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STUDIES IN RADAR CROSS SECTIONS XXXVI - DIFFRACTION OF A PLANE WAVE BY AN ALMOST CIRCULAR CYLINDER

by

P. C. Clemmow and V. H. Weston

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Summary

The two-dimensional problem of an E-polarized plane wave incident on a perfectly conducting cylinder of almost circular cross section is treated, the maximum deviation of the perimeter of the cross section from a strict circle being regarded mathematically as an infinitesimal quantity whose second and higher powers are neglected.

In the body of the paper the method of solution uses infinite Fourier transform techniques, but an analysis involving a Watson transformation, more traditional in this type of problem, is given in Appendix A. Various Bessel function results are required, some of which are derived in appendices.

Attention is for the most part directed to the case in which the mean radius is large compared to the wavelength, and the form of the solution then appropriate is examined in some detail. In particular, initial terms of asymptotic expansions in inverse powers of the mean radius to wavelength ratio are obtained both for the "specular" and for the "creeping" contributions to the far field. It is shown that the former contribution is in agreement with that derived by the Luneberg-Kline method, and the latter with the prescription proposed by Keller.

While this report was being printed the authors learned of a related effort at the Microwave Research Institute of the Polytechnic Institute of Brooklyn.

C. J. Marcinkowski and L. B. Felsen, "Diffraction by a Cylinder with a Variable Surface Impedance" (to be published).

This work has been summarized in Polytechnic Institute of Brooklyn reports

Quarterly Progress Report No. 14 - 15 October 1958 through 14 January 1959, Report R-452.14-59, Contract AF18(600)-1505.

Quarterly Progress Report No. 16 - 15 April 1959 through 14 September 1959, Report R-452.16-59, Contract AF18(600)-1505.

1. Introduction

Exact solutions of boundary value problems in the theory of wave diffraction are available only for certain specific bodies of relatively simple shape. On the other hand, various approximate methods have been proposed for treating bodies of rather general shapes. In the cases when a) the linear dimensions of the body are small compared with the wavelength, b) the body is convex and has a radius of curvature everywhere large compared with the wavelength, it is probably broadly true, at least for scalar waves, that there are approximate methods which yield the initial terms of an expansion in powers of a definitive parameter. Nevertheless, the theory of some aspects of these methods is as yet sufficiently lacking in rigour for one to welcome the opportunity to test their predictions against trustworthy results obtained by other means. The object of the present paper is to solve a new problem which has a degree of generality, and then to check against the solution methods that have been proposed for the case b) mentioned above.

The solution obtained is for a time-harmonic electromagnetic E-polarized plane wave incident on a perfectly conducting cylinder, the uniform cross section of which is almost circular. The equation of the cross section of the cylinder in the two-dimensional polar coordinates (r, θ) is

$$r = a + bf(\theta) \quad , \tag{1}$$

where $f(\theta)$ is a sufficiently smooth[+] function of θ, of maximum modulus unity, but otherwise arbitrary, and kb ($k = 2\pi$/wavelength) is a small parameter. Powers of kb higher than the first are neglected in the analysis.

If $ka \gg 1$, the method known as Luneberg-Kline (see Keller, Lewis and Seckler (1956) for references and applications) and the formula proposed by Keller (1956) should be valid. The result given by these in combination is therefore also worked out, and checked against that already obtained. Agreement is found to the extent expected.

Thus, it may be noted, a proposed solution which depends on one type of approximation (based on $ka \gg 1$) is verified in the case when another type of approximation (based on $kb \ll 1$) can be made. A technique of this kind does not seem previously to have been exploited in diffraction theory, where past checks of approximate solutions have been made only against exact solutions.

The available exact solutions for cylindrical bodies without sharp edges are limited to those with circular, elliptic or parabolic cross sections. Since Keller's prescription is based on the exact solution for a circular cylinder, it has been checked only by reference to elliptic and parabolic cylinders. The present analysis is therefore thought to be of interest in providing further evidence of the reliability of Keller's prescription, with a moderate degree of generality furnished by the arbitrariness of the function $f(\theta)$ in (1). Moreover

[+] It seems hard to state precisely the necessary restrictions on $f(\theta)$; some remarks on this point are offered in § 8.

the Bessel function analysis required does not go far beyond that used in the circular cylinder problem, which is now familiar from much discussion; the treatment is therefore somewhat less elaborate than one involving either Mathieu or parabolic cylinder functions.

In the interests of clarity and brevity, the treatment in the paper is confined to the simplest possible case of interest, namely that in which, with all other parameters prescribed, kb is taken to be so small that the first two terms in a power series expansion in kb give an adequate approximation. A second concession to brevity is the limitation of the detailed discussion to points of observation in the far field. On the other hand, the analysis could be handled in various less restricted ways, and it might well prove profitable subsequently to carry out more extensive investigations on the lines indicated in § 8.

The plan of the paper is as follows. After settling the delicate matter of notation (§ 2), the solution correct to the linear term in kb is derived, with emphasis on the form appropriate when ka is large (§ 3). The derivation uses infinite Fourier integral transforms in a manner similar to that advocated by Clemmow (1959) for the circular cylinder problem. Also given, in Appendix A, is an alternative derivation based on the Watson (1918) technique; this is included for the benefit of those who may prefer the more traditional approach. Then, in § 4, the integral expression for the part of the far field associated with specular reflection is evaluated asymptotically for large ka by the method of steepest descents; and, in § 5, the residue terms associated with the "creeping-wave" contributions to the far field are simplified with the help of an unfamiliar Bessel function relation obtained in Appendix B, and then asymptotically approximated for large ka. The asymptotic result of § 4 is shown in § 6 to agree with that given by the Luneberg-Kline method, and the asymptotic result of § 5 is shown in § 7 to agree with Keller's prescription. The paper ends with a discussion, in § 8, of the significance of the restrictive conditions which validate the analysis, and examines the extent to which they can profitably be relaxed without making the problem intractable.

2. Notation

Attention should be drawn to the use here of the particular Airy integrals defined and fully tabulated by Miller (1946). There are good grounds for regarding these as the standard forms (see, for example, Jeffreys and Jeffreys (1956)), and consequently they are adopted here. Unfortunately, they have hardly been used at all in the relevant literature on diffraction problems, with the result that some unnecessary computation has been done, and a confusing proliferation of notations exists.

The following is a list of the notations that will be used:

$\exp(i\omega t)$ is the time factor, left understood.

k is the propagation constant, 2π divided by the wavelength.

r, θ are the polar coordinates of a typical point of observation P; in general $-\pi < \theta \leqslant \pi$, but without loss of generality it is assumed that $0 \leqslant \theta \leqslant \pi$.

The direction of propagation of the incident plane wave is $\theta = 0$.

r', θ' are the polar coordinates of a typical point Q on the perimeter of the body, so that equation (1) states

$$r' = a + b\, f(\theta') \quad . \tag{2}$$

The configuration is shown in Figure 1.

Q_o is the "lower" of the two points on the perimeter of the body at which the tangent is parallel to the direction of propagation of the incident plane wave.

Q_1 is that one of the two points at which the tangent passes through P, reached first in travelling counter-clockwise round the perimeter from Q_o.

s is the distance $Q_1 P$.

r'_o, θ'_o, and r'_1, θ'_1 are the respective coordinates of Q_o and Q_1.

The configuration is shown in Figure 2.

t is the arc length along the perimeter of the body, measured from some fixed point to Q.

ρ is the radius of curvature of the perimeter at Q.

t_o, t_1 are the respective values of t at Q_o, Q_1.

ρ_o, ρ_1 are the respective values of ρ at Q_o, Q_1.

T is the total length of the perimeter of the body.

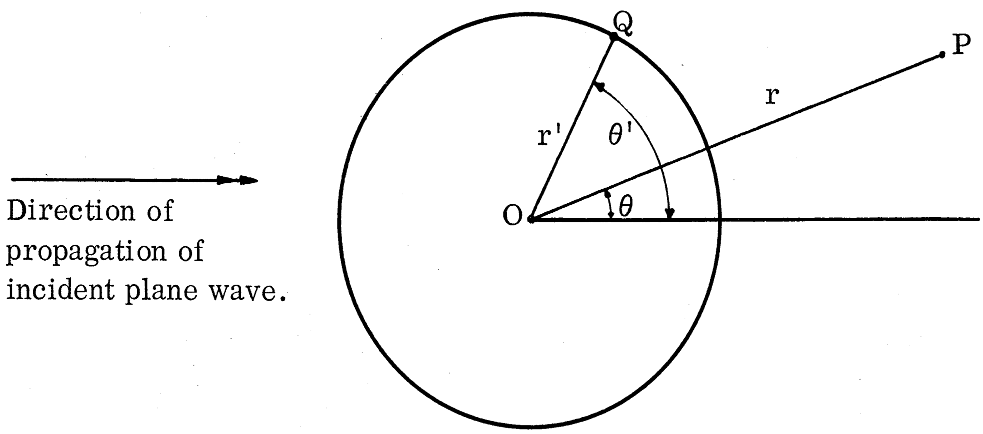
Q
P
r
r'
θ'
O
θ
Direction of
propagation of
incident plane wave.

FIG 1.

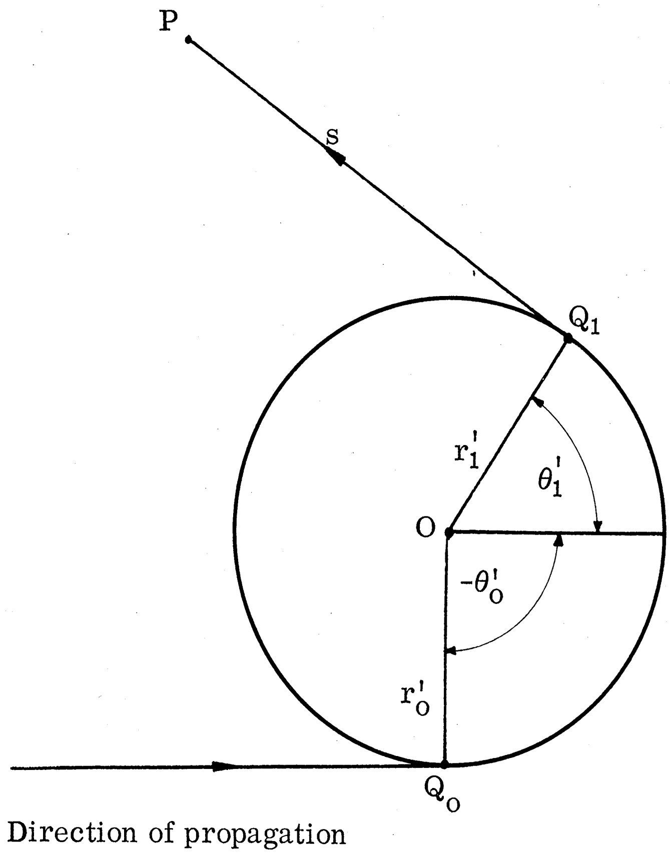
P
s
Q_1
r_1'
θ_1'
O
$-\theta_0'$
r_0'
Q_0
Direction of propagation
of incident plane wave.

FIG 2.

Ai(z), Bi(z) are the Airy integrals defined and tabulated by Miller (1946).

a_ℓ , $\ell = 1, 2, 3, \ldots$, are the zeros of Ai(z); they are negative real numbers, tabulated by Miller (1946).

ν_ℓ , $\ell = 1, 2, 3, \ldots$, are the zeros of $H_\nu^{(2)}(ka)$, qua function of ν, which lie in the lower half of the complex ν plane.

For large ka,

$$\nu_\ell \sim ka + \tau_\ell (ka)^{1/3} - \frac{a_\ell^2 \, e^{i\pi/3}}{2^{2/3} \cdot 30} \, \frac{1}{(ka)^{1/3}} \tag{3}$$

is an asymptotic approximation to ν_ℓ, where in fact τ_ℓ is just $-a_\ell \exp(-i\pi/3)\, 2^{1/3}$, but occurs sufficiently often to warrant designation by a separate symbol. The approximation (3) is that given, in a different notation, by Franz (1954, equation (A 17b)). An alternative derivation is noted in Appendix D.

m is a positive integer, which denotes the order of a Fourier component of $f(\theta')$.

3. The solution

In this section the solution, to the first order in kb, is obtained by the same procedure as that applied by Clemmow (1959) to the circular cylinder. The electromagnetic field is **E**-polarized, and U denotes the only non-zero component of the electric field, namely that parallel to the cylinder parameters. U is a solution of the scalar wave equation with Dirichlet boundary conditions on the surface of the cylinder.

3.1 The general form

The incident field is the plane wave

$$U^i = e^{-ikr\cos\theta} , \tag{4}$$

and the scattered field is taken in the form

$$U^s = -\int_{-\infty}^{\infty} P(\nu)\, H_\nu^{(2)}(kr)\, e^{i\theta\nu}\, d\nu \quad , \tag{5}$$

where $P(\nu)$ is to be found. The boundary condition, $U^i + U^s = 0$ on the surface (2), gives

$$\int_{-\infty}^{\infty} P(\nu)\, H_\nu^{(2)}\left\{k\left[a + b f(\theta')\right]\right\} e^{i\theta'\nu}\, d\nu = e^{-ik\left[a + b f(\theta')\right]\cos\theta'} \quad , \tag{6}$$

for all values of θ'.

Equation (6) is now solved to the first order in kb. Explicitly, the results

$$H_\nu^{(2)}\left\{k\left[a + b f(\theta')\right]\right\} = H_\nu^{(2)}(ka) + kb f(\theta')\, H_\nu^{(2)'}(ka) + O\left[(kb)^2\right], \tag{7}$$

$$e^{-ikb f(\theta')\cos\theta'} = 1 - ikb f(\theta')\cos\theta' + O\left[(kb)^2\right] \tag{8}$$

are used+; and the notation

$$P(\nu) = \frac{1}{H_\nu^{(2)}(ka)}\left[p(ka,\nu) + kb\, q(ka,\nu)\right] + O\left[(kb)^2\right] \tag{9}$$

introduced. In (9), $p(ka,\nu)$ is known from the case b = 0 when the cylinder is strictly circular, and is determined by the equation

$$\int_{-\infty}^{\infty} p(ka,\nu) e^{i\theta'\nu}\, d\nu = e^{-ika\cos\theta'} \quad ; \tag{10}$$

+The prime on the Hankel function in (7) indicates differentiation with respect to the argument, and is used in a similar sense subsequently. No confusion with the prime of θ' should arise.

it may be written formally

$$p(ka,\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(ka\cos\theta' + \nu\theta')}\, d\theta' . \tag{11}$$

The terms of order kb in equation (6) then yield

$$\int_{-\infty}^{\infty} f(\theta')\left\{ \frac{H_\nu^{(2)'}(ka)}{H_\nu^{(2)}(ka)}\, p(ka,\nu) + q(ka,\nu)\right\} e^{i\theta'\nu}\, d\nu = -i\, f(\theta')\cos\theta'\, e^{-ika\cos\theta'} , \tag{12}$$

which is to be solved for $q(ka, \nu)$.

Now $f(\theta')$ is periodic in θ', of period 2π. It can therefore be represented as a Fourier series, and the later detailed discussion is in fact confined to a consideration of a typical term in such a series. For the general theory, however, it is convenient, and may indeed be useful, to adopt once again a formal Fourier integral representation, thus

$$f(\theta') = \int_{-\infty}^{\infty} F(\mu)\, e^{i\theta'\mu}\, d\mu \quad . \tag{13}$$

The insertion in (12) of (13), together with the representation

$$\int_{-\infty}^{\infty} p'(ka,\nu) e^{i\theta'\nu}\, d\nu = -i\cos\theta'\, e^{-ika\cos\theta'} \tag{14}$$

obtained by differentiating (10) with respect to ka, is then readily seen to give

$$q(ka,\nu)=\int_{-\infty}^{\infty} F(\mu)\left[p'(ka,\nu-\mu)-\frac{H^{(2)'}_{\nu-\mu}(ka)}{H^{(2)}_{\nu-\mu}(ka)}\,p(ka,\nu-\mu)\right]d\mu \ . \qquad (15)$$

The scattered field is determined by the substitution of (15) and (9) into (5).

3.2 The form suitable when ka is not large

The specific expression for $q(ka,\nu)$ resulting from (15) depends on the specific representation adopted for the function $p(ka,\nu)$ given by (11). For the case when ka is not large the useful form of solution is that corresponding to the classical series expansion for the circular cylinder. This is obtained by taking the delta function representation

$$p(ka,\nu)=\sum_{n=-\infty}^{\infty}(-i)^n J_n(ka)\,\delta(\nu-n)\ , \qquad (16)$$

so that the incident field on r=a is expressed as a Fourier series. The formula obtained by substituting (16) into (15) is simplified with the help of the expression for the Bessel function Wronskian, and reads

$$q(ka,\nu)=\frac{2\,i}{\pi ka}\sum_{n=-\infty}^{\infty}\frac{(-i)^n}{H^{(2)}_n(ka)}\,F(\nu-n)\ . \qquad (17)$$

3.3 The form suitable when ka is large

When ka is large the useful form of solution is obtained by taking the formal representation (cf. Clemmow 1959)

$$p(ka,\nu) = e^{-\frac{1}{2} i\pi\nu} J_{\nu}(ka) \sum_{n=-\infty}^{\infty} e^{2i\pi n\nu} \quad . \tag{18}$$

The formula obtained by substituting (18) into (15) is simplified with the help of the Bessel function Wronskian, and reads

$$q(ka,\nu) = \frac{2i}{\pi ka} \int_{-\infty}^{\infty} F(\mu) \frac{e^{-\frac{1}{2} i\pi(\nu-\mu)}}{H^{(2)}_{\nu-\mu}(ka)} \left\{ \sum_{n=-\infty}^{\infty} e^{2i\pi n(\nu-\mu)} \right\} d\mu \ . \tag{19}$$

3.4 The case $f(\theta')=\cos(m\theta')$

It has already been noted that $f(\theta')$ can be represented as a Fourier series. Since the theory is only being worked to a linear approximation, the expression for $q(ka,\nu)$ can be obtained, in principle at least, by summing the contributions arising from the individual terms in this Fourier series. The process of superposition is evidenced in the mathematical formalism on writing

$$F(\mu) = \sum_{m=-\infty}^{\infty} A_m \, \delta(\mu - m) \ , \tag{20}$$

which is equivalent to

$$f(\theta') = \sum_{m=-\infty}^{\infty} A_m e^{im\theta'} , \qquad (21)$$

and substituting (20) into (15).

In the present paper the detailed discussion is confined to the case $f(\theta') = \cos(m\theta')$, where m is a positive integer, and can be regarded as treating a typical term in the Fourier series representation of a general distortion of the cylinder cross section from strict circularity. It is clear that results thus established which can be stated in a form independent of m are effectively established for a general small distortion.

Now $f(\theta') = \cos(m\theta')$ is equivalent to

$$F(\mu) = \frac{1}{2}\left[\delta(\mu - m) + \delta(\mu + m)\right] . \qquad (22)$$

When ka is not large, the appropriate expression for $q(ka, \nu)$ is obtained by substituting (22) into (17). This gives

$$q(ka, \nu) = \frac{i}{\pi ka} \sum_{n=-\infty}^{\infty} \frac{(-i)^n}{H_n^{(2)}(ka)} \left[\delta(\nu - n - m) + \delta(\nu - n + m)\right] . \qquad (23)$$

The excess of the field over that pertaining to the case b = 0, hereinafter called the <u>perturbation field</u>, is therefore, from (9) and (5),

$$U^p = -\frac{ib}{\pi a}\sum_{n=-\infty}^{\infty}\frac{i^{m-n}}{H_n^{(2)}(ka)}\left[\frac{1}{H_{n-m}^{(2)}(ka)}+\frac{(-)^m}{H_{n+m}^{(2)}(ka)}\right] H_n^{(2)}(kr)\, e^{in\theta} \quad . \qquad (24)$$

When ka is large the appropriate expression for $q(ka, \nu)$ is obtained by substituting (22) into (19). This gives

$$q(ka,\nu) = \frac{i^{m+1}}{\pi ka}\, e^{-\frac{1}{2}i\pi\nu}\left[\frac{1}{H_{\nu-m}^{(2)}(ka)}+\frac{(-)^m}{H_{\nu+m}^{(2)}(ka)}\right]\sum_{n=-\infty}^{\infty} e^{2i\pi n\nu} , \qquad (25)$$

with the corresponding expression for the perturbation field

$$U^p = -\frac{i^{m+1} b}{\pi a}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}i\pi\nu}}{H_\nu^{(2)}(ka)}\left[\frac{1}{H_{\nu-m}^{(2)}(ka)}+\frac{(-)^m}{H_{\nu+m}^{(2)}(ka)}\right] e^{2i\pi n\nu} H_\nu^{(2)}(kr) e^{i\theta\nu}\, d\nu \, . \qquad (26)$$

It is perhaps worth remarking that some check on expressions (24) and (26) is afforded by considering the two special cases m=0 and m=1. In both these cases the problem is in essence simply that of diffraction by a strictly circular cylinder: for m=0 yields the precisely circular cross section $r' = a+b$; whilst m=1 yields $r' = a + b\cos\theta'$, and to the linear approximation in b this is simply $r'=a$ displaced a distance b along the direction $\theta=0$. The solutions both for the cases m=0 and m=1 should therefore be deducible directly from that for a strictly circular cylinder. It can be verified that they can be so obtained, though when m=1 the verification is not entirely trivial.

3.5 The far field

The remainder of the paper is devoted to an examination of the perturbation field, for large ka, in the radiation region. For $kr \to \infty$, $H_\nu^{(2)}(kr)$ in (26) is replaced by the asymptotic approximation

$$H_\nu^{(2)}(kr) \sim \sqrt{\left(\frac{2}{\pi}\right)}\, e^{\frac{1}{4}i\pi}\, e^{\frac{1}{2}i\pi\nu}\, \frac{e^{-ikr}}{\sqrt{(kr)}} , \tag{27}$$

and then

$$U^p \sim \frac{\sqrt{2}\, i^m}{\pi\sqrt{\pi}}\, e^{-\frac{1}{4}i\pi}\, \frac{b}{a}\, \frac{e^{-ikr}}{a\sqrt{(kr)}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{H_\nu^{(2)}(ka)} \left[\frac{1}{H_{\nu-m}^{(2)}(ka)} + \frac{(-)^m}{H_{\nu+m}^{(2)}(ka)}\right] e^{i(\theta+2\pi n)\nu}\, d\nu . \tag{28}$$

Apart from the case $\theta=0$, which is not considered here, there is no loss of generality in taking $0<\theta\leqslant\pi$, and it is convenient from now on to assume that θ lies in this range.

The nature of the terms in the summation over n in (28) can be distinguished in a way made familiar by the corresponding circular cylinder analysis.

The term n=0, whose contribution to U^p is

$$\frac{\sqrt{2}\, i^m}{\pi\sqrt{\pi}}\, e^{-\frac{1}{4}i\pi}\, \frac{b}{a}\, \frac{e^{-ikr}}{\sqrt{(kr)}} \int_{-\infty}^{\infty} \frac{1}{H_\nu^{(2)}(ka)} \left[\frac{1}{H_{\nu-m}^{(2)}(ka)} + \frac{(-)^m}{H_{\nu+m}^{(2)}(ka)}\right] e^{i\theta\nu}\, d\nu , \tag{29}$$

corresponds to the specularly reflected ray of conventional geometrical optics. The evaluation of (29) for large ka can be accomplished by the method of steepest descents; this is done in § 4.

The terms $n = \pm 1, \pm 2, \pm 3$, whose contribution to U^p may, by a trivial transformation, be written

$$\frac{\sqrt{2}\, i^m}{\pi\sqrt{\pi}}\, e^{-\frac{1}{4}i\pi}\, \frac{b}{a}\, \frac{e^{-ikr}}{\sqrt{(kr)}} \sum_{n=o}^{\infty} \int_{-\infty}^{\infty} \frac{1}{H_{\nu}^{(2)}(ka)} \left[\frac{1}{H_{\nu-m}^{(2)}(ka)} + \frac{(-)^m}{H_{\nu+m}^{(2)}(ka)} \right]$$

$$\times\, e^{-2i\pi m\nu} \left[e^{-i\theta\nu} + e^{-i(2\pi-\theta)\nu} \right] d\nu, \tag{30}$$

correspond to rays which have "crept" round the body. The evaluation of (30) can be accomplished by closing the path of integration in the complex ν plane with an infinite semi-circle below the real axis, and calculating the residues of the enclosed poles. The details are worked out in § 5.

4. The "specular" contribution

In this section the contribution to U^p corresponding to the specularly reflected ray of conventional geometrical optics, namely (29), is worked out by evaluating the integral

$$\int_{-\infty}^{\infty} \frac{e^{i\theta\nu}}{H_{\nu}^{(2)}(ka)\; H_{\nu-m}^{(2)}(ka)}\, d\nu \tag{31}$$

by the method of steepest descents. The method provides an asymptotic expansion in inverse powers of ka, and the first two terms of this expansion are obtained explicitly. In § 6 it is shown that the result is in agreement with that derived by the Luneberg-Kline technique.

It is convenient to introduce a new variable of integration α via the relation

$$\nu = ka \cos \alpha \ . \tag{32}$$

Then the Debye asymptotic form of the Hankel function is

$$H_{\nu}^{(2)}(ka) = \frac{2}{\pi} e^{\frac{1}{4}i\pi} \frac{1}{\sqrt{(ka \sin\alpha)}} \left\{ 1 + i \frac{5 - 2\sin^2\alpha}{24 \sin^3\alpha} \frac{1}{ka} + O\left[\frac{1}{(ka)^2}\right] \right\} e^{-ika(\sin\alpha - \alpha\cos\alpha)} , \tag{33}$$

in which the terms shown explicitly are those whose retention is necessary to preserve the required degree of approximation.

Then if

$$\nu - m = ka \cos(\alpha + \alpha_m), \tag{34}$$

some trivial algebra shows that

$$\alpha_m = \frac{m}{\sin\alpha} \frac{1}{ka} \quad \frac{m^2 \cos\alpha}{2\sin^3\alpha} \frac{1}{(ka)^2} + O\left[\frac{1}{(ka)^3}\right] , \tag{35}$$

with the result that

$$ka\left[\sin(\alpha+\alpha_m) - (\alpha+\alpha_m)\cos(\alpha+\alpha_m)\right]$$

$$= ka(\sin\alpha - \alpha\cos\alpha) + m\alpha + \frac{m^2}{2\sin\alpha} \frac{1}{ka} + O\left[\frac{1}{(ka)^2}\right] , \tag{36}$$

and

$$H^{(2)}_{\nu-m}(ka) = \sqrt{\left(\frac{2}{\pi}\right)}\, e^{\frac{1}{4}i\pi} \frac{1}{\sqrt{(ka\sin\alpha)}} \left\{ 1 + \frac{i(5-2\sin^2\alpha) - 12m(\cos\alpha + im\sin\alpha)\sin\alpha}{24\sin^3\alpha} \frac{1}{ka} + O\left[\frac{1}{(ka)^2}\right] \right\}$$
$$\times e^{i\left[ka(\sin\alpha - \alpha\cos\alpha) + m\alpha\right]} . \tag{37}$$

Finally, using (33) and (37), it appears that when (31) is expressed in terms of the variable of integration α, the integrand is

$$\frac{\sin\alpha\, e^{ika\theta\cos\alpha}}{H^{(2)}_{\nu}(ka)\, H^{(2)}_{\nu-m}(ka)} = -\frac{1}{2} i\pi ka \sin^2\alpha\, e^{im\alpha} \left\{ 1 + \frac{6m(\cos\alpha + im\sin\alpha)\sin\alpha - i(5-2\sin^2\alpha)}{12\sin^3\alpha} \frac{1}{ka} + O\left[\frac{1}{(ka)^2}\right] \right\} e^{ika\Phi(\alpha)} , \tag{38}$$

where

$$\Phi(\alpha) = 2(\sin\alpha - \alpha\cos\alpha) + \theta\cos\alpha . \tag{39}$$

The saddle-point for the integral is therefore evidently at $\alpha = \theta/2$, and with the notation

$$\beta = \alpha - \frac{1}{2}\theta \tag{40}$$

$$s = \sin\frac{1}{2}\theta \quad , \qquad c = \cos\frac{1}{2}\theta \ , \tag{41}$$

it is adequate to write

$$e^{ika\Phi(\alpha)} = e^{2ikas}\left[1 + ika\left(\frac{2}{3}c\beta^2 - \frac{1}{4}s\,\beta^4\right) - \frac{2}{9}(ka)^2 c^2\beta^6\right] e^{ikas\beta^2} , \tag{42}$$

$$\sin^2\alpha\, e^{im\alpha} = s^2\, e^{\frac{1}{2}im\theta}\left[1 + (im + 2\frac{c}{s})\beta + \left(2im\frac{c}{s} + \frac{1}{s^2} - \frac{1}{2}m^2 - 2\right)\beta^2\right] \tag{43}$$

The procedure is now to substitute (42) and (43) into (38), to replace α in the 1/(ka) term in the curly bracket on the right hand side of (38) by $\theta/2$, and to integrate over β from $-\infty \exp(i\pi/4)$ to $\infty \exp(i\pi/4)$. The result, after the collection of some half dozen terms, is that the integral (31) is

$$\frac{1}{2}\pi\sqrt{\pi}\, e^{-\frac{1}{4}i\pi}\, e^{\frac{1}{2}im\theta}\left[ka\sin(\tfrac{1}{2}\theta)\right]^{3/2}\left\{1-i\,\frac{8-(4m^2-1)\sin^2(\frac{1}{2}\theta)}{16\sin^3(\frac{1}{2}\theta)}\,\frac{1}{ka}+O\left[\frac{1}{(ka)^2}\right]\right\} e^{2ika\sin(\frac{1}{2}\theta)}\,. \tag{44}$$

In order to evaluate (29), the integral identical with (31) except that m replaces -m is also required. It is, of course, obtained immediately from (44) on replacing m by -m.

The explicit expressions for (29) are distinguished according as to whether m is odd or even. If m is <u>odd</u>, the perturbation part of the far field associated with specular reflection is

$$(-)^{\frac{1}{2}(m-1)}\, i\,kb\sqrt{2}\left[\sin(\tfrac{1}{2}\theta)\right]^{3/2}\sin(\tfrac{1}{2}m\theta)$$
$$\times\sqrt{(ka)}\left\{1-i\,\frac{8-(4m^2-1)\sin^2(\frac{1}{2}\theta)}{16\sin^3(\frac{1}{2}\theta)}\,\frac{1}{ka}+O\left[\frac{1}{(ka)^2}\right]\right\} e^{2ika\sin(\frac{1}{2}\theta)}\,\frac{e^{-ikr}}{\sqrt{(kr)}}\,. \tag{45}$$

If m is <u>even</u>, the corresponding expression is

$$(-)^{\frac{1}{2}m-1}\, i\,kb\sqrt{2}\left[\sin\left(\tfrac{1}{2}\theta\right)\right]^{3/2}\cos\left(\tfrac{1}{2}m\theta\right)$$

$$\times\sqrt{(ka)}\left\{1-i\,\frac{8-(4m^2-1)\sin^2\left(\frac{1}{2}\theta\right)}{16\sin^3\left(\frac{1}{2}\theta\right)}\;\frac{1}{ka}+O\left[\frac{1}{(ka)^2}\right]\right\}e^{2ika\sin\left(\frac{1}{2}\theta\right)}\;\frac{e^{-ikr}}{\sqrt{(kr)}}\;. \tag{46}$$

5. The "creeping" contribution

Attention is now turned to expression (30), which is the contribution to U^p associated with "ray paths" which in part lie along the perimeter of the cylinder cross section (see Figure 2).

The only singularities, in the complex ν-plane, of the integrand in (30) are simple poles associated with the zeros of $H_\nu(ka)$, $H_{\nu-m}(ka)$ and $H_{\nu+m}(ka)$ <u>qua</u> functions of ν. A straightforward method for evaluating the integral in (30) is therefore to close the path of integration with an infinite semi-circle below the real axis (recalling that $0<\theta\leqslant\pi$), the result being $2\pi i$ times the sum of the residues of all the poles in the lower half of the complex ν plane. It is only necessary to write down explicitly the contribution to (30) associated with, say, the "counter-clockwise rays", one of which is shown diagrammatically in Figure 2; that is, the part of (30) arising from the term $\exp(-i\,\theta\nu)$ in the last factor of the integrand. The contribution of the "clockwise rays", arising from the term $\exp\left[-i(2\pi-\theta)\nu\right]$ in the last factor of the integrand, can then be deduced simply by writing $2\pi-\theta$ for θ.

The poles of $H^{(2)}_{\nu}(ka)$ in the lower half of the complex ν plane are ν_ℓ, $\ell = 1, 2, 3, \ldots$, with a functional dependence on ka of which the most convenient expression is the form (3). By the procedure just described, it therefore appears, after some simple rearrangement, that the perturbation part of the far field associated with the counter-clockwise creeping rays is

$$-2\sqrt{\left(\frac{2}{\pi}\right)}\, e^{\frac{1}{4}i\pi}\, i^m\, \frac{b}{a}\, \frac{e^{-ikr}}{\sqrt{(kr)}} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{\left\{\frac{d}{d\nu}\left[H^{(2)}_{\nu}(ka)\right]\right\}_{\nu=\nu_\ell}} \times \left[\frac{1+(-)^m e^{im\theta}}{H^{(2)}_{\nu_\ell-m}(ka)} + \frac{(-)^m + e^{-im\theta}}{H^{(2)}_{\nu_\ell+m}(ka)}\right] e^{-i\nu_\ell(\theta+2\pi n)} \,. \tag{47}$$

It is now convenient to appeal to Appendix B for the relation

$$H^{(2)}_{\nu_\ell \pm m}(ka) = g(\pm m, \nu, ka)\, H^{(2)'}_{\nu_\ell}(ka)\,, \tag{48}$$

where $g(\pm m, \nu, ka)$ is a polynomial in $1/(ka)$ of degree $m-1$, given by (B15) or (B17) respectively according as to whether m is even or odd. The Hankel function Wronskian in turn enables (48) to be written

$$H^{(2)}_{\nu_\ell \pm m} = -\frac{4i}{\pi ka}\, g(\pm m, \nu, ka)\, \frac{1}{H^{(1)}_{\nu_\ell}(ka)} \,. \tag{49}$$

The perturbation part of the far field associated with the counter-clockwise creeping rays is therefore

$$\sqrt{\left(\frac{\pi}{2}\right)}\, e^{-\frac{1}{4} i\pi}\, i^m\, kb\, \frac{e^{-ikr}}{(kr)} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{H^{(1)}_{\nu_\ell}(ka)}{\left\{ \frac{d}{d\nu} \left[H^{(2)}_{\nu}(ka) \right] \right\}_{\nu=\nu_\ell}}$$

$$\times \left[\frac{1+(-)^m e^{im\theta}}{g(-m,\nu,ka)} + \frac{(-)^m + e^{-im\theta}}{g(m,\nu,ka)} \right] e^{-i\nu_\ell(\theta+2\pi n)} . \qquad (50)$$

The expression (50) for the linear term in kb is exact. The final step is to simplify it by the use of asymptotic expansions for large ka of the terms in the double summation. The results are given explicitly only for m even .

It is shown in Appendix C, equation (C 11), that

$$g(\pm m, \nu_\ell, ka) = \mp m \left\{ 1 + \frac{1}{3}(m^2-1)\tau_\ell \frac{1}{(ka)^{2/3}} \pm \frac{1}{6} m(m^2-1) \frac{1}{ka} + O\left[\frac{1}{(ka)^{4/3}} \right] \right\} , \qquad (51)$$

where τ_ℓ are the numbers defined in (3). The factor in square brackets in the terms in the double summation of (50) is therefore, for m even,

$$\frac{2i}{m} \left[1 - \frac{1}{3}(m^2-1)\tau_\ell \frac{1}{(ka)^{2/3}} \right] \sin(m\theta) + \frac{1}{3}(m^2-1) \left[1+\cos(m\theta) \right] \frac{1}{ka} + O\left[\frac{1}{(ka)^{4/3}} \right] . \qquad (52)$$

It is shown in Appendix D that+

$$\frac{H^{(1)}_{\nu_\ell}(ka)}{\left\{\frac{d}{d\nu}\left[H^{(2)}_{\nu}(ka)\right]\right\}_{\nu=\nu_\ell}} = -\frac{\pi}{2^{4/3}} e^{i\pi/6} (ka)^{1/3}\left[Bi(a_\ell)\right]^2 \left\{1+\frac{\tau_\ell}{3(ka)^{2/3}} + O\left[\frac{1}{(ka)^{4/3}}\right]\right\}, \tag{53}$$

where Ai(z), Bi(z) are the Airy integrals tabulated by Miller (1946), and $Ai(a_\ell) = 0$.

The perturbation part of the far field associated with the counter-clockwise creeping rays is therefore, for large ka and m even,

$$(-)^{\frac{1}{2}m+1} \frac{\pi\sqrt{\pi}}{2^{5/6}} e^{5i\pi/12} kb \frac{e^{-ikr}}{\sqrt{(kr)}} (ka)^{1/3} \sum_{n=0}^{\infty}\sum_{\ell=1}^{\infty}\left[Bi(a_\ell)\right]^2\left\{1+\frac{1}{3}\tau_\ell\frac{1}{(ka)^{2/3}} + O\left[\frac{1}{(ka)^{4/3}}\right]\right\}$$

$$\times\left\{\left[\frac{1}{m}-\frac{1}{3}(m-\frac{1}{m})\tau_\ell\frac{1}{(ka)^{2/3}}\right]\sin(m\theta)-\frac{i}{6}(m^2-1)\left[1+\cos(m\theta)\right]\frac{1}{ka}+O\left[\frac{1}{(ka)^{4/3}}\right]\right\} e^{-i\nu_\ell(\theta+2\pi n)} \tag{54}$$

where for consistency the asymptotic approximation to ν_ℓ should also be invoked, indeed to one term beyond those given explicitly in (3).

The main object of retaining explicitly in (54) so many inverse powers of ka is not numerical accuracy. They are retained, and displayed as they are, in

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+ Apart from the notation, the first term in the asymptotic expansion (53) is a standard result in the relevant literature of diffraction theory; the authors are unable to find a reference for the second term. The proof in Appendix D appeals to the Bessel function expansions given by Olver (1954), which have not apparently been used before in this connection.

order to yield as full a check on the creeping ray theory as the problem and method allow. In fact, the whole of (54) is given by Keller's prescription. Since the prescription is in a sense just the first term in an asymptotic development, the extent of the agreement is, perhaps, at first sight rather surprising. The way in which it comes about is shown in detail in § 7.

6. Luneberg-Kline theory

The purpose of this section is to recover, by the Luneberg-Kline procedure, expressions (45) and (46) for the perturbation part of the far field associated with specular reflection.

For the two-dimensional scalar wave equation the procedure may be stated as follows (see, for example, Keller, Lewis and Seckler, 1956). The "specular" part of the total scattered field is written in the form[+]

$$e^{-ik\psi} \sum_{n=o}^{\infty} \frac{v_n(r,\theta)}{(-ik)^n} \quad , \tag{55}$$

where

$$\left| \nabla \psi \right|^2 = 1, \tag{56}$$

$$2 \nabla v_o \cdot \nabla \psi + v_o \nabla^2 \psi = 0 \ , \tag{57}$$

$$2 \nabla v_n \cdot \nabla \psi + v_n \nabla^2 \psi = -\nabla^2 v_{n-1}, \ n = 1, 2, 3, \ldots \quad . \tag{58}$$

[+] In the literature the Luneberg -Kline procedure is usually presented with the assumption of a time dependence exp(-iωt). In the application here of previous theory the sign of i is therefore changed.

In terms of the distance S along the specularly reflected ray, from the caustic (reflected ray envelope) to the point r, θ in the illuminated region, equations (56), (57) and (58) are equivalent to

$$\psi = \psi(S_0) + S - S_0, \tag{59}$$

$$\left.\begin{aligned} v_0(S) &= v_0(S_0)\sqrt{[G(S)/G(S_0)]} \;, \\ v_n(S) &= v_n(S_0)\sqrt{[G(S)/G(S_0)]} - \frac{1}{2}\int_{S_0}^{S} \sqrt{[G(S)/G(\sigma)]}\, \nabla^2 v_{n-1}(\sigma)\, d\sigma, \\ & \qquad n = 1, 2, 3, \ldots, \end{aligned}\right\} \tag{60}$$

where S_0 is the value of S at the point of intersection (r', θ') of the ray with the perimeter of the diffracting body, and G(S) is the curvature of the reflected wave front at r, θ.

For an incident plane wave, and with r, θ in the far field,

$$S_0 = \frac{1}{2}\rho \sin\left(\frac{1}{2}\theta\right) \quad , \tag{61}$$

where ρ is the radius of curvature of the perimeter at r', θ'; and

$$G(S) = 1/S \,. \tag{62}$$

If the incident wave is (4), the boundary condition that the total field vanish on the perimeter of the diffracting body gives

$$\psi(S_0) = r' \cos\theta' \quad , \tag{63}$$

$$v_0(S_0) = -1 \quad , \tag{64}$$

$$v_n(S_0) = 0 \quad , \qquad n = 1, 2, 3, \ldots \quad . \tag{65}$$

Hence, from (59),

$$\psi = S - S_0 + r' \cos \theta' \quad ; \tag{66}$$

and from (60),

$$v_0(S) = -\sqrt{(S_0/S)} \quad . \tag{67}$$

Now in order to check (55) against (45) and (46) it is only necessary to distinguish v_0, of all the v_n, from its unperturbed (b=0) value. For since the theory of § 3 shows that the perturbation part of the solution is linear in kb, any length d typical of the body, which is associated with the k in any term in the summation in (55), may be taken in the form

$$d = a + b\, O(1) \quad , \tag{68}$$

where the order term refers to the order in ka; whence

$$\frac{1}{(kd)^n} = \frac{1}{(ka)^n} + kb\, O\left[\frac{1}{(ka)^{n+1}}\right] \quad . \tag{69}$$

Furthermore, it is easy to establish that, if second and higher powers of kb are neglected,

$$\theta' = \frac{1}{2}\pi + \frac{1}{2}\theta - kb\, m \sin\left[\frac{1}{2}m(\pi+\theta)\right]\frac{1}{ka} \quad , \tag{70}$$

$$\rho = a\left\{1 - kb(m^2-1)\cos\left[\frac{1}{2}m(\pi+\theta)\right]\frac{1}{ka}\right\} \quad , \tag{71}$$

and, for a point in the far field,

$$S - S = r - a\sin\left(\frac{1}{2}\theta\right) - b\left\{\sin\left(\frac{1}{2}\theta\right)\cos\left[\frac{1}{2}m(\pi+\theta)\right] - m\cos\left(\frac{1}{2}\theta\right)\sin\left[\frac{1}{2}m(\pi+\theta)\right]\right\} \quad . \tag{72}$$

Hence, from (66),

$$\psi = r-2a\,\sin(\tfrac{1}{2}\,\theta)-2b\,\sin(\tfrac{1}{2}\,\theta)\,\cos\left[\tfrac{1}{2}\,m\,(\pi+\theta)\right]; \tag{73}$$

and, from (67) and (61),

$$v_0(S) = -\sqrt{\left[\tfrac{1}{2}\sin(\tfrac{1}{2}\,\theta)\right]}\,\sqrt{(ka)}\left\{1-\tfrac{1}{2}\,kb(m^2-1)\cos\left[\tfrac{1}{2}\,m(\pi+\theta)\right]\frac{1}{ka}\right\}\frac{1}{\sqrt{(kr)}}\,. \tag{74}$$

The substitution into (55) of the expressions (73) and (74), together with the unperturbed (b=0) value of $v_1(S)$, namely (Keller, Lewis and Seckler, 1956)

$$v_1(S) = \sqrt{\left[\tfrac{1}{2}\,\sin\,(\tfrac{1}{2}\,\theta)\right]}\,\frac{8-3\sin^2(\tfrac{1}{2}\,\theta)}{16\,\sin^3(\tfrac{1}{2}\,\theta)}\cdot\frac{1}{\sqrt{(ar)}}\,, \tag{75}$$

gives

$$-ikb\sqrt{2}\left[\sin(\tfrac{1}{2}\,\theta)\right]^{3/2}\cos\left[\tfrac{1}{2}\,m(\pi+\theta)\right]$$
$$\times\sqrt{(ka)}\left\{1-i\,\frac{8-(4m^2-1)\sin^2(\tfrac{1}{2}\,\theta)}{16\,\sin^3(\tfrac{1}{2}\,\theta)}\,\frac{1}{ka}+O\left[\frac{1}{(ka)^2}\right]\right\}e^{2ika\sin(\frac{1}{2}\theta)}\,\frac{e^{-ikr}}{\sqrt{(kr)}}\,. \tag{76}$$

Evidently (76) agrees with (45) when m is odd, and with (46) when m is even.

7. Creeping ray theory

In the notation of § 2 (with particular reference to Figure 2), the prescription for the far field associated with the counter-clockwise creeping rays which it is the purpose of this section to check is

$$\frac{\pi\sqrt{\pi}}{2^{5/6}}\, e^{-i\pi/12}\, \frac{e^{-iks}}{\sqrt{(ks)}}\, k^{1/3}\, \rho_o^{1/6}\, \rho_1^{1/6} \sum_{n=o}^{\infty} \sum_{\ell=1}^{\infty} \left[Bi(a_\ell)\right]^2$$

$$x\left\{1+O\left[\frac{1}{(k\rho)^{1/3}}\right]\right\}\exp\left\{-ik\left(t_1-t_o+nT+\tau_\ell\int_{t_o}^{t_1+nT}\frac{dt}{(k\rho)^{2/3}}\right)\right\}, \tag{77}$$

where d is some length typical of the body. This differs from what Keller (1956) has proposed only in notation and by the inclusion of an order term which makes the statement precise.

The expression (77) is now evaluated for the problem in hand, with the simplification that second and higher powers of kb are neglected. For the sake of brevity, only the case when m is <u>even</u> is considered. In particular, this implies

$$\theta_o' = -\frac{1}{2}\pi \quad . \tag{78}$$

It may also be noted that

$$\theta_1' = \theta - \frac{1}{2}\pi - (-)^{\frac{1}{2}m}\,\frac{b}{a}\,\sin(m\theta) \quad . \tag{79}$$

With the help of (78) and (79), the following results may be obtained with some straightforward algebra:

$$k^{1/3}\,\rho_o^{1/6}\,\rho_1^{1/6} = (ka)^{1/3}\left\{1-(-)^{\frac{1}{2}m}\,\frac{1}{6}\,kb(m^2-1)\left[1+\cos(m\theta)\right]\frac{1}{ka}\right\} \tag{80}$$

$$\exp\left\{-ik\left[t_1-t_o+nT+\tau_\ell\int_{t_o}^{t_1+nT}\frac{dt}{(k\rho)^{2/3}}\right]\right\}$$

$$= \left\{ 1-(-)^{\frac{1}{2}m} \quad ikb \left[\frac{1}{m} - m - \frac{1}{3}(m - \frac{1}{m}) \tau_{\ell} \frac{1}{(ka)^{2/3}} \right] \quad \sin(m\theta) \right\}$$

$$x \quad \exp \left\{ -ika \left[1 + \frac{\tau_{\ell}}{(ka)^{2/3}} \right] (\theta + 2\pi n) \right\} , \qquad (81)$$

$$e^{-iks} = e^{-ikr} \left[1 - (-)^{\frac{1}{2}m} \; ikb \; m \; \sin(m\theta) \right] . \qquad (82)$$

Furthermore, the argument in § 6 leading to (69) is equally applicable here. The expression in curly brackets in (77) containing d is therefore

$$1 + O\left[\frac{1}{(ka)^{1/3}} \right] \quad + kb \;\; O\left[\frac{1}{(ka)^{4/3}} \right] , \qquad (83)$$

where it should be stressed that everything included in the part of (83) independent of b is identical with the corresponding part of the solution for the case of the circular cylinder b = 0.

Thus, although no appeal has been made to any general knowledge of the order term in d in (77), the form which it takes under the particular conditions of the problem in hand is, in fact, adequate for present purposes. For the substitution of (80) - (83) into (77) leads at once to a linear term in kb identical with (54), the part of (83) independent of b accounting both for the first expression in curly brackets in (54) and for the third and subsequent terms of the asymptotic expansion of ν in inverse powers of ka.

8. Discussion

In this section some attempt is made first to discuss the possibilities and limitations of the theory as it has been presented, and secondly to examine the extent to which the restrictions validating the analysis can profitably be eased. It seems difficult to be precise, particularly in the latter task, and some of the remarks offered are conjectural.

The theory as it stands is clearly capable of treating aspects of the problem other than those explicitly considered; the calculation of the near field, for example, or the development of the case when ka is not large. Also, of course, in principle a perturbation on other "canonical" problems should be tractable, though only the analysis for diffraction by an almost spherical body is likely to be not appreciably harder than that for the almost circular cylinder.

The assumption most likely to lead to a convincing justification of the theory is that kb is a mathematically infinitesimal quantity. However, even with this assumption, which largely evades the question as to whether in given practical circumstances the theory will be useful, the proof of the general validity of the theory is not absolutely clear cut. The reason is that various of the linear Taylor expansion approximations in kb depend on other parameters which take values up to infinity; such parameters are γ in (7) and m in (70), for example. To meet this situation the following plausible argument is offered. If the integrations or summations involving ν and m are convergent, they may be truncated at some stage (which is independent of kb) without sensible error. Thus ν and m may be regarded as having maximum values, and the linear Taylor expansion approximations are unquestionably valid. This argument implies that the analysis holds for any "distortion" function $f(\theta')$ which has a convergent Fourier series representation.

The importance of the diffraction problem, though, rests mainly in its ability to provide answers to practical situations. From this point of view it is not of much interest to take kb small enough to validate the analysis; rather, given a problem in which the specified kb is <u>prima</u> <u>facie</u> small, it must be decided whether or not the analysis is valid, in the sense of being capable of yielding an acceptable numerical approximation to the true solution. The question turns, in fact, on the accuracy of the linearization in kb, when kb and the other parameters of the problem are prescribed. This point is now considered.

Apart from kb, the dimensionless parameters which completely specify the problem when $f(\theta') = \cos(m\theta')$ are ka and m. It would seem, therefore, that the accuracy of the linear Taylor expansions used in the analysis must be investigated in terms of the values of kb, ka and m. Two types of these Taylor expansions may be distinguished; namely, on the one hand (7) and (8) which are basic to the derivation of the solution in § 3, and on the other hand expansions which are introduced subsequently in the course of simplifying the solution, and checking it against results derived by alternative methods. The former are evidently fundamental, since without them no progress at all can be made on the present lines. The latter, in contrast, are quite unimportant; indeed, as explained in a moment, considerable interest attaches to situations in which they are not valid. Attention is therefore concentrated on (7), the validity of (8) being assured by $kb \ll 1$.

If ka is large, (7) looks fairly innocuous; but it must be remembered that the significant values of ν certainly involve m. If, then, the possibility of m increasing indefinitely be envisaged, which a priori is likely to cause trouble, (7) must be looked at for indefinitely large ν. In this case, each differentiation of $H_{\nu}^{(2)}(ka)$ with respect to ka brings in a factor of order $\nu/(ka)$. The suggestion is, therefore, that the maximum permissible value of m for which (7) holds is given roughly by the criterion

$$kb \frac{m}{ka} \ll 1 , \tag{84}$$

or

$$b \ll a/m . \tag{85}$$

The significance of (85) can be directly stated in words, thus: the amplitude of the "ripple" representing the distortion of the cross section of the cylinder from a strict circle must be much less than its wavelength. This seems a reasonable criterion, and leads to some rather interesting consequences which are now considered.

The conditions

$$kb \ll 1, \; ka \gg 1, \; b \ll a/m, \tag{86}$$

obviously permit

$$\frac{b}{a} m^2 = O(1) \; . \tag{87}$$

This implies that the solution of § 3 can hold in cases when the radius of curvature of the perimeter differs at some points appreciably from a, and might even, for example become negative, that is the cylinder might have concavities. This latter possibility is particularly interesting, in that it falls outside the scope of the Keller theory used in § 7. Indeed, there is even a peculiar feature of the Keller theory in its apparent failure for quite simple large convex cylinders if their cross sections have points at which the curvature is zero. Consider for example the plane wave exp(-ikx) incident on the body

$$x^4 + y^4 = a^4 \quad . \tag{88}$$

Keller theory gives an infinite result because ρ_o is infinite. A similar pecularity exists in the statements made in § 4 of a paper by Jones (1957) about the total scattering cross sections of general bodies. Jones claims that his formulae (16) and (17) are valid if (his) $k\rho_1$, $k\rho_2$ are both large. For the body (88) they are infinite, and Jones' (16) and (17) then lead respectively to infinite and negative infinite cross sections. Recent work by Franz[+], in which a further

[+]Reported at the 1959 URSI-Toronto Symposium on Electromagnetic Wave Theory.

term is added to the Keller prescription, also runs into trouble for the body (88). An investigation of diffraction by large bodies with convex surfaces having points at which the curvature is zero therefore seems to be required, and it looks as though this could be done by the perturbation analysis of § 3.

Appendix A

The purpose of this appendix is first to derive the perturbation field by the classical method, using a Fourier series in θ, and then to apply what is commonly known as the Watson transformation. The procedure is offered as an alternative to that of §3, on more traditional lines, and arrives at the same forms of solution.

The incident plane wave (4) is written in the Fourier series representation

$$U^i = \sum_{n=-\infty}^{\infty} (-i)^n J_n(kr)\, e^{in\theta} \quad ; \tag{A1}$$

and the scattered field, neglecting second and higher powers of kb, in the corresponding representation

$$U^s = -\sum_{n=-\infty}^{\infty} (-i)^n \frac{J_n(ka)}{H_n^{(2)}(ka)} (1+kb\, c_n)\, H_n^{(2)}(kr)\, e^{in\theta} \quad , \tag{A2}$$

with coefficients c_n as yet unknown. If b is set zero in (A2), the solution for the circular cylinder r'=a, with boundary condition $U^i + U^s = 0$, is obviously recovered.

The c_n are determined by the boundary condition $U^i + U^s = 0$ on the perimeter (2). If (7) is substituted into (A2), and the companion result for $J_n(kr')$ into (A1), the linear term in kb gives

$$\sum_{n=-\infty}^{\infty} (-i)^n c_n J_n(ka) e^{in\theta'} = f(\theta') \sum_{n=-\infty}^{\infty} (-i)^n \left[J_n'(ka) - J_n(ka) \frac{H_n^{(2)'}(ka)}{H_n^{(2)}(ka)} \right] e^{in\theta'}$$

$$= \frac{2i}{\pi ka} f(\theta') \sum_{n=-\infty}^{\infty} \frac{(-i)^n}{H_n^{(2)}(ka)} e^{in\theta'} \quad . \tag{A3}$$

If $f(\theta')$ itself has the Fourier series representation (21), it is easy to see that (A3) gives

$$c_n = \frac{2\, i^{n+1}}{\pi ka\, J_n(ka)} \sum_{m=-\infty}^{\infty} \frac{(-i)^m}{H_m^{(2)}(ka)} A_{n-m} \quad . \tag{A4}$$

This is equivalent to what would be obtained by substituting (20) into (17).

For the special case of interest, $f(\theta') = \cos(m\theta')$, (A4) gives

$$c_n = \frac{i^{m+1}}{\pi ka\, J_n(ka)} \left[\frac{1}{H_{n-m}^{(2)}(ka)} + \frac{(-)^m}{H_{n+m}^{(2)}(ka)} \right] , \tag{A5}$$

and the perturbation field is

$$U^p = -\frac{ib}{\pi a} \sum_{n=-\infty}^{\infty} \frac{i^{m-n}}{H_n^{(2)}(ka)} \left[\frac{1}{H_{n-m}^{(2)}(ka)} + \frac{(-)^m}{H_{n+m}^{(2)}(ka)} \right] H_n^{(2)}(kr)\, e^{in\theta} , \tag{A6}$$

which is identical to (24).

Since (A6) is poorly convergent for large ka, the Watson (1918) transformation is applied. The first step is to replace the summation in (A6) by an integral over a contour surrounding the poles at $\nu = 0, \pm 1, \pm 2, \ldots$ of $\sin(\pi\nu)$. Explicitly

$$U^p = -\frac{bi^m}{2\pi a} \int_{C_1+C_2} \frac{e^{\frac{1}{2} i\pi\nu}}{\sin(\pi\nu) H_\nu^{(2)}(ka)} \left[\frac{1}{H_{\nu-m}^{(2)}(ka)} + \frac{(-)^m}{H_{\nu+m}^{(2)}(ka)} \right] H_\nu^{(2)}(kr)\, e^{i\theta\nu} d\nu , \quad \text{(A7)}$$

where C_1 and C_2 are lines parallel to the real ν axis, running respectively above it from right to left and below it from left to right. If ν is replaced by $-\nu$ in the integral over C_1, (A7) gives

$$U^p = -\frac{bi^m}{\pi a} \int_{C_2} \frac{e^{\frac{1}{2} i\pi\nu}}{\sin(\pi\nu) H_\nu^{(2)}(ka)} \left[\frac{1}{H_{\nu-m}^{(2)}(ka)} + \frac{(-)^m}{H_{\nu+m}^{(2)}(ka)} \right] H_\nu^{(2)}(kr) \cos(\nu\theta)\, d\nu , \quad \text{(A8)}$$

where use is made of the fact that $\exp(-\frac{1}{2} i\pi\nu) H_\nu^{(2)}(kr)$ is an even function of ν, and the path C_2 is retained on the presumption that $0 < \theta \leqslant \pi$.

When the point of observation r, θ is in the shadow region of geometrical optics, the integral in (A8) can be evaluated by closing the path of integration by an infinite semi-circle in the lower half of the complex ν plane, and evaluating the residues of the poles of the integrand which are thereby enclosed; these poles being, of course, at the zeros of $H_\nu^{(2)}(ka)$, $H_{\nu-m}^{(2)}(ka)$, and $H_{\nu+m}^{(2)}(ka)$.

But when r, θ is in the illuminated region of geometrical optics, a transformation of the integral in (A8) is necessary before the residue representation is appropriate (see, for example, Franz , 1954). This case is now considered.

With the decomposition

$$\cos(\nu\theta) = e^{-i\pi\nu}\cos\left[(\theta-\pi)\nu\right] + i\, e^{i(\theta-\pi)\nu}\sin(\pi\nu) \qquad \text{(A9)}$$

(A8) becomes

$$U^p = -\frac{bi^{m+1}}{\pi a}\int_{C_2} \frac{e^{-\frac{1}{2}i\pi\nu}}{H_\nu^{(2)}(ka)}\left[\frac{1}{H_{\nu-m}^{(2)}(ka)} + \frac{(-)^m}{H_{\nu+m}^{(2)}(ka)}\right] H_\nu^{(2)}(kr)\, e^{i\theta\nu}\, d\nu$$

$$-\frac{bi^m}{\pi a}\int_{C_2} \frac{e^{-\frac{1}{2}i\pi\nu}}{\sin(\pi\nu) H_\nu^{(2)}(ka)}\left[\frac{1}{H_{\nu-m}^{(2)}(ka)} + \frac{(-)^m}{H_{\nu+m}^{(2)}(ka)}\right] H_\nu^{(2)}(kr)\cos\left[(\theta-\pi)\nu\right] d\nu \; . \qquad \text{(A10)}$$

Since the first integral in (A10) does not contain $1/\sin(\pi\nu)$, the path of integration C_2 may be replaced by the real ν axis from $-\infty$ to $+\infty$. Its contribution to U^p is thus identical to the n=0 term in (26); or to (29) for the far field case when $H_\nu^{(2)}(kr)$ is replaced by its asymptotic form (27).

Again, since the path C_2 lies in the lower half of the complex ν plane, the factor $1/\sin(\pi\nu)$ in the integrand of the second integral in (A10) may be replaced by the expansion

$$\frac{1}{\sin(\pi\nu)} = 2i\sum_{n=0}^{\infty} e^{-i\pi(2n+1)\nu} \; . \qquad \text{(A11)}$$

The corresponding contribution to U^p is thus identified with (26) with the n=0 term excluded; or with (30) for the far field case.

Appendix B

The object of this appendix is to show that, if m is a positive or negative integer, any cylinder function $Z_{\nu+m}(z)$ can be expressed in the form+

$$Z_{\nu+m}(z)=h(m,\nu,z)\,Z_{\nu}(z)+g(m,\nu,z)\,Z'_{\nu}(z) \quad , \tag{B1}$$

where $h(m,\nu,z)$ and $g(m,\nu,z)$ are polynomials in $1/z$, and to find the specific forms of these polynomials.

That the form of (B1) is correct can be seen from the relations

$$\left(\frac{d}{zdz}\right)^m \left[z^{\nu} Z_{\nu}(z)\right] = z^{\nu-m} Z_{\nu-m}(z) \qquad (m=0,1,2,3\ldots) , \tag{B2}$$

$$\left(\frac{d}{zdz}\right)^m \left[z^{-\nu} Z_{\nu}(z)\right] = (-)^m z^{-\nu-m} Z_{\nu+m}(z) \qquad (m=0,1,2,3\ldots) \tag{B3}$$

by carrying out the differentiations indicated on the left hand sides, at each stage reducing the second derivative by means of Bessel's equation

$$Z''_{\nu} = -\frac{1}{z} Z'_{\nu} - \left(1 - \frac{\nu^2}{z^2}\right) Z_{\nu} \ . \tag{B4}$$

This procedure shows readily enough that $h(m,\nu,z)$ and $g(m,\nu,z)$ are polynomials in $1/z$ of degree not exceeding $|m|$, but does not seem suited to deriving their explicit forms.

Further information can be obtained by taking Z_{ν} to be J_{ν}, and substituting into (B1) its expansion as a power series in z. This shows that $h(m,\nu,z)$ and $g(m,\nu,z)$ must be respectively of the same and opposite parity as m: or more

+Throughout this appendix the prime denotes differentiation with respect to z.

specifically, if $|m|$ is odd, $h(m, \nu, z)$ is $1/z$ times a polynomial in $1/z^2$ and $g(m, \nu, z)$ is a polynomial in $1/z^2$, whereas if m is even the statement holds if $h(m, \nu, z)$ and $g(m, \nu, z)$ are interchanged.

Explicit expressions for $h(m, \nu, z)$ and $g(m, \nu, z)$ can be derived in the following way. Substitute the right hand side of (B1) into Bessel's equation of order $\nu + m$, and eliminate Z_ν'' by means of (B4). The result is

$$p_1 Z_\nu' + p_2 Z_\nu = 0 \, , \qquad \text{(B5)}$$

where p_1, p_2 are polynomials in $1/z$. This implies $p_1 = p_2 = 0$; or explicitly

$$h'' + \frac{1}{z} h' - 2\left(1 - \frac{\nu^2}{z^2}\right) g' - \frac{m(2\nu+m)}{z^2} h - \frac{2\nu^2}{z^3} g = 0 \, , \qquad \text{(B6)}$$

$$g'' + 2h' - \frac{1}{z} g' + \frac{1 - m(2\nu+m)}{z^2} g = 0 \, . \qquad \text{(B7)}$$

If the operator

$$\delta = \frac{1}{z^2} \frac{d}{d(1/z^2)} \qquad \text{(B8)}$$

is introduced, (B6) and (B7) are

$$\delta^2 h + z\left(1 - \frac{\nu^2}{z^2}\right) \delta g - \frac{1}{4} m(2\nu+m) h - \frac{\nu^2}{2z} g = 0 \qquad \text{(B9)}$$

$$\delta^2 g + \delta g - z\delta h + \frac{1}{4}\left[1 - m(2\nu+m)\right] g = 0 \, . \qquad \text{(B10)}$$

Then the elimination of h from (B9) and (B10) gives, after some algebra,

$$\left\{\delta(\delta - \tfrac{1}{2}) + \frac{1}{z^2}\left(\delta - \frac{m-1}{2}\right)\left(\delta + \frac{m+1}{2}\right)\left(\delta + \frac{m+2\nu+1}{2}\right)\left(\delta - \frac{m+2\nu-1}{2}\right)\right\} g = 0 \, . \qquad \text{(B11)}$$

Now (B11) is a hypergeometric equation; and in fact implies, that by virtue of what has already been stated about the polynomial nature of g, if $|m|$ is odd

$$g = C_1 \, {}_4F_1\left(-\frac{m-1}{2}, \frac{m+1}{2}, \frac{m+2\nu+1}{2}, -\frac{m+2\nu-1}{2}; \frac{1}{2}; -\frac{1}{z^2}\right) \qquad \text{(B12)}$$

where C_1 is independent of z.

If $|m|$ is even it is convenient to use the equation for zg corresponding to (B11). This turns out to be

$$\left\{ \delta(\delta+\tfrac{1}{2}) + \frac{1}{z^2}(\delta + \frac{m+2}{2})(\delta - \frac{m-2}{2})(\delta + \frac{2\nu+m+2}{2})(\delta - \frac{2\nu+m-2}{2}) \right\}(zg) = 0 \ , \qquad \text{(B13)}$$

which implies

$$g = C_2 \frac{1}{z} \ {}_4F_1\left(\frac{1}{2}m+1, -\frac{1}{2}m+1, \frac{1}{2}m+\nu+1, -\frac{1}{2}m-\nu+1; \frac{3}{2}; -\frac{1}{z^2}\right) \ , \qquad \text{(B14)}$$

where C_2 is independent of z.

The function h can be found by integrating (B7) with respect to z, the arbitrary constant of integration being fixed by substitution into (B6).

It then only remains to determine C_1 and C_2. This is done by specifying Z_ν as $H_\nu^{(2)}$ in (B1), multiplying the equation by $\sqrt{z}\,\exp(iz)$, and considering the limit as $z \to \infty$.

The final formulae are, for $|m|$ even

$$g(m,\nu,z) = (-)^{\frac{1}{2}m} \frac{1}{2}m(m+2\nu)\frac{1}{z} \ {}_4F_1\left(\frac{1}{2}m+1, -\frac{1}{2}m+1, \frac{1}{2}m+\nu+1, -\frac{1}{2}m-\nu+1; \frac{3}{2}; -\frac{1}{z^2}\right) \ ,$$

(B15)

$$h(m,\nu,z)=\frac{1}{2}\left[-g'(m,\nu,z)+\frac{1}{z}g(m,\nu,z)\right]$$
$$+(-)^{\frac{1}{2}m}\;{}_4F_1\left(\tfrac{1}{2}m,-\tfrac{1}{2}m,\tfrac{1}{2}m+\nu,-\tfrac{1}{2}m-\nu;\tfrac{1}{2};-\frac{1}{z^2}\right)\quad, \tag{B 16}$$

and for $|m|$ odd

$$g(m,\nu,z)=(-)^{\frac{1}{2}(m+1)}\;{}_4F_1\left(-\frac{m-1}{2},\frac{m+1}{2},\frac{m+2\nu+1}{2},-\frac{m+2\nu-1}{2};\frac{1}{2};-\frac{1}{z^2}\right), \tag{B 17}$$

$$h(m,\nu,z)=\frac{1}{2}\left[-g'(m,\nu,z)+\frac{1}{z}g(m,\nu,z)\right]$$
$$+(-)^{\frac{1}{2}(m-1)}\;\frac{1}{2}m(m+2\nu)\frac{1}{z}\;{}_4F_1\left(-\frac{m-1}{2},\frac{m+1}{2},\frac{m+1}{2}+\nu,-\frac{m-1}{2}-\nu;\frac{3}{2};-\frac{1}{z^2}\right). \tag{B 18}$$

In particular, for m = 1,

$$g(1,\nu,z)=-1\quad, \tag{B 19}$$

$$h(1,\nu,z)=\frac{\nu}{z}\quad; \tag{B 20}$$

for m = 2,

$$g(2,\nu,z)=-2(\nu+1)\frac{1}{z}\quad, \tag{B 21}$$

$$h(2,\nu,z)=-1+2\nu(\nu+1)\frac{1}{z^2}\quad; \tag{B 22}$$

for m = 3,

$$g(3,\nu,z)=1-4(\nu+1)(\nu+2)\frac{1}{z^2}\quad, \tag{B 23}$$

$$h(3,\nu,z)=-(3\nu+4)\frac{1}{z}+4\nu(\nu+1)(\nu+2)\frac{1}{z^3}\;. \tag{B 24}$$

The formulae (B19) to (B24) have been verified by an independent method.

Appendix C

The purpose of this appendix is to obtain the initial terms of the asymptotic expansion for large ka of

$$H^{(2)}_{\nu_\ell+m}(ka) \Big/ H^{(2)'}_{\nu_\ell}(ka) \qquad \text{(C 1)}$$

where ν_ℓ is a zero of $H^{(2)}_\nu(ka)$ qua function of ν, m is an even positive or negative integer, and the prime denotes differentiation of the Hankel function with respect to its argument.

If in (B 1) $\nu = \nu_\ell$, z = ka, it appears that

$$H^{(2)}_{\nu_\ell+m}(ka) = g(m, \nu_\ell, ka)\, H^{(2)'}_{\nu_\ell}(ka) \quad . \qquad \text{(C 2)}$$

But according to Appendix B, with 2n introduced for convenience in place of m,

$$g(2n, \nu_\ell, ka) = (-)^n \frac{2n(\nu_\ell + n)}{ka} \; {}_4F_1\left[1+n, 1-n, 1+\nu_\ell+n, 1-\nu_\ell-n; \frac{3}{2}; -\frac{1}{(ka)^2}\right] . \qquad \text{(C 3)}$$

Since [cf. (3)],

$$\nu_\ell = ka + \tau_\ell (ka)^{1/3} + O\left[\frac{1}{(ka)^{1/3}}\right] , \qquad \text{(C 4)}$$

the only serious problem is the determination of the asymptotic form of the ${}_4F_1$ function in (C 3) (hereinafter simply written ${}_4F_1$) when ν_ℓ is given by (C 4).

Now it can be shown that+

$$(1+\nu_\ell+n)_p(1-\nu_\ell-n)_p=(-)^p\,\nu_\ell^{2p}\left[1+\frac{2pn}{\nu_\ell}+O\left(\frac{1}{\nu_\ell^2}\right)\right]; \qquad (C\,5)$$

so that

$$_4F_1=\sum_{p=o}^{n=1}\frac{(1+n)_p\,(1-n)_p}{p!\,(3/2)_p}\left(\frac{\nu_\ell}{ka}\right)^{2p}\left[1+\frac{2pn}{\nu_\ell}+O\left(\frac{1}{\nu_\ell^2}\right)\right], \qquad (C\,6)$$

or, using (C 4)

$$_4F_1=\,_2F_1(1+n,1-n;\tfrac{3}{2};1)+\left[\frac{\tau_\ell}{(ka)^{2/3}}+\frac{n}{ka}\right]\,_2F_1(2+n,2-n;\tfrac{5}{2};1)+O\left[\frac{1}{(ka)^{4/3}}\right] \qquad (C\,7)$$

But (Erdélyi 1953, pp 3, 104)

$$_2F_1(1+n,1-n;\tfrac{3}{2};1)=\frac{(1/2)!\,(-3/2)!}{(-\frac{1}{2}+n)!\,(-\frac{1}{2}-n)!}=(-)^{n+1}, \qquad (C\,8)$$

$$_2F_1(2+n,2-n;\tfrac{5}{2};1)=\frac{(3/2)!\,(-5/2)!}{(-\frac{1}{2}+n)!\,(-\frac{1}{2}-n)!}=(-)^{n}, \qquad (C\,9)$$

so that (C 7) gives

$$_4F_1=(-)^{n+1}\left\{1+\frac{4}{3}(n^2-1)\left[\frac{\tau_\ell}{(ka)^{2/3}}+\frac{n}{ka}\right]+O\left[\frac{1}{(ka)^{4/3}}\right]\right\}. \qquad (C\,10)$$

Finally, the substitution of (C10) and (C4) into (C3), and the replacement of n by $\frac{1}{2}$ m, give

$$g(m,\nu_\ell,ka)=-m\left\{1+\frac{1}{3}(m^2-1)\left[\frac{\tau_\ell}{(ka)^{2/3}}+\frac{m}{2ka}\right]+O\left[\frac{1}{(ka)^{4/3}}\right]\right\}. \qquad (C\,11)$$

+The notation is $(a)_o=1, (a)_n=a(a+1)(a+2)\ldots(a+n-1)$ for n=1, 2, 3... .

Appendix D

The purpose of this appendix is to obtain the initial terms of the asymptotic expansion for large ka of the function

$$H^{(1)}_{\nu_\ell}(ka) \Big/ \left\{ \frac{d}{d\nu} \left[H^{(2)}_{\nu}(ka) \right] \right\}_{\nu = \nu_\ell} , \tag{D1}$$

where ν_ℓ is a zero of $H^{(2)}_{\nu}(ka)$ qua function of ν.

It can be shown, either from the Sommerfeld integral representation of the Hankel functions, or from recurrence relations, that

$$\frac{d}{d\nu}\left[H^{(2)}_{\nu}(ka) \right] = -H^{'}_{\nu}(ka) + \frac{1}{6} H^{(2)'''}_{\nu}(ka) - \frac{3}{40} H^{(2)'''''}_{\nu}(ka) + \cdots , \tag{D2}$$

where the primes denote differentiation of the Hankel functions with respect to their argument.

Then if in (D2) Bessel's equation (B4) is used to express all Hankel function derivatives on the right hand side in terms of $H^{(2)}_{\nu}(ka)$ and $H^{(2)'}_{\nu}(ka)$, and if, further, ν is set equal to ν_ℓ, the result is

$$\left\{ \frac{d}{d\nu} \left[H^{(2)}_{\nu}(ka) \right] \right\}_{\nu = \nu_\ell} = -\left\{ 1 + \frac{1}{6}\left[1 - \frac{\nu_\ell^2 + 2}{(ka)^2} \right] + \frac{3}{40}\left[1 - \frac{2\nu_\ell^2 + 7}{(ka)^2} + \frac{\nu_\ell^4 + 35\nu_\ell^2 + 24}{(ka)^4} \right] \right\} H^{'}_{\nu_\ell}(ka) . \tag{D 3}$$

The substitution of (C 4) into (D3) now gives, with the help of the Hankel function

Wronskian,

$$\left\{ \frac{d}{d\nu} \left[H_{\nu}^{(2)}(ka) \right] \right\}_{\nu=\nu_{\ell}} = \frac{4i}{\pi ka} \left\{ 1 - \frac{\tau_{\ell}}{3(ka)^{2/3}} + O\left[\frac{1}{(ka)^{4/3}} \right] \right\} \frac{1}{H_{\nu_{\ell}}^{(1)}(ka)} . \qquad \text{(D 4)}$$

Furthermore

$$H_{\nu_{\ell}}^{(1)}(ka) = 2\, J_{\nu_{\ell}}(ka) , \qquad \text{(D5)}$$

and according to Olver (1954, equation 4·24)

$$J_{\nu_{\ell}}(ka) = \frac{e^{-i\pi/6}}{2^{2/3}} \, Bi(a_{\ell}) \frac{1}{(ka)^{1/3}} + O\left[\frac{1}{(ka)^{5/3}} \right] , \qquad \text{(D6)}$$

where Ai(z), Bi(z) are the Airy integrals tabulated by Miller (1946), and $Ai(a_{\ell}) = 0$.

From (D4), (D5), and (D6) the expression (D1) is evidently

$$- \frac{\pi}{2^{4/3}} e^{i\pi/6} (ka)^{1/3} \left[Bi(a_{\ell}) \right]^{2} \left\{ 1 + \frac{\tau_{\ell}}{3(ka)^{2/3}} + O\left[\frac{1}{(ka)^{4/3}} \right] \right\}, \qquad \text{(D 7)}$$

which is the required form.

Finally, it may be noted in parenthesis that another result due to Olver (1954, combining equations 4·24, 4·26), namely

$$H_{\nu}^{(2)}(ka) = \frac{2^{4/3} e^{i\pi/3}}{\nu^{1/3}} Ai(-\nu^{2/3} \zeta e^{i\pi/3}) + O\left[\frac{1}{\nu^{5/3}} \right] , \qquad \text{(D8)}$$

where

$$\zeta^{3/2} = \frac{3}{2} (\sigma - \tanh \sigma) , \qquad \text{(D9)}$$

$$\cosh \sigma = \nu/(ka) , \qquad \text{(D 10)}$$

leads rather easily to (3) from the approximate solution of

$$- \nu^{2/3} \zeta e^{i\pi/3} = a_{\ell} , \tag{D11}$$

for ν in terms of ka when ka is large.

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