SUBJECT: The Coupled Impedance Method of Q Measurement

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SUMMARY

This report is a reorganization and condensation of the theory underlying the earlier work of other investigators in the measurement of the Q of a resonant circuit, cavity, or antenna by the coupled impedance method. In this method, impedance measurements at several frequencies are taken on a coil which is coupled to an unknown resonant system, without knowledge of the degree of coupling, and the Q of the unknown system is computed from these measurements.

INTRODUCTION

It is sometimes desirable to obtain the Q of a resonant circuit, an antenna, or a cavity resonator without making any physical connection to it. This can be done by taking measurements at several frequencies (at least two) on a coil or loop which is magnetically coupled to the circuit in question. It is not necessary to know the degree of coupling, provided this is held constant during the measurements. A cavity resonator or antenna can be represented by an equivalent circuit having the same Q as that of the resonator or antenna; as far as the analysis is concerned, this has the advantage that the Q can be expressed in terms of the values of the elements in the equivalent circuit as well as in terms of the stored
and dissipated energies in the unknown distributed system.

CIRCUIT ANALYSIS

Consider the circuit of Fig. 1, where the unknown circuit is represented by the elements $L_2$, $R_2$ and $C_2$. The loop equations are

$$\begin{align*}
\dot{E}_1 &= Z_1 I_1 - Z_M I_2 \\
0 &= -Z_M I_1 + Z_2 I_2
\end{align*}$$

(1)

These can be combined to show that the impedance $Z = \frac{E_1}{I_1}$ seen by the source is

$$Z = Z_1 - Z_M^2 Z_2$$

(2)

where $Z_1$ is the impedance of the primary circuit alone and the last term is a coupled impedance

$$Z_c = -Z_M^2 Z_2$$

(3)

In all these equations $Z_M$ is equal to $+j\omega M$, so that

$$Z_M^2 = \omega^2 M^2$$

(4)

The impedance $Z_2$ is the total series impedance of the unknown secondary circuit:

$$Z_2 = R_2 + jX_2$$

(5)

where

$$X_2 = \omega L_2 - \frac{1}{\omega C_2}$$

(6)

The coupled impedance given by (3) can be separated into real and imaginary parts:

$$Z_c = R_c + jX_c$$

(7)

By substituting (4) and (5) in (3) and rationalizing the result, we obtain

$$R_c = \frac{\omega^2 M^2}{R_2^2 + X_2^2} R_2$$

(8)

$$X_c = -\frac{\omega^2 M^2}{R_2^2 + X_2^2} X_2 = \frac{-X_2}{R_2} R_c$$

(9)
The square of the magnitude of $Z_c$ is equal to the sum of the squares of (8) and (9):

$$Z_c^2 = R_c^2 + X_c^2 = \frac{\omega \mu h (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2}$$

(10)

Beginning with this quantity written as an identity

$$\frac{\omega \mu h (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} = \frac{\omega \mu h (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2}$$

(11)

we then rearrange the terms to obtain

$$\frac{\omega \mu h X_2^2}{(R_2^2 + X_2^2)^2} + \frac{\omega \mu h R_2^2}{(R_2^2 + X_2^2)^2} - \frac{\omega \mu h (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} = 0$$

(12)

We next complete the square of the last two terms by adding the term $\frac{\omega \mu h}{4R_2^2}$ to both sides of the equation:

$$\frac{\omega \mu h X_2^2}{(R_2^2 + X_2^2)^2} + \frac{\omega \mu h R_2^2}{(R_2^2 + X_2^2)^2} - \frac{\omega \mu h (R_2^2 + X_2^2)}{(R_2^2 + X_2^2)^2} + \frac{\omega \mu h}{4R_2^2} = \frac{\omega \mu h}{4R_2^2}$$

(13)

This is then equivalent to

$$\left[\frac{\omega \mu h X_2^2}{R_2^2 + X_2^2}\right]^2 + \left[\frac{\omega \mu h R_2^2}{R_2^2 + X_2^2} - \frac{\omega \mu h}{2R_2}\right]^2 = \left[\frac{\omega \mu h}{2R_2}\right]^2$$

(14)

By comparison with (8) and (9), we can see that this is the same as

$$\left[X_c\right]^2 + \left[R_c - \frac{\omega \mu h}{2R_2}\right]^2 = \left[\frac{\omega \mu h}{2R_2}\right]^2$$

(15)

If $\omega$ were constant, (15) would represent a family of circles in the plane of $R_c$ and $X_c$. Actually, if the $Q$ of the unknown circuit is sufficiently high, most of the significant change in $R_c$ and $X_c$ takes place at values of $\omega$ quite close to the resonant value $\omega_0$. Consequently, if (15) is rewritten with $\omega_0$ in place of $\omega$, we have an equation
\[
\left[ X_c \right]^2 + \left[ R_c - \frac{\omega_0^2 M^2}{2R_2} \right]^2 = \left[ \frac{\omega_0^2 M}{2R_2} \right]^2
\]  
(16)

which is approximately correct when \( Q \) is high and which has the analytical advantage of representing a family of true circles in the \( R_c - X_c \) plane.

One of the family of circles described by (16) is shown in Fig. 2. From inspection of the equation it is evident that all of the circles have centers on the \( R_c \) axis and pass through the origin. For each circle there is a point \( D \) where \( R_c = R_{c \text{ max}} \) and \( X_c = 0 \); by setting (9) equal to zero, we find that at \( D \)

\[
X_2 = 0.
\]  
(17)

For each circle there are two points where \( R_c = |X_c| \), located at the intersections of the circle with the lines \( X_c = \pm R_c \). These two points are labelled A and B in Fig. 2. By placing

\[
R_c = |X_c|
\]  
(18)

in (9), we find that at A and B

\[
X_2 = \pm R_2
\]  
(19)

Because the secondary is a simple resonant circuit, the usual resonant circuit relationships apply:

\[
\begin{align*}
X_2 &= 0 \quad \text{when } f = f_0 \\
X_2 &= -R_2 \quad \text{when } f = f_1 \approx f_0(1 - 1/2Q_2) \\
X_2 &= +R_2 \quad \text{when } f = f_2 \approx f_0(1 + 1/2Q_2)
\end{align*}
\]  
(20)

\[
Q_2 = \frac{f_0}{f_2 - f_1}
\]  
(21)

Because of (9), (17), and (19), we can identify the frequency \( f_0 \) with point D, the frequency \( f_1 \) with point A, and the frequency \( f_2 \) with point B. We then use (21) to find \( Q_2 \).
FIG. 1 THE CIRCUIT USED IN THE MEASUREMENT, WITH THE UNKNOWN RESONANT CIRCUIT AT THE RIGHT

FIG. 2 THE CIRCLE IN THE $R_C-X_C$ PLANE AS REPRESENTED BY (16)
MEASUREMENT PROCEDURE

Based on the foregoing analysis, the measurement procedure consists of the following steps:

1. At each of several frequencies near resonance,
   a. measure the impedance $Z = E_1/I_1$,
   b. measure $Z_1$ alone, by reducing $M$ to zero, and
   c. subtract to obtain $Z_c$.

2. Plot $Z_c$ in the complex impedance plane, as in Fig. 3 or Fig. 4,
   a. identify the frequency $f_0$ where $X_c = 0$ and $R_c = R_c$ max, and
   b. identify the frequencies $f_1$ and $f_2$ at points A and B where
      the curve intersects the lines $X_c = \pm R_c$.

3. Obtain $Q_2$ from (21).

SIMPLIFIED MEASUREMENT PROCEDURE

It is evident from Fig. 2 that the points A and B previously defined
by (18) could have been defined instead as points where $R_c = 0.5 R_c$ max.
In the case where it is known that $R_1$ (see Fig. 1) is negligible in
comparison with $R_c$, it may be assumed that the resistive component of
the measured impedance $Z$ is equal to $R_c$. In this case, reactance values may
be ignored entirely, and the frequencies $f_1$ and $f_2$ may be defined simply as
the frequencies where the resistive component of the measured impedance is
equal to half its maximum value. The simplified measurement procedure then
becomes:

1. At each of several frequencies, measure the resistive component of the impedance $Z = E_1/I_1$.

2. Plot a curve of this resistance versus frequency and identify the frequencies $f_1$ and $f_2$ where the resistance is equal to half of its maximum value.

3. Obtain $Q_2$ from (21).
Figure 3 shows two curves calculated for particular cases where \( Q_2 = 10 \) (dashed curve) and \( Q_2 = 100 \) (solid curve). The calculations are based on the exact equation (15) rather than the true circle equation (16), so that the circles are somewhat distorted. The values of \( M \) and \( R_2 \) have been selected so that the circles have the same diameters. The same curves are redrawn in the reflection-coefficient plane in Fig. 4; the relation between the two coordinate systems is such that a circle in one coordinate set transforms into a circle in the other. The lines \( X_c = \pm R_c \) in Fig. 3 transform into arcs of circles in Fig. 4.

In plotting the points in Fig. 3 and Fig. 4 from (8) and (9), it is convenient first to rewrite these equations in terms of \( \omega/\omega_0 \) and \( Q_2 \). Since \( \omega_0 \) is the value of \( \omega \) for which \( X_2 = 0 \), we use (6) to obtain

\[
\frac{\omega_0 L_2}{R_2} = \frac{1}{\omega_0 C_2} \quad (22)
\]

In terms of the equivalent circuit values, \( Q_2 \) may be defined as

\[
Q_2 = \frac{\omega_0 L_2}{R_2} = \frac{1}{\omega_0 C_2 R_2} \quad (23)
\]

We may then rewrite (6) as

\[
X_2 = \frac{\omega L_2}{\omega_0} - \frac{\omega_0}{\omega_0} \frac{\omega}{\omega_0} C_2 = Q_2 \frac{R_2}{R_2} (\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega_0}) \quad (24)
\]

If (24) is substituted for \( X_2 \) in (8) and (9) \( R_c \) and \( X_c \) can then be written as

\[
R_c = \frac{\omega_0^2 M^2}{R_2} x \frac{(\omega/\omega_0)^2}{1 + \left[Q_2 (\omega/\omega_0 - \omega_0/\omega)\right]^2} \quad (25)
\]

\[
X_c = -Q_2 (\omega/\omega_0 - \omega_0/\omega) R_c \quad (26)
\]

For simplicity, the curves in Fig. 3 and Fig. 4 are plotted with \( \omega_0^2 M^2/R_2 \) set equal to unity. In the case \( Q_2 = 100 \), the following table gives the computed values of \( R_c \) and \( X_c \) used in plotting:
FIG. 3 COUPLED RESISTANCE AND REACTANCE FOR $Q_2 = 10$ AND $Q_2 = 100$
\[\frac{\omega}{\omega_0} \quad 0.98 \quad 0.985 \quad 0.99 \quad 0.995 \quad 1.0 \quad 1.005 \quad 1.01 \quad 1.015 \quad 1.02
\]
\[R_c \quad 0.057 \quad 0.097 \quad 0.196 \quad 0.495 \quad 1.0 \quad 0.505 \quad 0.206 \quad 0.104 \quad 0.062
\]
\[X_c \quad +0.226 \quad +0.291 \quad +0.392 \quad +0.495 \quad 0 \quad -0.505 \quad -0.410 \quad -0.310 \quad -0.245
\]

In the case \(Q_2 = 10\), the computed values are

\[\frac{\omega}{\omega_0} \quad 0.8 \quad 0.85 \quad 0.9 \quad 0.95 \quad 1.0 \quad 1.05 \quad 1.1 \quad 1.15 \quad 1.2
\]
\[R_c \quad 0.030 \quad 0.062 \quad 0.149 \quad 0.437 \quad 1.0 \quad 0.564 \quad 0.261 \quad 0.140 \quad 0.100
\]
\[X_c \quad +0.135 \quad +0.202 \quad +0.313 \quad +0.450 \quad 0 \quad -0.550 \quad -0.498 \quad -0.407 \quad -0.366
\]

Upon inspection of Fig. 3 or Fig. 4, we note that the frequency dispersion is about 10 times as great for one curve as it is for the other, as would be expected from the difference in \(Q\). We note further that the points corresponding to the frequencies \(f_0(1 \pm 1/2Q)\) for the \(Q = 100\) curve fall almost exactly on the intersections with the \(X_c = \pm R_c\) lines, while for the \(Q = 10\) curve there seems to be a slight displacement. The reason for this displacement is that the frequencies \(f_0(1 \pm 1/2Q)\) given by (20) are not exact, the error becoming larger as \(Q\) becomes smaller.

In connection with the half-maximum resistance points mentioned in the simplified measurement procedure, we note that there is close agreement between these and the frequencies \(f_0(1 - 1/2Q)\) for the \(Q = 100\) curve, but that there is a considerable discrepancy in the case of the \(Q = 10\) curve. This discrepancy is caused by the substitution of \(\omega_0\) for \(\omega\) in (15) to obtain (16). However, since the frequency displacement is in the same direction and by about the same amount at both A and B, the error in using (21) is quite small, so that the simplified measurement procedure is still quite accurate for values of \(Q\) as small as 10.
FIG 4 THE CIRCLES OF FIG 3, IMPEDANCE OR ADMITTANCE COORDINATES PLOTTED IN REFLECTION-COEFFICIENT COORDINATES
REFERENCES


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