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ELECTRONIC DEFENSE GROUP TECHNICAL MEMORANDUM NO. 25

SUBJECT: The Coupled Impedance Method of Q Measurement

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SUMMARY

This report is a reorganization and condensation of the theory underlying the earlier work of other investigators in the measurement of the Q of a resonant circuit, cavity, or antenna by the coupled impedance method. In this method, impedance measurements at several frequencies are taken on a coil which is coupled to an unknown resonant system, without knowledge of the degree of coupling, and the Q of the unknown system is computed from these measurements.

INTRODUCTION

It is sometimes desirable to obtain the Q of a resonant circuit, an antenna, or a cavity resonator without making any physical connection to it. This can be done by taking measurements at several frequencies (at least two) on a coil or loop which is magnetically coupled to the circuit in question. It is not necessary to know the degree of coupling, provided this is held constant during the measurements. A cavity resonator or antenna can be represented by an equivalent circuit having the same Q as that of the resonator or antenna; as far as the analysis is concerned, this has the advantage that the Q can be expressed in terms of the values of the elements in the equivalent circuit as well as in terms of the stored

and dissipated energies in the unknown distributed system.

CIRCUIT ANALYSIS

Consider the circuit of Fig. 1, where the unknown circuit is represented by the elements L_2, R_2 and C_2. The loop equations are

$$\begin{aligned} \dot{E}_1 &= \dot{Z}_1\dot{I}_1 - \dot{Z}_M\dot{I}_2 \\ 0 &= -\dot{Z}_M\dot{I}_1 + \dot{Z}_2\dot{I}_2 \end{aligned} \qquad (1)$$

These can be combined to show that the impedance $\dot{Z} = \dot{E}_1/\dot{I}_1$ seen by the source is

$$\dot{Z} = \dot{Z}_1 - \dot{Z}_M{}^2/\dot{Z}_2 \qquad (2)$$

where $\dot{Z}_1$ is the impedance of the primary circuit alone and the last term is a coupled impedance

$$\dot{Z}_c = -\dot{Z}_M{}^2/\dot{Z}_2 \qquad (3)$$

In all these equations $\dot{Z}_M$ is equal to $\pm j\omega M$, so that

$$-\dot{Z}_M{}^2 = \omega^2 M^2 \qquad (4)$$

The impedance $\dot{Z}_2$ is the total series impedance of the unknown secondary circuit:

$$\dot{Z}_2 = R_2 + jX_2 \qquad (5)$$

where

$$X_2 = \omega L_2 - 1/\omega C_2 \qquad (6)$$

The coupled impedance given by (3) can be separated into real and imaginary parts:

$$\dot{Z}_c = R_c + jX_c \qquad (7)$$

By substituting (4) and (5) in (3) and rationalizing the result, we obtain

$$R_c = \frac{\omega^2 M^2}{R_2{}^2 + X_2{}^2} R_2 \qquad (8)$$

$$X_c = \frac{-\omega^2 M^2}{R_2{}^2 + X_2{}^2} X_2 = \frac{-X_2}{R_2} R_c \qquad (9)$$

The square of the magnitude of Z_c is equal to the sum of the squares of (8) and (9):

$$Z_c^2 = R_c^2 + X_c^2 = \frac{\omega^4 M^4 (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} \qquad (10)$$

Beginning with this quantity written as an identity

$$\frac{\omega^4 M^4 (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} = \frac{\omega^4 M^4 (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} \qquad (11)$$

we then rearrange the terms to obtain

$$\frac{\omega^4 M^4 X_2^2}{(R_2^2 + X_2^2)^2} + \frac{\omega^4 M^4 R_2^2}{(R_2^2 \ X_2^2)^2} - \frac{\omega^4 M^4 (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} = 0 \qquad (12)$$

We next complete the square of the last two terms by adding the term $\omega^4 M^4/4R_2^2$ to both sides of the equation:

$$\frac{\omega^4 M^4 X_2^2}{(R_2^2 + X_2^2)^2} + \frac{\omega^4 M^4 R_2^2}{(R_2^2 + X_2^2)^2} - \frac{\omega^4 M^4 (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} + \frac{\omega^4 M^4}{4R_2^2} = \frac{\omega^4 M^4}{4R_2^2} \qquad (13)$$

This is then equivalent to

$$\left[\frac{\omega^2 M^2 X_2}{R_2^2 + X_2^2}\right]^2 + \left[\frac{\omega^2 M^2 R_2}{R_2^2 + X_2^2} - \frac{\omega^2 M^2}{2R_2}\right]^2 = \left[\frac{\omega^2 M^2}{2R_2}\right]^2 \qquad (14)$$

By comparison with (8) and (9), we can see that this is the same as

$$\left[X_c\right]^2 + \left[R_c - \frac{\omega^2 M^2}{2R_2}\right]^2 = \left[\frac{\omega^2 M^2}{2R_2}\right]^2 \qquad (15)$$

If ω were constant, (15) would represent a family of circles in the plane of R_c and X_c. Actually, if the Q of the unknown circuit is sufficiently high, most of the significant change in R_c and X_c takes place at values of ω quite close to the resonant value ω_o. Consequently, if (15) is rewritten with ω_o in place of ω, we have an equation

$$\left[X_c\right]^2 + \left[R_c - \frac{\omega_o^2 M^2}{2R_2}\right]^2 = \left[\frac{\omega_o^2 M}{2R_2}\right]^2 \qquad (16)$$

which is approximately correct when Q is high and which has the analytical advantage of representing a family of true circles in the R_c-X_c plane.

One of the family of circles described by (16) is shown in Fig. 2. From inspection of the equation it is evident that all of the circles have centers on the R_c axis and pass through the origin. For each circle there is a point D where $R_c = R_{c\ max}$ and $X_c = 0$; by setting (9) equal to zero, we find that at D

$$X_2 = 0. \qquad (17)$$

For each circle there are two points where $R_c = \left|X_c\right|$, located at the intersections of the circle with the lines $X_c = \pm R_c$. These two points are labelled A and B in Fig. 2. By placing

$$R_c = \left|X_c\right| \qquad (18)$$

in (9), we find that at A and B

$$X_2 = \pm R_2 \qquad (19)$$

Because the secondary is a simple resonant circuit, the usual resonant circuit relationships apply:

$$\left.\begin{array}{lll} X_2 = 0 & \text{when } f = f_o & \\ X_2 = -R_2 & \text{when } f = f_1 & \cong f_o(1 - 1/2Q_2) \\ X_2 = +R_2 & \text{when } f = f_2 & \cong f_o(1 + 1/2Q_2) \end{array}\right\} \qquad (20)$$

$$Q_2 = \frac{f_o}{f_2 - f_1} \qquad (21)$$

Because of (9), (17), and (19), we can identify the frequency f_o with point D, the frequency f_1 with point A, and the frequency f_2 with point B. We then use (21) to find Q_2.

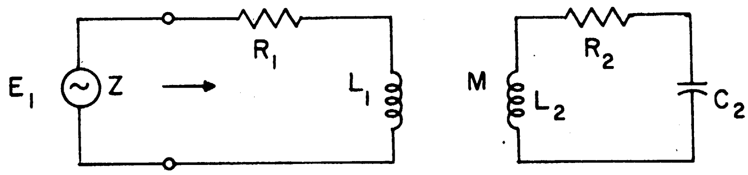

FIG. 1 THE CIRCUIT USED IN THE MEASUREMENT, WITH THE UNKNOWN RESONANT CIRCUIT AT THE RIGHT

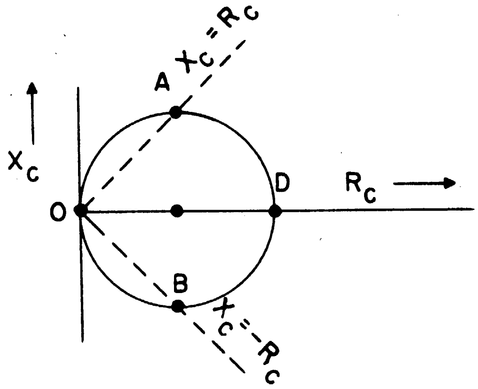

FIG. 2 THE CIRCLE IN THE $R_c - X_c$ PLANE AS REPRESENTED BY (16)

MEASUREMENT PROCEDURE

Based on the foregoing analysis, the measurement procedure consists of the following steps:

1. At each of several frequencies near resonance,
   a. measure the impedance $\dot{Z} = \dot{E}_1/\dot{I}_1$,
   b. measure $\dot{Z}_1$ alone, by reducing M to zero, and
   c. subtract to obtain $\dot{Z}_c$.
2. Plot $\dot{Z}_c$ in the complex impedance plane, as in Fig. 3 or Fig. 4,
   a. identify the frequency f_o where $X_c = 0$ and $R_c = R_c$ max, and
   b. identify the frequencies f_1 and f_2 at points A and B where the curve intersects the lines $X_c = \pm R_c$.
3. Obtain Q_2 from (21).

SIMPLIFIED MEASUREMENT PROCEDURE

It is evident from Fig. 2 that the points A and B previously defined by (18) could have been defined instead as points where $R_c = 0.5\ R_{c\ max}$. In the case where it is known that R_1 (see Fig. 1) is negligible in comparison with R_c, it may be assumed that the resistive component of the measured impedance $\dot{Z}$ is equal to R_c. In this case, reactance values may be ignored entirely, and the frequencies f_1 and f_2 may be defined simply as the frequencies where the resistive component of the measured impedance is equal to half its maximum value. The simplified measurement procedure then becomes:

1. At each of several frequencies, measure the resistive component of the impedance $\dot{Z} = \dot{E}_1/\dot{I}_1$.
2. Plot a curve of this resistance versus frequency and identify the frequencies f_1 and f_2 where the resistance is equal to half of its maximum value.
3. Obtain Q_2 from (21).

NUMERICAL EXAMPLES

Figure 3 shows two curves calculated for particular cases where $Q_2 = 10$ (dashed curve) and $Q_2 = 100$ (solid curve). The calculations are based on the exact equation (15) rather than the true circle equation (16), so that the circles are somewhat distorted. The values of M and R_2 have been selected so that the circles have the same diameters. The same curves are redrawn in the reflection-coefficient plane in Fig. 4; the relation between the two coordinate systems is such that a circle in one coordinate set transforms into a circle in the other. The lines $X_c = \pm R_c$ in Fig. 3 transform into arcs of circles in Fig. 4.

In plotting the points in Fig. 3 and Fig. 4 from (8) and (9), it is convenient first to rewrite these equations in terms of ω/ω_o and Q_2. Since ω_o is the value of ω for which $X_2 = 0$, we use (6) to obtain

$$\omega_o L_2 = 1/\omega_o C_2 \tag{22}$$

In terms of the equivalent circuit values, Q_2 may be defined as

$$Q_2 = \omega_o L_2/R_2 = 1/\omega_o C_2 R_2 \tag{23}$$

We may then rewrite (6) as

$$X_2 = \omega L_2 \omega_o/\omega_o - \omega_o/\omega_o \omega C_2 = Q_2 R_2(\omega/\omega_o - \omega_o/\omega) \tag{24}$$

If (24) is substituted for X_2 in (8) and (9) R_c and X_c can then be written as

$$R_c = \frac{\omega_o^2 M^2}{R_2} \times \frac{(\omega/\omega_o)^2}{1 + \left[Q_2(\omega/\omega_o - \omega_o/\omega)\right]^2} \tag{25}$$

$$X_c = -Q_2(\omega/\omega_o - \omega_o/\omega)\, R_c \tag{26}$$

For simplicity, the curves in Fig. 3 and Fig. 4 are plotted with $\omega_o^2 M^2/R_2$ set equal to unity. In the case $Q_2 = 100$, the following table gives the computed values of R_c and X_c used in plotting:

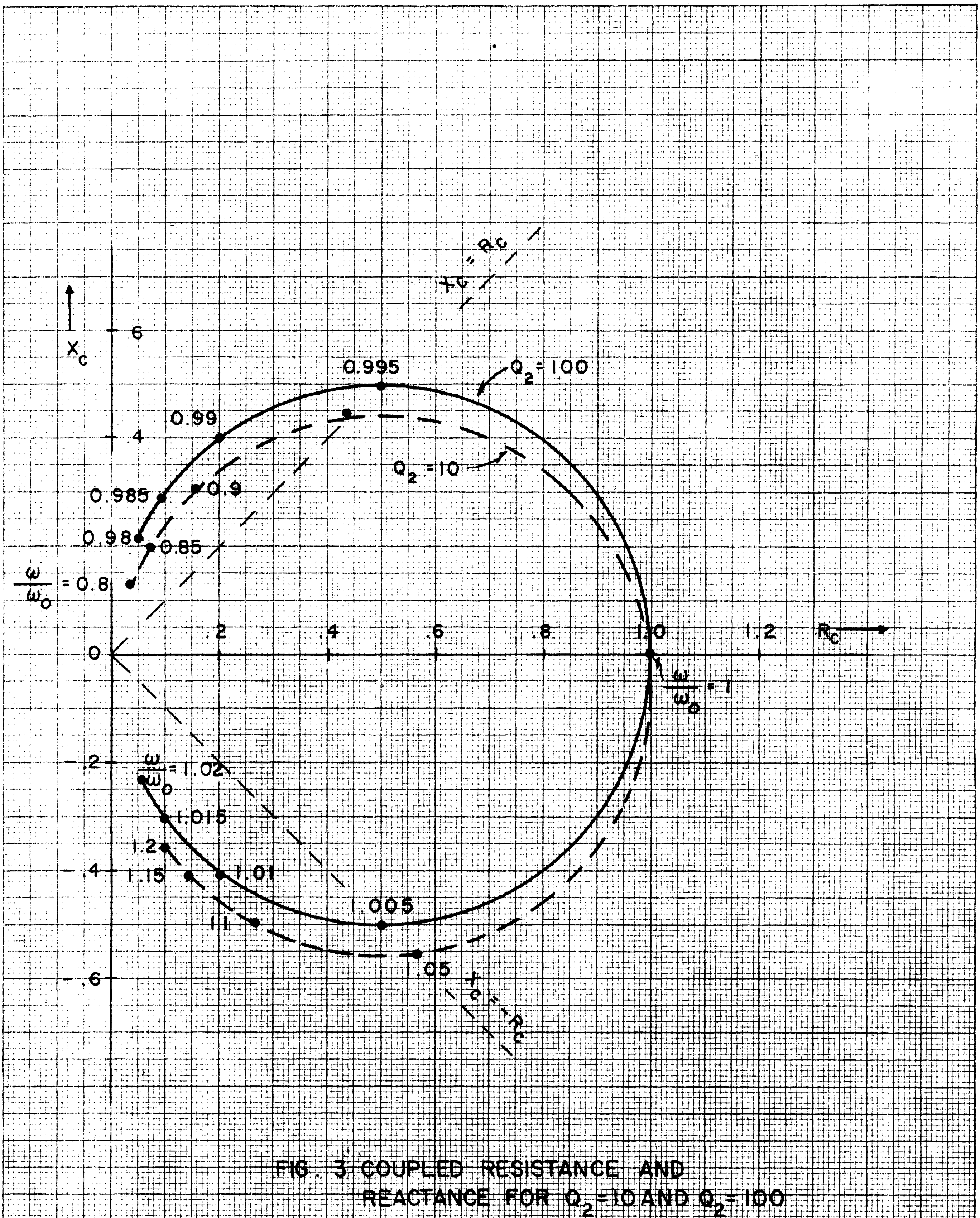

FIG. 3 COUPLED RESISTANCE AND REACTANCE FOR $Q_2 = 10$ AND $Q_2 = 100$

ω/ω_0	0.98	0.985	0.99	0.995	1.0	1.005	1.01	1.015	1.02
R_c	0.057	0.097	0.196	0.495	1.0	0.505	0.206	0.104	0.062
X_c	+0.226	+0.291	+0.392	+0.495	0	-0.505	-0.410	-0.310	-0.245

In the case Q_2 = 10, the computed values are

ω/ω_0	0.8	0.85	0.9	0.95	1.0	1.05	1.1	1.15	1.2
R_c	0.030	0.062	0.149	0.437	1.0	0.564	0.261	0.140	0.100
X_c	+0.135	+0.202	+0.313	+0.450	0	-0.550	-0.498	-0.407	-0.366

Upon inspection of Fig.3 or Fig. 4, we note that the frequency dispersion is about 10 times as great for one curve as it is for the other, as would be expected from the difference in Q. We note further that the points corresponding to the frequencies $f_o(1 \pm 1/2Q)$ for the Q = 100 curve fall almost exactly on the intersections with the $X_c = \pm R_c$ lines, while for the Q = 10 curve there seems to be a slight displacement. The reason for this displacement is that the frequencies $f_o(1 \pm 1/2Q)$ given by (20) are not exact, the error becoming larger as Q becomes smaller.

In connection with the half-maximum resistance points mentioned in the simplified measurement procedure, we note that there is close agreement between these and the frequencies $f_o(1 - 1/2Q)$ for the Q = 100 curve, but that there is a considerable discrepancy in the case of the Q = 10 curve. This discrepancy is caused by the substitution of ω_o for ω in (15) to obtain (16). However, since the frequency displacement is in the same direction and by about the same amount at both A and B, the error in using (21) is quite small, so that the simplified measurement procedure is still quite accurate for values of Q as small as 10.

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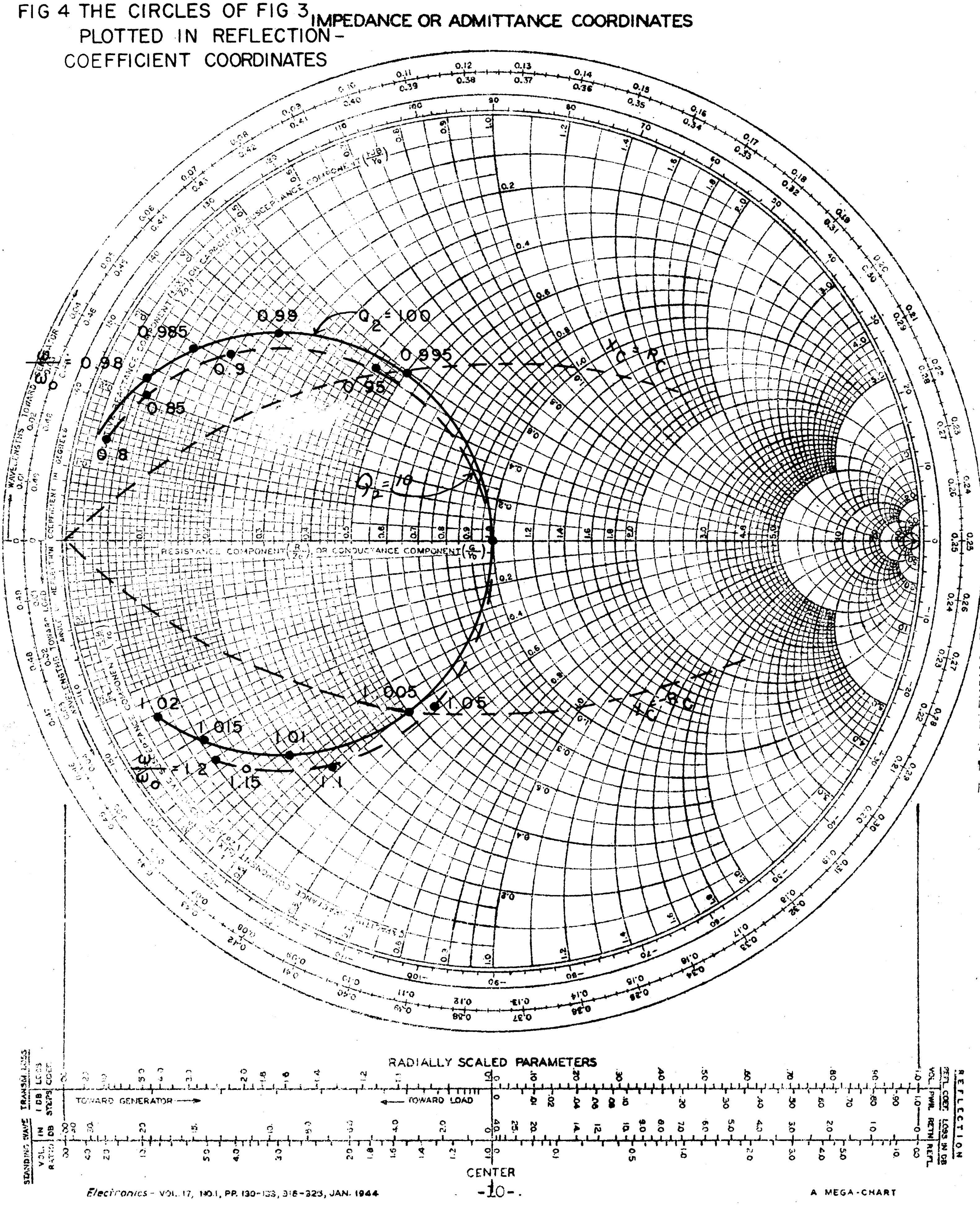
FIG 4 THE CIRCLES OF FIG 3 PLOTTED IN REFLECTION-COEFFICIENT COORDINATES



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