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TWO IRROTATIONAL SUPERSONIC FLOWS WITH HELICAL STREAM LINES

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The purpose of this research was to determine characteristic systems for steady, supersonic flows of a polytropic gas and to apply the results to the study of: (1) simple waves for plane rotational motion; (2) some irrotational three-dimensional motions. First, intrinsic forms of the characteristic systems were obtained. These equations relate the directional derivatives of the magnitude of the velocity, q, and the sound speed, c, and curvatures of the characteristic manifolds. With the aid of these relations, it was shown in our first report that: (1) for plane rotational isentropic motion of a polytropic gas, simple waves (straight-line bicharacteristics) exist only at Mach number one; (2) the bicharacteristics are radial straight lines in this case; (3) the flow is a vortex flow which possesses a limiting circle (or arc of a circle). Further, it was shown that a class of plane rotational flows exists for which the Mach number is constant along each bicharacteristic. In the second report, two supersonic irrotational flows whose stream lines are space curves are studied. These flows are characterized by the fact that one family of ∞^1 characteristic surfaces are parallel planes. The stream lines are helices in one case and winding curves lying on right circular cones in the other case. The Mach number in each case varies with the radius of the cross section of the cylinder or cone.

ABSTRACT

The case where one family of characteristic surfaces are ∞^1 parallel planes is considered. By use of the intrinsic form of the characteristic relations as derived in a previous report, it is shown that (in the present case) two flow patterns of a polytropic gas can be determined. These flows are three-dimensional, steady, supersonic, irrotational, and isentropic. In the first flow, the stream lines are helices, along which the Mach number is constant, and the bicharacteristics are concentric circles. For the second flow, the stream lines are space curves which wind along right circular cones; any given stream line, in the limit, makes a fixed angle with every generator of the cone. The Mach number varies with the radius of the cross section of any cone and the bicharacteristics are these circular cross sections. The relation between the Mach number and the radius of the cylinder of cross section of the cone is determined. It is shown that as this radius goes from some limiting value to infinity, the Mach number goes from infinity to one.

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1. INTRODUCTION

In a previous report, we obtained an intrinsic formulation of the characteristic relations for the steady supersonic flow of a compressible fluid.[1] The present report is concerned with a study of two classes of irrotational, isentropic flows both of which possess a family of parallel planes as characteristic surfaces. In the first class of flows, every family of ∞^1 bicharacteristics forms right circular cylinders with generators perpendicular to the family of plane characteristic surfaces. The resulting flows possess helical stream lines lying on these right circular cylinders; the magnitude of the velocity q is constant along a given helix; the value of q in terms of the radius r of the cylinder is determined. The second class of flows is characterized by the fact that every family of ∞^1 bicharacteristics forms right circular cones; the stream lines are helix-like curves lying on these cones; the value of q is determined in terms of the radius of a cross section of a cone.

2. A CLASS OF SPACE FLOWS

For plane flows, the two families of characteristic surfaces are right cylinders with parallel generators. Here, we consider _the case where one family of characteristic surfaces are parallel planes_. We shall show that this condition defines a family of space flows.

We recall[2] that in section 3* [see the equations following (3.1)*] we assumed that the _ordered_ triad of unit vectors (t^j, n^j, ℓ^j) form a right-hand system. Thus, if the characteristic planes are assumed to be perpendicular to the z-axis of an x,y,z Cartesian orthogonal coordinate system, and n^j is sensed in the positive direction of the z-axis, then the _ordered_ pair $(t^j, -\ell^j)$ form a right-hand set in _any plane orthogonal to the z-axis_. Let us introduce into any such plane, z = constant, a family of parameter curves, α = variable, along t^j and a family of parameter curves, β = variable, along ℓ^j. Then the arc-length element in any such plane is

$$ds^2 = (Ad\alpha)^2 + (Bd\beta)^2 , \qquad (2.1)$$

where A and B are functions of α, β, and z. If $\theta(\alpha,\beta,z)$ denotes the angle between the x-axis and t^j, then the unit vectors t^j, ℓ^j have Cartesian orthogonal components

$$
\begin{aligned}
t^j &\rightarrow (\cos\theta, \sin\theta, 0) \quad , \\
\ell^j &\rightarrow (\sin\theta, -\cos\theta, 0) \quad .
\end{aligned}
\tag{2.2}
$$

From these last equations, it follows that in any plane, z = constant, the α,β parameter curves are determined by

$$
\begin{aligned}
\frac{\partial x}{\partial \alpha} &= A\cos\theta, \quad \frac{\partial y}{\partial \alpha} = A\sin\theta \quad , \\
\frac{\partial x}{\partial \beta} &= B\sin\theta, \quad \frac{\partial y}{\partial \beta} = -B\cos\theta \quad .
\end{aligned}
\tag{2.3}
$$

Further, the curvatures $\kappa, \bar{\kappa}$ of the α = variable, β = variable, curves are

$$\kappa = -\frac{1}{A}\frac{\partial\theta}{\partial\alpha} = -\frac{1}{AB}\frac{\partial A}{\partial\beta} \quad , \tag{2.4}$$

$$\bar{\kappa} = \frac{1}{B}\frac{\partial\theta}{\partial\beta} = -\frac{1}{AB}\frac{\partial B}{\partial\alpha} \quad . \tag{2.5}$$

The last relations in (2.4),(2.5) are obtained by partial differentiation of (2.3). These relations imply that the integrability conditions of (2.3) are satisfied, or that the Riemann tensor for the plane vanishes. Note that in forming the partials of (2.4),(2.5), z is constant. Thus, if we write the variables which are kept constant, then

$$\frac{\partial\theta}{\partial\alpha} = \left.\frac{\partial\theta}{\partial\alpha}\right)_{z,\beta} \quad , \text{ etc.} \tag{2.6}$$

We digress briefly to consider the significance of (2.6). The relations (2.3) define a coordinate transformation

$$x = x(\alpha,\beta,\delta) \quad , \qquad y = y(\alpha,\beta,\delta) \quad , \qquad z = \delta \quad . \tag{2.7}$$

Thus, in (2.3) through (2.6), the differentiation is taken with respect to the variables of the set α,β,δ. We shall need to relate the operators

$$\left.\frac{\partial}{\partial z}\right)_{x,y} , \quad \left.\frac{\partial}{\partial \delta}\right)_{\alpha,\beta} , \tag{2.8}$$

where the last derivative is equivalent to $\partial/\partial z)_{\alpha,\beta}$. Comparing (2.3) and (2.7), we see that the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\ \frac{\partial x}{\partial \delta} & \frac{\partial y}{\partial \delta} & \frac{\partial z}{\partial \delta} \end{bmatrix} \tag{2.9}$$

is

$$\begin{bmatrix} A \cos \Theta & A \sin \Theta & 0 \\ B \sin \Theta & -B \cos \Theta & 0 \\ \frac{\partial x}{\partial \delta} & \frac{\partial y}{\partial \delta} & 1 \end{bmatrix} . \tag{2.10}$$

Since the matrix of $\partial\alpha/\partial x$, etc. consists of the reduced cofactors of the determinant of the matrix (2.9) or (2.10), we find that

$$\begin{bmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \beta}{\partial x} & \frac{\partial \delta}{\partial x} \\ \frac{\partial \alpha}{\partial y} & \frac{\partial \beta}{\partial y} & \frac{\partial \delta}{\partial y} \\ \frac{\partial \alpha}{\partial z} & \frac{\partial \beta}{\partial z} & \frac{\partial \delta}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\cos \Theta}{A} & \frac{\sin \Theta}{B} & 0 \\ \frac{\sin \Theta}{A} & \frac{-\cos \Theta}{B} & 0 \\ -a_{13} & -a_{23} & 1 \end{bmatrix} , \tag{2.11}$$

where

$$\begin{aligned} a_{13} &= \frac{\sin \Theta}{A} \frac{\partial y}{\partial \delta} + \frac{\cos \Theta}{A} \frac{\partial x}{\partial \delta} , \\ a_{23} &= \frac{-\cos \Theta}{B} \frac{\partial y}{\partial \delta} + \frac{\sin \Theta}{B} \frac{\partial x}{\partial \delta} . \end{aligned} \tag{2.12}$$

Thus, we find by use of the chain rule and (2.11),

$$\left.\frac{\partial}{\partial z}\right)_{x,y} = -a_{13}\frac{\partial}{\partial \alpha} - a_{23}\frac{\partial}{\partial \beta} + \frac{\partial}{\partial \delta} \quad . \qquad (2.13)$$

The significance of the coefficients a_{13}, a_{23} can be seen from the following arguments. Using (2.10) and (2.12), we find that

$$a_{13} = \frac{1}{A^2}\left[\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \delta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \delta} + (0)(1)\right] ,$$

$$a_{23} = \frac{1}{B^2}\left[\frac{\partial x}{\partial \beta}\frac{\partial x}{\partial \delta} + \frac{\partial y}{\partial \beta}\frac{\partial y}{\partial \delta} + (0)(1)\right] . \qquad (2.14)$$

If $\cos(\alpha,\delta)$ denotes the cosine of the angle between the curves β = constant, δ = constant and α = constant, β = constant, and $\cos(\beta,\delta)$ denotes the cosine of the angle between the curves α = constant, δ = constant and α = constant, β = variable, then

$$A\,a_{13} = \cos(\alpha,\delta) , \quad B\,a_{23} = \cos(\beta,\delta) \quad . \qquad (2.15)$$

Again, we note that if we solve (2.12) for

$$\frac{\partial y}{\partial \delta} , \ \frac{\partial x}{\partial \delta} ,$$

we obtain

$$\frac{\partial x}{\partial \delta} = B\,a_{23}\sin\Theta + A\,a_{13}\cos\Theta ,$$

$$\frac{\partial y}{\partial \delta} = A\,a_{13}\sin\Theta - B\,a_{23}\cos\Theta \quad . \qquad (2.16)$$

Using (2.3), the above becomes

$$\frac{\partial x}{\partial \delta} = a_{23}\frac{\partial x}{\partial \beta} + a_{13}\frac{\partial x}{\partial \alpha} ,$$

$$\frac{\partial y}{\partial \delta} = a_{23}\frac{\partial y}{\partial \beta} + a_{13}\frac{\partial y}{\partial \alpha} \quad . \qquad (2.17)$$

Thus, $x(\alpha,\beta,\delta)$, $y(\alpha,\beta,\delta)$ are two independent solutions of the partial differential equation

$$\frac{\partial u}{\partial \delta} - a_{23}\frac{\partial u}{\partial \beta} - a_{13}\frac{\partial u}{\partial \alpha} = 0 ,$$

or (as is evident)

$$\left.\frac{\partial u}{\partial z}\right)_{x,y} = 0 .$$

From (2.3) and (2.16), we obtain a set of integrability conditions which must be added to the equations (2.4) and (2.5). These are

$$\frac{\partial}{\partial \alpha}(B\, a_{23}) - A\, a_{13}\frac{\partial \theta}{\partial \alpha} = - A\frac{\partial \theta}{\partial \delta} , \tag{2.18}$$

$$\frac{\partial}{\partial \alpha}(A\, a_{13}) + B\, a_{23}\frac{\partial \theta}{\partial \alpha} = \frac{\partial A}{\partial \delta} , \tag{2.19}$$

$$\frac{\partial}{\partial \beta}(B\, a_{23}) - A\, a_{13}\frac{\partial \theta}{\partial \beta} = \frac{\partial B}{\partial \delta} , \tag{2.20}$$

$$\frac{\partial}{\partial \beta}(A\, a_{13}) + B\, a_{23}\frac{\partial \theta}{\partial \beta} = B\frac{\partial \theta}{\partial \delta} . \tag{2.21}$$

From (2.15) we see that if the angles between the coordinate lines, α = variable and δ = variable, β = variable and δ = variable, are $\pi/2$, then $A\, a_{13} = B\, a_{23} = 0$. The above relations show that in this case, θ, A, and B are all independent of δ (as is to be expected). The relations (2.4), (2.5), and (2.18) through (2.21) imply that the <u>Riemann tensor vanishes in the α,β,δ system</u>.

To determine formulas (4.3),* (4.9),* (4.10),* and (3.9)* in terms of the <u>congruences</u> t^j, ℓ^j, n^j, we must evaluate the curvature terms. Since the n^j congruence consists of straight lines parallel to the z-axis, the curvature vector of these curves, u_k, vanishes. Further, the second fundamental tensor of the planes (z = constant) is $s_{jk} = 0$; also, the mean curvature $\overline{M}$ of these planes vanishes. By definition, the curvature vector $\overline{u}_k$ of the ℓ_j (or $-\ell_j$) congruence is

$$\ell^k \partial_k \ell_j = \overline{u}_j = \overline{\kappa}\, t_j ,$$

or

$$\overline{u}_j t^j = \overline{\kappa} . \tag{2.22}$$

Finally, we consider the curvature K of $(3.13)^*$:

$$K = n^p \ell^k (\partial_p t_k - \partial_k t_p) \quad .$$

Since the vector t^j lies along curves in the planes z = constant,

$$n^p \ell^k \partial_k t_p = 0 \quad .$$

That is, the vector $\ell^k \partial_k t_p$ has no component along n^p. Further, $n^p \partial_p t_k$ represents the directional derivative of t_k in the z-direction. From geometric considerations and the fact that the ordered pair $(t_k, -\ell_k)$ form a right-hand set, it follows that

$$n^p \partial_p t_k = - \left.\frac{\partial\theta}{\partial z}\right)_{x,y} \ell_k \quad .$$

Using the above results and (2.13), the formula for K reduces to

$$K = \left(a_{13} \frac{\partial\theta}{\partial\alpha} + a_{23} \frac{\partial\theta}{\partial\beta} - \frac{\partial\theta}{\partial\delta} \right) \quad . \qquad (2.23)$$

Finally, we note that

$$\frac{\partial}{\partial t} = A^{-1} \frac{\partial}{\partial\alpha} \quad , \quad \frac{\partial}{\partial\ell} = B^{-1} \frac{\partial}{\partial\beta} \quad ,$$

$$\frac{\partial}{\partial n} = \left.\frac{\partial}{\partial z}\right)_{x,y} = - a_{13} \frac{\partial}{\partial\alpha} - a_{23} \frac{\partial}{\partial\beta} + \frac{\partial}{\partial\delta} \quad . \qquad (2.24)$$

Using (2.4), (2.5), (2.22), and (2.24), we find that $(4.3)^*$ becomes

$$\frac{\partial}{\partial\alpha} \frac{q^2}{\sqrt{q^2 - c^2}} = \frac{c^2}{\sqrt{q^2 - c^2}} \frac{\partial \ln B}{\partial\alpha} + \frac{1}{\sqrt{q^2 - c^2}} \left(\frac{\partial h_o}{\partial\alpha} - \frac{T\partial S}{\partial\alpha} \right) . \qquad (2.25)$$

Further, $(4.9)^*$ reduces to

$$\left.\begin{aligned}
&\frac{(\gamma - 3)\, q^2 + 4\, c^2}{(\gamma - 1)\sqrt{q^2 - c^2}} \left(-a_{13} \frac{\partial}{\partial \alpha} - a_{23} \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \delta}\right) \sqrt{q^2 - c^2} = \\
&= \frac{c\sqrt{q^2 - c^2}}{A} \frac{\partial \ln B}{\partial \alpha} + \frac{(\gamma-3)\, q^2 - (\gamma-5)\, c^2}{(\gamma-1)(q^2-c^2)} \left(-a_{13} \frac{\partial}{\partial \alpha} - a_{23} \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \delta}\right) h_o + \\
&+ \frac{2q^2 - (\gamma + 3)\, c^2}{(\gamma - 1)\,(q^2 - c^2)}\, T \left(-a_{13} \frac{\partial}{\partial \alpha} - a_{23} \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \delta}\right) S \quad .
\end{aligned}\right\} \tag{2.26}$$

In order to evaluate $(4.10)^*$, we note that

$$t^j\, t^k\, \partial_j\, \ell_k = -\ell^k\, t^j\, \partial_j\, t_k = -\ell^k\, u_k = -\kappa \quad .$$

Thus using (2.4) and (2.23), we find that $(4.10)^*$ becomes

$$\begin{aligned}
q\frac{\partial q}{\partial \beta} &= \frac{\partial h_o}{\partial \beta} - \frac{T\partial S}{\partial \beta} - (q^2 - c^2)\, \frac{\partial \ln A}{\partial \beta} + \\
&+ c\sqrt{q^2 - c^2}\, B\left(a_{13}\frac{\partial \theta}{\partial \alpha} + a_{23}\frac{\partial \theta}{\partial \beta} - \frac{\partial \theta}{\partial \delta}\right) \quad .
\end{aligned} \tag{2.27}$$

Finally, $(3.9)^*$ reduces to

$$\left(-c\; a_{13} + \frac{\sqrt{q^2 - c^2}}{A}\right)\frac{\partial h_o}{\partial \alpha} - c\; a_{23}\frac{\partial h_o}{\partial \beta} + c\,\frac{\partial h_o}{\partial \delta} = 0 \quad . \tag{2.28}$$

Evidently, the differential equations of the stream lines are

$$\frac{d\alpha}{-a_{13} + \dfrac{\sqrt{M^2 - 1}}{A}} = \frac{d\beta}{-a_{23}} = d\delta \quad , \tag{2.29}$$

where M is the Mach number, q/c. Since (2.29) may be written as

$$\frac{A d\alpha}{-A\; a_{13} + \sqrt{M^2 - 1}} = \frac{B d\beta}{-B\; a_{23}} = d\delta \quad .$$

and $Ad\alpha$, $Bd\beta$, $d\delta$ are the arc-length elements along the coordinate lines, and $A\; a_{13}$, $B\; a_{23}$ are $\cos\,(\alpha,\delta)$, $\cos\,(\beta,\delta)$, it follows that the stream lines are space curves. If cos (S,α) denotes the cosine of the angle between a stream line and the α = variable coordinate line, etc., then

$$\cos(S,\alpha) : \cos(S,\beta) : \cos(S,\delta) = -\cos(\alpha,\delta) + \sqrt{M^2-1} : -\cos(\beta,\delta) : 1 .$$

3. A FAMILY OF ISENTROPIC IRROTATIONAL FLOWS WITH PLANE CHARACTERISTIC SURFACES (HELICAL STREAM LINES, CONCENTRIC CIRCULAR BICHARACTERISTICS)

We shall consider those flows of section 2 for which

$$h_o = \text{constant} , \quad S = \text{constant}, \quad a_{13} = a_{23} = 0 . \tag{3.1}$$

As noted earlier, (2.18) through (2.21) imply that

$$\frac{\partial A}{\partial \delta} = \frac{\partial B}{\partial \delta} = \frac{\partial \theta}{\partial \delta} = 0 , \tag{3.2}$$

that is, the curves, δ = variable, are orthogonal to the planes, z = constant. We shall show that <u>every family of ∞^1 bicharacteristics (δ = variable, α = variable) forms right circular cylindrical surfaces with generators parallel to the z-axis</u>. From (2.26), (3.1), and (3.2), we see, by the use of a simple argument, that

$$\frac{\partial q}{\partial \delta} = 0 , \tag{3.3}$$

$$\frac{\partial B}{\partial \alpha} = 0 . \tag{3.4}$$

From (3.2) and (3.4), and the fact that B is independent of δ, we see that by the proper choice of a scale factor along the β = variable curves,

$$B = 1 . \tag{3.5}$$

Further, (2.5) implies that the <u>orthogonal trajectories of the bicharacteristics are straight lines</u>, or

$$\bar{\kappa} = 0 , \quad \theta = \theta(\alpha) . \tag{3.6}$$

Returning to (2.25) and (2.27), we see that (3.1), (3.3), and (3.5) imply that (for $M = q/c$)

$$q = q(\beta) \ . \tag{3.7}$$

$$\frac{M^2}{2(M^2-1)} \frac{d}{d\beta} \ln \frac{M^2}{2 + (\gamma-1) M^2} = \frac{\partial}{\partial\beta} \ln A \ . \tag{3.8}$$

Integrating (3.8), we obtain

$$F(\alpha) \left[\frac{2 + (\gamma-1) M^2}{M^2-1}\right]^{1/\gamma+1} = A \ , \tag{3.9}$$

where $F(\alpha)$ is an arbitrary function of α. By the proper choice of a scale factor along the curves α = variable, we can select $F(\alpha) = 1$. Hence, A is a function of β only, and M is determined by

$$A^{(\gamma+1)} = \left[\frac{2 + (\gamma-1) M^2}{M^2-1}\right] . \tag{3.10}$$

Again, since $\theta = \theta(\alpha)$, (2.4) implies that

$$A = c_1 \beta + c_2 \ , \tag{3.11}$$

$$\theta = c_1 \alpha + c_3 \ , \tag{3.12}$$

where c_1, c_2, c_3 are arbitrary constants. If $c_1 \neq 0$, then M is not constant and the relations (2.3) lead to

$$x = \frac{c_1\beta + c_2}{c_1} \sin (c_1\alpha + c_3) \ ,$$

$$y = \frac{-c_1\beta + c_2}{c_1} \cos (c_1\alpha + c_3) \ .$$

The bicharacteristics β = constant, δ = constant are concentric circles. From (2.29), the differential equations of the stream lines are

$$\frac{A d\alpha}{\sqrt{M^2-1}} = d\delta \ , \quad \beta = \text{constant} \ ,$$

or by integration (where c_4 is a constant),

$$A\alpha = \sqrt{M^2-1} \ \delta + c_4 \ , \quad \beta = \text{constant} \ .$$

Since $\alpha = c_1^{-1}(\Theta - c_3)$, $\delta = z$, and the surfaces β = constant are right circular cylinders, the stream lines are helices lying on these cylinders and cutting the generators of the cylinders in the constant angle ρ where

$$\cot \rho = \frac{1}{c_1 \sqrt{M^2-1}} .$$

Finally, using polar coordinates $\Theta = c_1\alpha$, $r = A/c_1$, we see that the Mach number M depends only on r. That is, for air, $\gamma = 1.4$, if

$$u = M^2 - 1 ;$$

then (3.10) leads to

$$(c_1\, r)^{2.4} = \left[\frac{2.4 + .4\, u}{u}\right] .$$

We note that for $c_1 = 1$ as $M \to 1$, $u \to 0$, $r \to \infty$; $M = 2$, $u = 3$, $r = (1.2)^{1/2.4}$. It is easily shown that as r decreases, M increases. Evidently q is constant along any stream line.

4. A SECOND FAMILY OF IRROTATIONAL, ISENTROPIC FLOWS WITH PLANE CHARACTERISTIC SURFACES (STREAM LINES LYING ON RIGHT CIRCULAR CONES, CONCENTRIC CIRCULAR BICHARACTERISTICS)

We shall now consider those flows of section 2 for which

$$h_o = \text{constant} , \quad S = \text{constant} ,$$

$$B\, a_{23} = k , \quad A\, a_{13} = 0 , \tag{4.1}$$

where k is a constant such that $0 < |k| < 1$. Since $B\, a_{23}$, $A\, a_{13}$ are the cosines of the angles between the β,δ and α,δ coordinate lines, respectively, the conditions (4.1) imply that the <u>coordinate surfaces β = constant are surfaces with straight-line generators (δ = variable) which make an angle ϕ with the z-axis</u> where

$$\cos \phi = \sqrt{1 - k^2} .$$

The relations (2.18) through (2.21) become

$$0 = \frac{A\partial\theta}{\partial\delta} , \tag{4.2}$$

$$k \frac{\partial\theta}{\partial\alpha} = \frac{\partial A}{\partial\delta} , \tag{4.3}$$

$$0 = \frac{\partial B}{\partial\delta} , \tag{4.4}$$

$$k \frac{\partial\theta}{\partial\beta} = \frac{B\partial\theta}{\partial\delta} . \tag{4.5}$$

From (4.5) and (4.2) we see that

$$\theta = \theta(\alpha) . \tag{4.6}$$

Further, the relation (2.5) becomes

$$\frac{\partial B}{\partial\alpha} = 0 . \tag{4.7}$$

The relations (4.4) and (4.7) imply

$$B = B(\beta) . \tag{4.8}$$

Thus, the <u>orthogonal trajectories of the bicharacteristics α = variable are straight lines</u>. The relation (2.4) becomes

$$\frac{d\theta}{d\alpha} = \frac{1}{B} \frac{\partial A}{\partial\beta} . \tag{4.9}$$

Integrating (4.9), we find that

$$A = \theta' \int^{\beta} B \, d\beta + G(\delta,\alpha) , \tag{4.10}$$

where $\theta' = d\theta/d\alpha$ and $G(\delta,\alpha)$ is an arbitrary function of δ,α. Differentiating (4.10) with respect to δ and using (4.3), we find that

$$G(\delta,\alpha) = k\delta \, \theta' + H(\alpha) , \tag{4.11}$$

where $H(\alpha)$ is an arbitrary function of α. Thus, (4.10) becomes

$$A = \theta' \int^{\beta} B \, d\beta + k\delta \, \theta' + H(\alpha) . \tag{4.12}$$

Since $a_{13} = 0$, $\Theta = \Theta(\alpha)$, $B = B(\beta)$, we find that (2.25) implies that

$$q = q\ (\beta, \delta) \quad . \tag{4.13}$$

From (2.27), we see that $H(\alpha)$ = constant = ℓ, and also that $\Theta' = d\Theta/d\alpha$ = constant = p. The relation (4.12) becomes

$$A = p\left[\int^{\beta} B\, d\beta + k\delta\right] + \ell \quad , \tag{4.14}$$

and (2.27) reduces to (in terms of M)

$$-\frac{M^2}{(M^2 - 1)}\left[\frac{1}{M} - \frac{(\gamma - 1)\, M}{2 + (\gamma - 1)\, M^2}\right]\frac{\partial M}{\partial \beta}$$

$$= \frac{Bp}{p\left[\int^{\beta} B\, d\beta + k\delta\right] + \ell} \quad . \tag{4.15}$$

Now the operator

$$-a_{13}\frac{\partial}{\partial \alpha} - a_{23}\frac{\partial}{\partial \beta} + \frac{\partial}{\partial \delta}$$

is [see (4.1)]

$$-\frac{k}{B}\frac{\partial}{\partial \beta} + \frac{\partial}{\partial \delta} \quad .$$

Thus, for any function F of

$$\left[p \int^{\beta} B\, d\beta + kp\delta\right] + \ell \quad ,$$

we find

$$-\frac{k}{B}\frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial \delta} \equiv \left(-\frac{kpB}{B} + kp\right) \equiv 0 \quad . \tag{4.16}$$

If we require that M, the Mach number, be a function of

$$A = p\int^{\beta} B\, d\beta + kp\delta + \ell \quad , \tag{4.17}$$

then (2.26) is identically satisfied. Integrating (4.5), we obtain [see (3.9)]

$$\frac{[2 + (\gamma - 1) M^2]}{M^2 - 1} = a\, A^{\gamma+1} \quad , \tag{4.18}$$

where a is a constant. As M goes from 1 to ∞, A goes from ∞ to $[(\gamma-1)/a]^{1/\gamma+1}$.

By integrating (2.3), we obtain

$$\left.\begin{aligned} x &= \frac{A(\delta,\beta)}{p} \sin p\,\alpha \quad , \\ y &= -\frac{A(\delta,\beta)}{p} \cos p\,\alpha \quad , \\ z &= \delta \quad . \end{aligned}\right\} \tag{4.19}$$

Thus the bicharacteristics (β = constant, δ = constant) are again concentric circles of radii A/p. The differential equation of the stream lines is [see (2.29)]

$$\frac{A d\alpha}{-k + \sqrt{M^2-1}} = d\delta \ , \quad \beta = \text{constant} \quad . \tag{4.20}$$

Since $A = A(\beta,\delta)$, we find that (4.20) reduces to

$$\alpha = \int \frac{-k + \sqrt{M^2-1}}{A} d\delta \ , \quad \beta = \text{constant} \quad . \tag{4.21}$$

From (4.14) and (4.19), it follows that the surfaces β = constant are right circular cones. Hence, the stream lines cut the generators (α = constant, β = constant) of the cones in the angle ρ where

$$\tan \rho = \frac{A d\alpha}{d\delta} = -k + \sqrt{M^2 - 1} \quad . \tag{4.22}$$

Since A/p is the radius of any cross section of a cone, as the radius increases, M decreases [see (4.18)]. Further, (4.22) shows that as the radius increases, the angle ρ decreases. Thus, every stream line winds along a right circular cone and always becomes steeper. In the limit (as the radius of a cone approaches infinity and M approaches one), the stream lines cut the generator in a constant angle.

5. SOME GENERAL REMARKS

In section 3, we showed that if the characteristic surfaces are parallel planes and if two families of ∞^1 cylinders, perpendicular to these planes, can be passed through the bicharacteristics, then one family of these cylinders consists of right circular cylinders. Further, in section 4, we showed that if two special families of ∞^1 cones can be passed through the bicharacteristics then one of these classes of cones consists of right circular cones. Two other cases of cones remain to be discussed. These are given by

$$a_{13} = 0 , \quad B\, a_{23} = k , \tag{5.1}$$

$$A\, a_{13} = \ell , \quad B\, a_{23} = k , \tag{5.2}$$

where k and ℓ are constants. In the first case, it is easily seen that the bicharacteristics are radial lines. However, <u>the equations of fluid flow cannot be satisfied in this case</u>.

6. REFERENCES

1. N. Coburn, Simple Waves in the Steady, Supersonic, Plane, Rotational Flow of a Compressible Polytropic Gas, Eng. Res. Inst., Univ. of Mich., Project 2201-3-P, Sept. 1954.

2. References to the above paper will be denoted by starring the equation or section number.