THE UNIVERSITY OF MICHIGAN

RADC TN-59-164
ASTIA AD-215 487

STUDIES IN RADAR CROSS SECTIONS XXXV - ON THE SCALAR THEORY OF THE DIFFRACTION OF A PLANE WAVE BY A LARGE SPHERE

by

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April 1959

Report No. 2778-6-T

on

CONTRACT AF30(602)-1853
PROJECT 5535
TASK 45773

Prepared for

ROME AIR DEVELOPMENT CENTER
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
GRIFFISS AIR FORCE BASE, NEW YORK

STUDIES IN RADAR CROSS SECTIONS

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PREFACE

This is the thirty-fifth in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers; and (d) low and high density ionization phenomena.

K. M. Siegel

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SUMMARY

The problem, with boundary condition either $U = 0$ or $\partial U/\partial r = 0$, is solved by a method different from those already given by other authors. The fundamental idea is analogous to that introduced by the present writer to treat diffraction by a circular cylinder, and is carried out by means of an infinite Legendre integral transform and its inverse. Useful integral representations for the field, previously obtained by other means, are rederived. First, a discussion of the radiation part of the scattered field is given; this includes the two special cases of forward and back scattering, the former of which yields the total scattering cross-section. Subsequently, the field at a finite distance is considered.

1. Introduction

The purpose of this paper is to treat the problem of diffraction by a large sphere by a method analogous to that recently used by the present writer (Clemmow 1959 a, hereinafter referred to as I) to solve the problem of diffraction by a large circular cylinder. For the cylinder, an infinite Fourier integral transform and its inverse were used; for the sphere, an infinite Legendre integral transform and its inverse (Clemmow 1959 b, hereinafter referred to as II) are appropriate. Otherwise, however, the technique in the latter case is closely parallel to that in the former. The present paper, therefore, follows much the same lines as I, and may conveniently be read in conjunction with it.

In the cylinder problem, the use of infinite Fourier transform analysis seems first to have been exploited by Friedlander (1954), though there is a hint of it in an early work by Debye (1908). On the other hand, an infinite Legendre transform analysis for the sphere problem appears to be new*, unless it be traced back in the history of the theory of radio propagation around the surface of the earth to papers by March (1912) and Rybczynski (1913). Admittedly, March introduced an integral representation involving Legendre functions, but the connection, if any, with the present theory is sufficiently disguised to make a pursuit of the comparison certainly

* Though there is some similarity to the techniques used by Felsen; see, for example, Felsen (1957).

tedious and probably unprofitable. It may, however, be remarked in passing that the justification of the dismissal of March's work by Love (1915) and Watson (1918) is perhaps questionable. For the criticism was chiefly levelled at the fact that March's representation was invalid at all points on the radial line from the center of the earth pointing directly away from the transmitter; and yet this in itself, as the present treatment, for example, bears witness, is no evidence of the unsoundness of the theory.

There is no need here, then, to add further to the discussion given in I of the general background to the problem and of the relevant literature. References in the body of the paper draw attention to previous work mainly for the sake of avoiding duplication of specific calculations.

The incident field is taken to be a plane wave, and for the most part the development is presented in terms of the Dirichlet problem, in which the wave function vanishes on the surface of the sphere. In § 2 the fundamental idea is stated and expressed mathematically by means of an infinite Legendre integral transform and its inverse. In § 3, the radiation field is considered; the discussion includes both that in the forward direction, from which the total scattering cross-section is obtained, and that in the backward direction, which happens to be a special case as regards the mathematical treatment. In § 4, attention is turned to the field at any finite distance from the center of the sphere. In § 5, the closely analogous Neumann problem, in which the normal derivative of the wave function vanishes on the surface of the sphere, is briefly discussed.

2. The General Nature of the Solution

The problem considered is that of the scalar plane wave specified by*

$$U^i = e^{-ikr\cos\theta} \tag{1}$$

falling on the sphere $r = a$, where (r, θ, ϕ) are spherical polar coordinates (see Fig. 1).. The convention adopted is that the physical

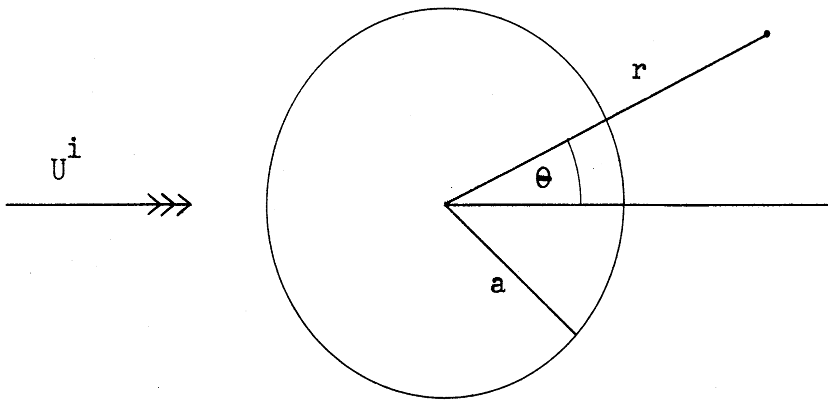

Figure 1

The configuration

space is embraced by the range of values of θ between 0 and π, and the analysis is evidently independent of ϕ. The Dirichlet boundary condition $U=0$ on $r=a$ is chosen, where U is the wave function of the total field.

*The suppressed time factor is $\exp(i\omega t)$.

Following the argument of I, the aim is to obtain a convenient form of the solution by expressing it in terms of functions which are not individually periodic in θ. This is easily done with the use of the infinite Legendre integral transform defined and discussed in II.

In the notation of II, the function

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \nu\, p\,(\theta_1, \theta_2; \nu)\cot(\pi\nu)\, E_{\nu-\frac{1}{2}}(\theta)\, d\nu, \qquad (2)$$

where

$$p\,(\theta_1, \theta_2; \nu) = -\frac{i}{\pi}\int_{\theta_1}^{\theta_2} e^{-ika\cos\psi} E_{-\nu-\frac{1}{2}}(\psi)\sin\psi\, d\psi, \qquad (3)$$

is equal to

$$\begin{cases} e^{-ika\cos\theta} & \text{for } \theta_1 < \theta < \theta_2 \\ \frac{1}{2}\, e^{-ika\cos\theta} & \text{for } \theta = \theta_1,\ \theta = \theta_2, \\ 0 & \text{for } \theta < \theta_1,\ \theta > \theta_2, \end{cases} \qquad (4)$$

for arbitrary values of θ_1, θ_2 ($\theta_2 > \theta_1$). Two supplementary remarks have to be made. First, (2) is meaningless when θ has one of the values $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$, because $E_{\nu-\frac{1}{2}}(\theta)$ is then logarithmically singular. Secondly, the path of integration in (2) avoids the poles at $\nu = \pm 1, \pm 2, \pm 3, \ldots$ by running parallel to the real ν axis; whether it lies entirely above or entirely below the real axis is immaterial, since the residue of the pole at $\nu = n$ cancels that of the pole at $\nu = -n$.

Then, likewise, the function

$$-\frac{1}{\pi}\sqrt{\left(\frac{a}{r}\right)}\int_{-\infty}^{\infty} \nu \; \frac{p(\theta_1, \theta_2; \nu)}{H_\nu^{(2)}(ka)} \; H_\nu^{(2)}(kr) \cot(\pi\nu) E_{\nu-\frac{1}{2}}(\theta) d\nu \qquad (5)$$

has the value minus (4) on $r = a$. It is thus an outgoing field which on the surface of the sphere cancels the incident field for θ between θ_1 and θ_2, and is zero there for θ less than θ_1 or greater than θ_2. Clearly, then, an exact representation of the scattered field is given by the superposition of all functions (5) corresponding to non-overlapping ranges $[\theta_1, \theta_2]$ which together span the full range $-\infty$ to ∞.

Again following the argument in I, it may be said in general terms that for $ka \gg 1$ the scattered field is given to a good approximation by the expression (5) with the range $[\theta_1, \theta_2]$ roughly spanning that of physical space, namely $[0, \pi]$; and that the further the range $[\theta_1, \theta_2]$ is from some such "primary" range, the smaller, at a given point in physical space, the expression (5) becomes. Furthermore, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$,

$$E_{\nu-\frac{1}{2}}(\theta + 2n\pi) = i^n e^{i\pi n\nu} E_{\nu-\frac{1}{2}}(\theta) \tag{6}$$

(see II, equation (8)), and it follows from (3) that

$$p(\theta_1 - 2n\pi, \theta_2 - 2n\pi; \nu) = (-)^n e^{2i\pi n\nu} p(\theta_1, \theta_2; \nu). \tag{7}$$

A decrease of $2n\pi$ in both θ_1 and θ_2 is therefore equivalent in (5) to an increase of $2n\pi$ in θ, so that an interpretation in terms of rays travelling around the sphere becomes evident if the solution is built up by taking successive $\left[\theta_1, \theta_2\right]$ ranges to be of width 2π.

3. The Radiation Field

3.1. The Case $\theta \neq 0$

In this section the radiation field, that is, the part of the scattered field of order $1/(kr)$ for $kr \gg 1$, is considered for all directions other than that in which the incident wave is travelling.

For any fixed ν, as $kr \to \infty$

$$H_\nu^{(2)}(kr) \sim \sqrt{\left(\frac{2}{\pi}\right)} e^{\frac{1}{4} i\pi} e^{\frac{1}{2} i\pi\nu} \frac{e^{-ikr}}{\sqrt{(kr)}}, \tag{8}$$

and (5) is consequently asymptotic to

$$- \frac{\sqrt{(2\,ka)}}{\pi\sqrt{\pi}}\, e^{\frac{1}{4} i\pi}\, \frac{e^{-ikr}}{kr} \int_{-\infty}^{\infty} \nu\, \frac{p(\theta_1,\, \theta_2;\, \nu)}{H_{\nu}^{(2)}(ka)}\, e^{\frac{1}{2} i\pi\nu} \cot\,(\pi\nu) E_{\nu-\frac{1}{2}}(\theta) d\nu\,. \tag{9}$$

The justification for substituting (8) into the integrand in (5), where the variable of integration ν runs to infinity, is essentially that given for the corresponding step in I.

The behaviour in the complex ν-plane of the integrand in (9) is now considered. As noted in II, the only simgularities of $E_{-\nu-\frac{1}{2}}(\theta)$ are simple poles at $\nu = 1/2, 3/2, 5/2, \ldots$; and it is evident, from (3), that $p(\theta_1,\, \theta_2;\, \nu)$ can have no others. But the integrand in (9) contains the factor $\cos(\pi\nu)$, so that its only singularities in addition to the irrelevant poles at $\nu = 0, \pm 1, \pm 2, \pm 3, \ldots$ are simple poles arising from the zeros of $H_{\nu}^{(2)}(ka)$. Furthermore, for any fixed value of θ ($\sin\theta \neq 0$), and any fixed value of $\arg\nu$ in the range $-\pi < \arg\nu < \pi$, as $|\nu| \to \infty$

$$E_{\nu-\frac{1}{2}}(\theta) \sim \sqrt{\left(\frac{\pi}{2}\right)}\; e^{\frac{1}{4} i\pi}\; \frac{e^{i\nu\theta}}{\sqrt{(\nu \sin\theta)}}\,, \tag{10}$$

where $\sqrt{\nu}$ has a positive real part, and $\sqrt{(\sin\theta)}$ is positive for real values of θ between 0 and π.

It follows that the path of integration in (9) may be replaced, if $\theta_2 < -\pi/2$, by one enclosing the poles of $1/H_\nu^{(2)}(ka)$ in the upper half-plane, and if $\theta_1 > 3\pi/2$, by one enclosing the corresponding poles in the lower half-plane. For any range $[\theta_1, \theta_2]$ outside $[-\pi/2, 3\pi/2]$, the integral (9) can therefore be evaluated by the calculation of residues.

The main contribution to the scattered field is contained in the expression (9) with $\theta_1 = -\pi/2$, $\theta_2 = 3\pi/2$. In this case the appropriate method of evaluation is that of steepest descents. In order to apply this method, $p(-\pi/2, 3\pi/2; \nu)$ is expressed as the sum of terms whose individual behaviour can be represented, for the most part, by exponential functions with comparatively simple exponents. As a first step in achieving this (cf. I) the ψ path of integration in (3) is distorted in the way indicated in Figure 2, where the regions of convergence at infinity are shown shaded. The contribution to $p(-\pi/2, 3\pi/2; \nu)$ of the curved portions of the new ψ path of integration is*

$$-\sqrt{\left(\frac{\pi}{2ka}\right)}\, e^{\frac{1}{4} i\pi}\, e^{-\frac{1}{2} i\pi\nu} \left[H_\nu^{(1)}(ka) - H_\nu^{(2)}(ka)\right] . \qquad (11)$$

* In essence, this result is to be found in Watson (1944) p. 175. But since the integration is now in the complex ψ-plane, in contrast to the more standard complex $\cos\psi$-plane used by Watson, it seems worthwhile to give a direct derivation. This is done in Appendix A.

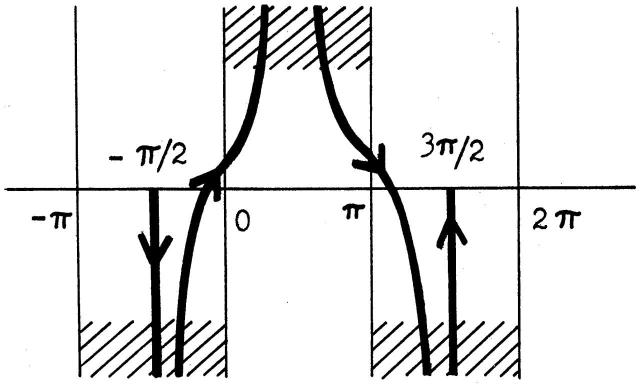

Figure 2

Distortion of the Ψ path of integration

in p(- π/2, 3 π/2; ν) defined by (3)

Furthermore, the contributions from the straight portions of the path are conveniently taken in conjunction with the functions $p(-\pi/2 \pm 2n\pi, 3\pi/2 \pm 2n\pi; \nu)$, $n = 1, 2, 3, \ldots$. In effect, then, these functions can be replaced respectively by

$$(-)^{n+1}\sqrt{\left(\frac{\pi}{2ka}\right)}\, e^{\frac{1}{4}i\pi}\; e^{\mp 2i\pi n\nu}\; e^{-\frac{1}{2}i\pi\nu}\left[H_{\nu}^{(1)}(ka)-H_{\nu}^{(2)}(ka)\right]. \tag{12}$$

The radiation part of the scattered field now appears in the form

$$U^{s} = P(\theta)\,\frac{e^{-ikr}}{kr}\,, \tag{13}$$

where

$$P(\theta) = \frac{i}{\pi} \int_{-\infty}^{\infty} \nu \left[\frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} - 1 \right] \cot(\pi\nu)\, E_{\nu-\frac{1}{2}}(\theta)\, d\nu$$

$$+ \frac{i}{\pi} \sum_{n=1}^{\infty} (-)^n \int_{-\infty}^{\infty} \nu \left[\frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} - 1 \right] \cot(\pi\nu)(e^{-2i\pi n\nu} + e^{2i\pi n\nu}) E_{\nu-\frac{1}{2}}(\theta) d\nu . \tag{14}$$

Equation (14) should be compared to equation (12) of I. In both cases complete rigour has been sacrificed because the integrals involved are not strictly convergent. However, there is no fear of error if a procedure analogous to that in I is adopted, and equation (14) is replaced by

$$P(\theta) = \frac{i}{\pi} \int_{-\infty}^{\infty} \nu \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \cot(\pi\nu)\, E_{\nu-\frac{1}{2}}(\theta)\, d\nu$$

$$- \frac{i}{\pi} \sum_{n=0}^{\infty} (-)^n \int_{-\infty}^{\infty} \nu \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \cot(\pi\nu)\, e^{-2i\pi n\nu} \left[E_{-\nu-\frac{1}{2}}(\theta) + e^{-2i\pi\nu} E_{\nu-\frac{1}{2}}(\theta) \right] d\nu . \tag{15}$$

As far as the stages in its derivation are concerned, equation (15) corresponds to equation (13) of I. In contrast to the latter, however, equation (15) must undergo a further slight transformation before it can be evaluated by standard techniques, because the factor cot $(\pi\nu)$ in the integrand of the first integral spoils an immediate application of the method of steepest descents. The difficulty is overcome by writing in that integrand

$$\cot(\pi\nu) = i + \left[1 - i\tan(\pi\nu)\right]\cot(\pi\nu)$$

$$= i - 2\cot(\pi\nu)\sum_{n=1}^{\infty}(-)^n e^{-2i\pi n\nu} \text{ for } \mathrm{Im}\,\nu < 0; \qquad (16)$$

so that, if the path of integration is now definitely specified to run below the real axis, the result is

$$P(\theta) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\nu\,\frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)}\,E_{\nu-\frac{1}{2}}(\theta)\,d\nu$$

$$+\frac{i}{\pi}\sum_{n=0}^{\infty}(-)^n\int_{-\infty}^{\infty}\frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)}\cot(\pi\nu)\,e^{-2i\pi n\nu}\left[e^{-2i\pi\nu}E_{\nu-\frac{1}{2}}(\theta)-E_{-\nu-\frac{1}{2}}(\theta)\right]d\nu.$$

(17)

The integral in the first term of (17) can be evaluated by the method of steepest descents. If (10) is used, together with the Debye asymptotic form for the Hankel functions, the saddle-point is seen to be located at $\nu = ka \cos(\theta/2)$. The steepest descents path is somewhat as shown in Figure 3, being asymptotic at infinity to the lines of zeros of $H_\nu^{(1)}(ka)$; and it should be observed that the poles of $E_{\nu-\frac{1}{2}}(\theta)$, which

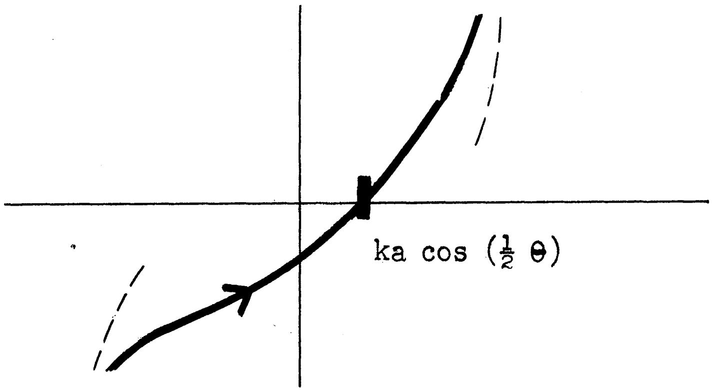

Figure 3

The steepest descents path in the complex ν-plane for the first integral in equation (17)

lie along the negative real axis, do not interfere with the path distortion. The resulting contribution to $P(\theta)$ is (see Appendix B)

$$-\frac{1}{2} ka\, e^{2ika\sin(\frac{1}{2}\theta)} \left\{ 1 - \frac{i}{2\, ka \sin^3(\frac{1}{2}\theta)} + 0\left[(ka)^{-2}\right] \right\}. \tag{18}$$

Derivations of (18) by different methods have been given by Franz and Depperman (1954) and by Keller, Lewis and Seckler (1956).

The integrals in the sum in (17) can be evaluated by closing the contours around the poles in the lower half-plane, these being at the zeros, $\nu = \nu_s$ say, s = 1, 2, 3, ..., of $H_\nu^{(2)}(ka)$. If it is noted that (cf. equations (9) and (11) of II)

$$e^{-2i\pi\nu} E_{\nu-\frac{1}{2}}(\theta) - E_{-\nu-\frac{1}{2}}(\theta) = -i\pi e^{-i\pi\nu} \tan(\pi\nu) P_{\nu-\frac{1}{2}}(-\cos\theta), \quad (19)$$

the resulting contribution to P(θ) is seen to be

$$-2\pi i \sum_{n=0}^{\infty} (-)^n \sum_{s=1}^{\infty} \nu_s \frac{H_{\nu_s}^{(1)}(ka)}{\left\{\frac{d}{d\nu}\left[H_\nu^{(2)}(ka)\right]\right\}_{\nu=\nu_s}} e^{-i\pi(2n+1)\nu_s} P_{\nu_s-\frac{1}{2}}(-\cos\theta). \quad (20)$$

Expression (20) agrees with the analogous one given for a point source by Franz (1954).

It is worth making a further remark in connection with the particular case θ = π (back scattering). As it stands, the first integral in (17) is meaningless at θ = π, because of the logarithmic singularity there of $E_{\nu-\frac{1}{2}}(\theta)$. Furthermore, in the steepest descents evaluation described, the asymptotic representation of $E_{\nu-\frac{1}{2}}(\theta)$ requires $|\nu \sin \theta|$ to be much greater than unity at the saddle-point $\nu = ka \cos(\theta/2)$. It would seem, therefore, that as θ approaches π, the calculation is only valid for increasingly large values of ka. Nevertheless, the result (18) is well behaved at $\theta = \pi$, and on physical grounds no startling variations of field-strength would be expected in the vicinity of the back scattering direction. It may therefore be plausibly anticipated that (18) does, in fact, hold for all values of θ, other than those too close to zero. To establish this by a rigorous mathematical treatment of the first term of (17) would appear to be quite difficult. On the other hand it is possible to rewrite this term in a form which remains valid at $\theta = \pi$, and it is then a relatively straightforward matter to evaluate it for this particular case. For if the path of integration, shown in Figure 3, is displaced so that it runs "symmetrically" through the origin, a sign change of the variable of integration ν gives the form

$$-\frac{1}{2\pi}\int \nu \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)}\left[E_{\nu-\frac{1}{2}}(\theta) - e^{2i\pi\nu}E_{-\nu-\frac{1}{2}}(\theta)\right] d\nu, \quad (21)$$

which from (19) is

$$\frac{1}{2}\, i \int \nu \tan(\pi\nu)\, e^{i\pi\nu}\, \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)}\, P_{\nu-\frac{1}{2}}(-\cos\theta)\, d\nu\,. \quad (22)$$

And at $\theta = \pi$ the expression (22) is

$$\frac{1}{2} i \int \nu \tan(\pi\nu) \, e^{i\pi\nu} \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \, d\nu . \tag{23}$$

A direct steepest descents evaluation of (23) is spoilt by the presence of the factor $\tan(\pi\nu)$ in the integrand. But a modified steepest descent treatment can be applied, and the expression (18) with $\theta = \pi$ recovered (see Appendix C). This gives added support to the contention that the result of a direct evaluation of the first term of (17) holds uniformly up to $\theta = \pi$.

3.2. The Case $\theta = 0$

The analysis of the previous section is now supplemented by a consideration of the radiation field in the forward direction. This gives the total scattering cross-section, σ say, through the relation

$$\sigma = -\frac{4\pi}{k^2} \operatorname{Im} P(0). \tag{24}$$

The required modification to the treatment in § 3.1 is analogous to that made at the corresponding stage in I. The symmetry now present in the problem is preserved by taking the original primary range for $[\theta_1, \theta_2]$ as

$\left[-3\pi/2,\ 3\pi/2\right]$ rather than $\left[-\pi/2,\ 3\pi/2\right]$, and the ψ path of integration in the integral (3) for p $(-3\pi/2, 3\pi/2; \nu)$ is distorted as shown in Figure 4. The result is

$$P(\theta) = \frac{i}{\pi} \int_{-\infty}^{\infty} \nu \left[(1 - e^{2i\pi\nu}) \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} - 1 \right] \cot(\pi\nu)\, E_{\nu-\frac{1}{2}}(\theta)\, d\nu$$

$$+ \frac{i}{\pi} \sum_{n=1}^{\infty} (-)^n \int_{-\infty}^{\infty} \nu \left\{ \left[\frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} - 1 \right] e^{-2i\pi n\nu} \right.$$

$$\left. - \left[1 + e^{2i\pi\nu} \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \right] e^{2i\pi n\nu} \right\} \cot(\pi\nu) E_{\nu-\frac{1}{2}}(\theta)\, d\nu \; . \tag{25}$$

In order to put $\theta = 0$, (25) must be transformed so that the only Legendre functions appearing are $P_{\nu-\frac{1}{2}}(\cos\theta)$. The procedure is similar to that which converts (15) into the sum of (22) and (20). Since (equation (12) of II)

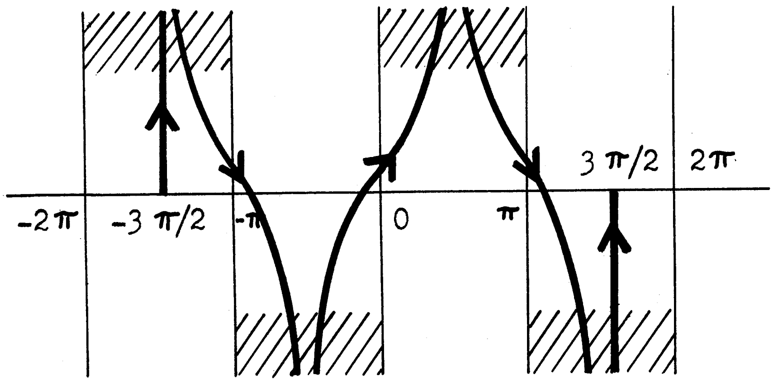

Figure 4

Distortion of the ψ path of integration

in p(-3 π/2, 3 π/2; ν) defined by (3)

$$E_{\nu-\frac{1}{2}}(\theta) - E_{-\nu-\frac{1}{2}}(\theta) = -\pi\tan(\pi\nu)P_{\nu-\frac{1}{2}}(\cos\theta), \tag{26}$$

the aim must be to make the factors multiplying $E_{\nu-\frac{1}{2}}(\theta)$ in the integrands odd functions of ν. This is achieved by using the identity (16) in that part of the integrand of the first integral in (25) which does not have this character, namely the part not involving the Hankel functions. If, then, the paths of integration are taken to run below the real axis, (25) can be written

$$P(\theta) = \frac{i}{\pi} \int_{-\infty}^{\infty} \nu \left[(1-e^{2i\pi\nu}) \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \cot(\pi\nu) - i \right] E_{\nu-\frac{1}{2}}(\theta)\, d\nu$$

$$+ \frac{i}{\pi} \sum_{n=1}^{\infty} (-)^n \int_{-\infty}^{\infty} \nu \left\{ \left[\frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} + 1 \right] e^{-2i\pi n\nu} \right.$$

$$\left. - \left[1 + e^{2i\pi\nu} \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \right] e^{2i\pi n\nu} \right\} \cot(\pi\nu) E_{\nu-\frac{1}{2}}(\theta)\, d\nu \;. \quad (27)$$

Consider, first, the summation part of (27). A sign change of ν in part of the integrands, and use of (26), shows that it is

$$-i \sum_{n=1}^{\infty} (-)^n \int_{-\infty}^{\infty} \nu \left[1 + \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \right] e^{-2i\pi n\nu} P_{\nu-\frac{1}{2}}(\cos\theta)\, d\nu \; ; \quad (28)$$

and (28) in turn can be evaluated by closing the path of integration around the poles ν_s, s = 1, 2, 3, ..., of $1/H_\nu^{(2)}(ka)$ in the lower half-plane, with

$$-2\pi \sum_{n=1}^{\infty} (-)^n \sum_{s=1}^{\infty} \nu_s \frac{H^{(1)}_{\nu_s}(ka)}{\left\{\frac{d}{d\nu}\left[H^{(2)}_{\nu}(ka)\right]\right\}_{\nu=\nu_s}} e^{-2i\pi n \nu_s} P_{\nu_s - \frac{1}{2}}(\cos\theta) \tag{29}$$

as the resulting contribution to $P(\theta)$.

The isolated integral in (27) is handled as follows. It is noted that, by changing the sign of ν, it can equally well be written with $E_{\nu-\frac{1}{2}}(\theta)$ replaced by $E_{-\nu-\frac{1}{2}}(\theta)$, provided the path of integration runs above, instead of below, the real axis. If, now, half the sum of the two integrals is formed, each path of integration can be taken to run "symmetrically" through the origin from the third to the first quadrant in the complex ν-plane; for only the poles of cot $(\pi\nu)$ are thereby crossed, and the contributions of the residues of those at $\nu = \pm n$ are easily seen to cancel. Again appealing to (26), the resulting contribution to $P(\theta)$ can be written

$$\frac{1}{2} i \int \nu \left[i \tan(\pi\nu) - (1 - e^{2i\pi\nu}) \frac{H^{(1)}_{\nu}(ka)}{H^{(2)}_{\nu}(ka)}\right] P_{\nu-\frac{1}{2}}(\cos\theta)\, d\nu. \tag{30}$$

The combination of (30) and (29), with $\theta = 0$, gives

$$P(0) = \frac{1}{2} i \int \nu \left[i \tan(\pi \nu) - (1 - e^{2i\pi\nu}) \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \right] d\nu$$

$$- 2\pi \sum_{n=1}^{\infty} (-)^n \sum_{s=1}^{\infty} \nu_s \frac{H_{\nu_s}^{(1)}(ka)}{\left\{ \frac{d}{d\nu} \left[H_\nu^{(2)}(ka) \right] \right\}_{\nu=\nu_s}} e^{-2i\pi n \nu_s} . \tag{31}$$

This is just the form obtained by Franz and Beckmann (1957), who derive from the integral several terms of the asymptotic approximation in inverse powers of ka.

4. The Field at an Arbitrary Distance From the Sphere

In this section the problem is considered for unrestricted values of kr, so that (5) cannot be approximated by (9). Two methods are given for obtaining expressions for the field, corresponding respectively to those designated I and III in §4 of Reference I.

In the first method the scattered field is deduced directly from the final expression obtained for the radiation part in § 3.1. This expression is given by (13) and (17), where the path of integration in the first integral in (17) is that shown in Figure 3, and the summation part of (17) is replaced by (20). Loosely stated, the procedure is to write

$$\sqrt{\left(\frac{\pi}{2}\right)}\, e^{-\frac{1}{4}i\pi}\, \frac{e^{-\frac{1}{2}i\pi\nu}}{\sqrt{(kr)}}\, H_{\nu}^{(2)}(kr), \tag{32}$$

with the appropriate value of ν, in place of exp (-ikr)/(kr), which is the asymptotic approximation to (32) as $r \to \infty$. In the region where the result converges, it is an outgoing solution of the wave equation free of singularities for $r > a$, whose asymptotic form for $kr \to \infty$ is precisely that of U^s. It must therefore be U^s, which thus appears in the form

$$U^s = -\frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}}\, \frac{1}{\sqrt{(kr)}} \int \nu\, \frac{H_{\nu}^{(1)}(ka)}{H_{\nu}^{(2)}(ka)}\, H_{\nu}^{(2)}(kr)\, e^{-\frac{1}{2}i\pi\nu}\, E_{\nu-\frac{1}{2}}(\theta)\, d\nu$$

$$-\frac{\pi\sqrt{(2\pi)}}{\sqrt{(kr)}}\, e^{\frac{1}{4}i\pi} \sum_{n=0}^{\infty} (-)^n \sum_{s=1}^{\infty} \nu_s \frac{e^{-\frac{1}{2}i\pi\nu_s} H_{\nu_s}^{(1)}(ka)\, H_{\nu_s}^{(2)}(kr)}{\left\{\frac{d}{d\nu}\left[H_{\nu}^{(2)}(ka)\right]\right\}_{\nu=\nu_s}}\, e^{-i\pi(2n+1)\nu_s}\, P_{\nu_s-\frac{1}{2}}(-\cos\theta). \tag{33}$$

The path of integration in the integral is that shown in Figure 3, and the integral is only convergent for $\pi/2 < \theta < 3\pi/2$. Hence, (33) gives the scattered field in the half-space on the illuminated side of the plane through the center of the sphere perpendicular to the direction of propagation of the incident wave.

The half-space $\theta < \pi/2$ includes the shadow region, where the total field can become very small. This suggests that a consideration of the total field, rather than the scattered field alone, might yield an expression which is valid for points of observation not catered for by (33). The inclusion of the incident field, in a suitable representation is the basis of the second method. The representation is obtained merely by writing r for a in (2), (3) and (4). If θ is allowed to lie in the range 0 to π, the primary range for $[\theta_1, \theta_2]$ is $[-\pi/2, 3\pi/2]$, and the corresponding contribution to the total field is

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \nu \left\{ p_r(-\pi/2, 3\pi/2;\nu) - \sqrt{\left(\frac{a}{r}\right)}\, \frac{p(-\pi/2, 3\pi/2;\nu)}{H_\nu^{(2)}(ka)}\, H_\nu^{(2)}(kr) \right\} \cot(\pi\nu) E_{\nu-\frac{1}{2}}(\theta)\, d\nu, \tag{34}$$

where

$$p_r(-\pi/2, 3\pi/2;\nu) = -\frac{i}{\pi}\int_{-\pi/2}^{3\pi/2} e^{-ikr\cos\psi} E_{-\nu-\frac{1}{2}}(\psi)\sin\psi\, d\psi\,. \tag{35}$$

The analysis can now proceed in a manner closely parallel to that of §3.1. The ψ path of integration in $p(-\pi/2, 3\pi/2; \nu)$ and $p_r(-\pi/2, 3\pi/2; \nu)$ is distorted as in Figure 2, and the contributions to (34) associated respectively with the curved and straight portions of the ψ path can be treated separately without giving divergent integrals. For the former, $p(-\pi/2, 3\pi/2; \nu)$ in (35) is replaced by (11), and $p_r(-\pi/2, 3\pi/2; \nu)$ by (11) with r written for a. The latter is taken in conjunction with the contributions arising from ranges $[\theta_1, \theta_2]$ outside the primary range. The total field thus appears in the form

$$U = -\frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}}\frac{1}{\sqrt{(kr)}}\int_{-\infty}^{\infty}\nu e^{-\frac{1}{2}i\pi\nu}\left[H_\nu^{(1)}(kr) - \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)}H_\nu^{(2)}(kr)\right]\cot(\pi\nu)E_{\nu-\frac{1}{2}}(\theta)d\nu$$

$$-\frac{e^{\frac{1}{4}i\pi}}{\sqrt{(2\pi)}}\frac{1}{\sqrt{(kr)}}\sum_{n=1}^{\infty}(-)^n\int_{-\infty}^{\infty}\nu e^{-\frac{1}{2}i\pi\nu}\left[H_\nu^{(1)}(kr) - \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)}H_\nu^{(2)}(kr)\right]$$

$$\times \cot(\pi\nu)(e^{-2i\pi n\nu} + e^{2i\pi n\nu})E_{\nu-\frac{1}{2}}(\theta)d\nu. \quad (36)$$

The evaluation of (36) can proceed in either of two ways, useful respectively if the point of observation is outside or inside the shadow region.

For points outside the shadow region, the technique is essentially the same as that employed in § 3.1 for the evaluation of (15). The paths of integration are taken to run below the real axis, and the identity (16) is used for the factor $\cot(\pi\nu)$ in the integrand of the first integral of (36). The result is

$$U = \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}}\frac{1}{\sqrt{(kr)}}\int_{-\infty}^{\infty} \nu e^{-\frac{1}{2}i\pi\nu}\left[H_\nu^{(1)}(kr) - \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)}H_\nu^{(2)}(kr)\right] E_{\nu-\frac{1}{2}}(\theta)\, d\nu$$

$$-\pi\sqrt{(2\pi)}e^{\frac{1}{4}i\pi}\frac{1}{\sqrt{(kr)}}\sum_{n=0}^{\infty}(-)^n\sum_{s=1}^{\infty}\nu_s\frac{e^{-\frac{1}{2}i\pi\nu_s}H_{\nu_s}^{(1)}(ka)H_{\nu_s}^{(2)}(kr)}{\left\{\frac{d}{d\nu}\left[H_\nu^{(2)}(ka)\right]\right\}_{\nu=\nu_s}} e^{-i\pi(2n+1)\nu_s}P_{\nu_s-\frac{1}{2}}(-\cos\theta). \tag{37}$$

For $\pi/2 < \theta < 3\pi/2$, the path of integration in the integral in (37) can be taken to be that shown in Figure 3, and the result then evidently agrees with (33) provided

$$\frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}}\frac{1}{\sqrt{(kr)}}\int \nu e^{-\frac{1}{2}i\pi\nu} H_\nu^{(1)}(kr)\, E_{\nu-\frac{1}{2}}(\theta)\, d\nu = e^{-ikr\cos\theta}. \tag{38}$$

Equation (38) is the analogue of equation (33) of I. It can be shown to hold for those values of θ for which the integral converges, namely $\pi/2$ to $3\pi/2$.*

For $0 < \theta < \pi/2$ (but still outside the shadow region), the path of integration in the integral in (37) can be deformed so that both terminations at infinity are in the lower half of the complex ν-plane. The corresponding contribution to U is then conveniently written

$$-\frac{e^{-\frac{1}{4}i\pi}}{\sqrt{(2\pi)}}\frac{1}{\sqrt{(kr)}}\int_C \nu \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} H_\nu^{(2)}(kr)\, e^{-\frac{1}{2}i\pi\nu}\, E_{\nu-\frac{1}{2}}(\theta)\, d\nu, \tag{39}$$

where the path C (Figure 5) traverses two saddle-points; one near $\nu = ka \cos(\theta/2)$ associated with the reflected wave, and one near $\nu = kr \sin\theta$ associated with the incident wave (Franz and Beckmann, 1956).

For points inside the shadow region the evaluation of (36) is more straightforward. The sign of ν is changed in the parts of the integrands of the integrals in the sum which contain the factor $\exp(2i\pi n\nu)$, and the identity (26) then leads immediately to the well known result

$$U = -\pi\sqrt{(2\pi)}e^{-\frac{1}{4}i\pi}\frac{1}{\sqrt{(kr)}}\sum_{n=0}^{\infty}(-)^n\sum_{s=1}^{\infty}\nu_s \frac{e^{-\frac{1}{2}i\pi\nu_s}H_{\nu_s}^{(1)}(ka)H_{\nu_s}^{(2)}(kr)}{\left\{\frac{d}{d\nu}\left[H_\nu^{(2)}(ka)\right]\right\}_{\nu=\nu_s}} e^{-2i\pi n\nu_s} P_{\nu_s-\frac{1}{2}}(\cos\theta). \tag{40}$$

*A proof of (38) can be constructed with the help of the infinite Legendre integral transform and its inverse.

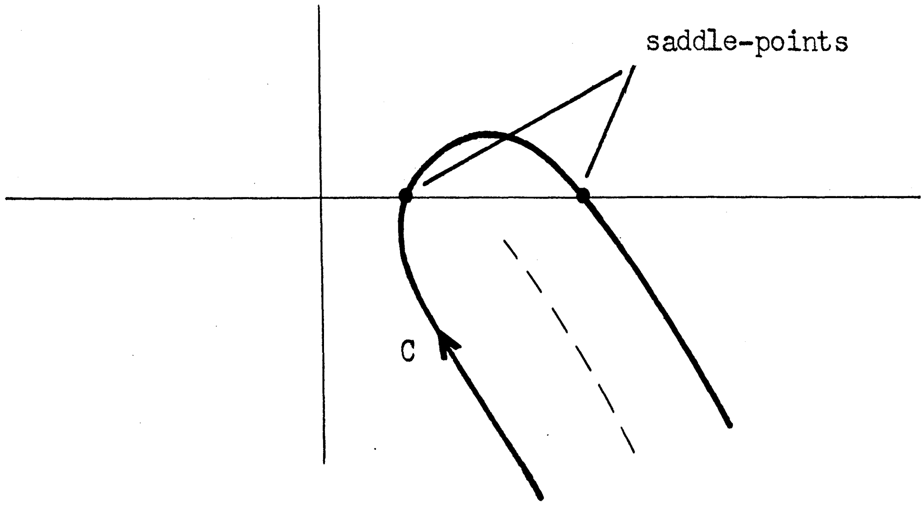

Figure 5

The path C in the complex ν-plane for the integral in (39)

5. The Neumann Problem

The foregoing analysis needs only slight modification for the case when the boundary condition is $\partial U/\partial r = 0$ on $r = a$. The contribution to U^s analogous to (5) is

$$-\frac{1}{\pi\sqrt{(ka)}} \int_{-\infty}^{\infty} \nu \frac{q\,(\theta_1,\theta_2;\nu\,)}{\left[\frac{1}{\sqrt{(ka)}}\, H_\nu^{(2)}(ka)\right]'} H_\nu^{(2)}(kr) \cot(\pi\nu) E_{\nu-\frac{1}{2}}(\theta)\, d\nu, \tag{41}$$

where the dash denotes differentiation with respect to the argument of the Hankel function, and

$$q(\theta_1,\, \theta_2;\nu) = -\frac{1}{\pi} \int_{\theta_1}^{\theta_2} \cos\psi\, e^{-ika\cos\psi} E_{-\nu-\frac{1}{2}}(\psi) \sin\psi\, d\psi\,. \tag{42}$$

For it is clear that on r = a the part of $\partial U^s / \partial r$ corresponding to (41) cancels the normal derivative of the incident field for $\theta_1 < \theta < \theta_2$, and is zero for $\theta < \theta_1$, $\theta > \theta_2$.

Furthermore, from (3) and (42),

$$q(\theta_1, \theta_2; \nu) = \frac{\partial}{\partial (ka)} \left[p(\theta_1, \theta_2; \nu) \right] . \qquad (43)$$

Evidently, then, the analysis for the Dirichlet problem applies also to the Neumann problem if $H_\nu^{(1)}(ka)/\sqrt{(ka)}$, $H_\nu^{(2)}(ka)/\sqrt{(ka)}$ are throughout replaced respectively by $\left[H_\nu^{(1)}(ka)/\sqrt{(ka)} \right]'$, $\left[H_\nu^{(2)}(ka)/\sqrt{(ka)} \right]'$.

Appendix A

The object of this Appendix is to show that

$$-\frac{i}{\pi}\int_{(-1)}^{(1)} e^{-iz\cos\psi} E_{-\nu-\frac{1}{2}}(\psi)\sin\psi\, d\psi = \sqrt{\left(\frac{\pi}{2}\right)}\, e^{\frac{1}{4}i\pi}\,\frac{e^{-\frac{1}{2}i\pi\nu}}{\sqrt{z}}\, H_{\nu}^{(2)}(z), \qquad (A.1)$$

$$-\frac{i}{\pi}\int_{(1)}^{(2)} e^{-iz\cos\psi} E_{-\nu-\frac{1}{2}}(\psi)\sin\psi\, d\psi = -\sqrt{\left(\frac{\pi}{2}\right)}\, e^{\frac{1}{4}i\pi}\,\frac{e^{-\frac{1}{2}i\pi\nu}}{\sqrt{z}}\, H_{\nu}^{(1)}(z), \qquad (A.2)$$

where the paths of integration run between certain regions of convergence at infinity in the complex ψ-plane. For real positive values of z the situation is depicted in Figure (A-1), where the regions of convergence at infinity are shown shaded. It is tacitly understood that the paths of integration do not intersect the branch cuts of $E_{-\nu-\frac{1}{2}}(\psi)$, which start from the branch points at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ and travel to infinity along straight lines parallel to the negative imaginary axis.

Now an expression for $E_{\nu-\frac{1}{2}}(\psi)$ valid in the upper half of the complex ψ-plane is (see equation (7) of II)

$$E_{\nu-\frac{1}{2}}(\psi) = \sqrt{\pi}\,\frac{(\nu-\frac{1}{2})!}{\nu!}\, e^{i(\nu+\frac{1}{2})\psi}\, F(\tfrac{1}{2}, \nu+\tfrac{1}{2}; \nu+1; e^{2i\psi})$$

$$= e^{i(\nu+\frac{1}{2})\psi}\sum_{n=0}^{\infty}\frac{(n-\frac{1}{2})!}{n!}\,\frac{(\nu+n-\frac{1}{2})!}{(\nu+n)!}\, e^{2in\psi}\,. \qquad (A.3)$$

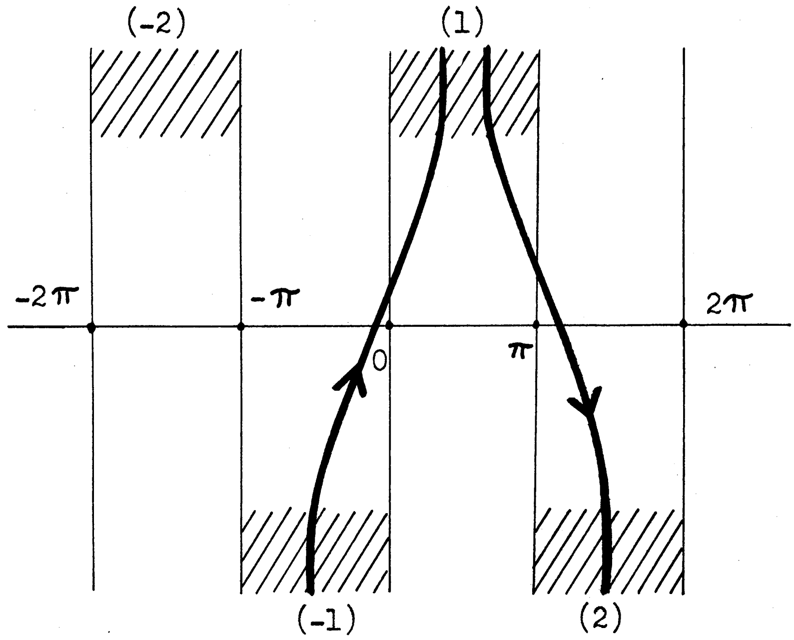

Figure A-1

Paths of integration and regions of convergence at infinity in the complex ψ-plane

Also

$$\int_{(-2)}^{(1)} e^{i\nu\psi} e^{-iz\cos\psi} \sin\psi \, d\psi$$

$$= \frac{1}{iz} \int_{(-2)}^{(1)} e^{i\nu\psi} \frac{d}{d\psi} (e^{-iz\cos\psi}) \, d\psi$$

$$= - \frac{\nu}{z} \int_{(-2)}^{(1)} e^{i\nu\psi} e^{-iz\cos\psi} \, d\psi$$

$$= - 2\pi \frac{\nu}{z} e^{-\frac{1}{2}i\pi\nu} J_{\nu}(z). \tag{A.4}$$

Hence, from (A.3) and (A.4),

$$-\frac{i}{\pi}\int_{(-2)}^{(1)} e^{-iz\cos\psi}\, E_{\nu-\frac{1}{2}}(\psi)\sin\psi\, d\psi$$

$$= 2e^{\frac{1}{4}i\pi}\,\frac{e^{-\frac{1}{2}i\pi\nu}}{z}\sum_{n=0}^{\infty}(-)^n\,\frac{(n-\frac{1}{2})!}{n!}\,\frac{(\nu+n-\frac{1}{2})!}{(\nu+n)!}\cdot(\nu+2n+\tfrac{1}{2})\,J_{\nu+2n+\frac{1}{2}}(z)$$

$$= \sqrt{(2\pi)}\,e^{\frac{1}{4}i\pi}\,\frac{e^{-\frac{1}{2}i\pi\nu}}{\sqrt{z}}\,J_\nu(z), \tag{A.5}$$

where the last identity is a special case of the Gegenbauer addition theorem (Watson (1944), p. 370, equation (9)).

But (A.5) can be written

$$-\frac{i}{\pi}\left\{\int_{(-2)}^{(-1)}+\int_{(-1)}^{(1)} e^{-iz\cos\psi}\, E_{\nu-\frac{1}{2}}(\psi)\,\sin\psi\, d\psi\right\}$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\,e^{\frac{1}{4}i\pi}\,\frac{e^{-\frac{1}{2}i\pi\nu}}{\sqrt{z}}\left[H_\nu^{(1)}(z)+H_\nu^{(2)}(z)\right]. \tag{A.6}$$

Now allow z to be temporarily complex, $z = |z| \exp(i\zeta)$ say, where $0 < \zeta < \pi$. Then it is not difficult to see that the path of integration of the first integral on the left hand side of (A.6) can be chosen so that, for all values of ψ on it, $\exp(-iz\cos\psi)$ becomes exponentially small as $|z| \to \infty$ (it is simplest to consider $\zeta = \pi/2$). The first term on the left hand side of (A.6) is thereby identified with the first term on the right hand side. Thus

$$-\frac{i}{\pi}\int_{(-2)}^{(-1)} e^{-iz\cos\psi}\, E_{\nu-\frac{1}{2}}(\psi)\sin\psi\, d\psi = \sqrt{\left(\frac{\pi}{2}\right)}\, e^{\frac{1}{4}i\pi}\, \frac{e^{-\frac{1}{2}i\pi\nu}}{\sqrt{z}}\, H_\nu^{(1)}(z), \qquad (A.7)$$

$$-\frac{i}{\pi}\int_{(-1)}^{(1)} e^{-iz\cos\psi}\, E_{\nu-\frac{1}{2}}(\psi)\sin\psi\, d\psi = \sqrt{\left(\frac{\pi}{2}\right)}\, e^{\frac{1}{4}i\pi}\, \frac{e^{-\frac{1}{2}i\pi\nu}}{\sqrt{z}}\, H_\nu^{(2)}(z). \qquad (A.8)$$

Finally, (A.1) follows from (A.8) by changing the sign of ν; and (A.2) follows from (A.7) by changing the sign of ν and changing the variable of integration from ψ to $\psi + 2\pi$.

Appendix B

The object of this Appendix is to show that, for ka$\gg$1,

$$-\frac{1}{\pi}\int \nu \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} E_{\nu-\frac{1}{2}}(\theta)\, d\nu$$

$$= -\tfrac{1}{2}\, ka\, e^{2ik \sin(\frac{1}{2}\theta)} \left\{ 1 - \frac{i}{2\, ka \sin^3(\frac{1}{2}\theta)} + 0\left[(ka)^{-2}\right] \right\}, \qquad (B.1)$$

where the path of integration is that shown in Figure 3.

If a new variable of integration α is introduced via the relation

$$\nu = ka \cos\alpha\,, \qquad (B.2)$$

the first two terms of the Debye asymptotic forms for the Hankel functions give

$$\frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} \sim -i\left[1 - \frac{i(5 - 2\sin^2\alpha)}{12\, ka \sin^3\alpha}\right] e^{2ika(\sin\alpha - \alpha\cos\alpha)}, \qquad (B.3)$$

and the first two terms of the relation (see equation (6) of II)

$$E_{\nu-\frac{1}{2}}(\theta) = \sqrt{\left(\frac{\pi}{2}\right)}\, e^{\frac{1}{4}i\pi}\, \frac{(\nu-\frac{1}{2})!}{\nu!}\, \frac{e^{i\nu\theta}}{\sqrt{(\sin\theta)}}\, F(\tfrac{1}{2}, \tfrac{1}{2}; \nu+1; -\frac{ie^{i\theta}}{2\sin\theta}) \qquad (B.4)$$

give

$$E_{\nu-\frac{1}{2}}(\theta) \sim \sqrt{\left(\frac{\pi}{2}\right)}\, e^{\frac{1}{4}i\pi} \frac{1}{\sqrt{(ka \sin\theta \cos\alpha)}} \left[1 - \frac{i \cot\theta}{8\, ka \cos\alpha}\right] e^{ika\,\theta \cos\alpha} . \tag{B.5}$$

When (B.3) and (B.5) are substituted into the left hand side of (B.1), the exponential part of the integrand is exp $\left[ika\, f(\alpha)\right]$, where

$$f(\alpha) = 2(\sin\alpha - \alpha\cos\alpha) + \theta \cos\alpha . \tag{B.6}$$

The saddle-point, then, is evidently at $\alpha = \theta/2$, and with the notation

$$\beta = \alpha - \tfrac{1}{2}\theta, \tag{B.7}$$

$$s = \sin(\tfrac{1}{2}\theta), \; c = \cos(\tfrac{1}{2}\theta), \tag{B.8}$$

it is adequate to write

$$e^{ika\, f(\alpha)} = e^{2ikas}\left[1 + ika\,(\tfrac{2}{3}c\beta^3 - \tfrac{1}{4}s\beta^4) - \tfrac{2}{9}k^2a^2c^2\beta^6\right] e^{ikas\beta^2}, \tag{B.9}$$

$$\sin\alpha\sqrt{(\cos\alpha)} = s\sqrt{c}\left[1 + \frac{2-3s^2}{2sc}\beta - \frac{10-9s^2}{8c^2}\beta^2\right], \tag{B.10}$$

$$\sin\alpha\sqrt{(\cos\alpha)}\, e^{ikaf(\alpha)} = s\sqrt{c}\, e^{2ikas}\left[1 - \frac{10-9s^2}{8c^2}\beta^2 + ika\left(\frac{2}{3s} - \frac{5}{4}s\right)\beta^4 - \frac{2}{9}k^2a^2c^2\beta^6\right]. \tag{B.11}$$

The procedure is now standard. Expressions (B.3), (B.5) and (B.11) are substituted into the integral of (B.1), the path of integration for β is taken from $-\infty \exp(i\pi/4)$ to $\infty \exp(i\pi/4)$, and the integration is performed term by term. To the accuracy required, the second term in the square brackets in (B.3) and (B.5) need only be taken in conjunction with the first term (unity) in the square bracket of (B.11). The result appears explicitly in the form

$$-\tfrac{1}{2}kae^{2ikas}\left[1-\frac{i}{2kas}\,\frac{10-9s^2}{8c^2}-\frac{3i}{4kas^2}\left(\frac{2}{3s}-\frac{5s}{4}\right)+\frac{5ic^2}{12kas^2}-\frac{i}{4ka}\left(\frac{1-2s^2}{4c^2s}+\frac{5-2s^2}{3s^3}\right)\right], \tag{B.12}$$

which simplifies to the right hand side of (B.1), the order term in (B.1) being evident from inspection.

The algebra in this derivation has been written out at some length for two reasons. First, to show how the terms which would individually blow up at $\theta = \pi$, namely those in (B.12) with c^2 in the denominator (which originate in (B.5) and (B.10)), in combination remain finite there. Secondly, to give an indication of the labour involved. Since a simple answer is obtained as the sum of a considerable number of terms, it might be thought that a better technique would be available, and it is interesting to question whether the completely different approach of Keller, Lewis and Seckler (1956) has here an advantage. The author understands, however, from private conversation with J. B. Keller, that their Luneberg-Kline method, which obtains the asymptotic form for $ka \gg 1$ without recourse to an exact solution, also becomes laborious if higher order terms are sought.

Appendix C

The object of this Appendix is to show that

$$\frac{1}{2}i\int \nu \tan(\pi\nu) e^{i\pi\nu} \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} d\nu = -\frac{1}{2}\, ka\, e^{2ika}\left\{1 - \frac{i}{2ka} + 0\left[(ka)^{-2}\right]\right\}, \tag{C.1}$$

where the path of integration is that shown in Figure 3, but with $\theta = \pi$ so that it runs through the origin.

The method adopted is based on an argument of the steepest descents type, but direct application of the standard technique in the manner of Appendix B is spoilt by the factor $\tan(\pi\nu)$ in the integrand, However, in view of the symmetry in ν of the integrand, only one half of the path of integration need be considered. If this is taken as the portion for which $\mathrm{Im}\,\nu < 0$, the identity

$$\tan(\pi\nu) = -i - 2i\sum_{n=1}^{\infty}(-)^n e^{-2i\pi n\nu} \tag{C.2}$$

can be used, and the left hand side of (C.1) written as

$$I_0 + \sum_{n=1}^{\infty}{}' I_n, \tag{C.3}$$

where

$$I_o = \int^{0} \nu e^{i\pi\nu} \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} d\nu \qquad (C.4)$$

$$I_n = (-)^n 2 \int^{0} \nu e^{i\pi\nu} \frac{H_\nu^{(1)}(ka)}{H_\nu^{(2)}(ka)} e^{-2i\pi n\nu} d\nu, \; n=1,2,3,\ldots, \qquad (C.5)$$

the lower limits of integration being at infinity in the third quadrant of the complex ν-plane.

To evaluate I_o and I_n, the new variable of integration α is introduced as in Appendix (B) (equation (B.2)), and the asymptotic form (B.3) used for the quotient of the Hankel functions. The evaluation of I_o then follows closely the pattern of Appendix B, the calculation being, in fact, considerably less laborious since, effectively, $s = 1$, $c = 0$, and the function $E_{\nu-\frac{1}{2}}(\theta)$ is absent. The result is

$$I_o \sim - \tfrac{1}{2}\, ka\, e^{2ika} \left(1 - \frac{5i}{12\, ka}\right). \qquad (C.6)$$

For I_n, $n = 1,2,3,\ldots,$ the same type of argument gives, in the first place,

$$I_n \sim (-)^n\, 2ie^{ika} (ka)^2 \int_0^{\infty e^{\frac{1}{4}i\pi}} \beta\, e^{ika\beta^2}\, e^{2i\pi n\, ka\beta}\, d\beta \,. \qquad (C.7)$$

Here, $\beta = \alpha - \pi/2$ (cf. equation (B.7)), and the factor $\exp(-2i\pi n \nu)$, which distinguishes the integrand of I_n from that of I_o, has been replaced by exp $(2i\pi n\, ka\beta)$. It can be seen, though hardly without some further working, that the inclusion of the next approximation to exp $(-2i\pi n\, \nu)$, which would introduce the further factor $1 - i\pi\, n\, ka\beta^3/3$, would only yield terms negligible to the present order of approximation.

Now in (C.7) write

$$\gamma = \beta + \pi n, \tag{C.8}$$

to obtain

$$I_n \sim (-)^n\, 2i\, e^{ika}(ka)^2\, e^{-ika\,\pi^2 n^2} \int_{\pi n}^{\infty e^{\frac{1}{4}i\pi}} (\gamma - \pi n)\, e^{ika\,\gamma^2}\, d\gamma$$

$$\sim (-)^{n+1}\, \frac{i}{2\pi^2 n^2}\, e^{2ika}. \tag{C.9}$$

Since

$$\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^2} = \frac{\pi^2}{12}, \tag{C.10}$$

the substitution of (C.6) and (C.9) into (C.3) gives

$$-\tfrac{1}{2}\, ka\, e^{2ika} \left(1 - \frac{i}{2ka}\right), \tag{C.11}$$

which is in agreement with (C.1).

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