THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING
Department of Chemical and Metallurgical Engineering
Multi-phase Fluids Laboratory

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A MATHEMATICAL MODEL FOR THE POPPET NOZZLE

K. H. Coats
J. L. York
M. R. Tek

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NOMENCLATURE

a_o	radius of pintle
A	area
b_o	radius of sleeve
C_o	orifice coefficient
c	viscous damping coefficient for dashpot
D	diameter
F	force on pintle
F_o	force exerted by spring when nozzle is closed
f	frequency of pintle vibration
k	spring constant
lw	lost work
L	length of contact between dashpot piston and cylinder
m	mass of pintle plus moving parts
p	pressure
Δp	$p_1 - p_2$ or $p_2 - p_1$
q	nozzle flow rate
q_e	average nozzle flow rate
Re	Reynolds number at cross section 2
t	time
u	fluid velocity
x	pintle opening
x_e	equilibrium or average position of pintle

Subscripts

1	cross section 1 (see Fig. 1 of report 2931-1-P)
2	cross section 2 (see Fig. 1 of report 2931-1-P)
i	inner
o	outer

Greek

θ	angle of conical pintle head
ρ	density of flowing liquid
μ	viscosity of flowing liquid

I. INTRODUCTION

A mathematical model for the poppet nozzle was formulated and presented in the previous progress report 2931-1-P of September, 1959. This report presents results pertinent to the extension, refinement, and application of this model. First, the model has been adapted to treat the case of a nozzle with a dashpot. Second, a test nozzle with two different pintles has been built and tested to determine the orifice coefficient (or lost work) as a function of pintle opening and Reynolds number. The manner in which the orifice coefficient is incorporated into the equations governing nozzle performance is shown. Finally, a comparison is given between the actual performance of a nozzle as determined experimentally by Mr. Richard Wilcox of Delavan and the performance predicted by the mathematical model.

Throughtout this report the word "poppet" is used in a descriptive sense, referring to the type of nozzle. The word "pintle" is employed to designate the central component of the nozzle, the moving rod and its integral head piece.

II. SUMMARY OF RESULTS

Calculations have been carried out on the IBM 704 digital computer to determine the performance of a spring-loaded poppet nozzle with a dashpot. The results show that a stable, nonoscillatory pintle position is attained almost instantaneously for a sufficiently large dashpot viscous damping coefficient. Figure 1 shows pintle position and flow rate as a function of time for a nozzle operating at an upstream pressure of 480 psia and having a dashpot damping coefficient of .05 pound force-second per inch. An essentially stable pintle opening and flow rate is attained within .04 sec after start-up. Figure 2 presents the calculated performance of the same nozzle with the exception of the viscous damping coefficient, which has been increased to .3 pound force-second per inch. The same equilibrium values of pintle opening and flow rate are attained considerably faster due to the larger damping coefficient.

The relationship between equilibrium flow rate and pressure drop across the nozzle is presented in Fig. 3. This plot shows that the flow rate in gallons per minute increases with Δp at a rate slightly greater than that corresponding to a straight-line relationship. All the results plotted on Figs. 1, 2, and 3 were calculated with the assumption of negligible lost work.

The important conclusion to be drawn from these results is that a nonoscillatory pintle position can be attained for each upstream pressure p_1 if a dashpot with a sufficiently large damping coefficient is built into the nozzle. The stable pintle position means, of course, that wear due to collision between the pintle head and seat or sleeve is avoided.

A relation has been derived giving the viscous damping coefficient as a function of design dimensions of the dashpot. This relation

$$c = \frac{8\pi \mu L D_i^4}{g_c (D_o-D_i)^2 (D_o^2-D_i^2)} \tag{1}$$

should provide a basis for design of an effective dashpot. The terms in Eq. (1) are defined in the nomenclature and in Fig. 4, a sketch of the dashpot. If a dashpot piston with a rounded edge is to be employed to avoid sticking, Eq. (1) will not be strictly accurate and the value of c should in that case be determined experimentally.

A test nozzle, sketched in Fig. 5, has been built and tested to determine the orifice coefficient for poppet nozzles as a function of Reynolds number. This coefficient is defined by the relation

$$lw + \frac{\Delta p}{\rho} = C_o^2 \frac{\Delta p}{\rho} , \tag{2}$$

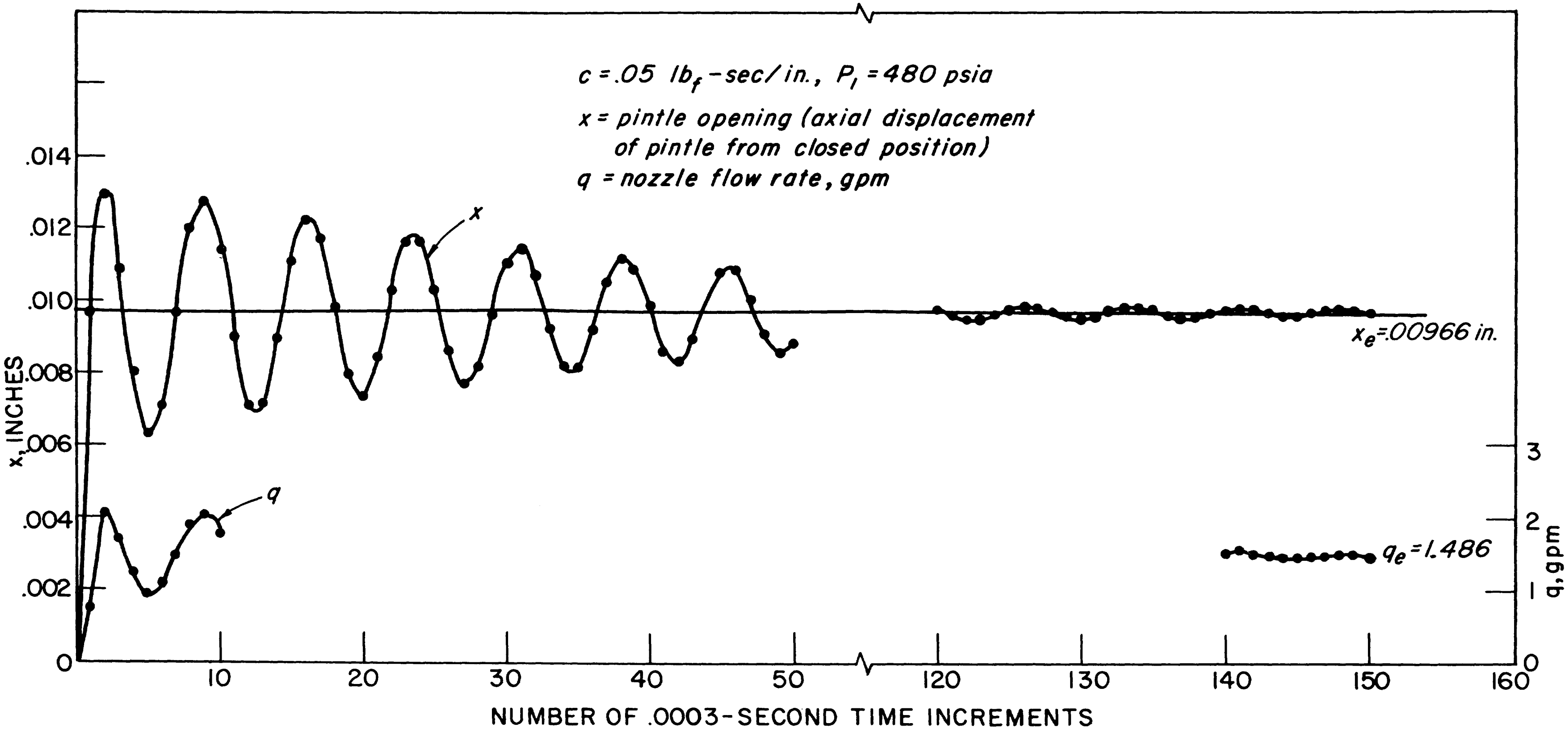

Fig. 1. Pintle opening and nozzle flow rate vs. time for nozzle with dashpot.

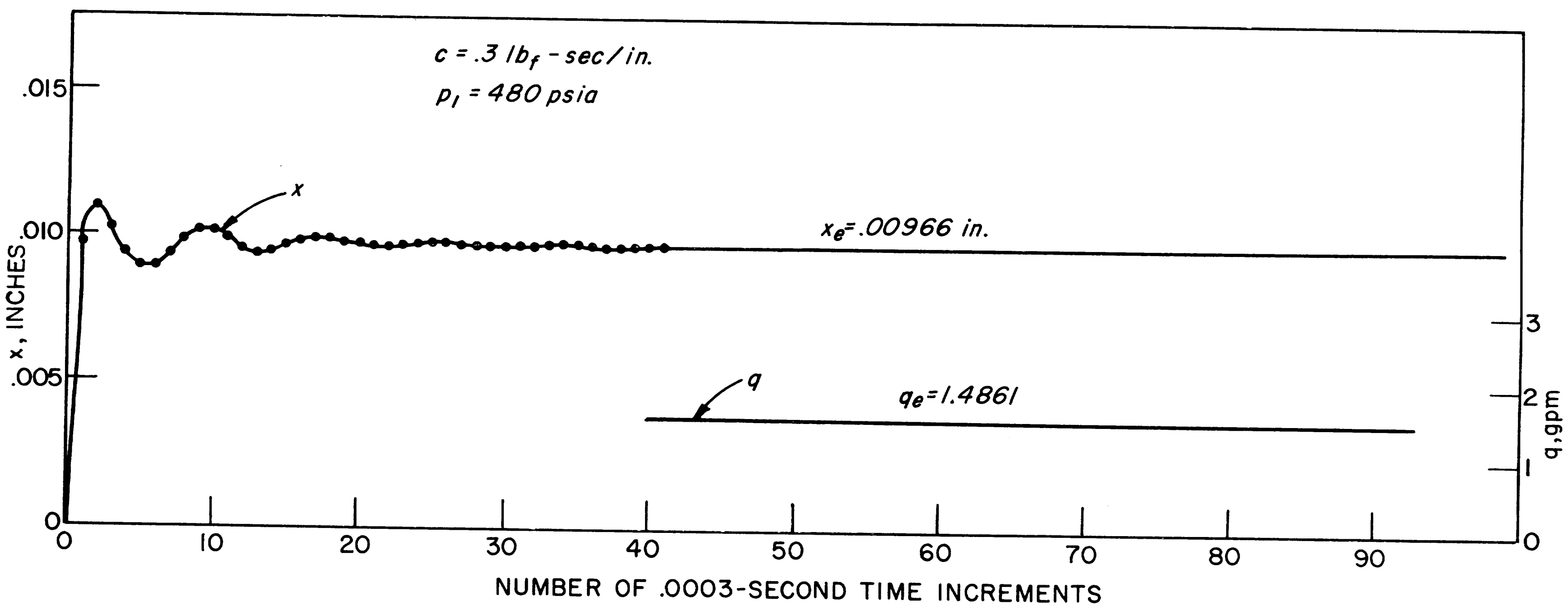

Fig. 2. Pintle opening and nozzle flow rate vs. time for nozzle with dashpot.

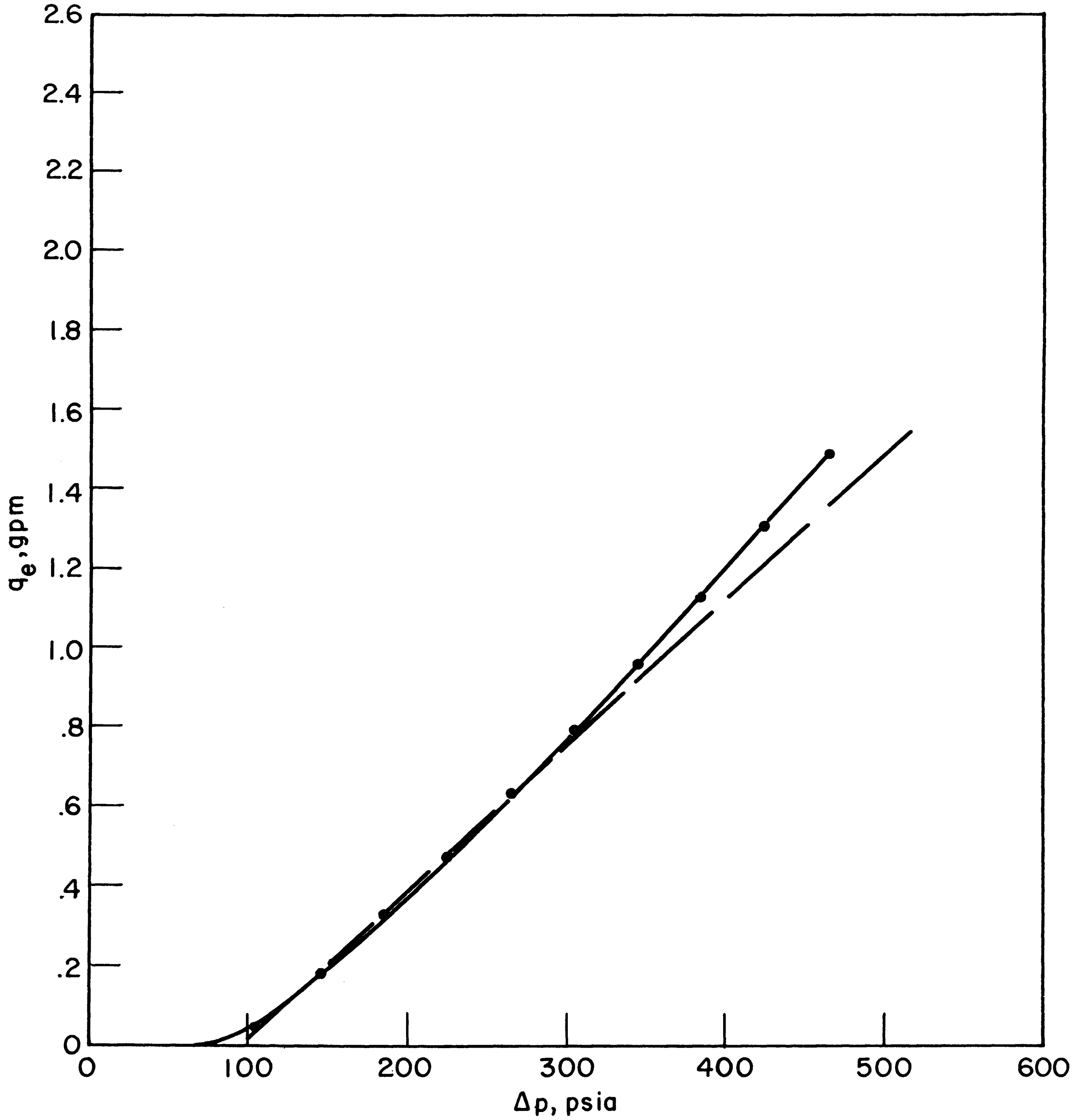

Fig. 3. Flow rate vs. Δp for spring-loaded poppet nozzle with dashpot.

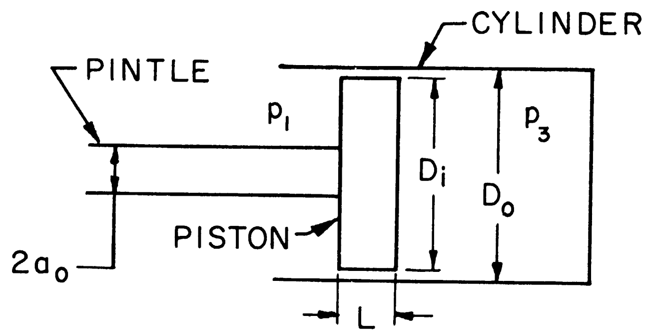

Fig. 4. Sketch of dashpot.

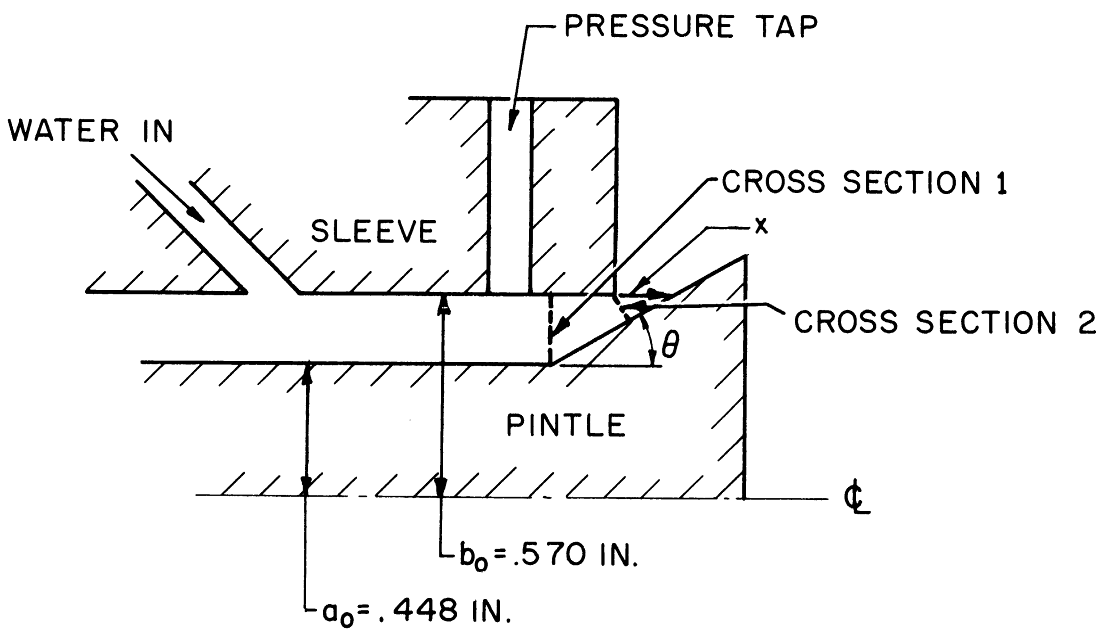

Fig. 5. Sketch of test nozzle.

where lw is the energy lost by friction and Δp is the pressure drop through the orifice. The experimentally determined values of C_o are plotted in Figs. 6 and 7. The plots show that C_o is a function of pintle opening x and pintle head angle θ as well as Reynolds number. More data need to be taken in the low Reynolds number region to determine whether C_o tends toward unity or zero (as does C_o for a fixed area orifice) at small Re.

The results given in Figs. 6 and 7 indicate that C_o decreases (i.e., lost work increases) as the angle of the pintle head increases (x and Re held constant) and that C_o decreases as the pintle opening x increases. The data, although incomplete, yield the important fact that lost work in the poppet nozzle orifice is not negligible, i.e., the orifice coefficient C_o is significantly less than 1.0.

Delavan nozzle DMC 2360 has been tested by Mr. Richard Wilcox and the flow rate obtained as a function of pressure drop. The design dimensions and other necessary data were employed along with the equations developed in report 2931-1-P to calculate the performance of this nozzle. The predicted and observed flow rates are compared in Fig. 8. The results show that, when lost work is assumed to be negligible ($C_o = 1.0$), the predicted flow rates are considerably above the observed values. The experimental data on lost work discussed above show that a relation $C_o = a - bx$ gives a qualitatively correct relationship for C_o. When the assumed relation $C_o = .61 - 12x$ is employed in the mathematical model, good agreement is obtained between the observed and predicted flow rates, as shown in Fig. 8. A frequency of 380-390 cps was recorded on the sonic analyzer during the testing of the nozzle. The frequency of pintle vibration can be predicted from Eq. (3) of report 2931-1-P. The slope of the F vs x curve must be calculated from the design dimensions of the nozzle. The slope for nozzle DMC 2360 was calculated by the 704 computer as 113 pounds force per inch; this slope yields a calculated frequency of 390 cps, in good agreement with the observed 380-390 cps.

During the testing of the nozzle a loud noise and erratic spray was reported at a flow rate of 288 lb/hr and upstream pressure of about 300 psig. The computer calculations for the nozzle showed that the conical pintle head would become completely exposed (beyond the seating or sleeve) at a pressure between 250 and 300 psig. At this large pintle opening, the bearing or positioning ring (ridge) on the pintle will come in line with the fuel entry ports or holes, as is evident from Fig. 9. Thus the noise and erratic spray at the upper pressure range of operation could be due to pintle oscillations induced by partial blocking of the fuel entry holes by the pintle positioning ring. If this is the cause of the trouble, then moving the positioning ridge back farther on the pintle should remedy the situation.

A nozzle of second design was tested by Mr. Wilcox and within a certain flowrate range the conical pintle head was observed sliding around the periphery of the orifice, yielding a nonuniform spray. This situation could be remedied either by reducing the clearance at the pintle positioning surfaces (with attending problem of inducing "sticking") or by increasing the axial distance between the positioning surfaces (see Fig. 9). The latter is suggested as a feasible method of keeping the pintle centered during operation.

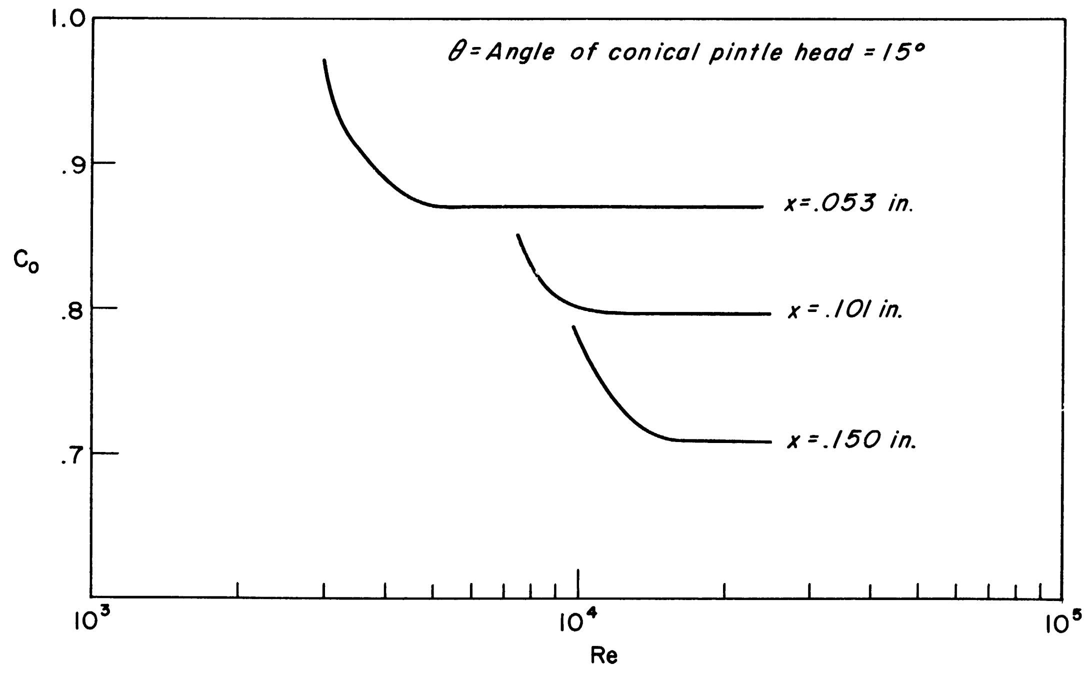

Fig. 6. Orifice coefficient C_o vs. Reynolds number Re.

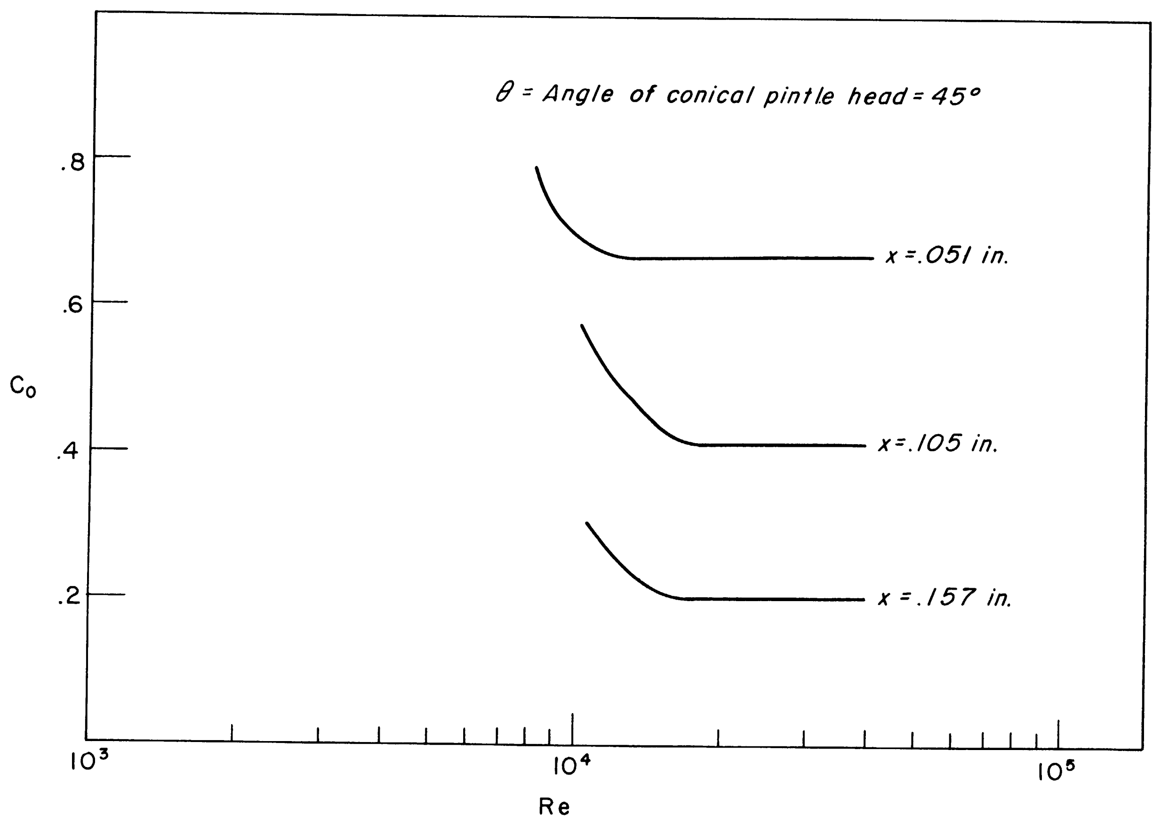

Fig. 7. Orifice coefficient C_o vs. Reynolds number Re.

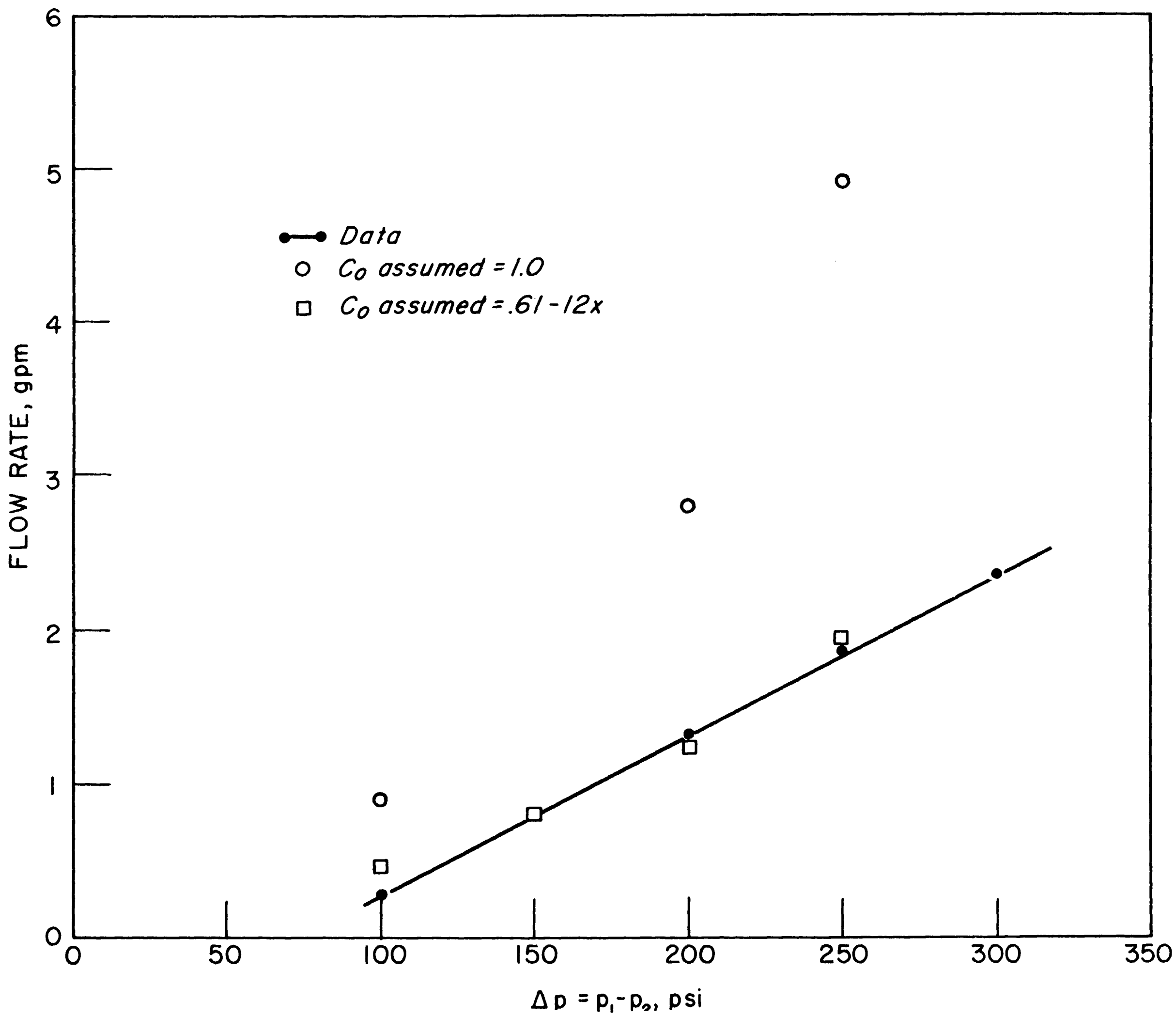

Fig. 8. Flow rate vs. pressure drop for Delavan nozzle DMC 2360.

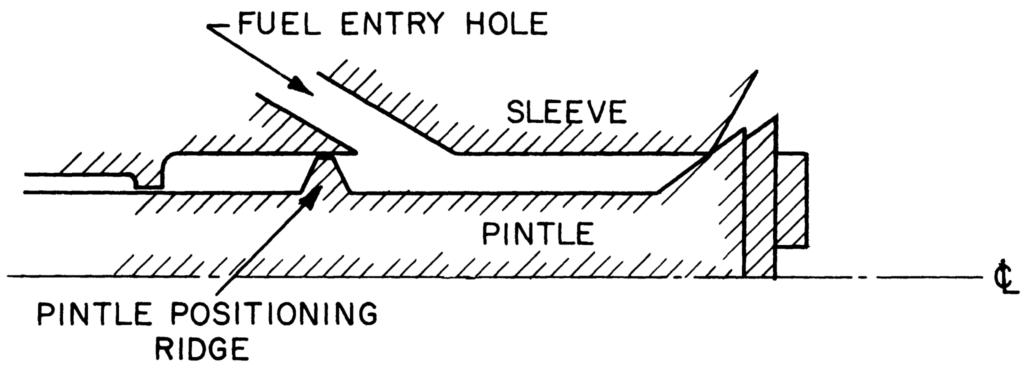

Fig. 9. Sketch of test nozzle.

III. REVIEW OF PROGRESS AND RECOMMENDATIONS FOR FUTURE WORK

A mathematical model has been formulated for the poppet nozzle which allows prediction of amplitude and frequency of pintle vibration and of flow rate as a function of pressure drop. The model shows that pintle vibration is inevitable for a spring-loaded poppet nozzle operating without a dashpot. The model has been adapted to treat the case of a nozzle with a dashpot and shows that a stable nonoscillatory pintle position is attained very quickly for a sufficiently large damping coefficient. A relation has been derived as a guide in designing an effective dashpot. Calculated curves of flow rate versus pressure drop yield a flow rate which increases with Δp at a greater rate than that corresponding to a straight-line relationship. This agrees well qualitatively with Delavan data for various nozzles.

Experimental data taken at Michigan show that the orifice coefficient C_o is significantly less than unity for the poppet nozzle in the operating range of Reynolds number. This necessity of accounting for the lost work is confirmed by application of the mathematical model to an actual test case: ignoring the lost work results in poor agreement between observed and predicted performance while choice of a realistic C_o less than unity produces close agreement between observed and predicted nozzle flow rates.

Concerning recommended future work, the major difficulty remaining in the development of an accurate mathematical method of predicting nozzle performance is the relationship between orifice coefficient C_o and Reynolds number. At present, insufficient data have been taken in the low Reynolds number range. Since C_o seems to be a function of pintle opening and conical head angle as well as Re, a correlative problem of significant magnitude exists. Once C_o has been correlated, a step-by-step outline, aided to maximum extent by generalized tables and charts, of the method of calculation for prediction of nozzle performance should be formulated. This method of calculation would allow determination of a nozzle's performance without building and testing it and would allow selection of one of several proposed nozzles on a basis of optimum performance.

The very important intermediate step between determination of C_o and formulation of the general method of calculation is of course the testing of the mathematical method by application to actual test cases. The excellent data already gathered by Mr. Wilcox on the DMC 2360 nozzle are sufficient for partial evaluation of the calculational method.

The feasibility of designing an effective dashpot to obtain a stable pintle opening and of lengthening the distance between pintle positioning surfaces should be investigated.

In the formulation of a long-range research and development program concerning variable-area devices, radical departures from the current design such as pneumatic control of pintle position, hydraulic centering through swirl flow, and magnetic pintle-centering devices should be entertained.

IV. PREDICTED PERFORMANCE OF NOZZLE WITH DASHPOT

The predicted performance of a spring-loaded poppet nozzle without a dashpot was presented in the previous report 2931-1-P. The calculated results showed that this type of nozzle will operate with an oscillatory pintle vibration of definite amplitude and frequency. The question arises as to whether introduction of a dashpot into the nozzle assembly will eliminate the oscillatory pintle movement and, if so, how such a dashpot should be designed. The dashpot has been taken into account mathematically by adding a force on the pintle proportional to its velocity; that is,

$$\text{Force}_{\text{dashpot}} = - c \frac{dx}{dt} ,$$

where c is the dashpot viscous damping coefficient and dx/dt is the velocity of the pintle.

The values of design dimensions and parameters* used in all calculations reported in this section are:

a_o = .055 in.
b_o = .07 in.
F_o = .36 pound force
m = .0003 slug
c = .05, .3, or .6 lb force-sec/in., as noted
p_2 = 14.7 psia
k = 120 lb force/in.
θ = 25°
p_1 = as noted

Figure 1 presents the calculated pintle displacement and nozzle flow rate as a function of time for a damping coefficient of c = .05 pound force-sec/in. and p_1 = 480 psia. The results were calculated by a numerical solution of the equation governing the pintle motion. This equation (I-3)** and its finite difference equivalents (I-5) and (I-6) are derived in Appendix I. The oscillatory pintle motion is seen to damp out rapidly, establishing a steady-state pintle opening of x_e = .0096 in. The term x_e denotes the pintle opening at which the fluid pressure forces exactly counterbalance the spring force so that the total force on the pintle is zero. A steady state is attained essentially at about

*Nomenclature and geometry of the nozzle are the same in this report as in report 2931-1-P. The reader is referred to page 1 and Fig. 1 of that report for this information, as well as to page vii of this report.

**(I-3) means Appendix I, Eq. (3).

time t = .04 sec. The nozzle flow rate attains a steady value of 1.486 gallons per minute.

The results plotted in Fig. 2 are for a nozzle identical to that for which the results in Fig. 1 were calculated, except that the viscous damping coefficient has been increased from .05 to .3. The steady-state pintle opening and flow-rate values are identical, but the oscillatory motion is damped much more rapidly.

Figure 3 presents the equilibrium or steady-state values of flow rate q as a function of pressure drop $\Delta p = p_1 - p_2$. These results are valid for any non-zero damping coefficient c since the magnitude of c affects only the rate of damping of the oscillatory motion or rate of approach to steady state. Actually the digital computer results showed no change in the fourth decimal place of x after t = about .012 sec for c = .3 and t = about .006 sec for c = .6. Figure 3 shows that flow rate increases with Δp at a slightly greater rate than that corresponding to a straight-line relationship.

A derivation given in Appendix II relates the viscous damping coefficient c to dashpot design dimensions as

$$c = \frac{8\pi \ \mu \ L \ D_i^4}{g_c \ (D_o - D_i)^2 \ (D_o^2 - D_i^2)}$$

where μ is the liquid viscosity and D_o, D_i, and L are as noted in Fig. 4. This relation assumes laminar liquid flow through the annular region between the dashpot piston and cylinder wall; it should give an accurate estimation of c for known L, D_o, and D_i, and thus give some basis for design of an effective dashpot.

V. EXPERIMENTAL DETERMINATION OF ORIFICE COEFFICIENTS

Whenever a viscous fluid flows through a pipe, some energy is lost from the fluid due to friction at the pipe wall. The increased fluid velocity in the region of a constriction in the pipe amplifies the lost energy or lost work. Such a constriction theoretically results in a conversion from pressure energy to kinetic energy; in actual practice a portion of the pressure energy is lost through friction, the remainder appearing as increased kinetic energy of the fluid.

The lost work or energy is usually accounted for by a coefficient in the case of orifices, as

$$lw + \frac{\Delta p}{\rho} = C_o^2 \frac{\Delta p}{\rho} , \qquad (2)$$

where lw is the energy lost by friction, Δp is the pressure drop through the orifice, and C_o is the "orifice coefficient." This coefficient appears in the equation relating velocity and pressure drop as

$$u_1 = C_o \sqrt{\frac{2g_c(-\Delta p)/\rho}{(A_1/A_2)^2-1}} , \qquad (3)$$

where u_1 is the fluid velocity at the entrance to the constriction, ρ is fluid density, and A_1 and A_2 are the flow areas at the entrance and exit, respectively, of the constriction. Since u_1 can be directly related to the flow rate in gallons per minute through the orifice, C_o can be easily calculated if Δp and the flow rate are measured and the dimensions necessary to calculate A_1 and A_2 are known.

A nozzle, sketched in Fig. 5, has been built and tested for the purpose of determining C_o for converging annular orifices peculiar to poppet nozzles. Two pintles were employed with the same sleeve, one with $\theta = 15°$, the other with $\theta = 45°$. The ends of the pintles were threaded so that the pintle could be set at any desired opening. The body of the nozzle was threaded to fit a 2-in. pipe which began at a water line and connected to a gear pump and rotameter. A pressure gage was fitted to the nozzle as shown in Fig. 5.

The data were taken in the following manner. The pintle was screwed out to an opening which was measured with a depth gage and recorded as x. Valves in the water line were set to give a range of flow rates. At each valve setting, the flow rate was read from the rotameter and the pressure at the nozzle was read and recorded. The pintle opening was then changed and recorded and flow rates and corresponding nozzle pressures were again recorded. The pressure gage and rotameter were calibrated prior to insertion into the apparatus, and depth-gage readings on the pintle position were taken both before and after each run. For low-pressure runs, the pressure gage was replaced by a mercury manometer.

The values of C_o for each run were calculated from Eq. (3), where

$$A_1 = \pi(b_o^2 - a_o^2),$$

$$A_2 = \pi x \sin\theta (2b_o - x \tan\theta), \text{ and}$$

$$u_1 = q(\text{gpm}) (8.33/A_1\rho).$$

The orifice coefficient C_o is plotted in Figs. 6 and 7 versus Reynolds number where Reynolds number is evaluated at cross section 2 (see Fig. 2 of report 2931-1-P) as

$$Re = \frac{D_2 u_2 \rho}{\mu} = C_1 \frac{q}{2\pi b_o - \pi x \sin\theta \cos\theta} \frac{1}{\mu},$$

where C_1 is a conversion constant to obtain a dimensionless Reynolds number, q is in gpm, b_o and x in inches and μ in centipoises.

VI. COMPARISON BETWEEN PREDICTED AND OBSERVED NOZZLE PERFORMANCE

The force on the pintle can be calculated from the equation

$$F = p_1A_1 - p_2A_2 \cos\theta - 2A_1 C_o^2 (p_1 - p_2) \frac{(A_1/A_2)\cos\theta - 1}{(A_1/A_2)^2 - 1}$$

$$- \pi p_2 (b_o - x \sin\theta \cos\theta)^2 + p_1 \pi a_o^2 - F_o - kx \quad , \qquad (4)$$

where C_o is the orifice coefficient accounting for lost work. The areas A_1 and A_2 are calculated as

$$A_1 = \pi (b_o^2 - a_o^2)$$

$$A_2 = \pi x \sin\theta (2b_o - x \tan\theta) \quad .$$

The mean or average value of x (pintle opening) can be calculated by setting $F = 0$ in Eq. (4); this mean value of x, denoted by x_e, can then be employed to calculate the average flow rate q_e through the nozzle as

$$q_e = 3.11 A_1 C_o \sqrt{\frac{288 g_c (p_1 - p_2)}{\rho} \frac{1}{(A_1/A_2)^2 - 1}} \quad , \qquad (5)$$

where A_2 is calculated for $x = x_e$.

The value of F_o, the initial spring loading (force exerted by spring when $x = 0$ or nozzle is closed), is determined by setting x equal to 0 in Eq. (4). The total force F is also equal to 0 at the critical value of upstream pressure p_1 which is sufficient just to crack the nozzle open. Thus setting x and F equal to 0 and p_1 equal to p_c in Eq. (4), one obtains

$$0 = p_c A_1 - \pi p_2 b_o^2 + p_c \pi a_o^2 - F_o$$

or

$$F_o = p_c \pi (b_o^2 - a_o^2) + p_c \pi a_o^2 - \pi p_2 b_o^2 = \pi b_o^2 (p_c - p_2) \quad .$$

Thus F_o is determined by the value of p_1 necessary to initiate flow through the nozzle.

The data given on DMC 2360 nozzle are as follows:

a_o = .055 in.
b_o = .07025 in.
k = spring constant = 114.2 pounds force/inch
p_c = 3 psig
m = 3.2822 grams = .000225 slug
θ = 25°
ρ = 47.5 lb/ft^3
μ = 1.145 centistokes
f = frequency = 380-390 cps, from sonic analyzer

Pressure Drop, p_1-p_2	Flow Rate, lb/hr
100 psig	110
200	502
250	713
300	898

A computer program was written to employ the design dimensions listed above in solving Eq. (4) for F as a function of x at each of several p_1 values. The value of x for which F = 0 (x_e) was employed in Eq. (5) to calculate the average flow rate q_e. This calculation process was carried out twice, once with C_o assumed equal to 1.0 (equivalent to assumption of no lost work) and again with C_o set equal to .61 - 12x. The observed and calculated flow rates are plotted versus Δp on Fig. 8. The assumption of no lost work is seen to yield excessively high flow rates, while the relation C_o = .61 - 12x results in close agreement between observed and predicted flow rates.

The assumption of a relation of the form C_o = a - bx is qualitatively correct if the Reynolds number at the orifice is sufficiently high so that at a given value of x the operating point in Fig. 6 or 7 is located on the flat portion of the curve. Clearly, more data on C_o are necessary and a correlative relationship must be derived so that C_o may be expressed as a function of x in Eqs. (4) and (5).

The frequency of pintle vibration may be predicted from Eq. (3) of report 2931-1-P,

$$f = 1/2\pi \sqrt{-s/m} ,$$

where s is the slope of the F versus x curve in pounds force per foot. The computer yielded this slope as minus 1358 so that

$$f = 1/2\pi \sqrt{1358/.000225} = 390 \text{ cps} ,$$

which compares well with the 380-390 cps noted on the sonic analyzer during testing of the nozzle.

APPENDIX I

EQUATIONS GOVERNING PERFORMANCE OF A POPPET NOZZLE WITH A DASHPOT

Progress Report No. 2931-1-P presented a mathematical model for the poppet nozzle. The nomenclature, equations, and geometrical sketches included in that report are referred to below but are not repeated here.

The basic equation in the mathematical model is simply Newton's law,

$$F_T = ma \quad , \tag{I-1}$$

where F_T is the total force on the pintle, m is the mass of the pintle, and a is the acceleration. When the poppet nozzle includes a dashpot, a viscous damping force acts on the pintle. Thus the total force on the pintle is given by

$$F_T(x) = F(x) - c(dx/dt) \quad , \tag{I-2}$$

where F(x) is given by Eq. (13P)* and c is the viscous damping coefficient. Equation (I-1) may now be written

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} = \frac{1}{m}F(x) \quad . \tag{I-3}$$

The variables whose values must be known to solve this equation for x as a function of time are a_o, b_o, θ, c, m, F_o, and the spring constant k. The damping coefficient c is given by

$$c = \frac{8\pi\ \mu\ L\ D_i^4}{g_c\ (D_o - D_i)^2\ (D_o^2 - D_i^2)} \quad , \tag{I-4}$$

where μ is the fluid viscosity, L the length of contact between the dashpot piston and cylinder, and D_o and D_i are the diameters of the cylinder and piston, respectively. This expression is derived below with the assumption of laminar liquid flow through the annular region between cylinder and piston.

Equation (I-3) can be solved by numerical techniques in which the derivative d^2x/dt^2 is replaced by a finite difference form and x(t) is calculated stepwise at time $t = \Delta t, 2\Delta t, 3\Delta t, \ldots$. An approximate solution to (I-3) can be obtained by substituting a linear function of x for F(x). The numerical technique involves solution of the equations

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*All equation numbers followed by letter P refer to equations in Progress Report No. 2931-1-P.

$$x_1 = \frac{12(\Delta t)^2}{m} \frac{F(x_o)}{M+2} \text{ , and} \tag{I-5}$$

$$x_{n+1} = \frac{M+2}{M+1} x_n - \frac{1}{M+1} x_{n-1} + \frac{12(\Delta t)^2}{M} \frac{F(x_n)}{M+1} \text{ ,} \tag{I-6}$$

where x_n = value of x at time $t = n\Delta t$, and $M = c\Delta t/m$. The initial conditions for which (I-5) is valid are

$$x_o = (x)_{t=o} = 0 \tag{I-7}$$

$$\left(\frac{dx}{dt}\right)_{t=o} = 0 \text{ ;} \tag{I-8}$$

that is, the displacement x and velocity dx/dt are both initially zero.

The analytical solution to (I-3) for the same initial conditions appears as

$$x = x_e \left[1 - e^{-(c/2m)t} \left(\cosh dt + \frac{c}{2m_d} \sinh dt\right)\right] \text{ ,} \tag{I-9}$$

where the force F(x) [from Eq. (13P)] has been represented by

$$F(x) = a\,(1 - x/x_e) \text{ .} \tag{I-10}$$

In Eqs. (I-9) and (I-10),

$$d = \frac{1}{2}\sqrt{\frac{c^2}{m^2} - \frac{4a}{mx_e}} \text{ , and}$$

$$x_e = \text{value of x for which } F(x) = 0 \text{ .}$$

The flow rate in gallons per minute through the nozzle is a function of x and is given by Eq. (27P). The area A_2 in that equation is a function of x as given in Eq. (15P).

APPENDIX II

VISCOUS DAMPING COEFFICIENT FOR DASHPOT

A relation is derived below for the viscous damping coefficient of a dashpot. Figure 4 is a sketch of the dashpot mechanism considered. The piston of diameter D_i slides in a cylinder of diameter D_o. The length of contact or width of the piston is L. The liquid pressures on the two sides of the piston are denoted by p_1 and p_2 as shown. The displacement of the piston from some reference position is denoted by x.

The damping coefficient c is calculated from the equation

$$F_{dashpot} = - c \frac{dx}{dt} - p_1 \pi a_o^2 , \qquad \text{(II-1)}$$

where $F_{dashpot}$ denotes the force on the piston due to the difference in pressures p_1 and p_2; thus

$$F_{dashpot} = (p_2-p_1)\pi \frac{D_i^4}{4} - p_1 \pi a_o^2 . \qquad \text{(II-2)}$$

The pressure difference p_2-p_1 can be related to the lost work occurring due to friction in the annulus.

$$lw = - \frac{\Delta p}{\rho} = - \frac{(p_1-p_2)}{\rho} = \frac{f L v^2}{2g_c D} \qquad \text{(II-3)*}$$

For laminar flow through the annulus,

$$f = \frac{64}{Re} = \frac{64 \mu}{Dv \rho} . \qquad \text{(II-4)}$$

Combining Eqs. (II-3) and (II-4), one obtains

$$p_2-p_1 = \frac{f\rho L v^2}{2g_c D} = \frac{32 \mu L v}{g_c D^2} . \qquad \text{(II-5)}$$

If the fluid is assumed incompressible, the velocity v through the annulus can be related to the velocity dx/dt of the piston as

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*This is Eq. (63) of Unit Operations, Brown, G. G., et al., Wiley, 1953.

$$v \frac{\pi}{4} (D_o^2 - D_i^2) = - \frac{dx}{dt} \frac{\pi}{4} D_i^2$$

or

$$v = - \frac{dx}{dt} \frac{D_i^2}{D_o^2 - D_i^2} . \qquad \text{(II-6)}$$

Also, the term D in Eq. (II-5) should be replaced by $(D_o - D_i)$, for an annulus. Combination of (II-5) and (II-6) now yields

$$p_2 - p_1 = - \frac{32 \mu L}{g_c (D_o - D_i)^2} \frac{D_i^2}{D_o^2 - D_i^2} \frac{dx}{dt} ,$$

so that $F_{dashpot}$, from Eq. (II-2), becomes

$$F_{dashpot} = - \frac{8\pi \mu L D_i^4}{g_c (D_o - D_i)^2 (D_o^2 - D_i^2)} \frac{dx}{dt} - p_1 \pi a_o^2 \qquad \text{(II-7)}$$

or

$$F_{dashpot} = - c \frac{dx}{dt} - p_1 \pi a_o^2 , \qquad \text{(II-8)}$$

where

$$c = \frac{8\pi \mu L D_i^4}{g_c (D_o - D_i)^2 (D_o^2 - D_i^2)} .$$