Optimum Rocket Trajectories

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I

INTRODUCTION

For the purpose of this paper, the flight path of a rocket will be divided into three sections: (a) launching, possibly vertically; (b) the flight path after the launching stage; (c) the homing phase. The present work is concerned only with the second portion. It is assumed that the craft can be considered as a particle of variable mass, and that the flight path lies in a plane.

Our problem can be stated as follows. At the end of the launching stage the craft possesses a velocity vector--that is, a speed and a direction of motion can be assigned to the craft. These data furnish the initial conditions of the problem. The forces that will be considered as acting on the craft are: (a) thrust due to the burning of fuel; (b) lift; (c) resistance; (d) gravity. In order to furnish an analytic treatment of the problem, we shall here assume that lift and drag forces can be approximated by expressions that depend on characteristic constants and the first power of the velocity. Thrust will be considered constant in magnitude--that is, the mass of the craft is assumed to decrease linearly with time because of the uniform rate of burning. However, the direction of thrust is to be varied continuously if necessary, so as to obtain the trajectory that is optimum in the sense defined below.

We shall now define the optimum trajectory. First, we note that two types of motion will be considered: either the craft burns fuel throughout the motion or it burns fuel for a specified time and then coasts with fuel jets turned off. We shall fix the horizontal distance attained by the craft and require that the vertical distance at this fixed horizontal distance be stationary, or, alternatively, we shall fix the vertical distance attained and require that the horizontal distance at this fixed vertical distance be stationary. If a trajectory satisfies this requirement, it shall be called an optimum trajectory.

The above approach is to be considered as a first attack on the problem of optimum trajectories. The essential idealization is the use of linear drag and lift. One could use quadratic drag or some more complicated drag function as well as a non-linear lift function. The introduction of such functions would lead to problems of the Bolza type in the Calculus of Variations. Because of the non-linear character of the resulting system of differential equations, very little insight into the theory of optimum trajectories could be obtained. It is for this reason that we have here attacked the simplified problem. Further, if the optimum trajectories for a realistic drag function do not depart too widely from the optimum trajectories for linear drag, then our results will prove valuable. With this in mind, we have developed the theory of optimum trajectories in two directions: (1) extensions of the idealized theory to various other criteria for optimum trajectories, such as minimum boost velocity, or maximum terminal velocity; (2) attempts to study the departure of the idealized linear theory from the actual non-linear theory, by perturbations or by stability studies. These will be reported in later papers.

II

SUMMARY

The general equations of motion are stated in Section IV, and general formulas for the solution of these equations, to be verified in succeeding sections, are furnished. With the aid of these formulas, the variational equations for the optimum trajectory are determined.

In Section V, the case of flight with neither lift nor drag is examined, and solutions of the differential equations of motion obtained. By use of the variational equations, it is shown that for either of the two types of motion mentioned previously--that is, continuous consumption of fuel or consumption of fuel for a specified time followed by a coasting phase--the direction of thrust is fixed in the plane of motion for an optimum trajectory. The relation of thrust direction to burning time for the first type of motion and the relation of thrust to total flight time for various burning times in the second type are shown by figures in the text.

Section VI considers the case of flight with linear resistance and no lift. Again, it is seen that the direction of thrust is fixed in the plane of motion for an optimum trajectory, and graphs similar to those of the preceding section are given. For the drag coefficients examined, the effect of drag on the desired thrust direction is negligible.

The case of flight with linear lift and no resistance is treated in Section VII. For an optimum trajectory the thrust direction is no longer fixed in the plane of motion.

In fact, the thrust direction at various points of the trajectory is obtained by solving a particular equation.

Finally, in Section VIII another type of optimum trajectory is discussed. We seek to determine the thrust schedule so that in a fixed burning time and coasting time the horizontal or vertical distance attained by the craft is a maximum, and in maximizing this horizontal or vertical distance we place no restriction on the vertical or horizontal distance, respectively. The thrust schedule for such optimum trajectories is immediately apparent when a linear drag law and no lift are considered. However, when a linear lift law and no drag are assumed, the thrust schedule is not obvious.

The introduction of modified variables is found to simplify the approach to the several phases of the problem, and to permit easier handling of the relations involved. However, one may not be inclined to follow each modification through, and as a result some difficulty may be encountered. To avoid this possibility, the analysis of one case--that of zero drag, zero lift, with coasting following the burning stage--is carried through in the original variables in Appendix A.

III

NOTATION

c the horizontal component $P \cos \phi$ of the modified craft thrust.

$\bar{g}$ the acceleration of gravity: 32 ft/sec^2.

g a modified gravitation coefficient $\bar{g}/\dot{r}^2$, a constant.

n a modified time variable, $1 - \dot{r}t$.

n_1 the value of n at time $t = t_1$.

n_2 the value of n at time $t = t_2$.

q an independent variable used in solving the equations of motion.

r specific burning rate, a constant: time^{-1}.

s the vertical component $P \sin \phi$ of the modified craft thrust.

t time, measured from the end of the launching stage.

t_1 fuel burning time.

t_2 total time of flight, burning time plus coasting time.

x horizontal distance of the craft at any modified time n; a subscript on x denotes a value of n.

y vertical distance of the craft at any modified time n; a subscript on y denotes a value of n.

$\overline{K}$ a characteristic constant in the linear resistance law.

K the modified drag coefficient $\frac{-\overline{K}}{M_o\dot{r}}$, a constant.

$\overline{L}$ a characteristic constant in the linear lift law.

L the modified lift coefficient $\frac{-\overline{L}}{M_o\dot{r}}$, a constant.

M_o the initial mass of the craft, including fuel.

P the modified craft thrust $\frac{T}{M_o\dot{r}^2}$.

T the magnitude of the craft thrust acting during the burning time, a constant.

ϕ the angle of inclination of thrust to the horizontal axis at any time.

$\frac{dx}{dn}$ the modified horizontal component of craft velocity; a subscript denotes a value of n.

$\frac{dy}{dn}$ the modified vertical component of craft velocity; a subscript denotes a value of n.

Note:

At time $t = 0$, $n = n_o = 1$. The expressions $x)_{n_o}$, $\left(\frac{dx}{dn}\right)_{n_o}$ refer to the values of the expressions at modified time $n = 1$; the expressions $x)_{n_1}$, $\left(\frac{dx}{dn}\right)_{n_1}$ refer to the values of the expressions at modified time $n = n_1$, and so forth. We assume that $x)_{n_o} = y)_{n_o} = 0$.

IV

THE GENERAL EQUATIONS OF MOTION AND THE VARIATIONAL EQUATIONS

Let the plane of the flight path be specified, and let us introduce a co-ordinate set in this plane in such a way that the rocket is initially at the origin. Then the equations of motion are

$$M_o(1 - \dot{r}t)\frac{d^2x}{dt^2} = -\bar{K}\frac{dx}{dt} - \bar{L}\frac{dy}{dt} + T\cos\phi ,$$

$$M_o(1 - \dot{r}t)\frac{d^2y}{dt^2} = -\bar{K}\frac{dy}{dt} + \bar{L}\frac{dx}{dt} + T\sin\phi - M_o(1 - \dot{r}t)\bar{g} ,$$

which can be simplified by introducing the new variable $n = 1 - \dot{r}t$. Computations show that

$$\frac{dx}{dt} = -\dot{r}\frac{dx}{dn} , \qquad \frac{d^2x}{dt^2} = \dot{r}^2\frac{d^2x}{dn^2} ,$$

and that similar expressions are valid for $\frac{dy}{dt}, \frac{d^2y}{dt^2}$. Combining the above relations, we may write

$$(4.1) \qquad \frac{d^2x}{dn^2} + \frac{K}{n}\frac{dx}{dn} + \frac{L}{n}\frac{dy}{dn} = \frac{c}{n} ,$$

$$(4.2) \qquad \frac{d^2y}{dn^2} + \frac{K}{n}\frac{dy}{dn} - \frac{L}{n}\frac{dx}{dn} = \frac{s}{n} - g .$$

In order to discuss the variational equations, it is necessary that we know the form of the solutions of the last expressions. In the case of burning for a specified time and then coasting, the values of x and y at the end of the total flight time will be shown by integration to be of the form

$$(4.3)\qquad x(n_2,\phi) = f(n_1,n_2) + \int_1^{n_1} j(n_1,n_2,n)\ c\ dn + \int_1^{n_1} h(n_1,n_2,n)\ s\ dn\ ,$$

$$(4.4)\qquad y(n_2,\phi) = \overline{f}(n_1,n_2) + \int_1^{n_1} \overline{j}(n_1,n_2,n)\ c\ dn + \int_1^{n_1} \overline{h}(n_1,n_2,n)\ s\ dn\ .$$

where f, j, h, and so forth, are known functions of their respective arguments: f and $\overline{f}$ are solutions of equations (4.1) and (4.2) when s = c = 0; and j, $\overline{j}$, h, $\overline{h}$ are integrating factors of the left-hand sides of those equations. It should be noted that x and y depend on n_2, when n_1 is specified, and also on the thrust direction ϕ which is arbitrarily chosen. In the case of continuous burning the values of x and y at the end of the burning time, which is also the flight time, will be shown to be of the form

$$(4.5)\qquad x(n_1,\phi) = u(n_1) + \int_1^{n_1} v(n_1,n)\ c\ dn + \int_1^{n_1} w(n_1,n)\ s\ dn\ ,$$

$$(4.6)\qquad y(n_1,\phi) = \overline{u}(n_1) + \int_1^{n_1} \overline{v}(n_1,n)\ c\ dn + \int_1^{n_1} \overline{w}(n_1,n)\ s\ dn\ .$$

The quantities u and $\bar{u}$ are solutions of equations (4.1) and (4.2) when $s = c = 0$, and v, $\bar{v}$, w, $\bar{w}$ are integrating factors of the left-hand sides of those equations. It will be shown that these functions have the property

$$(4.7) \qquad v(n_1,n_1) = \bar{v}(n_1,n_1) = w(n_1,n_1) = \bar{w}(n_1,n_1) = 0 ,$$

and it is of importance to note that $0 < n_1 < 1$, and also that $-\infty < n_2 < n_1 < 1$.

We shall first form the variational equations for the case of burning and coasting. Using relations (4.3) and (4.4) we find that

$$(4.8) \qquad \delta x = \left[f_{n_2} + \int_1^{n_1} (j_{n_2} c + h_{n_2} s) dn \right] \delta n_2 - \int_1^{n_1} (js - hc) \delta \phi \, dn ,$$

$$(4.9) \qquad \delta y = \left[\bar{f}_{n_2} + \int_1^{n_1} (\bar{j}_{n_2} c + \bar{h}_{n_2} s) dn \right] \delta n_2 - \int_1^{n_1} (\bar{j} s - \bar{h} c) \delta \phi \, dn ,$$

where the subscripts n_2 denote partial derivatives with respect to n_2. Equating δx, δy to zero in equations (4.8) and (4.9), and solving the expression (4.8) for δn_2, we obtain

$$\delta n_2 = \frac{\int_1^{n_1} (js - hc) \delta \phi \, dn}{f_{n_2} + \int_1^{n_1} (j_{n_2} c + h_{n_2} s) dn} .$$

Combining this last relation with equation (4.9), we may write

$$(4.10)\quad \frac{\bar{f}_{n_2} + \int_1^{n_1} (\bar{j}_{n_2} c + \bar{h}_{n_2} s)\,dn}{f_{n_2} + \int_1^{n_1} (j_{n_2} c + h_{n_2} s)\,dn} \int_1^{n_1} (js - hc)\,\delta\phi\, dn - \int_1^{n_1} (\bar{j}s - hc)\,\delta\phi\, dn = 0 .$$

If we assume that there is an optimum path, then along this path ϕ is known as a function of n_1, n_2, and n. Thus, the fractional expression in equation (4.10) is a function of n_1 and n_2. We denote this expression by $\lambda(n_1,n_2)$ so that

$$(4.11)\quad \left[\bar{f}_{n_2} + \int_1^{n_1} (\bar{j}_{n_2} c + \bar{h}_{n_2} s)\,dn\right] - \lambda\left[f_{n_2} + \int_1^{n_1} (j_{n_2} c + h_{n_2} s)\,dn\right] = 0 ,$$

and expression (4.10) becomes

$$(4.12)\quad \int_1^{n_1} (\bar{j}s - \bar{h}c)\,\delta\phi\, dn - \lambda\int_1^{n_1} (js - hc)\,\delta\phi\, dn = 0 .$$

It is to be noted that the derivation of equations (4.11) and (4.12) from (4.8) and (4.9) could have been made directly by use of the Lagrange multiplier λ. Since λ is a function of n_1 and n_2, it may be introduced under the integral sign in the second expression in equation (4.12) to give

$$(4.13)\quad \int_1^{n_1} \left[(\bar{j}s - \bar{h}c) - \lambda(js - hc)\right]\delta\phi\, dn = 0 .$$

Further, $\delta\phi$ is an arbitrary but integrable function of n. From this it follows that the bracketed expression in the integrand of equation (4.13) vanishes, that is,

$$(4.14)\qquad s = \frac{(\bar{h} - \lambda h)}{(\bar{j} - \lambda j)}\, c\ .$$

From this we readily obtain

$$c = \frac{\pm P(\bar{j} - \lambda j)}{\sqrt{(\bar{j} - \lambda j)^2 + (\bar{h} - \lambda h)^2}}\ ,$$

$$s = \frac{\pm P(\bar{h} - \lambda h)}{\sqrt{(\bar{j} - \lambda j)^2 + (\bar{h} - \lambda h)^2}}\ ;$$

the meanings of the symbols are given in the section on notation. By use of the above equations, we obtain the expression for λ:

$$(4.15)\qquad \bar{f}_{n_2} - \lambda f_{n_2} \pm$$

$$\pm\, p \int_1^{n_1} \frac{(\bar{j}_{n_2} - \lambda j_{n_2})(\bar{j} - \lambda j) + (\bar{h}_{n_2} - \lambda h_{n_2})(\bar{h} - \lambda h)}{\sqrt{(\bar{j} - \lambda j)^2 + (\bar{h} - \lambda h)^2}}\, dn = 0\ .$$

Fortunately, in most of our work the fractional expression in equation (4.14) is independent of n, that is, <u>the thrust direction is fixed in the plane of motion</u>. Hence

$$(4.16)\qquad \lambda = \frac{\bar{j}s - \bar{h}c}{js - hc}\ ,$$

where λ, c and s are independent of n. So, introducing into equation (4.11) this expression for λ, we get an algebraic relation between s and c:

$$(4.17)\qquad Ac^2 + Bs^2 + Csc + Dc + Es = 0\ ,$$

where

$$A = \bar{h}\int_1^{n_1} j_{n_2}dn - h\int_1^{n_1} \bar{j}_{n_2}dn ,$$

$$B = j\int_1^{n_1} \bar{h}_{n_2}dn - \bar{j}\int_1^{n_1} h_{n_2}dn ,$$

$$C = j\int_1^{n_1} \bar{j}_{n_2}dn - \bar{j}\int_1^{n_1} j_{n_2}dn -$$

$$- h\int_1^{n_1} \bar{h}_{n_2}dn + \bar{h}\int_1^{n_1} h_{n_2}dn ,$$

$$D = \bar{h}\, f_{n_2} - h\, \bar{f}_{n_2} ,$$

$$E = j\, \bar{f}_{n_2} - \bar{j}\, f_{n_2} .$$

By use of equation (4.17) and the relation $s^2 + c^2 = P^2$, we can obtain c, s and ϕ.

The variational equations for the case of continuous burning can be formed in exactly the same manner. Because of the relations (4.7), the theory is exactly the same as in the previous case. For convenience in future use, we include the formulas obtained. Corresponding to equations (4.14) and (4.16) we have

$$(4.18) \quad s = \frac{\bar{w} - \lambda w}{\bar{v} - \lambda v}\, c ,$$

and

$$\lambda = \frac{\bar{v}s - \bar{w}c}{vs - wc} ;$$

corresponding to (4.17), we have

$$(4.19) \quad A'c^2 + B's^2 + C'sc + D'c + E's = 0 ,$$

where

$$A' = \bar{w}\int_1^{n_1} v_{n_1}dn - w\int_1^{n_1} \bar{v}_{n_1}dn ,$$

$$B' = v\int_1^{n_1} \bar{w}_{n_1}dn - \bar{v}\int_1^{n_1} w_{n_1}dn ,$$

$$C' = v\int_1^{n_1} \bar{v}_{n_1}dn - \bar{v}\int_1^{n_1} v_{n_1}dn -$$

$$- w\int_1^{n_1} \bar{w}_{n_1}dn + \bar{w}\int_1^{n_1} w_{n_1}dn ,$$

$$D' = \bar{w}\ u_{n_1} - w\ \bar{u}_{n_1} ,$$

$$E' = v\ \bar{u}_{n_1} - \bar{v}\ u_{n_1}$$

Note: in this case we vary burning time n_1 and thrust inclination ϕ.

V

ZERO LIFT AND DRAG

Continuous Burning

When both lift and drag are absent, equations (4.1) and (4.2) become

$$\frac{d^2x}{dn^2} = \frac{c}{n} ,$$

$$\frac{d^2y}{dn^2} = \frac{s}{n} - g .$$

The integrals of these equations are respectively

$$(5.1) \qquad x(n_1,\phi) = (n_1 - 1)\left(\frac{dx}{dn}\right)_{n_0} + \int_1^{n_1} (n_1 - n)\,\frac{c}{n}\,dn ,$$

$$(5.2) \qquad y(n_1,\phi) = (n_1 - 1)\left(\frac{dy}{dn}\right)_{n_0} - \frac{g}{2}(n_1 - 1)^2 + \int_1^{n_1} (n_1 - n)\,\frac{s}{n}\,dn .$$

Comparing these with expressions (4.5) and (4.6), we see that for the present case

$$u = (n_1 - 1)\left(\frac{dx}{dn}\right)_{n_0} , \quad v = \frac{n_1 - n}{n} , \quad w = 0 ,$$

$$\bar{u} = (n_1 - 1)\left(\frac{dy}{dn}\right)_{n_0} - \frac{g}{2}(n_1 - 1)^2 , \quad \bar{v} = 0 ,$$

$$\bar{w} = \frac{n_1 - n}{n} .$$

Thus, equation (4.18) becomes

$$s = -\frac{c}{\lambda} ,$$

indicating that thrust is fixed in the plane of motion. Equation (4.19) is reduced to the form

$$P^2 \ln n_1 + s\left[\left(\frac{dy}{dn}\right)_{n_0} - g(n_1 - 1)\right] + c\left(\frac{dx}{dn}\right)_{n_0} = 0 ,$$

and since by definition $s = P \sin \phi$ and $c = P \cos \phi$, we may write this as

$$\text{(5.3)} \quad \sin(\phi + \alpha) = \frac{-P \ln n_1}{\sqrt{\left(\frac{dx}{dn}\right)^2_{n_0} + \left[\left(\frac{dy}{dn}\right)_{n_0} - g(n_1 - 1)\right]^2}} ,$$

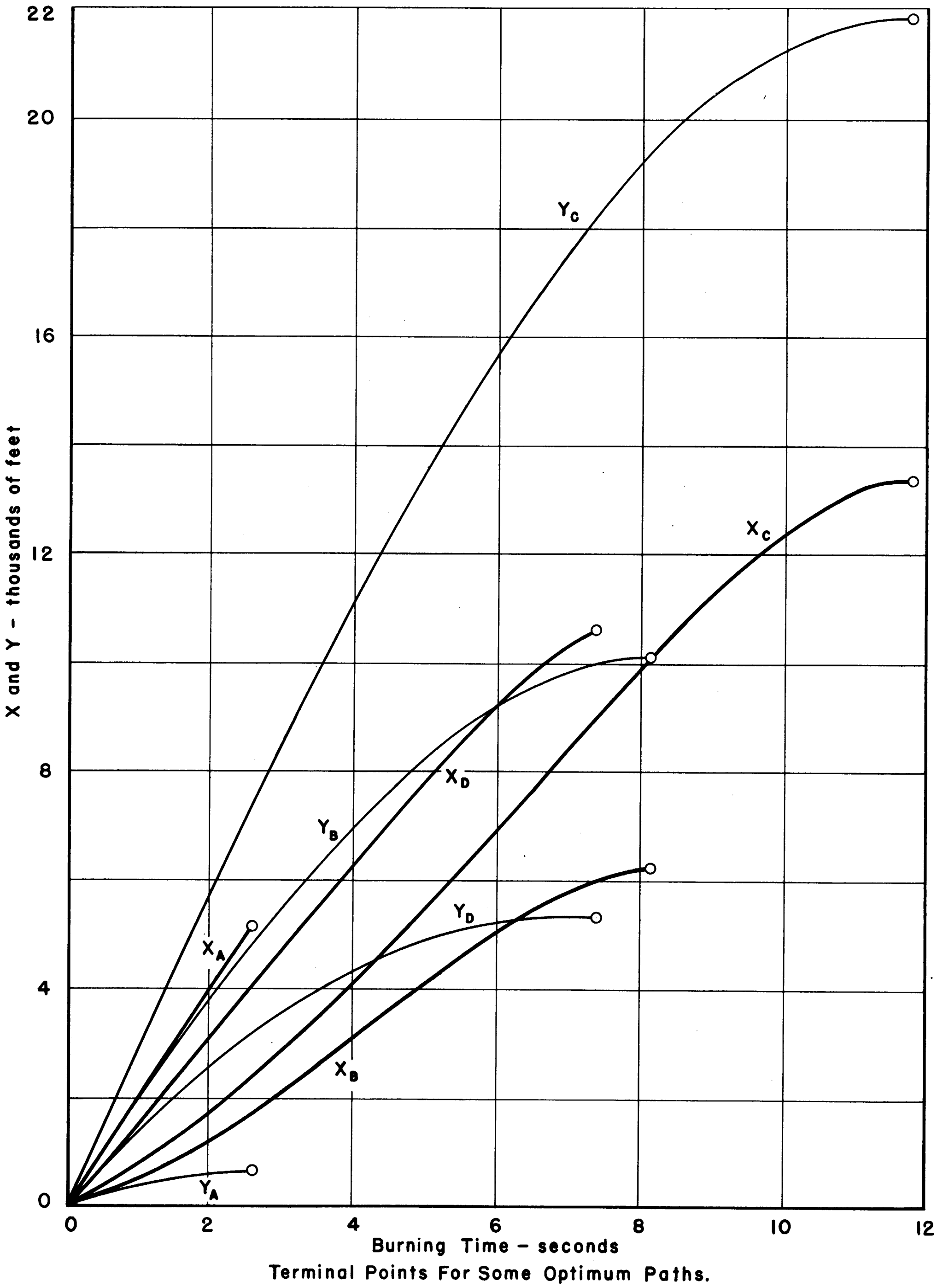
where

$$\text{(5.4)} \quad \cos \alpha = \frac{\left(\frac{dy}{dn}\right)_{n_0} - g(n_1 - 1)}{\sqrt{\left(\frac{dx}{dn}\right)^2_{n_0} + \left[\left(\frac{dy}{dn}\right)_{n_0} - g(n_1 - 1)\right]^2}} ,$$

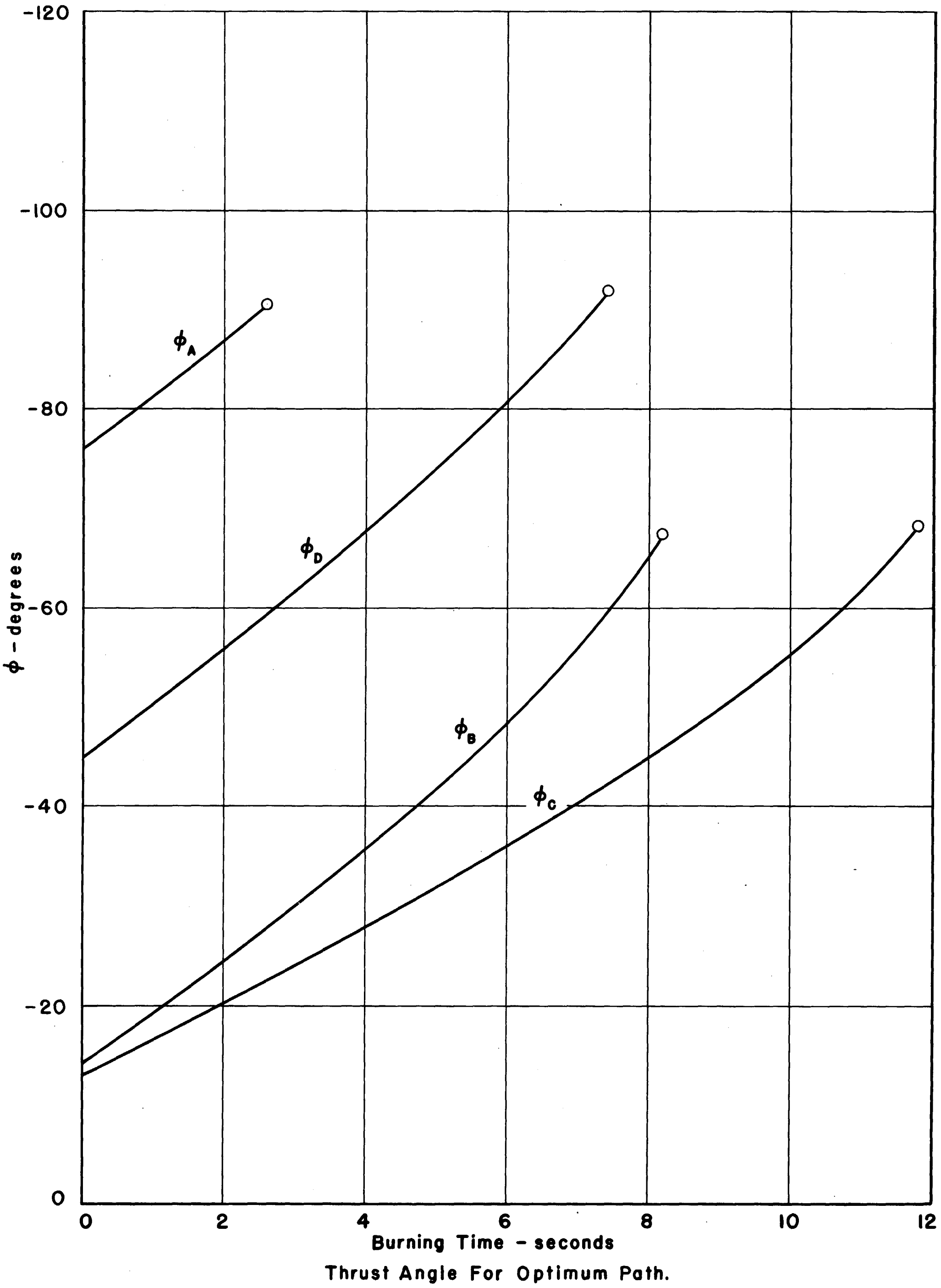
$$\text{(5.5)} \quad \sin \alpha = \frac{\left(\frac{dx}{dn}\right)_{n_0}}{\sqrt{\left(\frac{dx}{dn}\right)^2_{n_0} + \left(\frac{dy}{dn}\right)_{n_0} - g(n_1 - 1)^2}} .$$

Evidently, ϕ will exist only if

$$\text{(5.6)} \quad \frac{(P \ln n_1)^2}{2} \leq \frac{1}{2}\left\{\left(\frac{dx}{dn}\right)^2_{n_0} + \left[\left(\frac{dy}{dn}\right)_{n_0} - g(n_1 - 1)\right]^2\right\} .$$



Terminal Points For Some Optimum Paths.
Figure I

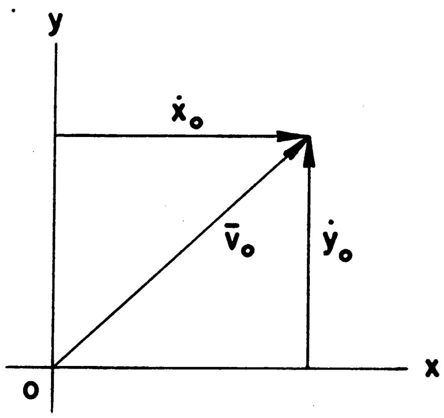


Thrust Angle For Optimum Path.
Figure II

The left-hand side of equation (5.6), when multiplied by the mass of the craft at time $t = t_1$, represents the kinetic energy at time $t = t_1$ of a craft, of initial velocity zero, moving in a vacuum under a fixed thrust. Similarly, the right-hand side when multiplied by the mass of the craft at time $t = t_1$, represents the kinetic energy at time $t = t_1$ of a craft whose initial velocity vector has components $\left(\frac{dx}{dt}\right)_o$, $\left(\frac{dy}{dt}\right)_o$, and which moves in vacuum under the force of gravity. Hence, equation (5.6) implies that an optimum trajectory exists only when the latter kinetic energy is larger than the former. This is clearly a restriction, in the form of a lower bound, on the initial components of velocity.

For the case of continuous burning of fuel, some examples of optimum results are shown in Figure I. These assume neither drag nor lift, and give maximum horizontal distance for fixed vertical distance. In the examples we assume five to be the ratio of thrust to total craft weight, inclusive of fuel, and assume a burning rate $\dot{r}$ of 0.02/sec. The co-ordinates of craft position at the end of the burning time are given as functions of burning time for these values of the craft velocity vector as existing at the end of the launching stage:

Case A: $\dot{x}_o = 2000$ ft/sec, $\dot{y}_o = 500$ ft/sec.
Case B: $\dot{x}_o = 500$ ft/sec, $\dot{y}_o = 2000$ ft/sec.
Case C: $\dot{x}_o = 750$ ft/sec, $\dot{y}_o = 3000$ ft/sec.
Case D: $\dot{x}_o = \dot{y}_o = 1460$ ft/sec.



As we can see from the figure at the left, these values imply for Case A that the craft velocity vector is initially inclined at an angle of about 14° from the horizontal. Corresponding to the values of the co-ordinates as shown, the necessary angle ϕ of thrust inclination to the horizontal is given as a function of burning time in Figure II.

Burning and Coasting

During the coasting phase $n_2 < n < n_1$, the equations of motion are

$$(5.7) \qquad \frac{d^2x}{dn^2} = 0 ,$$

$$(5.8) \qquad \frac{d^2y}{dn^2} = - g .$$

The conditions for x and y at modified time $n = n_1$ are to be obtained from equations (5.1) and (5.2), and by differentiating these equations with respect to n_1 we obtain the velocity components

$$(5.9) \qquad \left(\frac{dx}{dn}\right)_{n_1} = \left(\frac{dx}{dn}\right)_{n_o} + \int_1^{n_1} \frac{c}{n} \, dn ,$$

$$(5.10) \qquad \left(\frac{dy}{dn}\right)_{n_1} = \left(\frac{dy}{dn}\right)_{n_o} - g(n_1 - 1) + \int_1^{n_1} \frac{s}{n} \, dn .$$

Integration of equations (5.7) and (5.8) yields

$$x(n_2,\phi) = x(n_1,\phi) + \left(\frac{dx}{dn}\right)_{n_1} (n_2 - n_1) ,$$

$$y(n_2,\phi) = y(n_1,\phi) + \left(\frac{dy}{dn}\right)_{n_1} (n_2 - n_1) - g \frac{(n_2 - n_1)^2}{2} ,$$

and by substituting into these the co-ordinates of position at the end of burning from equations (5.1) and (5.2), and velocity from equations (5.9) and (5.10), we obtain

$$x(n_2,\phi) = (n_2 - 1) \left(\frac{dx}{dn}\right)_{n_o} + \int_1^{n_1} \frac{(n_2 - n)}{n} c \, dn ,$$

$$y(n_2,\phi) = (n_2 - 1)\left(\frac{dy}{dn}\right)_{n_0} - \frac{g}{2}(n_2 - 1)^2 + $$
$$+ \int_1^{n_1} \frac{(n_2 - n)}{n}\, s\, dn \, .$$

Comparison of these latter equations with the relations (4.3) and (4.4) shows that

$$f = (n_2 - 1)\left(\frac{dx}{dn}\right)_{n_0} , \qquad j = \frac{n_2 - n}{n} , \qquad h = 0 ,$$

$$\bar{f} = (n_2 - 1)\left(\frac{dy}{dn}\right)_{n_0} - \frac{g}{2}(n_2 - 1)^2 , \qquad \bar{j} = 0 ,$$

$$\bar{h} = \frac{n_2 - n}{n} .$$

Consequently, equation (4.14) becomes

$$s = -\frac{c}{\lambda} ,$$

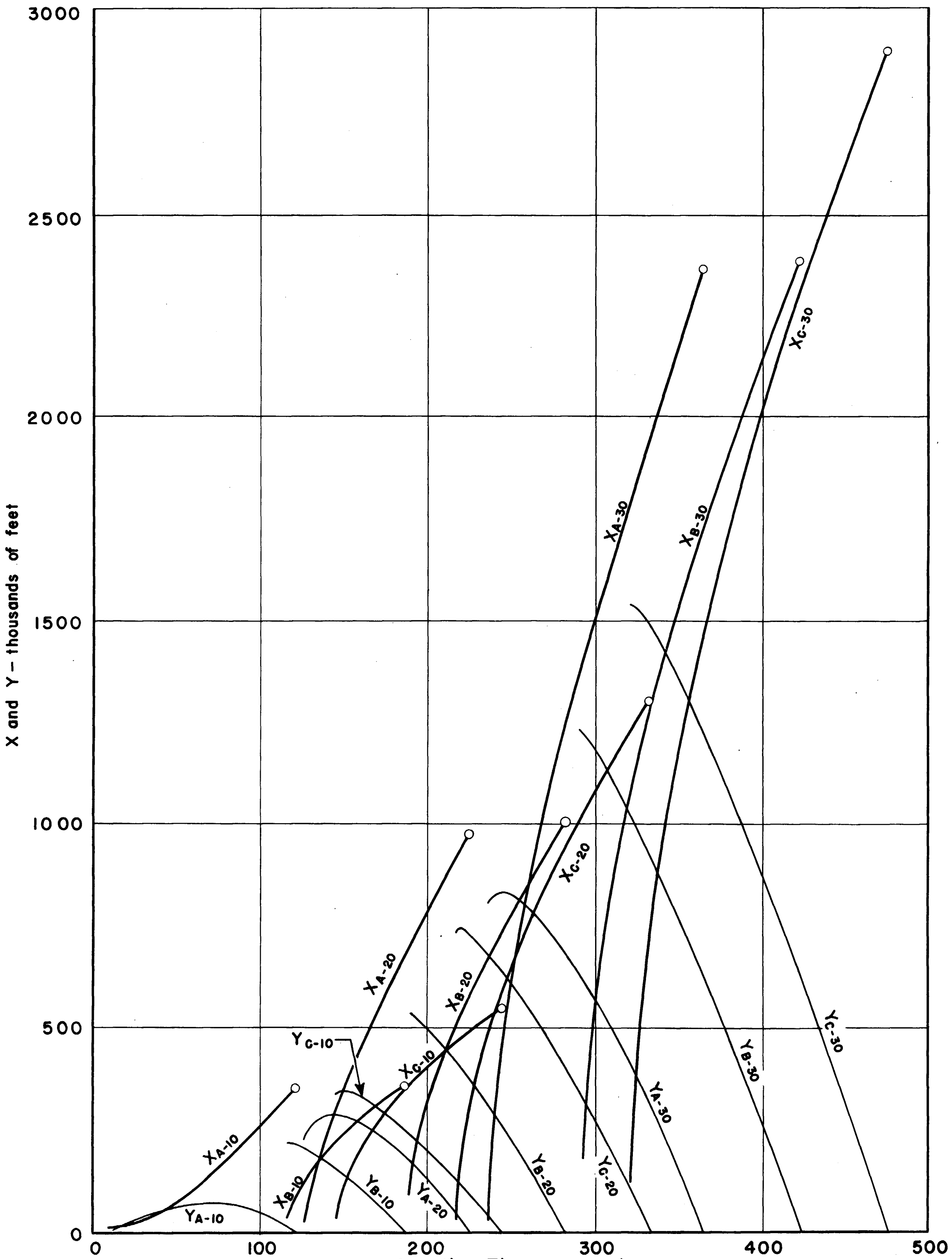
from which we conclude that <u>again thrust direction is fixed in the plane of motion</u>. Equation (4.18) furnishes the relation

$$P2 \ln n_1 + s \left[\left(\frac{dy}{dn}\right)_{n_0} - g(n_2 - 1)\right] + c \left(\frac{dx}{dn}\right)_{n_0} = 0 .$$

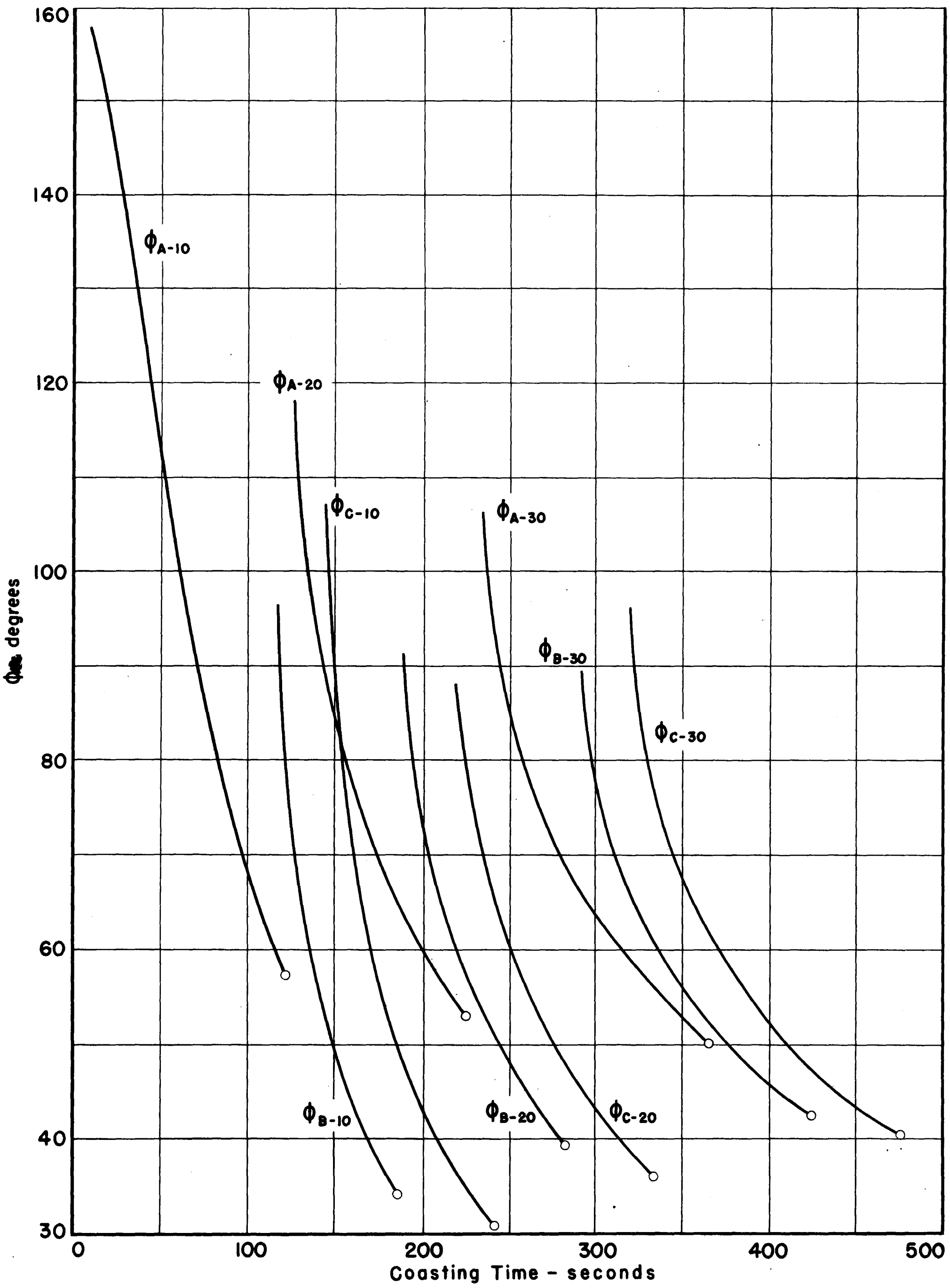
It is clear that for this case formulas (5.3) through (5.6) are valid with n_1 replaced by n_2 in the various denominators and in the right-hand side of equation (5.6).

Examples of optimum results for vertical paths are given in Figure III for cases of burning followed by coasting. Values of the co-ordinates are shown as functions of total flight time. Again, lift and drag are assumed negligible, and the conditions at the end of the launching stage are the same as those used for the examples of Figures I and II. Burning times are 10, 20 and 30 seconds, as indicated on the graphs; the label x_{A-20}, for instance, defines the curve representing the variation of the x co-ordinate with flight time for conditions A--that is, for $\dot{x}_o$ = 2000 ft/sec, $\dot{y}_o$ = 500 ft/sec--and for a burning time of 20 seconds. Corresponding angles of necessary thrust inclination, measured from the horizontal, are indicated in Figure IV. The figures have been drawn so that some indication of maximum vertical distances attained by the rocket may be seen. Actually, at very large vertical or horizontal distances, the effect of the variation in gravity, curvature of earth, etc. should be taken into consideration.

In order to make the methods employed more readily understandable, the process of optimization in the case of zero lift and zero drag, burning followed by coasting, is discussed in unmodified terms in Appendix A.



Terminal Points For Some Optimum Paths.
Figure III



Thrust Angle For Optimum Path.

Figure IV

VI

ZERO LIFT AND LINEAR DRAG

Continuous Burning

When lift is negligible and a linear drag is assumed, the equations of motion (4.1) and (4.2) become

$$\frac{d^2x}{dn^2} + \frac{K}{n}\frac{dx}{dn} = \frac{c}{n} , \tag{6.1}$$

$$\frac{d^2y}{dn^2} + \frac{K}{n}\frac{dy}{dn} = \frac{s}{n} - g . \tag{6.2}$$

In order to integrate these relations we introduce the function $G = G(n,q)$ which satisfies the adjoint differential equation

$$\frac{d^2G}{dn^2} - \frac{K}{n}\frac{dG}{dn} + \frac{KG}{n^2} = 0 . \tag{6.3}$$

It is easily verified that the value

$$G(n,q) = \frac{1}{K-1}\left[n^K q^{1-K} - n\right] , \qquad K \neq 0,1 , \tag{6.4}$$

satisfies equation (6.3) and has the useful properties

$$G(q,q) = 0 , \qquad \left(\frac{dG}{dn}\right)_{q,q} = 1 .$$

The case $K = 0$ has been covered in the previous sections. If $K = 1$, then the function G is

$$G(n,q) \quad = \quad n \ln (n/q) \ .$$

Because of the fact that the case K = 1 is rarely encountered in practice, we omit the discussion of this case and treat equation (6.4). We shall now apply the standard method for treating a differential equation and its adjoint. Combining equation (6.3) successively with equations (6.1) and (6.2), we find that

$$(6.5) \qquad \frac{d}{dn}\left(G\frac{dx}{dn} - x\frac{dG}{dn}\right) + \frac{d}{dn}\left(\frac{KGx}{n}\right) = \frac{cG}{n} ,$$

and

$$(6.6) \qquad \frac{d}{dn}\left(G\frac{dy}{dn} - y\frac{dG}{dn}\right) + \frac{d}{dn}\left(\frac{KGy}{n}\right) = \left(\frac{s}{n} - g\right) G .$$

On integrating these equations between the limits $q \leqq n \leqq 1$ and replacing q by n_1, we obtain

$$(6.7) \qquad x(n_1,\phi) = \frac{1}{1-K}(n_1{}^{1-K} - 1)\left(\frac{dx}{dn}\right)_{n_o} + \frac{1}{1-K}\int_1^{n_1}\left[n^{K-1}n_1{}^{1-K} - 1\right] c \, dn ,$$

$$(6.8) \qquad y(n_1,\phi) = \frac{1}{1-K}(n_1{}^{1-K} - 1)\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K-1}\left[\frac{1}{2} - \frac{n_1{}^{1-K}}{K+1} - \frac{1}{2}\,\frac{K-1}{K+1}\,n_1{}^2\right] + \frac{1}{1-K}\int_1^{n_1}\left[n^{K-1}n_1{}^{1-K} - 1\right] s \, dn .$$

Note that $K \neq 0$, 1 or -1 in relation (6.8); in future work, we will omit these special values of K. Comparing these latter expressions with equations (4.5) and (4.6), we find

$$u = \frac{1}{1-K}(n_1^{\,1-K} - 1)\left(\frac{dx}{dn}\right)_{n_o},$$

$$v = \frac{n^{K-1}n_1^{\,1-K} - 1}{1-K},$$

$$w = 0,$$

$$\bar{u} = \frac{1}{1-K}(n_1^{\,1-K} - 1)\left(\frac{dy}{dn}\right)_{n_o} + $$

$$+ \frac{g}{K-1}\left[\frac{1}{2} - \frac{n_1^{\,1-K}}{K+1} - \frac{K-1}{K+1}n_1^2\right],$$

$$\bar{v} = 0,$$

$$\bar{w} = \frac{n^{K-1}n_1^{\,1-K} - 1}{1-K}.$$

Again, <u>thrust direction is fixed in the plane of motion</u>. By use of relation (4.19), we find that

$$(6.9) \qquad \frac{P^2}{K}(n_1^{\,K} - 1) + s\left[\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K+1} - \frac{2g}{K+1}n_1^{\,K+1}\right] +$$

$$+ c\left(\frac{dx}{dn}\right)_{n_o} = 0.$$

The formulas corresponding to expressions (5.3) through (5.5) are

$$(6.10) \qquad \sin(\phi + \alpha) =$$

$$= \frac{-P(n_1^{\,K} - 1)}{K\sqrt{\left(\frac{dx}{dn}\right)_{n_o}^2 + \left[\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K+1} - \frac{2g}{K+1}n_1^{\,K+1}\right]^2}},$$

(6.11) $\cos\alpha$ =

$$= \frac{\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K+1} - \frac{2g}{K+1} n_1^{K+1}}{\sqrt{\left(\frac{dx}{dn}\right)^2_{n_o} + \left[\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K+1} - \frac{2g}{K+1} n_1^{K+1}\right]^2}} ,$$

(6.12) $\sin\alpha$ =

$$= \frac{\left(\frac{dx}{dn}\right)_{n_o}}{\sqrt{\left(\frac{dx}{dn}\right)^2_{n_o} + \left[\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K+1} - \frac{2g}{K+1} n_1^{K+1}\right]^2}} .$$

The inequality

(6.13) $$\frac{\left[P(n_1^K - 1)\right]^2}{2K^2} \leqq$$

$$\leqq \frac{1}{2}\left\{\left(\frac{dx}{dn}\right)^2_{n_o} + \left[\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K+1} - \frac{2g}{K+1} n_1^{K+1}\right]^2\right\}$$

must be satisfied; this can be interpreted physically in a manner analogous to relation (5.6).

Burning and Coasting

During the coasting phase $n_2 < n < n_1$, the equations of motion are

$$\frac{d^2x}{dn^2} + \frac{K}{n_1}\frac{dx}{dn} = 0 ,$$

$$\frac{d^2y}{dn^2} + \frac{K}{n_1}\frac{dy}{dn} = -g ,$$

which can be immediately integrated to give

$$(6.14)\quad x(n_2,\phi) = -\frac{n_1}{K}\left[e^{K(1-\frac{n_2}{n_1})} - 1\right]\left(\frac{dx}{dn}\right)_{n_1} + x(n_1,\phi) ,$$

$$(6.15)\quad y(n_2,\phi) = -\frac{n_1}{K}\left[e^{K(1-\frac{n_2}{n_1})} - 1\right]\left(\frac{dy}{dn}\right)_{n_1} + y(n_1,\phi) + $$
$$+ \frac{gn_1}{K}\left[\frac{n_1}{K} - n_2 + n_1 - \frac{n_1}{K}e^{K(1-\frac{n_2}{n_1})}\right] .$$

The conditions at modified time $n = n_1$ for x and y and their derivatives are obtained from equations (6.7) and (6.8) and their derivatives. We find that

$$(6.16)\quad \left(\frac{dx}{dn}\right)_{n_1} = n_1{}^{-K}\left(\frac{dx}{dn}\right)_{n_0} + \int_1^{n_1} n_1{}^{-K}n^{K-1}\, c\, dn ,$$

$$(6.17)\quad \left(\frac{dy}{dn}\right)_{n_1} = n_1{}^{-K}\left(\frac{dy}{dn}\right)_{n_0} + \frac{g}{K+1}(n_1{}^{-K} - n_1) + $$
$$+ \int_1^{n_1} n_1{}^{-K}n^{K-1}\, s\, dn .$$

Substituting the results from equations (6.7), (6.8), (6.16) and (6.17) into (6.14) and (6.15), we obtain

$$x(n_2,\phi) =$$
$$= \left[\frac{-n_1{}^{1-K}}{K}e^{K(1-\frac{n_2}{n_1})} - \frac{n_1{}^{1-K}}{K(K-1)} + \frac{1}{K-1}\right]\left(\frac{dx}{dn}\right)_{n_0} + $$
$$+ \int_1^{n_1} R\, c\, dn ,$$

$$y(n_2,\phi) =$$

$$= \left[\frac{-n_1^{1-K}}{K} e^{K(1 - \frac{n_2}{n_1})} - \frac{n_1^{1-K}}{K(K-1)} + \frac{1}{K-1}\right]\left(\frac{dy}{dn}\right)_{n_0} +$$

$$+ g\left[\frac{-n_1^{1-K} e^{K(1 - \frac{n_2}{n_1})}}{K(K+1)} - n_1^2 e^{K(1 - \frac{n_2}{n_1})} \frac{1}{K^2(K+1)} -\right.$$

$$- \frac{n_1^{1-K}}{K(K+1)(K-1)} + \frac{n_1^2(K^2 + 2K + 2)}{2K^2(K+1)} -$$

$$\left. - \frac{n_1 n_2}{K} + \frac{1}{2(K-1)}\right] + \int_1^{n_1} R\, s\, dn ,$$

$$R = \frac{1}{K(1-K)} n_1^{1-K} n^{K-1} - \frac{1}{1-K} - \frac{n_1^{1-K} n^{K-1}}{K} e^{K(1 - \frac{n_2}{n_1})} .$$

Comparing these expressions with equations (4.3) and (4.4), we find that

$$f = \left[\frac{-n_1^{1-K}}{K} e^{K(1 - \frac{n_2}{n_1})} - \frac{n_1^{1-K}}{K(K-1)} + \frac{1}{K-1}\right]\left(\frac{dx}{dn}\right)_{n_0} ,$$

$$j = n^{K-1}\left[\frac{-n_1^{1-K}}{K} e^{K(1 - \frac{n_2}{n_1})} - \frac{n_1^{1-K}}{K(K-1)}\right] - \frac{1}{1-K} ,$$

$$h = 0 ,$$

$$\bar{f} = \left[\frac{-n_1^{1-K}}{K} e^{K(1 - \frac{n_2}{n_1})} - \frac{n_1^{1-K}}{K(K-1)} + \frac{1}{1-K}\right]\left(\frac{dy}{dn}\right)_{n_o} +$$

$$+ g \left[\frac{-n_1^{1-K} e^{K(1 - \frac{n_2}{n_1})}}{K(K+1)} - \frac{n_1^2 e^{K(1 - \frac{n_2}{n_1})}}{K^2(K+1)} -\right.$$

$$- \frac{n_1^{1-K}}{K(K+1)(K-1)} + n_1^2 \frac{K^2 + 2K + 2}{2K^2(K+1)} - \frac{n_1 n_2}{K} +$$

$$\left. + \frac{1}{2(K-1)}\right] ,$$

$$\bar{j} = 0 ,$$

$$\bar{h} = n^{K-1}\left[- \frac{n_1^{1-K}}{K} e^{K(1 - \frac{n_2}{n_1})} - \frac{n_1^{1-K}}{K(K-1)}\right] - \frac{1}{1-K} .$$

Again equation (4.14) shows that <u>the thrust direction is fixed in the plane of motion</u>. From the relation (4.17) we have

$$\frac{P^2}{K}(n_1^K - 1) + \tag{6.18}$$

$$+ s \left[\left(\frac{dy}{dn}\right)_{n_o} + \frac{g}{K+1} + \frac{g n_1^{K+1}}{K(K+1)} - \frac{g n_1^{K+1}}{K} e^{-K(1 - \frac{n_2}{n_1})}\right] +$$

$$+ c \left(\frac{dx}{dn}\right)_{n_o} = 0 .$$

Formulas similar to (6.10) through (6.13) may be written. The only difference between equations (6.18) and (6.9) lies in the multiplier of s. We shall not explicitly list these results.

VII

ZERO DRAG AND LINEAR LIFT

Continuous Burning

When linear lift and negligible drag are assumed, the equations of motion may be written

$$(7.1) \qquad \frac{d^2x}{dn^2} + \frac{L}{n}\frac{dy}{dn} = \frac{c}{n} ,$$

$$(7.2) \qquad \frac{d^2y}{dn^2} - \frac{L}{n}\frac{dx}{dn} = \frac{s}{n} - g .$$

Eliminating y from equation (7.1) and x from equation (7.2), we obtain the relations

$$(7.3) \qquad \frac{d^3x}{dn^3} + \frac{1}{n}\frac{d^2x}{dn^2} + \frac{L^2}{n^2}\frac{dx}{dn} = -\frac{Ls}{n^2} + \frac{gL}{n} + \frac{1}{n}\frac{dc}{dn} ,$$

$$(7.4) \qquad \frac{d^3y}{dn^3} + \frac{1}{n}\frac{d^2y}{dn^2} + \frac{L^2}{n^2}\frac{dy}{dn} = \frac{Lc}{n^2} - \frac{g}{n} + \frac{1}{n}\frac{ds}{dn} .$$

The adjoint is

$$\frac{d^3G}{dn^3} - \frac{d^2}{dn^2}\left(\frac{G}{n}\right) + \frac{d}{dn}\left(\frac{L^2G}{n^2}\right) = 0 .$$

Simplifying this, we may write

$$(7.5) \qquad \frac{d^3G}{dn^3} - \frac{1}{n}\frac{d^2G}{dn^2} + \frac{2 + L^2}{n^2}\frac{dG}{dn} - \frac{2 + 2L^2}{n^3}G = 0 .$$

We shall attempt to find solutions of equation (7.5) of the form

$$(7.6)\qquad G = n^{\alpha} .$$

Combining equations (7.5) and (7.6), we find that α must satisfy the characteristic equation

$$\alpha^3 - 4\alpha^2 + (5 + L^2)\alpha - (2 + 2L^2) = 0 ,$$

which has solutions

$$\alpha = 2 , \quad 1 + iL , \quad 1 - iL .$$

Hence, the solutions (7.6) are of the form

$$n^2 , \quad n^{1+iL} , \quad n^{1-iL} ,$$

with real parts

$$n^2 , \quad n \cos(L \ln n) , \quad n \sin(L \ln n) .$$

If we choose

$$(7.7)\qquad G(n,q) = \frac{-qn \cos(L \ln \frac{n}{q})}{(L^2 + 1)} - \frac{qn \sin(L \ln \frac{n}{q})}{L(L^2 + 1)} + \frac{n^2}{L^2 + 1}$$

as the solution of equation (7.5), this satisfies the conditions

$$(7.8)\qquad G(q,q) = \left(\frac{dG}{dn}\right)_{q,q} = 0 , \qquad \left(\frac{d^2G}{dn^2}\right)_{q,q} = 1 .$$

By use of relation (7.8) and the standard technique for solving an equation with the aid of the adjoint--see equations (6.5), (6.6)--we obtain

$$(7.9) \quad x(n_1,\phi) = G(1,n_1)\left[c - L\frac{dy}{dn}\right]_{n=1} + \left[G(1,n_1) - \left(\frac{dG}{dn}\right)_{1,n_1}\right]\left(\frac{dx}{dn}\right)_1 + \int_1^{n_1} G(n,n_1)\left[\frac{-Ls}{n^2} + \frac{gL}{n} + \frac{1}{n}\frac{dc}{dn}\right] dn \;,$$

$$(7.10) \quad y(n_1,\phi) = G(1,n_1)\left[s - g + L\frac{dx}{dn}\right]_{n=1} + \left[G(1,n_1) - \left(\frac{dG}{dn}\right)_{1,n_1}\right]\left(\frac{dy}{dn}\right)_1 + \int_1^{n_1} G(n,n_2)\left[\frac{Lc}{n^2} - \frac{g}{n} + \frac{1}{n}\frac{ds}{dn}\right] dn \;.$$

These equations can be simplified. Integration by parts and use of relation (7.8) shows that

$$(7.11) \quad \int_1^{n_1} \frac{G(n,n_1)}{n}\frac{dc}{dn}\, dn = - G(1,n_1)(c)_1 - \int_1^{n_1} c\frac{d}{dn}\left(\frac{G}{n}\right) dn \;,$$

and an equation similar to this is valid if c is replaced by s. Use of equations (7.7) and (7.11) shows that relations (7.9) and (7.10) may be written

(7.12) $x(n_1,\phi) =$

$$= \frac{1}{L^2+1}\left[n_1 L\cos(L\ln n_1) - n_1\sin(L\ln n_1) - L\right]\left(\frac{dy}{dn}\right)_{n_0} +$$

$$+ \frac{1}{L^2+1}\left[n_1\cos(L\ln n_1) + n_1 L\sin(L\ln n_1) - 1\right]\left(\frac{dx}{dn}\right)_{n_0} -$$

$$- \frac{gn_1}{L^2+1}\sin(L\ln n_1) -$$

$$- \frac{1}{L^2+1}\int_1^{n_1}\left[-n_1\cos\left(L\ln\frac{n}{n_1}\right) - \frac{n_1}{L}\sin\left(L\ln\frac{n}{n_1}\right) + n\right]\frac{s}{n}\,dn -$$

$$- \frac{1}{L^2+1}\int_1^{n_1}\left[n_1 L\sin\left(L\ln\frac{n}{n_1}\right) - n_1\cos\left(L\ln\frac{n}{n_1}\right) + n\right]\frac{c}{n}\,dn,$$

and

(7.13) $y(n_1,\phi) =$

$$= \frac{-1}{L^2+1}\left[n_1 L\cos(L\ln n_1) - n_1\sin(L\ln n_1) - L\right]\left(\frac{dx}{dn}\right)_{n_0} +$$

$$+ \frac{1}{L^2+1}\left[n_1\cos(L\ln n_1) + n_1 L\sin(L\ln n_1) - 1\right]\left(\frac{dy}{dn}\right)_{n_0} +$$

$$+ \frac{g}{L^2+1}\left[n_1\cos(L\ln n_1) - 1\right] +$$

$$+ \frac{L}{L^2+1}\int_1^{n_1}\left[-n_1\cos\left(L\ln\frac{n}{n_1}\right) - \frac{n_1}{L}\sin\left(L\ln\frac{n}{n_1}\right) + n\right]\frac{c}{n}\,dn -$$

$$- \frac{1}{L^2+1}\int_1^{n_1}\left[n_1 L\sin\left(L\ln\frac{n}{n_1}\right) - n_1\cos\left(L\ln\frac{n}{n_1}\right) + n\right]\frac{s}{n}\,dn$$

Comparing these with equations (4.5) and (4.6), we find that

(7.14)

$$u = \frac{1}{L^2+1}\left[n_1 L\cos(L\ln n_1) - n_1\sin(L\ln n_1) - L\right]\left(\frac{dy}{dn}\right)_{n_0} +$$

$$+ \frac{1}{L^2+1}\left[n_1\cos(L\ln n_1) + n_1 L\sin(L\ln n_1) - 1\right]\left(\frac{dx}{dn}\right)_{n_0} -$$

$$- g\,\frac{n_1}{L^2+1}\sin(L\ln n_1)\ ,$$

$$(7.15)\quad v = -\frac{1}{L^2+1}\left[\frac{n_1 L\sin(L\ln\frac{n}{n_1}) - n_1\cos(L\ln\frac{n}{n_1}) + n}{n}\right],$$

$$(7.16)\quad w = \frac{-1}{L^2+1}\left[\frac{-n_1 L\cos(L\ln\frac{n}{n_1}) - n_1\sin(L\ln\frac{n}{n_1}) + nL}{n}\right],$$

(7.17)

$$\overline{u} = \frac{1}{L^2+1}\left[n_1 L\cos(L\ln n_1) - n_1\sin(L\ln n_1) - L\right]\left(\frac{dx}{dn}\right)_{n_0} +$$

$$+ \frac{1}{L^2+1}\left[n_1\cos(L\ln n_1) + n_1 L\sin(L\ln n_1) - 1\right]\left(\frac{dy}{dn}\right)_{n_0} +$$

$$+ \frac{g}{L^2+1}\left[n_1\cos(L\ln n_1) - 1\right]\ ,$$

$$(7.18)\quad \overline{v} = \frac{1}{L^2+1}\left[\frac{-n_1 L\cos(L\ln\frac{n}{n_1}) - n_1\sin(L\ln\frac{n}{n_1}) + nL}{n}\right],$$

$$\overline{w} = \frac{-1}{L^2+1}\left[\frac{n_1 L\sin(L\ln\frac{n}{n_1}) - n_1\cos(L\ln\frac{n}{n_1}) + n}{n}\right].$$

From equation (4.18), we have

(7.19)

$$\frac{s}{c} = \frac{-n_1(L+\lambda)\sin(L \ln \frac{n}{n_1}) + n_1(1-\lambda L)\cos(L \ln \frac{n}{n_1}) - n(1-\lambda L)}{-n_1(L+\lambda)\cos(L \ln \frac{n}{n_1}) - n_1(1-\lambda L)\sin(L \ln \frac{n}{n_1}) + n(L+\lambda)} .$$

Thus, we see that the thrust direction varies from point to point on the optimum trajectory. In order to obtain the value of λ, one must solve the algebraic equation corresponding to equation (4.15) for the case of continuous burning, which is

(7.20)
$$\bar{u}_{n_1} - \lambda u_{n_1} \pm$$

$$\pm P \int_1^{n_1} \frac{(\bar{v}_{n_1} - \lambda v_{n_1})(\bar{v}-\lambda v) + (\bar{w}_{n_1} - \lambda w_{n_1})(\bar{w}-\lambda w)}{\sqrt{(\bar{v}-\lambda v)^2 + (\bar{w}-\lambda w)^2}} \, dn = 0 .$$

Because of the complexity of the formulas for u, $\bar{u}$, etc., there seems to be little hope of finding the general solution of this equation. However, if we require that λ be constant, then equation (7.20) may be simplified. First, we note that in general

(7.21)
$$\frac{d}{dn_1} \sqrt{(\bar{v}-\lambda v)^2 + (\bar{w}-\lambda w)^2} =$$

$$= \frac{\left[(\bar{v}_{n_1} - \lambda v_{n_1})(\bar{v}-\lambda v) + (\bar{w}_{n_1} - \lambda w_{n_1})(\bar{w}-\lambda w)\right] - \lambda_{n_1}\left[v\bar{v} + w\bar{w} - \lambda(v^2+w^2)\right]}{\sqrt{(\bar{v}-\lambda v)^2 + (\bar{w}-\lambda w)^2}} .$$

Since λ is assumed to be constant, the expression under the integral sign in expression (7.20) can be simplified by use of equation (7.21). Substituting equations (7.14) through (7.18) and (7.21) into the relation (7.20), we may write

$$(7.22)\quad \left[\sin(L \ln n_1) - \lambda\cos(L \ln n_1)\right]\left(\frac{dx}{dn}\right)_{n_o} +$$

$$+ \left[\cos(L \ln n_1) + \lambda\sin(L \ln n_1)\right]\left(\frac{dy}{dn}\right)_{n_o} +$$

$$+ \frac{g}{L^2+1}\left[(1 + \lambda L)\cos(L \ln n_1) + (-\lambda + L)\sin(L \ln n_1)\right] +$$

$$+ \frac{P}{L^2+1}\sqrt{L^2+\lambda^2+\lambda^2L^2+1}\int_1^{n_1}\frac{d}{dn_1}\sqrt{n_1^2+1-2n_1 n\cos(L \ln \frac{n}{n_1})}\, dn = 0.$$

For arbitrarily chosen constants λ and n_1 $(0 < n_1 \leqq 1)$, the expression (7.22) determines a linear relation between the initial components of velocity, so that the craft may follow an optimum trajectory. By substituting this value of λ into equation (7.19), one obtains the desired thrust direction schedule for the optimum trajectory.

Burning and Coasting

During the coasting phase $n_2 < n < n_1$, the equations of motion are

$$\frac{d^2x}{dn^2} + \frac{L}{n_1}\frac{dy}{dn} = 0 ,$$

$$\frac{d^2y}{dn^2} - \frac{L}{n_1}\frac{dx}{dn} = - g .$$

The equations corresponding to (7.3) and (7.4) are

$$\frac{d^3x}{dn^3} + \frac{L^2}{n_1^2}\frac{dx}{dn} = \frac{Lg}{n_1} ,$$

$$\frac{d^3y}{dn^3} + \frac{L^2}{n_1}\frac{dy}{dn} = 0 ,$$

of which the integrals are easily found to be

(7.23) $x(n_2,\phi) =$

$$= -\left[\frac{n_1}{L} - \frac{n_1}{L}\cos\frac{L}{n_1}(n_1-n_2)\right]\left(\frac{dy}{dn}\right)_{n_1} - \frac{n_1}{L}\sin\frac{L}{n_1}(n_1-n_2)\left(\frac{dx}{dn}\right)_{n_1} +$$

$$+ x(n_1,\phi) + g\left[\frac{n_1}{L}(n_2-n_1) + \frac{n_1^2}{L^2}\sin\frac{L}{n_1}(n_1-n_2)\right] ,$$

(7.24) $y(n_2,\phi) =$

$$= \left[\frac{n_1}{L} - \frac{n_1}{L}\cos\frac{L}{n_1}(n_1-n_2)\right]\left(\frac{dx}{dn}\right)_{n_1} - \frac{n_1}{L}\sin\frac{L}{n_1}(n_1-n_2)\left(\frac{dy}{dn}\right)_{n_1} +$$

$$+ y(n_1,\phi) - g\left[\frac{n_1^2}{L^2} - \frac{n_1^2}{L^2}\cos\frac{L}{n_1}(n_1-n_2)\right] .$$

The boundary conditions at modified time $n = n_1$ for x, y and their derivatives are obtained from equations (7.12) and (7.13) and their derivatives. Thus, by differentiating equations

(7.12) and (7.13), we get

$$\left(\frac{dx}{dn}\right)_{n_1} = -\sin(L \ln n_1)\left(\frac{dy}{dn}\right)_{n_0} + \cos(L \ln n_1)\left(\frac{dx}{dn}\right)_{n_0} +$$

$$+\int_1^{n_1}\left[\frac{s}{n}\sin(L \ln \frac{n}{n_1}) + \frac{c}{n}\cos(L \ln \frac{n}{n_1})\right] dn -$$

$$-\frac{g}{L^2+1}\left[\sin(L \ln n_1) + L\cos(L \ln n_1)\right] ,$$

and

$$\left(\frac{dy}{dn}\right)_{n_1} = \sin(L \ln n_1)\left(\frac{dx}{dn}\right)_{n_0} + \cos(L \ln n_1)\left(\frac{dy}{dn}\right)_{n_0} +$$

$$+\int_1^{n_1}\left[\frac{s}{n}\cos(L \ln \frac{n}{n_1}) - \frac{c}{n}\sin(L \ln \frac{n}{n_1})\right] dn +$$

$$+\frac{g}{L^2+1}\left[\cos(L \ln n_1) - L\sin(L \ln n_1)\right] .$$

Substituting these into equations (7.23) and (7.24), we obtain

(7.25) $x(n_2,\phi) =$

$$= \left(\frac{dx}{dn}\right)_{n_0}\left\{\frac{-n_1}{L}\sin(L\ln n_1) - \frac{n_1}{L}\sin\left[\frac{-L}{n_1}(n_1-n_2)-L\ln n_1\right] + \right.$$

$$\left. + \frac{1}{L^2+1}\left[n_1\cos(L\ln n_1) + n_1 L\sin(L\ln n_1) - 1\right]\right\} +$$

$$+ \left(\frac{dy}{dn}\right)_n \left\{\frac{-n_1}{L}\cos(L\ln n_1) + \frac{n_1}{L}\cos\left[\frac{L}{n_1}(n_1-n_2)-L\ln n_1\right] + \right.$$

$$\left. + \frac{1}{L^2+1}\left[n_1 L\cos(L\ln n_1) - n_1\sin(L\ln n_1) - L\right]\right\} +$$

$$+ \int_1^{n_1}\frac{s}{n}\left\{\frac{-n_1}{L}\cos(L\ln\frac{n}{n_1}) + \frac{n_1}{L}\cos\left[\frac{L}{n_1}(n_1-n_2)+L\ln\frac{n}{n_1}\right]\right\}dn +$$

$$+ \frac{g}{L^2+1}\left\{\frac{-n_1}{L}\cos(L\ln n_1) + \frac{n_1}{L}\cos\left[\frac{L}{n_1}(n_1-n_2)-L\ln n_1\right] - \right.$$

$$\left. - n_1\sin\left[\frac{L}{n_1}(n_1-n_2)-L\ln n_1\right]\right\} +$$

$$+ g\left[\frac{n_1}{L}(n_2-n_1) + \frac{n_1^2}{L^2}\sin\frac{L}{n_1}(n_1-n_2)\right] +$$

$$+ \frac{1}{L^2+1}\int_1^{n_1}\frac{s}{n}\left[n_1 L\cos(L\ln\frac{n}{n_1})+n_1\sin(L\ln\frac{n}{n_1})-nL\right]dn -$$

$$- \frac{1}{L^2+1}\int_1^{n_1}\frac{c}{n}\left[n_1 L\sin(L\ln\frac{n}{n_1})-n_1\cos(L\ln\frac{n}{n_1})+n\right]dn +$$

$$+ \int_1^{n_1}\frac{c}{n}\left\{\frac{n_1}{L}\sin(L\ln\frac{n}{n_1}) - \frac{n_1}{L}\sin\left[\frac{L}{n_1}(n_1-n_2)+L\ln\frac{n}{n_1}\right]\right\}dn,$$

and

(7.26) $y(n_2,\phi) =$

$$= \left(\frac{dx}{dn}\right)_{n_0}\left\{\frac{n_1}{L}\cos(L\ln n_1) - \frac{n_1}{L}\cos\left[\frac{L}{n_1}(n_1-n_2)-L\ln n_1\right] - \right.$$

$$\left. - \frac{1}{L^2+1}\left[n_1 L\cos(L\ln n_1) - n_1\sin(L\ln n_1) - L\right]\right\} +$$

$$+ \left(\frac{dy}{dn}\right)_{n_0}\left\{\frac{-n_1}{L}\sin(L\ln n_1) - \frac{n_1}{L}\sin\left[\frac{+L}{n_1}(n_1-n_2)-L\ln n_1\right] + \right.$$

$$\left. + \frac{1}{L^2+1}\left[n_1\cos(L\ln n_1) + n_1 L\sin(L\ln n_1) - 1\right]\right\} +$$

$$+ \int_1^{n_1}\frac{s}{n}\left\{\frac{n_1}{L}\sin(L\ln\frac{n}{n_1}) - \frac{n_1}{L}\sin\left[\frac{L}{n_1}(n_1-n_2)+L\ln\frac{n}{n_1}\right]\right\} dn +$$

$$+ \int_1^{n_1}\frac{c}{n}\left\{\frac{n_1}{L}\cos(L\ln\frac{n}{n_1}) - \frac{n_1}{L}\cos\left[\frac{L}{n_1}(n_1-n_2)+L\ln\frac{n}{n_1}\right]\right\} dn +$$

$$+ \frac{g}{L^2+1}\left\{\frac{-n_1}{L}\sin(L\ln n_1) + n_1\cos\left[\frac{L}{n_1}(n_1-n_2) - L\ln n_1\right] - \right.$$

$$\left. - \frac{n_1}{L}\sin\left[\frac{L}{n_1}(n_1-n_2) - L\ln n_1 - 1\right]\right\} -$$

$$- g\left[\frac{n_1^2}{L^2} - \frac{n_1^2}{L^2}\cos\frac{L}{n_1}(n_1-n_2)\right] -$$

$$- \frac{1}{L^2+1}\int_1^{n_1}\frac{s}{n}\left[n_1 L\sin(L\ln\frac{n}{n_1})-n_1\cos(L\ln\frac{n}{n_1})+n\right] dn +$$

$$+ \frac{1}{L^2+1}\int_1^{n_1}\frac{c}{n}\left[-n_1 L\cos(L\ln\frac{n}{n_1})-n_1\sin(L\ln\frac{n}{n_1})+nL\right] dn .$$

It is clear from the form of equations (7.25) and (7.26) that the thrust direction varies from point to point on the optimum trajectory. The determination of optimum path thrust schedules is very complicated in the general case, for equation (4.15) must be solved. However, particular solutions of equation (4.15) corresponding to a constant λ can be obtained. The final formula is slightly more complicated than (7.22); hence, we shall not derive it explicitly.

VIII

OTHER VARIATIONAL PROBLEMS OF THE SAME TYPE

We saw in Section VII that the problem of maximizing x for a fixed y, or vice versa, either at the end of the total flight time or at the end of the burning period, with variable burning time and variable thrust direction assumed, is rather complicated when lift is involved. A simpler problem of the same type, involving lift, is that of maximizing x or y at the end of the burning period or at the end of the total flight time when both burning time and coasting time are preassigned and thrust direction is variable. The thrust direction schedule can be obtained immediately by applying direct variations to equations (7.12) and (7.25), or (7.13) and (7.26).

Maximum x for Continuous Burning, Zero Drag and Linear Lift

From equation (7.12) we find that

$$\int_1^{n_1} \left\{ \left[n_1 \; L \cos(L \ln \frac{n}{n_1}) + n_1 \sin(L \ln \frac{n}{n_1}) - Ln \right] \frac{s}{n} + \right.$$

$$\left. + \left[n_1 \; L \sin L(\ln \frac{n}{n_1}) - n_1 \cos(L \ln \frac{n}{n_1}) + n \right] \frac{c}{n} \right\} \delta\phi \; dn = 0.$$

If the integrand of this relation is to vanish for arbitrary $\delta\phi$, then it follows that

$$\tan \phi_x = - \frac{n_1 \; L \sin(L \ln \frac{n}{n_1}) - n_1 \cos(L \ln \frac{n}{n_1}) + n}{n_1 \; L \cos(L \ln \frac{n}{n_1}) + n_1 \sin(L \ln \frac{n}{n_1}) - Ln},$$

and this formula furnishes the desired thrust direction schedule for the optimum path.

Maximum y for Continuous Burning, Zero Drag and Linear Lift

From equation (7.13) we have the result

$$\tan \phi_y = - \cot \phi_x .$$

Maximum x for Burning and Coasting, Zero Drag and Linear Lift

From equation (7.25) we see that

$$\int_1^{n_1} \frac{c}{n} \left\{ \frac{-n_1}{L} \cos\left(L \ln \frac{n}{n_1}\right) + \frac{n_1}{L} \cos \left[\frac{L}{n_1}(n_1 - n_2) + L \ln \frac{n}{n_1}\right] + \right.$$

$$\left. + \frac{1}{L^2 + 1}\left[n_1 L \cos\left(L \ln \frac{n}{n_1}\right) + n_1 \sin\left(L \ln \frac{n}{n_1}\right) - nL\right]\right\} \delta\phi \, dn +$$

$$+ \frac{s}{n} \left\{ \frac{-n_1}{L} \sin\left(L \ln \frac{n}{n_1}\right) + \frac{n_1}{L} \sin \left[\frac{L}{n_1}(n_1 - n_2) + L \ln \frac{n}{n_1}\right] + \right.$$

$$\left. + \frac{1}{L^2 + 1}\left[n_1 L \sin\left(L \ln \frac{n}{n_1}\right) - n_1 \cos\left(L \ln \frac{n}{n_1}\right) + n\right]\right\} \delta\phi \, dn = 0.$$

Consequently,

$$\tan \phi_x =$$

$$= -\left\{\frac{-n_1}{L}\cos\left(L \ln \frac{n}{n_1}\right) + \frac{n_1}{L}\cos\left[\frac{L}{n_1}(n_1 - n_2) + L \ln \frac{n}{n_1}\right] + \right.$$

$$\left. + \frac{1}{L^2+1}\left[n_1 L \cos\left(L \ln \frac{n}{n_1}\right) + n_1 \sin\left(L \ln \frac{n}{n_1}\right) - nL\right]\right\} \div$$

$$\div \left\{\frac{-n_1}{L}\sin\left(L \ln \frac{n}{n_1}\right) + \frac{n_1}{L}\sin\left[\frac{L}{n_1}(n_1 - n_2) + L \ln \frac{n}{n_1}\right] + \right.$$

$$\left. + \frac{1}{L^2+1}\left[n_1 L \sin\left(L \ln \frac{n}{n_1}\right) - n_1 \cos\left(L \ln \frac{n}{n_1}\right) + n\right]\right\} .$$

Maximum y for Burning and Coasting, Zero Drag and Linear Lift

From equation (7.26) we find that

$$\tan \phi_y = - \cot \phi_x .$$

APPENDIX A

THE PROBLEM OF ZERO DRAG AND LIFT IN UNMODIFIED TERMS; BURNING AND COASTING

For the sake of simplicity, we shall consider optimization in the case of zero lift and drag, burning and coasting, directly in unmodified terms. We may write the equations of motion in the form

$$M_o(1 - \dot{r}t)\,\frac{d^2x}{dt^2} = T\cos\phi \;,$$

$$M_o(1 - \dot{r}t)\,\frac{d^2y}{dt^2} = T\sin\phi - \bar{g} \;,$$

and integrate to yield the respective velocity components at time t

$$\dot{x} = \dot{x}_o + \int_0^t \frac{T\cos\phi}{M_o(1 - \dot{r}t)}\,dt \;,$$

$$\dot{y} = \dot{y}_o - \bar{g}t + \int_0^t \frac{T\sin\phi}{M_o(1 - \dot{r}t)}\,dt \;;$$

and the co-ordinates of instanteneous position

$$x = \dot{x}_o t + \int_0^t \int_0^{\tau} \frac{T\cos\phi}{M_o(1 - \dot{r}\sigma)}\,d\sigma\,d\tau \;,$$

$$y = \dot{y}_0 t - \frac{1}{2}\bar{g}t^2 + \int_0^t \int_0^{\tau} \frac{T \sin \phi}{M_0(1 - \dot{r}\sigma)} \, d\sigma \, d\tau \; .$$

If the rocket burns fuel during the interval $(0, t_1)$, integration by parts gives, as the position co-ordinates of the rocket at some later time $t = t_2$,

$$\text{(A.1)} \qquad x_2 = \dot{x}_0 t_2 + \int_0^{t_1} (t_2 - t) \frac{T \cos \phi}{M_0(1 - \dot{r}t)} \, dt \; ,$$

$$\text{(A.2)} \qquad y_2 = \dot{y}_0 t_2 - \frac{1}{2}\bar{g}t_2^2 + \int_0^{t_1} (t_2 - t) \frac{T \sin \phi}{M_0(1 - \dot{r}t)} \, dt \; .$$

For an optimum path we require that x_2 be a maximum for fixed y_2 or y_2 be a maximum for fixed x_2. Along the path the thrust angle is some function of time t and flight time t_2. Thus, for an optimum path we require that

$$\delta x_2 = \left[\dot{x}_0 + \int_0^{t_1} \frac{T \cos \phi}{M_0(1 - \dot{r}t)} \, dt\right] \delta t_2 - \int_0^{t_1} (t_2 - t) \frac{T \sin \phi}{M_0(1 - \dot{r}t)} \, \delta\phi \, dt = 0 \; ,$$

$$\delta y_2 = \left[\dot{y}_0 - \bar{g}t_2 + \int_0^{t_1} \frac{T \sin \phi}{M_0(1 - \dot{r}t)} \, dt\right] \delta t_2 + \int_0^{t_1} (t_2 - t) \frac{T \cos \phi}{M_0(1 - \dot{r}t)} \, \delta\phi \, dt = 0 \; .$$

From these equations we get

(A.3)
$$\frac{\dot{y}_0 - \bar{g}t_2 + \int_0^{t_1} \frac{T \sin \phi}{M_0(1 - \dot{r}t)} dt}{\dot{x}_0 + \int_0^{t_1} \frac{T \cos \phi}{M_0(1 - \dot{r}t)} dt} \int_0^{t_1} (t_2 - t)\frac{T \sin \phi}{M_0(1-\dot{r}t)} \delta\phi \, dt + \int_0^{t_1} (t_2 - t) \frac{T \cos \phi}{M_0(1 - \dot{r}t)} \delta\phi \, dt = 0 .$$

The substitution

(A.4)
$$\lambda = \frac{\dot{y}_0 - \bar{g}t_2 + \int_0^{t_1} \frac{T \sin \phi}{M_0(1 - \dot{r}t)} dt}{\dot{x}_0 + \int_0^{t_1} \frac{T \cos \phi}{M_0(1 - \dot{r}t)} dt} ,$$

where λ is independent of t, permits us to write equation (A.3) in the form

$$\int_0^{t_1} (t_2 - t) \frac{T}{M_0(1 - \dot{r}t)} \left[\cos \phi + \lambda \sin \phi \right] \delta\phi \, dt = 0 .$$

Since $\delta\phi$ is an arbitrary function of t the bracketed expression above vanishes, so that

$$\cot \phi = - \lambda .$$

Consequently, thrust angle ϕ is independent of t, and therefore for an optimum path thrust direction is fixed in the plane of motion.

Now equations (A.1), (A.2) and (A.4) become respectively

$$\text{(A.5)} \qquad x_2 = \dot{x}_0 t_2 + \frac{T \cos \phi}{M_0} \int_0^{t_1} \frac{t_2 - t}{1 - \dot{r}t}\, dt \,,$$

$$\text{(A.6)} \qquad y_2 = \dot{y}_0 t_2 + \frac{1}{2}\bar{g} t_2^2 + \frac{T \sin \phi}{M_0} \int_0^{t_1} \frac{t_2 - t}{1 - \dot{r}t}\, dt \,,$$

$$\text{(A.7)} \qquad \dot{x}_0 \cos \phi + (\dot{y}_0 - \bar{g} t_2) \sin \phi = \frac{T}{M_0 \dot{r}} \ln (1 - \dot{r} t_1) \,.$$

For a given launching velocity and a specified burning time we have three equations in x_2, y_2, t_2 and ϕ. For fixed x_2 or fixed y_2 equations (A.5), (A.6) and (A.7) give respectively the values for y_2, t_2, ϕ or x_2, t_2, ϕ for the optimum path.

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