads at once to the case of a rectangular pulse, and this in turn to that of a delta function; finally, and now only formally, the delta function is used to establish the transform relations for the general case. As the origin of its conception suggests, the proposed Legendre transform has some affinity with the Fourier transform, and the analysis in the present paper is carried through on lines similar to those just indicated. No attempt is made to establish existence conditions.

Attention is confined to Legendre's equation of order zero, and in § 2 the appropriate solution is given, together with those of its properties to which appeal is made in the later discussion; chapter III, on Legendre functions, of the book Higher Transcendental Functions by Erdelyi, Magnus, Oberhettinger and Tricomi, which is part of the Bateman Project, is used as a standard reference and abbreviated to B.P. In § 3 the infinite Legendre integral transform and its inverse are stated, and then confirmed in the cases when the original function is successively a step function, a rectangular pulse, and a delta function. Finally, in § 4, the transform relations are tested by considering the case when the original function is even and of period 2π in θ, and the familiar representation as a series of Legendre polynomials is recovered; the coefficients in this series have been regarded [see, for example, Tranter (1951), Churchill (1954)] as a finite Legendre integral transform, the kernel here being simply a Legendre polynomial. The present theory is related to that of Legendre series in the same sort of way as the theory of Fourier integral transforms is related to that of Fourier series.

The manner in which the transform relations were established in the first place was essentially that just indicated; in brief, a guess verified. Subsequently, however, it was recognized that almost certainly they could be rigorously derived from a general theory, for example that described by Titchmarsh (1946). Titchmarsh works out many special cases, and it might be conjectured that the one discussed here has only escaped attention so far because superficially there would seem to be no requirement for the consideration of an infinite range of θ.

Evidently an appeal to a general theory would have led to a more logical presentation. Nevertheless, it is felt that the details of the procedure of verification are sufficiently instructive in themselves to be worth retaining. The body of the paper has therefore been left substantially in its original form (although in the light of the existence of a general theory one or two remarks are perhaps a little naive); and a short Appendix has been added, in which a methodical derivation of the transform relations is outlined, based on a formal procedure developed by Marcuvitz (1951).

SECTION II

A NON-PERIODIC SOLUTION OF LEGENDRE'S EQUATION

2.1 The function $E_{\nu-1/2}(\theta)$

As indicated in the introduction, what is sought is a non-periodic solution of Legendre's equation, in some sense analogous to $\exp(i\nu\theta)$, which is defined at least over a region of the complex θ-plane which includes the whole of the real axis. It proves convenient to consider Legendre's equation with degree $\nu - 1/2$, rather than ν; with order zero, and independent variable θ, this is

$$\frac{d}{d\theta}\left(\sin\theta \frac{dU}{d\theta}\right) + \left(\nu^2 - \frac{1}{4}\right)\sin\theta\, U = 0 \; . \qquad (1)$$

An equation with no first derivative term is obtained by the substitution

$$V = U\sqrt{(\sin\theta)} \; ; \qquad (2)$$

namely

$$\frac{d^2V}{d\theta^2} + \left(\nu^2 + \frac{1}{4\sin^2\theta}\right)V = 0 \; . \qquad (3)$$

In general terms, it is clear that equation (3) has solutions which behave like $\exp(i\nu\theta)$ for sufficiently large values of $|\nu|$. In fact, the corresponding solution of (1) is*

$$\frac{e^{i\nu\theta}}{\sqrt{(\sin\theta)}}\, F\left(\frac{1}{2}, \frac{1}{2}\, ;\, \nu + 1;\, -\frac{i e^{i\theta}}{2\sin\theta}\right) \; . \qquad (4)$$

* That (4) is a solution of equation (1) is a known result [see B.P. (1953)]; its precise relation to the standard solutions is given subsequently.

It is easy to see that the series for the hypergeometric function in (4) converges for all values of θ in the complex θ-plane, except those in the shaded regions shown in Figure 1, these regions being delimited by the curves defined by

$$\exp(2 \operatorname{Im} \theta) = 2 \cos(2 \operatorname{Re} \theta) \quad . \tag{5}$$

The value of $\sqrt{(\sin\theta)}$ in (4) is therefore determined throughout the unshaded region in Figure 1 by prescribing that it is positive, say, when θ has real values between $\pi/6$ and $5\pi/6$.

If the expression (4) is regarded as a function of ν, the only singularities which it has in the finite part of the complex ν-plane are simple poles at $\nu = -1, -2, -3, \ldots$.

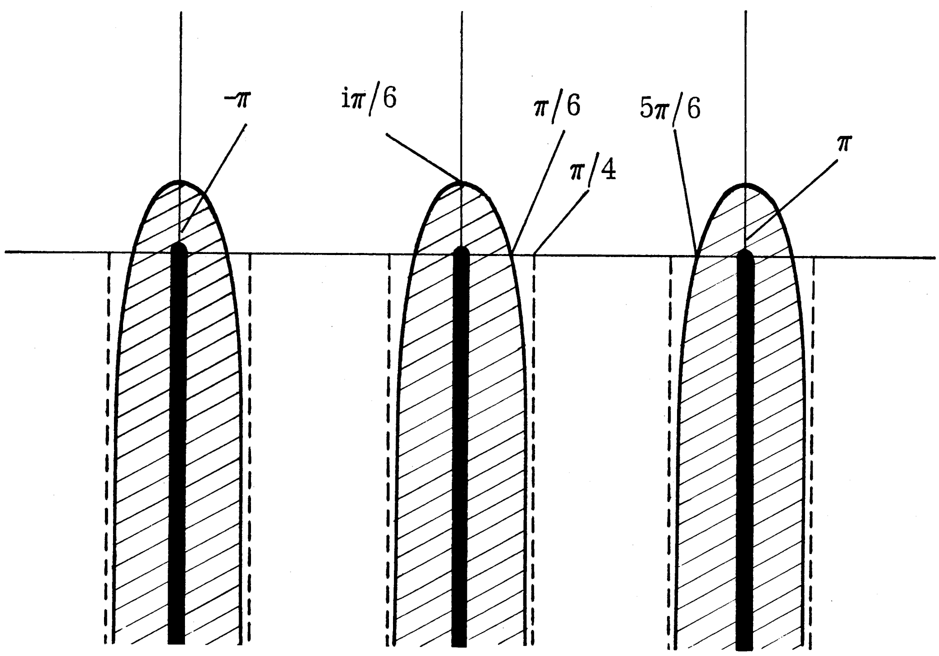

Figure 1. The complex θ-plane, showing the region (unshaded) in which expression (4) converges, and the branch cuts (thick lines) of the function $E_{\nu-1/2}(\theta)$.

The standard treatment of Legendre functions [see e.g. Hobson (1931), B.P. (1953)] proceeds in terms of an argument $\cos\theta$ rather than θ. However, when the real part of θ is between 0 and π, the solution (4) can be identified with a standard Legendre function by multiplying it by an appropriate function of ν. It is obviously convenient to keep as close as possible to the standard treatment, and the basic solution to equation (1) to be used hereinafter is therefore taken to be

$$E_{\nu-1/2}(\theta) = \sqrt{\left(\frac{\pi}{2}\right)}\, e^{i\pi/4}\, \frac{(\nu-1/2)!}{\nu\,!}\, \frac{e^{i\nu\theta}}{\sqrt{(\sin\theta)}}\, F\left(\frac{1}{2}, \frac{1}{2}; \nu+1; -\frac{i\,e^{i\theta}}{2\sin\theta}\right), \quad (6)$$

which for $0 < \mathrm{Re}\ \theta < \pi$ is the same as $Q_{\nu-1/2}(\cos\theta - i0)$ [B.P. (1953), p. 146, equation (4)].

It should be noted that the expression (6) can be continued analytically throughout the entire complex θ-plane cut by straight lines running parallel to the negative imaginary axis to infinity from branch points at $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ (see Figure 1). For example, another particularly useful form is [B.P. (1953), p. 146, equation (1)]

$$E_{\nu-1/2}(\theta) = \sqrt{\pi}\, \frac{(\nu-1/2)!}{\nu\,!}\, e^{i(\nu+1/2)\theta}\, F\left(\frac{1}{2}, \nu+\frac{1}{2}; \nu+1; e^{2i\theta}\right), \quad (7)$$

where here the series for the hypergeometric function converges for all values of θ (excluding the branch points) on and above the real axis.

Because of the factor $(\nu - 1/2)!/\nu!$, expressions (6) and (7), regarded as functions of ν, have simple poles, no longer at $\nu = -1, -2, -3, \ldots$, but instead at $\nu = -1/2, -3/2, -5/2, \ldots$; and these latter are the only singularities of $E_{\nu-1/2}(\theta)$ in the finite part of the complex ν-plane.

Expression (7) exhibits explicitly the way in which $E_{\nu-1/2}(\theta)$ is non-periodic in θ; namely, for $m = 0, \pm 1, \pm 2, \pm 3, \ldots$,

$$E_{\nu-1/2}(\theta + m\pi) = i^m e^{i\pi m\nu} E_{\nu-1/2}(\theta) \quad . \tag{8}$$

This important relation shows that the behavior of $E_{\nu-1/2}(\theta)$ throughout the entire complex θ-plane is specified in terms of its behavior in the region $0 \leqslant \text{Re}\,\theta < \pi$.

2.2 Relationships with $P_{\nu-1/2}(\cos\theta)$

It is useful to relate the function $E_{\nu-1/2}(\theta)$ to the other standard solution, $P_{\nu-1/2}(\cos\theta)$, of Legendre's equation. $P_{\nu-1/2}(\cos\theta)$ is defined in the range $0 < \text{Re}\,\theta < \pi$, and, in particular, one such relation can be used to give its analytic continuation throughout the entire complex θ-plane cut as in Figure 1. Because the only singularities in the complex ν-plane which arise are simple poles (at $\nu = -1/2, -3/2, -5/2, \ldots$) it follows that all the relations to be given hold throughout the entire complex ν-plane.

A convenient starting point, from which most of the results required later can be obtained, is the formula

$$E_{\nu-1/2}(\theta) = \frac{\pi}{2\cos(\pi\nu)} \left[i\, e^{i\pi\nu} P_{\nu-1/2}(\cos\theta) + P_{\nu-1/2}(-\cos\theta) \right] , \tag{9}$$

which can be shown to hold for $0 < \text{Re}\,\theta < \pi$ [B.P. (1953), p.140, equation (11)].

Since [B.P. (1953), p. 140, equation (1)]

$$P_{\nu-1/2}(\cos\theta) = P_{-\nu-1/2}(\cos\theta) \quad , \tag{10}$$

it follows from (9) that

$$E_{-\nu-1/2}(\theta) = \frac{\pi}{2\cos(\pi\nu)} \left[i e^{-i\pi\nu} P_{\nu-1/2}(\cos\theta) + P_{\nu-1/2}(-\cos\theta) \right] ; \tag{11}$$

and, by substracting (11) from (9), that

$$E_{\nu-1/2}(\theta) - E_{-\nu-1/2}(\theta) = -\pi \tan(\pi\nu) P_{\nu-1/2}(\cos\theta) \quad . \tag{12}$$

It is thus appropriate to define a new function

$$\tilde{P}_{\nu-1/2}(\theta) = -\frac{\cot(\pi\nu)}{\pi} \left[E_{\nu-1/2}(\theta) - E_{-\nu-1/2}(\theta) \right] \quad , \tag{13}$$

which is identical with $P_{\nu-1/2}(\cos\theta)$ for $0 < \mathrm{Re}\ \theta < \pi$, and which therefore gives its analytic continuation throughout the entire cut complex θ-plane.

Evidently

$$\tilde{P}_{\nu-1/2}(\theta) = \tilde{P}_{-\nu-1/2}(\theta) \quad . \tag{14}$$

Also from (9), for $0 < \mathrm{Re}\ \theta < \pi$,

$$E_{\nu-1/2}(\pi-\theta) = \frac{\pi}{2\cos(\pi\nu)} \left[i e^{i\pi\nu} P_{\nu-1/2}(-\cos\theta) + P_{\nu-1/2}(\cos\theta) \right] . \tag{15}$$

But, from (8),

$$E_{\nu-1/2}(\pi-\theta) = i e^{i\pi\nu} E_{\nu-1/2}(-\theta) \quad ;$$

whence (15) may be written

$$E_{\nu-1/2}(-\theta) = \frac{\pi}{2\cos(\pi\nu)} \left[P_{\nu-1/2}(-\cos\theta) - i e^{-i\pi\nu} P_{\nu-1/2}(\cos\theta) \right] . \tag{16}$$

Subtraction of (16) from (9) gives, for $0 < \mathrm{Re}\ \theta < \pi$, [cf. B.P. (1953), p. 144, equation (8)]

$$\tilde{P}_{\nu-1/2}(\theta) = P_{\nu-1/2}(\cos\theta) = -\frac{i}{\pi} \left[E_{\nu-1/2}(\theta) - E_{\nu-1/2}(-\theta) \right] . \tag{17}$$

Again, since

$$\tilde{P}_{\nu-1/2}(-\theta) = -\frac{\cot(\pi\nu)}{\pi}\left[E_{\nu-1/2}(-\theta) - E_{-\nu-1/2}(-\theta)\right] ,$$

use of (16) shows that, for $0 < \mathrm{Re}\,\theta < \pi$,

$$\tilde{P}_{\nu-1/2}(-\theta) = \tilde{P}_{\nu-1/2}(\theta) \quad . \tag{18}$$

However, it should perhaps be emphasized that $\tilde{P}_{\nu-1/2}(\theta)$ is not an even function of θ outside the region $-\pi < \mathrm{Re}\,\theta < \pi$.

2.3 Further results

As θ tends up to π through positive real values, the behavior of $P_{\nu-1/2}(\cos\theta)$ is well known to exhibit a logarithmic singularity [B.P. (1953), p.164, (16)]. A corresponding logarithmic behavior of $E_{\nu-1/2}(\theta)$ as θ tends down to zero or up to π is implicit in the formula (9). From this, in conjunction with (8), it is not difficult to establish that, for m = 0, ±1, ±2, ±3, ...,

$$\underset{\theta \to m\pi}{\mathrm{Limit}} \quad \sin\theta \, E'_{\nu-1/2}(\theta) = -e^{i\pi m(\nu-1/2)} , \tag{19}$$

where the dash denotes differentiation with respect to θ.

Now consider the Wronskian of the two solutions $E_{\nu-1/2}(\theta)$, $E_{-\nu-1/2}(\theta)$, defined as

$$W(\nu,\theta) = E_{\nu-1/2}(\theta)\, E'_{-\nu-1/2}(\theta) - E_{-\nu-1/2}(\theta)\, E'_{\nu-1/2}(\theta) \quad . \tag{20}$$

It is easily seen from equation (1) that

$$\frac{d}{d\theta}\left[\sin\theta \;\; W(\nu,\theta)\right] = 0 \quad ,$$

so that $\sin\theta\, W(\nu, \theta)$ is independent of θ. To find its actual value, let θ tend to zero. Then, from (19), $\sin\theta\, W(\nu, \theta)$ tends to the limit as θ tends to zero of the expression

$$-E_{\nu-1/2}(\theta) + E_{-\nu-1/2}(\theta) \quad .$$

But, for $0 < \mathrm{Re}\ \theta < \pi$, this expression, from (13), is just $\pi \tan(\pi\nu)\, \tilde{P}_{\nu-1/2}(\theta)$. And since $\tilde{P}_{\nu-1/2}(0) = 1$, it follows that

$$W(\nu, \theta) = \frac{\pi \tan(\pi\nu)}{\sin\theta} \quad . \tag{21}$$

$E_{\nu-1/2}(\theta)$ and $E_{-\nu-1/2}(\theta)$ are therefore independent solutions of Legendre's equation (1) unless $\nu = n$, where n has one of the values 0, ±1, ±2, ±3, ... In this latter case, from (9), for $0 < \mathrm{Re}\ \theta < \pi$,

$$E_{n-1/2}(\theta) = \frac{\pi}{2}\left[i\, P_{n-1/2}(\cos\theta) + (-)^n P_{n-1/2}(-\cos\theta)\right] = E_{-n-1/2}(\theta) \quad ;$$

also, from (8), for m = 0, ±1, ±2, ±3, ... ,

$$E_{\pm n-1/2}(\theta + m\pi) = i^m (-)^{mn} E_{\pm n-1/2}(\theta) \quad ;$$

hence, for all θ, if n is an integer

$$E_{n-1/2}(\theta) = E_{-n-1/2}(\theta) \quad . \tag{22}$$

This result implies, in particular, that by virtue of its definition (13), $\tilde{P}_{\nu-1/2}(\theta)$, regarded as a function of ν, is free of singularities throughout the entire finite part of the complex ν-plane.

Another result to which reference is made later is

$$\lim_{\nu \to 1/2} \sin\theta\, E'_{-\nu-1/2}(\theta) = -\cos\theta \quad . \tag{23}$$

This can be proved with some straightforward algebra from, say, the expression (7). More briefly, from (13),

$$\underset{\nu \to 1/2}{\text{Limit}} \; (\nu - \tfrac{1}{2}) \; E_{-\nu-1/2}(\theta)$$

$$= \underset{\nu \to 1/2}{\text{Limit}} \; \pi(\nu - \tfrac{1}{2}) \tan(\pi\nu) \; \tilde{P}_{\nu-1/2}(\theta)$$

$$= -1 \; , \text{ since } \tilde{P}_0(\theta) = 1 \; ;$$

so equation (1) shows that

$$\underset{\nu \to 1/2}{\text{Limit}} \; \frac{d}{d\theta} \left[\sin\theta \; E'_{-\nu-1/2}(\theta) \right] = \sin\theta \quad ,$$

and (23) follows by integration with respect to θ, where the fact that the integration constant is zero can be discovered by putting $\theta = 0$ and appealing to (19).

Also required is the result

$$\sin\theta \; E'_0(\theta) = -1 \quad . \tag{24}$$

This likewise may be easily deduced from equations (1) and (19).*

Finally, vital to the subsequent analysis is the asymptotic form of $E_{\nu-1/2}(\theta)$ as $|\nu| \to \infty$. For any fixed value of $\theta(\sin\theta \neq 0)$, and any fixed value of $\arg\nu$ in the range $-\pi < \arg\nu < \pi$,

$$E_{\nu-1/2}(\theta) \sim \sqrt{\frac{\pi}{2}} \; e^{i\pi/4} \; \frac{e^{i\nu\theta}}{\sqrt{(\nu \sin\theta)}} \qquad \text{as } |\nu| \to \infty \quad , \tag{25}$$

where $\sqrt{\nu}$ has a positive real part, and $\sqrt{(\sin\theta)}$ is positive for real values of θ between 0 and π.

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*In fact [cf. B.P. (1953), top of p. 152]

$$E_0(\theta) = \frac{1}{2} i\pi - \log\left[\tan(\frac{\theta}{2})\right] \quad ,$$

but this is not used in the sequel.

SECTION III

THE INTEGRAL TRANSFORM AND ITS INVERSE

3.1 A statement of the general transform relations

First, the general result is stated, with a few explanatory remarks.

The infinite Legendre integral transform of a function $F(\theta)$ is

$$f(\nu) = -\frac{i}{\pi} \int_{-\infty}^{\infty} F(\theta)\, E_{-\nu-1/2}(\theta) \sin\theta \, d\theta \; ; \qquad (26)$$

and the inverse of this transformation is

$$F(\theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \nu\, f(\nu) \cot(\pi\nu)\, E_{\nu-1/2}(\theta)\, d\nu \; , \qquad (27)$$

provided $\theta \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

No special points need be made in connection with (26), except to note that the $\sin\theta$ factor in the integrand is suggested by comparison with the simple finite transform in which the kernel is a Legendre polynomial*, and that (as appears explicitly in § 4) the constant factor is also chosen to facilitate comparison with this case. The path of integration in (26) can be along the real axis, or any equivalent path in the complex θ-plane which pays due regard to the branch cuts.

However, some features of (27) certainly call for comment.

First, $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ must be excluded, since for these values of θ $E_{\nu-1/2}(\theta)$ becomes logarithmically infinite and the right hand side of (27) is meaningless. Secondly, $\nu \cot(\pi\nu) E_{\nu-1/2}(\theta)$, regarded as a function of ν, has simple poles at $\nu = \pm 1, \pm 2, \pm 3, \ldots$. The path of integration in (27) must

* The derivation of (26) and (27) given in the Appendix shows that the factor $\sin\theta$ appears in the integrand of (26) because it multiplies $(\nu^2 - 1/4)U$ in the second term on the left hand side of equation (1).

therefore be specified with some precision. The convenient specification, herewith adopted, is that it runs parallel to the real axis; and in conjunction with this specification it is important to note that, aside from the effect of any singularities which $f(\nu)$ may have, it is immaterial whether the path lies above or below the real axis. This is because the residue of the integrand in (27) at the pole $\nu = n$ (n an integer) is

$$\frac{1}{\pi} \; n\, f(n)\; E_{n-1/2}(\theta) \quad , \tag{28}$$

whereas that at $\nu = -n$ is

$$-\frac{1}{\pi} \; n\, f(-n)\; E_{-n-1/2}(\theta) \quad ; \tag{29}$$

and (29) is just the negative of (28), since $E_{n-1/2}(\theta) = E_{-n-1/2}(\theta)$ $\left(\text{see } (22)\right)$, and this in turn, by virtue of the definition (26), implies that $f(n) = f(-n)$. On the tacit assumption, then, that the ν path of integration in (27) runs parallel to the real axis, the poles of the integrand associated with the factor $\cot(\pi\nu)$ are henceforth ignored.

It is not easy to account plausibly for the factor $\nu \cot(\pi\nu)$ in the integrand of (27). The ν part is again suggested by comparison with the finite transform, for which the inverse is a series with nth term of the form $a_n(n + 1/2)\, P_n(\cos\theta)$. But the presence of $\cot(\pi\nu)$ is more startling; about all that can be said is that the poles of $E_{\nu-1/2}(\theta)$ are removed by the $\cos(\pi\nu)$ in the numerator, that the disturbance this factor causes in the asymptotic behavior for large $|\nu|$ is annulled by the $\sin(\pi\nu)$ in the denominator, and finally, as just established, the poles associated with the $\sin(\pi\nu)$ can be left out of consideration.*

*In the derivation given in the Appendix, the factor $\cot(\pi\nu)$ is accounted for by the presence in the integrand of the factor $1/(W \sin\theta)$, where W is the Wronskian (20).

A last trivial remark; $E_{-\nu-1/2}(\theta)$ in (26) and $E_{\nu-1/2}(\theta)$ in (27) are interchangeable.

As explained in the introduction, the procedure now is to verify the inverse relation (27) in the case when $F(\theta)$ is a step function.

3.2 The step function

Consider the case

$$F(\theta) = \begin{cases} 0 & \text{for } \theta < \phi \ , \\ 1 & \text{for } \theta > \phi \ . \end{cases} \tag{30}$$

Then the transform (26) is

$$f(\nu) = -\frac{i}{\pi} \int_{\phi}^{\infty} E_{-\nu-1/2}(\theta) \sin\theta \, d\theta \ . \tag{31}$$

But the differential equation (1) can be written

$$E_{-\nu-1/2}(\theta) \sin\theta = -\frac{1}{\nu^2 - 1/4} \frac{d}{d\theta} \left[\sin\theta \ E'_{-\nu-1/2}(\theta) \right] ,$$

so that (31) gives

$$f(\nu) = -\frac{i}{\pi} \sin\phi \ \frac{E'_{-\nu-1/2}(\phi)}{\nu^2 - 1/4} \ , \qquad \text{Im}\,\nu < 0 \ . \tag{32}$$

It is clear from (8) that the stipulation Im $\nu < 0$ is necessary in order to make the integral in (31) converge at the upper limit. This means that the ν path of integration in (27) must be taken to lie below the real axis. It is therefore to be verified that

$$-\frac{i}{\pi^2} \int_{-\infty}^{\infty} \frac{\nu \sin\phi}{\nu^2 - 1/4} \cot(\pi\nu) \ E'_{-\nu-1/2}(\phi) \ E_{\nu-1/2}(\theta) \, d\nu \ , \quad \text{Im}\,\nu < 0, \tag{33}$$

represents the step function (30). This is accomplished by contour integration. The argument is first given in broad terms and then amplified by some further remarks.

Apart from the irrelevant poles at $\nu = \pm 1, \pm 2, \pm 3, \ldots,$ the only singularity of the integrand of (33) in the finite part of the complex ν-plane is a simple pole at $\nu = -1/2$. This is because the only singularities of $E'_{-\nu-1/2}(\phi)\ E_{\nu-1/2}(\theta) / (\nu - 1/2)$ are simple poles at $\nu = 1/2$ [from the factor $1/(\nu - 1/2)$], at $\nu = 3/2, 5/2, 7/2, \ldots$ [from the factor $E'_{-\nu-1/2}(\phi)$, which, by virtue of (23), does not have a pole at $\nu = 1/2$], and at $\nu = -1/2, -3/2, -5/2, \ldots$ [from the factor $E_{\nu-1/2}(\theta)$]; and all these poles are annulled by the factor $\cot(\pi\nu)$. If, therefore, the path of integration can be closed by an infinite semi-circle below the real axis, the result is zero; whereas, if it can be closed by an infinite semi-circle above the real axis, the result is $2/\pi$ times the residue of the integrand at the pole $\nu = -1/2$.

Now the way in which the path of integration may be closed is determined by the asymptotic form of the integrand as $|\nu| \to \infty$. From (25), by taking account in particular of the specification of $\sqrt{\nu}$, it can be seen that, as $|\nu| \to \infty$,

$$\frac{\nu \sin\phi}{\nu^2 - 1/4} \cot(\pi\nu)\ E'_{-\nu-1/2}(\phi)\ E_{\nu-1/2}(\theta) \sim \frac{\pi}{2}\sqrt{\left(\frac{\sin\phi}{\sin\theta}\right)}\ \frac{e^{i\nu(\theta-\phi)}}{\nu} ,$$

for any fixed $\arg\nu$ in the range $0 < |\arg\nu| < \pi$, and any fixed values of θ and ϕ other than $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. By Jordan's lemma (Whittaker and Watson (1927), p. 115) the value of (33) is therefore unaltered by closing the path of integration with an infinite semi-circle, below the real axis when $\theta < \phi$ and above the real axis when $\theta > \phi$. Furthermore, from (13),

$$\underset{\nu \to -1/2}{\text{Limit}} \quad \cot(\pi\nu)\ E_{\nu-1/2}(\theta) = -\pi\ \tilde{P}_{-1}(\theta) = -\pi \quad , \tag{35}$$

and from (24),

$$\sin\phi\ E_0'(\phi) = -1\ ;$$

so the residue of the integrand of (33) at the pole at $\nu = -1/2$ is $\pi/2$. Hence the required confirmation that (33) is zero for $\theta < \phi$ and unity for $\theta > \phi$.

A few comments on the proof just given should be made. First, the asymptotic form (34) does not hold if either θ or ϕ has one of the values $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. However, if ϕ (but not θ) has one of these values, the argument can be carried through as before; for, from (19), if $\phi = m\pi$,

$$\sin\phi\ E'_{-\nu-1/2}(\phi) = -e^{-i(\nu+1/2)\phi} \quad ,$$

and (34) is replaced by

$$\frac{\nu\sin\phi}{\nu^2 - 1/4}\ \cot(\pi\nu)\ E'_{-\nu-1/2}(\phi)\ E_{\nu-1/2}(\theta) \sim \mp\sqrt{\left(\frac{\pi}{2}\right)}\ e^{-1/4\, i\pi}\ e^{-1/2\, i\phi}\ \frac{e^{i\nu(\theta-\phi)}}{\nu\sqrt{\nu\sin\theta}} \quad , \tag{36}$$

with the upper or lower sign according as to whether $0 < \arg\nu < \pi$ or $-\pi < \arg\nu < 0$ respectively; the statements about the closure of the path of integration therefore still hold, and the rest of the argument is unaffected.

The argument breaks down, of course, if θ has one of the values $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. It has already been pointed out that (27) has no meaning for these values of θ.

Secondly, the asymptotic form (25) does not hold if $\arg \nu = \pm\pi$. It therefore fails to give the asymptotic form of the integrand of (33) at the extremes of the original path of integration, since the path runs parallel to the real axis. In order to establish strictly that the conditions for Jordan's lemma are fulfilled, the general nature, at least, of this asymptotic behavior must be found. The matter is not pursued here, but a more detailed investigation has been made which confirms the validity of the procedure.

Finally, it is instructive to evaluate (33) when $\theta = \phi$. In (30), $F(\theta)$ is not defined there, because its value at that single point does not affect its transform. But it can be shown that the representation (33) has the value 1/2 at $\theta = \phi$, thus exhibiting another feature in common with the Fourier representation.

As a preliminary it is noted that, for any fixed value of $\arg\nu$ in the range $0 < |\arg\nu| < \pi$,

$$E_{\nu-1/2}(\phi)\, E'_{-\nu-1/2}(\phi) + E_{-\nu-1/2}(\phi)\, E'_{\nu-1/2}(\phi) \rightarrow 0 \qquad \text{as } |\nu| \rightarrow \infty . \tag{37}$$

Hence

$$\int_{-\infty}^{\infty} \frac{\nu\cot(\pi\nu)}{\nu^2 - 1/4} \left[E_{\nu-1/2}(\phi_2)\, E'_{-\nu-1/2}(\phi) + E_{-\nu-1/2}(\phi)\, E'_{\nu-1/2}(\phi) \right] d\nu = 0 . \tag{38}$$

For the path of integration can be closed by an infinite semi-circle, either above or below the real axis; in the former case it encloses no singularities; in the latter it encloses the two poles at $\nu = 1/2$ and $\nu = -1/2$, the residues of which cancel.

The expression (33) when $\theta = \phi$ can therefore be written

$$- \frac{i \sin\phi}{2\pi^2} \int_{-\infty}^{\infty} \frac{\nu \cot(\pi\nu)}{\nu^2 - 1/4} \, W(\nu, \phi) \, d\nu \; , \qquad \text{Im}\, \nu < 0 \, , \tag{39}$$

where $W(\nu, \phi)$ is the Wronskian defined by (20). Using (21), it is seen that (39) is

$$- \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\nu}{\nu^2 - 1/4} \, d\nu \; , \qquad \text{Im}\, \nu < 0 \, ,$$

$$= - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\nu}{\nu^2 - 1/4} \, d\nu \; , \qquad \text{Im}\, \nu > 0 \, ,$$

$$= - \frac{i}{4\pi} \oint \frac{\nu}{\nu^2 - 1/4} \, d\nu \; ,$$

where the contour encloses both the poles in the anticlockwise sense. The value is therefore 1/2.

To sum up, it has been shown that the Legendre transform of the step function that is zero for $\theta < \phi$ and unity for $\theta > \phi$ is

$$- \frac{i}{\pi} \sin\phi \; \frac{E'_{-\nu-1/2}(\phi)}{\nu^2 - 1/4} \; , \qquad \text{Im}\, \nu < 0 \; ; \tag{40}$$

and that the inverse representation (27), undefined at $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots,$ is otherwise the original step function with, in addition, the particular value 1/2 at $\theta = \phi$.

It may be remarked in passing that, by virtue of (19), the Legendre transform of the step function that is zero for $\theta < 0$ and unity for $\theta > 0$ is

$$\frac{i}{\pi} \; \frac{1}{\nu^2 - 1/4} \; , \qquad \text{Im}\, \nu < 0 \, . \tag{41}$$

3.3 The rectangular pulse

It follows at once from the substraction of two step functions like (30) that the Legendre transform of the rectangular pulse defined by

$$F(\theta) = \begin{cases} 0 & \text{for } \theta < \phi \quad , \\ 1 & \text{for } \phi < \theta < \psi \quad , \\ 0 & \text{for } \theta > \psi \quad , \end{cases} \tag{42}$$

is

$$f(\nu) = \frac{i}{\pi(\nu^2 - 1/4)} \left[\sin\psi \; E'_{-\nu-1/2}(\psi) - \sin\phi \; E'_{-\nu-1/2}(\phi) \right] ; \tag{43}$$

and that the inverse representation (27), undefined at $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.,$ is otherwise the original rectangular pulse (42) with, in addition, the particular value 1/2 at both $\theta = \phi$ and $\theta = \psi$.

It is remarked that $f(\nu)$ given by (43) is free of singularities throughout the finite part of the complex ν-plane.

For the symmetrical case $\phi = -\psi$, and with $0 < \psi < \pi$, appeal to (17) and (18) shows that (43) may be written

$$f(\nu) = -\frac{\sin\psi}{\nu^2 - 1/4} \; \tilde{P}'_{\nu-1/2}(\psi) \quad ; \tag{44}$$

and since [B.P. (1953),p.161, formula (19)]

$$-\sin\psi \frac{d}{d\psi}\left[P_{\nu-1/2}(\cos\psi)\right] = (\nu + \frac{1}{2})\left[\cos\psi \; P_{\nu-1/2}(\cos\psi) - P_{\nu+1/2}(\cos\psi)\right]$$

$$= -(\nu - \frac{1}{2})\left[\cos\psi \; P_{\nu-1/2}(\cos\psi) - P_{\nu-3/2}(\cos\psi)\right],$$

expression (44) may in turn be written in either of the forms

$$f(\nu) = \frac{1}{\nu - 1/2}\left[\cos\psi\, P_{\nu-1/2}(\cos\psi) - P_{\nu+1/2}(\cos\psi)\right], \tag{45}$$

$$f(\nu) = -\frac{1}{\nu + 1/2}\left[\cos\psi\, P_{\nu-1/2}(\cos\psi) - P_{\nu-3/2}(\cos\psi)\right]. \tag{46}$$

3.4 The delta function

Here the case is considered in which the width of the rectangular pulse becomes infinitesimal. This is achieved by taking, in (42),

$$\psi = \phi + \delta\phi ,$$

where $\delta\phi$ is infinitesimal. To the first order in $\delta\phi$ the transform (43) is then

$$f(\nu) = \frac{i}{\pi(\nu^2 - 1/4)} \frac{d}{d\phi}\left[\sin\phi\, E'_{-\nu-1/2}(\phi)\right] \delta\phi$$

$$= -\frac{i}{\pi} \sin\phi\, E_{-\nu-1/2}(\phi)\, \delta\phi, \qquad \text{from (1).} \tag{47}$$

If now the area under the rectangular pulse is made unity, by changing the height of the pulse from unity to $1/\delta\phi$, the following result is obtained: the Legendre transform of the delta function, $\delta(\theta - \phi)$, is

$$f(\nu) = -\frac{i}{\pi} \sin\phi\, E_{-\nu-1/2}(\phi) , \tag{48}$$

and the representation (27), undefined at $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.,$ otherwise reproduces $\delta(\theta - \phi)$.*

*If the delta function is conceived as the limit as $\epsilon \to 0$ of a rectangular pulse of width ϵ and height $1/\epsilon$, it would seem desirable to specify a pulse height of $1/2\epsilon$ at the points of discontinuity, so that the sum of two such rectangular pulses, identical save that one is displaced a distance ϵ relative to the other, should in all respects reproduce a rectangular pulse of width 2ϵ. It must be allowed, however, that many text books admit no need for such a refinement in the conception of a delta function.

3.5 <u>The general case</u>

The confirmation of the transform relations (26) and (27) for the case when $F(\theta)$ is a delta function is now used in a purely formal way to give the confirmation for the general case. A general function $F(\theta)$ is written

$$F(\theta) = \int_{-\infty}^{\infty} F(\phi)\, \delta(\theta - \phi)\, d\phi \; ; \tag{49}$$

hence, from the result just proved for the transform (48), the representation (27), undefined for $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$, otherwise yields the original function if

$$f(\nu) = -\frac{i}{\pi} \int_{-\infty}^{\infty} F(\phi) \sin\phi \; E_{-\nu-1/2}(\phi)\, d\phi \; . \tag{50}$$

But (50) is just the transform (26), so the required confirmation has been obtained.

SECTION IV

THE LEGENDRE POLYNOMIAL SERIES

It is well known that, in general terms, for an even function H(θ) of period 2π the transform defined by

$$h(n) = \int_0^\pi H(\theta)\ P_n\,(\cos\theta)\ \sin\theta\ d\theta\ , \qquad (51)$$

for n = 0, 1, 2, 3, ..., yields the representation

$$H(\theta) = \sum_{n=0}^{\infty} (n + \tfrac{1}{2})\ h(n)\ P_n\,(\cos\theta)\ . \qquad (52)$$

It is instructive to test the transform relations (26) and (27) of the present paper by recovering (51) and (52) from a consideration of the case when F(θ) is even and of period 2π in θ.

First, the even function G(θ), which is H(θ) for $-\pi < \theta < \pi$ and zero for $|\theta| > \pi$, is considered. Its Legendre transform (26) is

$$g(\nu) = -\frac{i}{\pi} \int_{\pi}^{\pi} G(\theta)\ E_{-\nu-1/2}(\theta) \sin\theta\ d\theta$$

$$= -\frac{i}{\pi} \int_0^{\pi} H(\theta) \left[E_{-\nu-1/2}(\theta) - E_{-\nu-1/2}(-\theta)\right] \sin\theta\ d\theta$$

$$= \int_0^{\pi} H(\theta)\ P_{\nu-1/2}\,(\cos\theta)\ \sin\theta\ d\theta\ , \qquad \text{from (17)}\ . \qquad (53)$$

Again, the Legendre transform of G(θ - 2nπ) is

$$-\frac{i}{\pi} \int_{(2n-1)\pi}^{(2n+1)\pi} G(\theta - 2n\pi)\ E_{-\nu-1/2}(\theta) \sin\theta\ d\theta = (-)^n\, e^{-2i\pi n\nu}\, g(\nu)\ , \qquad (54)$$

for n = 0, ±1, ±2, ±3, ..., using (8).

Hence the transform of

$$\sum_{n=0}^{\infty} G(\theta - 2n\pi) \tag{55}$$

is

$$g(\nu) \sum_{n=0}^{\infty} (-)^n e^{-2i\pi n\nu} = \frac{e^{i\pi\nu} g(\nu)}{2\cos(\pi\nu)}, \quad \text{Im}\,\nu < 0, \tag{56}$$

the stipulation that the imaginary part of ν be negative being necessary in order to make the series converge; and similarly the transform of

$$\sum_{n=-1}^{-\infty} G(\theta - 2n\pi) \tag{57}$$

is

$$-\frac{e^{i\pi\nu} g(\nu)}{2\cos(\pi\nu)}, \qquad \text{Im}\,\nu > 0. \tag{58}$$

But $H(\theta)$ is obtained by adding (55) and (57). Its representation (27) is therefore

$$\frac{i}{2\pi} \oint \frac{\nu e^{i\pi\nu} g(\nu)}{\cos(\pi\nu)} \cot(\pi\nu)\, E_{\nu-1/2}(\theta)\, d\nu, \tag{59}$$

where the contour runs first from $-\infty$ to ∞ parallel to and below the real axis, and then from ∞ to $-\infty$ parallel to and above the real axis.

Since $P_{\nu-1/2}(\cos\theta)$, as a function of ν, is free of singularities throughout the finite part of the complex ν-plane, so also is $g(\nu)$, from (53); and the only relevant singularities of the integrand of (59) are simple poles at $\nu = -1/2, -3/2, -5/2, \ldots$. From an evaluation of the residues of these poles the expression (59) is evidently

$$\sum_{n=0}^{\infty} (n + \tfrac{1}{2})\, h(n)\, P_n(\cos\theta),$$

where

$$h(n) = g(-n - \tfrac{1}{2}) = \int_0^{\pi} H(\theta)\, P_n(\cos\theta) \sin\theta\, d\theta, \quad \text{from (53).}$$

Thus (51) and (52) are recovered.

REFERENCES

Bateman Project, (1953), Higher Transcendental Functions, Vol. 1, McGraw-Hill.

Churchill, R.V., (1954), Jour. Math. and Phys. 33, 165-175.

Hobson, E.W., (1931), The Theory of Spherical and Ellipsoidal Harmonics, Cambridge.

Marcuvitz, N., (1951), Comm. Pure Appl. Maths. 4, 263.

Titchmarsh, E.C., (1946), Eigenfunction Expansions Associated with Second-order Differential Equations, Oxford, Clarendon Press.

Whittaker, E.T. and Watson, G.N., (1927), Modern Analysis, Cambridge.

APPENDIX

In the main body of the paper the transform relations (26), (27) are stated and then verified. The purpose of this Appendix is to indicate a methodical derivation of the relations. The mathematical procedure adopted is that set out by Marcuvitz (1951), and is evidently quite formal. Presumably, as Marcuvitz indicates, a rigorous proof could be constructed, for example, on the basis of the method expounded by Titchmarsh (1946).

Equation (1) is rewritten

$$\frac{d}{d\theta}\left(\sin\theta \frac{dU}{d\theta}\right) + \lambda \sin\theta\, U = 0 \quad , \qquad \text{(A1)}$$

where

$$\lambda = \nu^2 - \frac{1}{4} \quad . \qquad \text{(A2)}$$

A Green's function $G(\theta, \phi)$, which satisfies the inhomogeneous equation

$$\frac{d}{d\theta}\left(\sin\theta \frac{dG}{d\theta}\right) + \lambda \sin\theta\, G = -\delta(\theta - \phi) \, , \qquad \text{(A3)}$$

is constructed in terms of the two independent solutions $E_{\nu-1/2}(\theta)$, $E_{-\nu-1/2}(\theta)$ of the homogeneous equation (A1).

If the condition $\mathrm{Im}\, \nu > 0$ is imposed,

$$E_{-\nu-1/2}(\theta) \longrightarrow 0 \qquad \text{as } \theta \longrightarrow -\infty \, ,$$

$$E_{\nu-1/2}(\theta) \longrightarrow 0 \qquad \text{as } \theta \longrightarrow \infty \, ,$$

and a representation of the Green's function is [cf. Marcuvitz (1951) p. 290, equations (3.13), (3.14)]

$$G(\theta,\phi) = \begin{cases} \dfrac{E_{-\nu-1/2}(\theta)\, E_{\nu-1/2}(\phi)}{W \sin\theta} & \text{for } \theta < \phi \, , \quad \text{(A4)} \\[2ex] \dfrac{E_{-\nu-1/2}(\phi)\, E_{\nu-1/2}(\theta)}{W \sin\theta} & \text{for } \theta > \phi \, , \quad \text{(A5)} \end{cases}$$

where W is the Wronskian defined by (20). Hence, from (21),

$$G(\theta,\phi) = \begin{cases} \dfrac{1}{\pi} \cot(\pi\nu)\, E_{-\nu-1/2}(\theta)\, E_{\nu-1/2}(\phi) & \text{for } \theta < \phi \, , \quad \text{(A6)} \\[2ex] \dfrac{1}{\pi} \cot(\pi\nu)\, E_{-\nu-1/2}(\phi)\, E_{\nu-1/2}(\theta) & \text{for } \theta > \phi \, . \quad \text{(A7)} \end{cases}$$

The representation [cf. Marcuvitz (1951) p. 287, equation (3.8)]

$$\frac{\delta(\theta - \phi)}{\sin\phi} = -\frac{1}{2\pi i} \oint G(\theta,\phi)\, d\lambda \, , \tag{A8}$$

where the contour of integration is closed anti-clockwise round all the singularities of $G(\theta,\phi)$ in the complex λ-plane, is now used.

From the discussion of $E_{\nu-1/2}(\theta)$ given in § 2 it is evident that the singularities of $G(\theta,\phi)$ in the complex λ-plane are a branch point at $\lambda = -1/4$ and simple poles at $\lambda = -1/4 + n^2$, $n = 1,2,3,\ldots$.

In order to keep the imaginary part of $\nu = \sqrt{(\lambda + 1/4)}$ positive the branch cut is taken to run, in effect, along the real λ axis to $+\infty$. The substitution of (A7) into (A8) then gives, for $\theta > \phi$,

$$\frac{\delta(\theta - \phi)}{\sin\phi} = \frac{i}{2\pi^2} \int \cot(\pi\nu)\, E_{\nu-1/2}(\theta)\, E_{-\nu-1/2}(\phi)\, d\lambda \, , \tag{A9}$$

where the contour of integration is shown in Figure A-1.

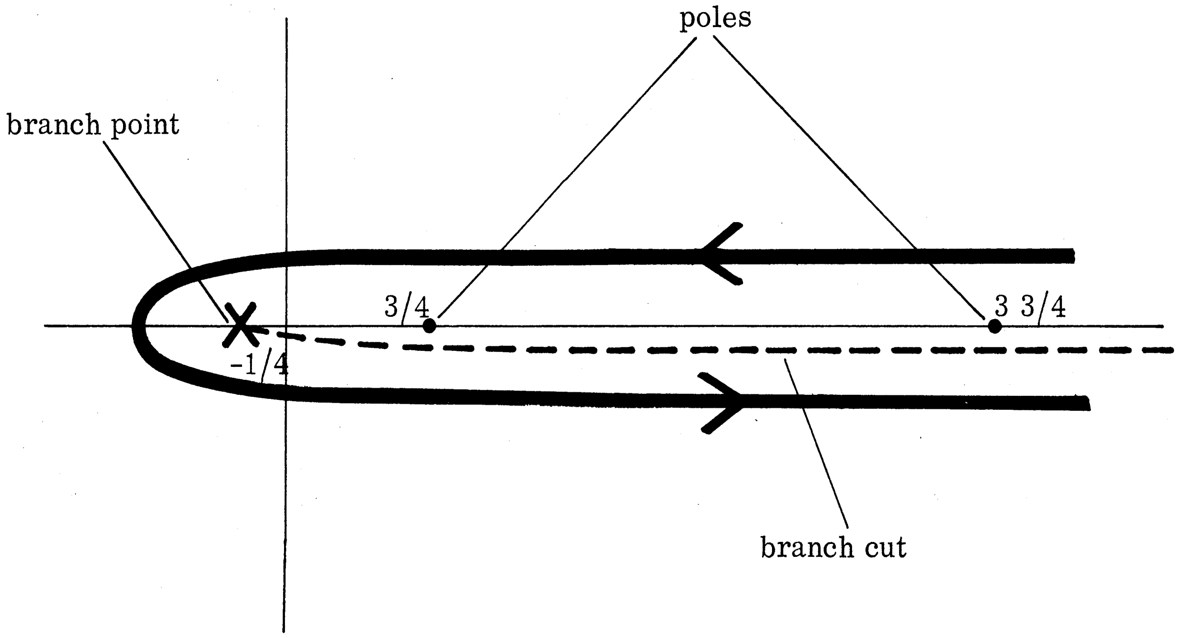

Figure A-1. The path of integration in the complex λ-plane for the integral in (A9).

Now change the variable of integration from λ to ν . Then (A9) gives,

for $\theta > \phi$,

$$\frac{\delta(\theta - \phi)}{\sin \phi} = -\frac{i}{\pi^2} \int_{-\infty}^{\infty} \nu \cot(\pi\nu)\; E_{\nu - 1/2}(\theta)\; E_{-\nu - 1/2}(\phi)\, d\nu , \qquad (A10)$$

where the path of integration runs above the real axis.

For $\theta < \phi$, the substitution of (A6) into (A8) leads to (A10), except for the interchange of θ and ϕ in the integrand. However, as noted in § 3.1, there is freedom to translate the ν path of integration across the real axis. A reversal of the sign of ν therefore shows that (A10) is equally valid for $\theta < \phi$.

The representation of $\delta(\theta - \phi)$ given by (27) and (48) is thus derived, and the transform relations (26), (27) may now be inferred in the manner of § 3.5.

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