ANALYTICAL METHODS IN THE THEORY OF NON-LINEAR OSCILLATIONS

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ABSTRACT

In this work methods are developed for constructing approximate solutions of non-linear differential equations which describe nearly periodic or nearly multiply periodic phenomena. The concern is primarily with systems of differential equations which can be generated from a Hamiltonian. The methods employed are, however, not restricted to Hamiltonian systems.

First a brief review of classical perturbation theory as developed by Poincaré is given. Following this a formal perturbation method is presented. This method was developed by Bogoliubov for generating asymptotic solutions of a system of differential equations which describe the motion of a dynamical system which is nearly periodic in a single phase. The Bogoliubov method is applied to a class of one dimensional Hamiltonian systems. For this one dimensional case it is demonstrated that the Bogoliubov method is equivalent to the classical perturbation theory of Poincaré.

The Bogoliubov method is generalized to deal with systems which are nearly periodic in two phases. This generalization is presented in the forms of a non-degenerate theory and of a degenerate theory. The manner in which the degenerate theory is formulated permits one to avoid the classic problem of small divisors to all orders.

The non-degenerate and degenerate perturbation theories are applied to a class of Hamiltonian systems. It is shown that non-degenerate perturbation theory is equivalent to the classical perturbation theory of Poincaré. This is not so with the degenerate theory.

Degenerate perturbation theory is employed to construct solutions of the equations which describe the motion of a charged particle which moves in a uniform magnetic field on which is superimposed a weak, transverse, hydromagnetic wave. It is shown that the effect of the wave is to cause energy, whose amount depends on the initial conditions, to be periodically exchanged between the transverse and the longitudinal motions of the particle. The relation between these results and those of previously published work is discussed.

The ideas of the main text are extended to more general systems in several appendices. A discussion of the convergence of degenerate perturbation theory is given. It is shown that degenerate perturbation theory provides an asymptotically correct solution for finite, but possibly quite long, times.
CHAPTER 1
INTRODUCTION

The analytical theory of oscillations has its origins in Newton's discussion of the motion of a pendulum.\(^1\) Unfortunately, until the latter part of the nineteenth century only linear oscillations were well understood; a theory of nonlinear oscillations was nonexistent. It was the astronomers of the nineteenth century who, through their studies of planetary motion, made the first significant contributions to a theory of nonlinear oscillations. The analytical theory of periodic motion in weakly nonlinear Hamiltonian systems reached its most highly developed state in the fundamental researches of Poincaré.\(^2\)

Between the publication of Poincaré's work and the advent of quantum mechanics there was little interest in advancing the theory of nonlinear oscillations. The realization of the intimate connection between periodic phenomena and quantization, however, brought about a sudden renewed interest. This interest was short lived; the matrix mechanics of Heisenberg and the wave mechanics of Schrödinger proved to be the more fruitful approaches to quantum phenomena. It was not long, however, before van der Pol again renewed interest in the theory of nonlinear oscillations by successfully describing the solutions of the equation which now bears his name.\(^3\) Van der Pol's work proved to be the seed from which developed a new analytical approach to the study of nonlinear oscillations. It was an attempt to understand the solutions of the van der Pol equation subjected to external forces which led Krylov and Bogoliubov to introduce their now famous method of averaging.\(^4\) This method, in its most general form, is applied to a system of differential equations which has the following "standard form":

\[ \dot{x}_i = \varepsilon A_i(x,t) \quad (i=1,\ldots,N), \quad (1.1) \]


\(^2\)H. Poincaré, Les Méthodes nouvelles de la Mécanique céleste, Volumes 1,2,3 (Gauthier-Villars, Paris, 1892-1899).

\(^3\)B. van der Pol, Nonlinear Theory of Electrical Oscillations (Communications Publishing House, Moscow, 1955).

where $\epsilon$ is a small parameter and $A_I$ is an analytic function of $x$ and $t$. It consists of making a transformation

$$x_i = u_i + \epsilon F_i(u,t), \quad (1.2)$$

such that

$$\dot{u}_i = \epsilon B_i(u). \quad (1.3)$$

Here $F_i$ and $B_i$ are formal infinite series in powers of $\epsilon$. Equations (1.2) and (1.3) are supplemented by the assumption that the

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\partial F_i}{\partial t} dt = 0, \quad (1.4)$$

where the $u_i$'s are held fixed during the integration. This assumption permits one to construct the $F_i$'s and $B_i$'s to an arbitrarily high power of $\epsilon$. It is clear from (1.4) why this technique is referred to as the method of averaging. $^5$

To this day the method of averaging remains one of the most useful analytical tools on the theory of weakly nonlinear phenomena. During the past thirty years it has been of great practical use in diverse fields of study. Attempts at understanding the theoretical foundations of the method have, however, had little success. Although one can demonstrate that it yields exact solutions for particular problems, general conditions for convergence have not yet been established. To date only proofs of asymptotic convergence have been presented and these proofs are only valid for times of order $1/\epsilon$. In some cases it can be shown by direct calculation that the asymptotic convergence is valid for times of order $\epsilon^{-N}$, $N$ being some positive integer. This state of affairs makes it clear that the foundations of the method of averaging are not well understood.

The techniques which we shall present in the following chapters are themselves forms of averaging. The averages are, however, over angles rather than over time. We begin by reviewing the classical perturbation theory of Poincare. Following this we present a one dimensional version of a perturbation method which was developed by Bogoliubov and Mitropolsky. $^6$ This technique constructs


$^6$Bogoliubov and Mitropolsky, op cit., p. 412 et seq.
asymptotic solutions of the equations of motion of a system which is nearly periodic in a single phase. We apply the Bogoliubov method to a one dimensional Hamiltonian system. As a next step we generalize the Bogoliubov method to deal with systems which are nearly periodic in two phases. This generalization is done in two steps: first we deal with nondegenerate systems, then with degenerate systems. We then apply the generalized perturbation theory to Hamiltonian systems.

Having explored the structure of the perturbation theories we develop the concept of asymptotic invariance. We demonstrate the usefulness of the perturbation theories in constructing asymptotic invariants and hierarchies of asymptotic invariants. Finally we employ degenerate perturbation theory to construct the solutions of the simple yet nontrivial problem of the motion of a charged particle in a uniform magnetic field on which is superimposed a weak transverse hydromagnetic wave.

Following the text are several appendices in which the ideas and techniques presented on the text are extended to systems of higher dimensions. These appendices are somewhat more advanced and considerably more formal than the main text. Consequently they are more difficult to read than the text. By restricting the text to one and two dimensional systems it was easier to focus attention on important ideas and conclusions. These ideas and conclusions are carried over to the general discussions given in the appendices. Thus an understanding of the text will greatly facilitate an understanding of the appendices.

Whenever possible an attempt is made to relate the results which we obtain to those obtained by Poincaré. For certain systems we can demonstrate that the present techniques are equivalent to those of Poincaré. In general, however the present techniques differ from those of Poincaré in several respects: Firstly, they are not restricted to Hamiltonian systems. Secondly, they are not restricted to multiply periodic solutions. Thirdly, they are such that one can avoid the classic problem of small divisors to an arbitrarily high order of perturbation theory. This possibility of avoiding small divisors is the key to explaining the asymptotic convergence of perturbation theory. A discussion of asymptotic convergence is given in Appendix E.
CHAPTER 2

CLASSICAL PERTURBATION THEORY*

In this chapter we shall review the more important ideas of classical perturbation theory. It should be made clear at the outset that classical perturbation theory seeks only the periodic or multiply periodic solutions of the equations of motion.

2.1 HAMILTON-JACOBI THEORY

An n-dimensional Hamiltonian system is one whose configuration can be described by n independent coordinates $q_i$ and n conjugate moments $p_i$, such that

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (2.1a)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (2.1b)$$

where $H$ is a function of the $q_i$'s, $p_i$'s and the time $t$. The function $H(q,p,t)$ is called the Hamiltonian.

A transformation to new coordinates $Q_i$ and new moments $P_i$ is said to be canonical if

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad (2.2a)$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i}, \quad (2.2b)$$

where $K$ is a function of the $Q_i$'s, $P_i$'s and the time $t$. The function $K(Q,P,t)$ is the Hamiltonian which is appropriate to the new coordinates and moments.

*The reader who is familiar with classical perturbation theory may omit this chapter.

7A detailed exposition of classical perturbation theory will be found in Poincaré, op cit.
For our purposes\(^8\) a canonical transformation can be described by a function \(S(q,P,t)\) in the following way:

\[

d_{i} = \frac{\partial S}{\partial q_{i}},
\]

\[

Q_{i} = \frac{\partial S}{\partial P_{i}},
\]

\[

K = H + \frac{\partial S}{\partial t}.
\]

The function \(S(q,P,t)\) is called a generating function.

It is often desirable to choose the canonical transformation such that the new coordinates \(Q_{i}\) and the new momenta \(P_{i}\) are constants of the motion. This can be accomplished if the new Hamiltonian \(K(Q,P,t)\) is identically zero. Thus, according to (2.3c), we require that

\[

H(q,p,t) + \frac{\partial S}{\partial t} = 0.
\]

However, equation (2.3a) says that

\[

p_{i} = \frac{\partial S}{\partial q_{i}}.
\]

We may, therefore, rewrite equation (2.4) in the form

\[

H(q,\frac{\partial S}{\partial q},t) + \frac{\partial S}{\partial t} = 0.
\]

Equation (2.5), known as the Hamilton-Jacobi equation, is a partial differential equation in \(n+1\) variables, \(q_{1},...,q_{n},t\), for the desired generating function.\(^9\) Equation (2.5) and the methods of solving it constitute what is known as Hamilton-Jacobi theory.

---

\(^8\)For a discussion of more general canonical transformations see H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1959), Chapter 8.

\(^9\)For a discussion of some properties of the Hamilton-Jacobi equation see Goldstein, *op cit.*, Chapter 9.
2.2 ACTION AND ANGLE VARIABLES

We consider a system which can be described by a Hamiltonian \( H(q_1, p_1, \ldots, q_N, p_N) \). This system is said to be multiply periodic if we can introduce new canonical momenta \( J_k \) and coordinates \( \gamma_k \) \((k=1, \ldots, N)\) such that:

(A) The configuration of the system is periodic in each of the \( \gamma_k \)'s with fundamental period \( 2\pi \).

(B) The Hamiltonian becomes a function of the \( J_k \)'s only.

The new variables \( J_k \) and \( \gamma_k \) are called action and angle variables respectively.

Let the generator of the canonical transformation \( q-p \rightarrow J-\gamma \) be a function \( S(q, J) \). It can be shown\(^{10}\) that the variables \( J_k \) are uniquely determined, apart from an integral linear transformation among the \( J_k \)'s with determinant \(+ 1\), by imposing the following additional requirement:

(C) The function

\[
S^*(q, \gamma) = S - \sum_{k=1}^{N} \gamma_k J_k
\]

shall be a periodic function of the \( \gamma_k \)'s with period \( 2\pi \). Condition (C) is not necessary for the introduction of action and angle variables; it is, however, useful in the development of a canonical perturbation theory.

2.3 PERTURBATIONS OF A NONDEGENERATE SYSTEM

Let us suppose that the Hamiltonian of a system can be written as

\[
H = H_0 + \sum_{n=1}^{R} \epsilon^n H_n,
\]  
(2.6)

where \( \epsilon \) is a small, dimensionless parameter and \( R \) is an unspecified upper limit.\(^{11}\) Let us further suppose that when \( \epsilon = 0 \) the system can be described by action and angle variables \( J_k^0 \) and \( \gamma_k^0 \) \((k=1, \ldots, N)\). The motion obtained by setting \( \epsilon = 0 \) is called the unperturbed motion.


\(^{11}\)We assume, of course, that the summation exists for a sufficiently large range of values of the coordinates and momenta.
If $\epsilon$ is sufficiently small it is reasonable to assume that the true motion will closely approximate the unperturbed motion. In particular, let us assume the true motion to be multiply periodic. If this is the case then it must be possible to introduce true action and angle variables $J_k$ and $\gamma_k$ through a generating function $S(\gamma^0, J)$. It is the task of classical perturbation theory to construct this function.

Since the unperturbed motion and the true motion are multiply periodic, it follows that the Cartesian coordinates of the system are periodic function of both the $\gamma_k^0$'s and the $\gamma_k$'s. This means that, apart from an integral linear transformation of the $\gamma_k^0$'s among themselves with determinant $\pm 1$, we have

$$\gamma_k = \gamma_k^0 + F_k(\gamma^0), \quad (2.7)$$

where $F_k$ is a periodic function of the $\gamma_k^0$'s. It follows from (2.7) and (C) that

$$S = \sum_k \gamma_k^0 J_k$$

is a periodic function of the $\gamma_k^0$'s with period $2\pi$.

Since the true motion is supposed to closely approximate the unperturbed motion, we shall assume that the function $S$ is expressible in the form

$$S = S_0 + \sum_{n=1}^{\infty} \epsilon^n S_n; \quad (2.9)$$

here $S_0$ generates the identity transformation and, because of (2.8), the $S_n$'s are periodic functions of the $\gamma_k^0$'s.

The formal perturbation theory consists of substituting the right hand side of (2.9) into the following Hamilton-Jacobi equation:

$$H_0(\partial S/\partial \gamma^0) + \sum_{n=1}^{R} \epsilon^n H_n(\gamma^0, \partial S/\partial \gamma^0) = \sum_{n=0}^{\infty} \epsilon^n W_n(J). \quad (2.10)$$

After this substitution the left hand side of (2.10) is expanded in a multiple Taylor series about the points $J_k(k=1,..,N)$. This expansion results in a formal infinite series in powers of $\epsilon$. The equality (2.10) can be formally satisfied by equating the coefficient of $\epsilon^n$ from the left hand side of (2.10) with the corresponding coefficient from the right hand side. When this is done for all
values of \( n \) there results an infinite set of coupled differential equations. The first two members of this set are as follows:

\[
H_0(J) = W_0(J), \quad (2.11a)
\]

\[
\sum_{k=1}^{N} \frac{\partial H_0}{\partial J_k} \frac{\partial S_1}{\partial \gamma_k^0} + H_1(\gamma^0, J) = W_1(J). \quad (2.11b)
\]

The zero'th order contribution to the energy is simply \( H_0(J) \). Since \( S_1 \) is to be a periodic function of the \( \gamma_k^0 \)'s, the first order contribution \( W_1 \) to the energy is found from (2.11b) to be

\[
W_1 = < H_1(\gamma^0, J) > \quad (2.12)
\]

where

\[
<H_1> = \frac{1}{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} H_1 d\gamma_1 \cdots d\gamma_N \quad (2.13)
\]

It follows from (2.12) that

\[
\sum \omega_k^0 \frac{\partial S_1}{\partial \gamma_k^0} = < H_1 > - H_1, \quad (2.14)
\]

where

\[
\omega_k^0 = \frac{\partial H_0}{\partial J_k}. \quad (2.15)
\]

The right hand side of (2.14) is a known, finite Fourier series having no zero harmonic. We may, therefore, write

\[
<H_1> - H_1 = \sum_{p} A_p(J) e^{i(p_1 \gamma_1^0 + \cdots + p_N \gamma_N^0)}, \quad (2.16)
\]

where the prime on the summation sign excludes the zero harmonic. It follows from (2.14) that
\[ S_1 = -i \sum_p \frac{A_p}{p_1^{\omega_1} + \ldots + p_N^{\omega_N}} e^{i(p_1\gamma_1 + \ldots + p_N\gamma_N)} \quad (2.17) \]

\[ + s_1(J), \]

where the prime on the summation sign again excludes the zero harmonic. The function \( s_1(J) \) is arbitrary and may be set equal to zero.

The functions \( W_2, S_2, W_3, S_3, \) etc., are obtained in a fashion analogous to that described above. It is clear that this technique fails when the frequencies are such that there exist some nonzero integers \( p_i \) for which the following equality is satisfied:

\[ p_1^{\omega_1} + \ldots + p_N^{\omega_N} = 0. \quad (2.18) \]

A system for which \( M \) relations of the form (2.18) exist is said to be \( M \)-fold degenerate. It is also clear that, when a denominator of the form

\[ p_1^{\omega_1} + \ldots + p_N^{\omega_N} = O(\epsilon) \quad (2.19) \]

arises in the course of the perturbation calculation, it renders the perturbation theory meaningless at the point where it occurs. We shall discuss this situation in the following sections.

2.4 PERTURBATION OF AN INTRINSICALLY DEGENERATE SYSTEM

Consider a system which has \( N \) degrees of freedom and which is multiply periodic when unperturbed. The system is said to have an \( M \)-fold intrinsic degeneracy if \( M \) relations of the form

\[ p_1^{\omega_1} + \ldots + p_M^{\omega_M} = 0, \quad (2.20) \]

the \( p_i \)'s being integers, exist identically in the unperturbed action variables \( J_0^a \). For such a system it is possible to introduce new canonical variables \( J_0^a \), \( \gamma_0^a \) \( (a = 1, \ldots, N-M) \) and \( J_0^r \), \( \gamma_0^r \) \( (r = N-M+1, \ldots, N) \) such that the unperturbed Hamiltonian \( H_0 \) depends only on the \( J_0^a \)'s; the frequencies \( \omega^a_0 \) are nondegenerate and
the \( w^k_\alpha \)'s vanish.\(^{12}\)

Let us suppose that we introduce the new variables \( J^0_\alpha, \gamma^0_\alpha \) and \( J^0_\tau, \gamma^0_\tau \) and then proceed as we did in section 2.3. We find that the first order contribution \( W_1 \) to the energy is given by the expression

\[
W_1 = \langle H_1 \rangle_a \tag{2.21}
\]

where \( \langle \rangle_a \) indicates the average over the nondegenerate angle variables \( \gamma^0_\alpha \). The \( W_1 \) given by (2.21) will depend upon the \( \gamma^0_\alpha \)'s; this violates condition (B) of section 2.2. Suppose, however, that we can find a canonical transformation to new variables \( \overline{J}^0_\alpha, \overline{\gamma}^0_\alpha, \overline{J}^0_\tau, \overline{\gamma}^0_\tau \) such that \( \langle H_1 \rangle \) is a function of the \( \overline{J}^0_\alpha \)'s only. Such a transformation would remove the violation of condition (B). The generator \( V \) of this transformation can be written as

\[
V = \sum_{k=1}^{N} \gamma^0_\alpha \overline{J}^0_k + V_1(\gamma^0_\alpha, \overline{J}^0_\tau) \tag{2.22}
\]

The function \( V_1 \) is found by solving the following Hamilton-Jacobi equation:

\[
\langle H_1 \rangle_a (\gamma^0_\alpha, \frac{\partial V}{\partial \gamma^0_\alpha}) = W_1(\overline{J}^0) \tag{2.23}
\]

This differential equation may or may not have a solution depending on the form of \( H_1 \).

If equation (2.23) has a solution then we may write the Hamiltonian \( H \) in terms of the variables \( \overline{J}^0_\alpha \) and \( \overline{\gamma}^0_\alpha \). We then assume that the true action and angle variables \( J_k \) and \( \gamma_k \) are related to the \( \overline{J}^0_\alpha \)'s and \( \overline{\gamma}^0_\alpha \)'s through the generating function

\[
S = S_0 + \sum_{n=1}^{\infty} \epsilon^n S_n(\gamma^0, J) \tag{2.24}
\]

where \( S_0 \) generates the identity transformation. From this point we proceed as we did for non-degenerate perturbation theory. In so doing we obtain an in-

\(^{12}\)For the remainder of this chapter the subscript \( a \) refers to the non-degenerate variables, the subscript \( r \) to the degenerate variables, and the subscript \( k \) to all the variables.
finite set of coupled differential equations. The first three members of this set are as follows:

\[ H_0(J_a) = W_0(J), \quad (2.25a) \]
\[
\sum_a \frac{\partial H_0}{\partial J_a} \frac{\partial S_1}{\partial \gamma_a^0} + H_1(J, \gamma^0) = W_1(J), \quad (2.25b) 
\]
\[
\sum_a \frac{\partial H_0}{\partial J_a} \frac{\partial S_2}{\partial \gamma_a^0} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 H_0}{\partial J_{a_i} \partial J_{a_j}} \frac{\partial S_1}{\partial \gamma_{a_i}^0} \frac{\partial S_1}{\partial \gamma_{a_j}^0} + H_2(J, \gamma^0) = W_2(J). \quad (2.25c) 
\]

Since \( S_1 \) is to be a periodic function of the \( \gamma_k^0 \)'s it follows from (2.25b) that

\[ W_1(J) = \langle H_1 \rangle_a. \quad (2.26) \]

With \( W_1 \) known then equation (2.25b) determines \( S_1 \) to within an additive function \( R_1 \) of the \( J_k \)'s and the \( \gamma_k^0 \)'s. Thus we may write

\[ S_1 = S_1^0(\gamma_0^0, J) + R_1(\gamma_0^0, J), \quad (2.27) \]

where \( S_1^0 \) is uniquely determined by (2.25b).

The function \( W_2(J) \) is found by averaging the left hand side of (2.25c) over all the angle variables. Once \( W_2 \) is known then the function \( R_1 \) can be determined by averaging (2.25c) over the non-degenerate angle variables and then integrating the remaining differential equation for \( R_1 \). When both \( W_2 \) and \( R_1 \) are known then equation (2.25c) determines \( S_2 \) to within an additive function \( R_2(\gamma_0^0, J) \).

The function \( R_2 \) is determined by the third order equation. This process can be continued to any desired order. The final result is an expression for the Hamiltonian in the form

\[ H = W_0(J_0) + \sum_{n=1}^{\infty} \epsilon^n W_n(J), \quad (2.28) \]
If the function $H_1$ has no zero harmonic in the $\gamma^O_s$’s then the method described above fails because the left hand side of (2.23) vanishes identically. This difficulty is easily removed by making a canonical transformation to new variables $\tilde{J}_k^O$ and $\tilde{\gamma}_k^0$ which completely eliminates the first order contribution to the Hamiltonian. The generator $F_1$ of this transformation will have the form

$$F(\gamma^O, \tilde{J}) = \sum_{k=1}^{N} \gamma_k^O \tilde{J}_k^O + \epsilon F_1(\gamma^O, \tilde{J}^O), \quad (2.29)$$

where $F_1$ is to be determined.

In order to find $F_1$ we write the Hamiltonian in terms of the $\gamma^O$’s and the $\tilde{J}_k^O$’s. Thus

$$H = h_0(\tilde{J}_s^O) + \sum_{n=1}^{\infty} \epsilon^n h_n(\gamma^O, \tilde{J}_k^O), \quad (2.30)$$

where

$$h_0 = H_0(\tilde{J}_s^O), \quad (2.31a)$$

$$h_1 = H_1(\gamma^O, J^O) + \sum_{s} \frac{\partial H_0}{\partial J_s^O} \frac{\partial F_1}{\partial J_s^O}, \quad (2.31b)$$

$$h_2 = H_2(\gamma^O, J^O) + \frac{1}{2} \sum_{a_i, a_j} \frac{\partial^2 H_0}{\partial J_{a_i}^O \partial J_{a_j}^O} \frac{\partial F_1}{\partial \gamma_{a_i}^O} \frac{\partial F_1}{\partial \gamma_{a_j}^O} \quad (2.31c)$$

$$+ \sum_{k} \frac{\partial H_1}{\partial \gamma_k^O} \frac{\partial F_1}{\partial \gamma_k^O}.$$

Since $H_1$ is assumed to have no zero harmonic, we may choose $F_1$ such that

$$h_1 = H_1(\gamma^O, \tilde{J}^O) + \sum_{s} \frac{\partial H_0}{\partial J_s^O} \frac{\partial F_1}{\partial \gamma_s^O} = 0. \quad (2.32)$$

Equation (2.32) determines $F_1$ to within an arbitrary additive function of the $\tilde{J}_s^O$’s and the $\gamma_s^O$’s. This arbitrariness may be set equal to zero.

Once $F_1$ is known we can express the Hamiltonian in terms of the $\tilde{J}_k^O$’s and the $\gamma_k^O$’s. Thus

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\[ H = I_0(\gamma^0) + \sum_{n=2}^{\infty} \epsilon_n I_n(\gamma^0, \gamma^0), \]  

(2.33)

where

\[ I_0 = h_0(\gamma^0), \]  

(2.34a)

\[ I_2 = h_2(\gamma^0, \gamma^0), \]  

(2.34b)

\[ I_3 = h_3(\gamma^0, \gamma^0) - \sum_{k=1}^{N} \frac{\partial \omega_2}{\partial \gamma^0} \frac{\partial F_1}{\partial \gamma^0}. \]  

(2.34c)

The methods of degenerate perturbation theory may now be applied to the Hamiltonian (2.33) unless \( I_2 \) has no zero harmonic in the \( \gamma^0 \)'s. If this should be the case then we simply eliminate \( I_2 \) by a canonical transformation just as we eliminated \( H_1 \), and then apply degenerate perturbation theory.

### 2.5 Perturbations of an Accidentally Degenerate System

As in the previous section we consider a system of \( N \) degrees of freedom which is multiply periodic in the unperturbed state. The system is said to have a \( M \)-fold accidental degeneracy if \( M \) relations of the form

\[ p_{11} \omega_1^0 + \ldots + p_{1M} \omega_M^0 = 0, \]  

(2.35)

where the \( p_{ij} \)'s are integers, exist for certain values of the unperturbed action variables \( \omega_1^0 \). For such a system it is possible to introduce new action and angle variables \( \omega_1^0, \omega_2^0, (s=1, \ldots, N-M), \gamma_r^0, (r=N-M+1, \ldots, N) \) such that

\[ \frac{\partial H_0}{\partial \omega_1^0} = \omega_1^0, \]  

(2.36)

while

\[ \frac{\partial H_0}{\partial \gamma_r^0} = 0 \]  

(2.37)

Equation (2.37) will hold only at the degenerate points. In general the equality

\[ \frac{\partial^2 H_0}{\partial \gamma_r \partial \gamma_s} = 0 \]  

(2.38)
will not be satisfied even at the degenerate points. This type of degeneracy cannot be treated by the method of section 2.5 since, in that section, the equality (2.38) was assumed to hold. It can be shown, \(^{13}\) that in order to deal with the perturbations of an accidentally degenerate system, the generating function \(S(\gamma^0, J)\) must have the form

\[
S = S_0 + \sum_{n=1}^{\infty} \mu^n S_n(\gamma^0, J),
\]

where \(S_0\) generates the identity transformation and where

\[
\mu = \epsilon^{1/2}.
\]

The generating function (2.39) is substituted into the Hamilton-Jacobi equation

\[
H(\gamma^0, \partial S / \partial \gamma^0) = \mathcal{W}_0(J) + \sum_{n=1}^{\infty} \mu^{2n} \mathcal{W}_{2n}(J).
\]

As before the left hand side of (2.41) is expanded in a multiple Taylor series about the \(J_k\)'s. Upon equating equal powers of \(\mu\) from both sides of (2.41) we obtain an infinite set of coupled differential equations. The first three members of this set are:

\[
H_0(J_8) = \mathcal{W}_0(J_8)
\]

\[
\sum \frac{\partial H_0}{\partial J_8} \frac{\partial S_1}{\partial \gamma_8^0} = 0,
\]

\[
\sum \frac{\partial H_0}{\partial J_8} \frac{\partial S_2}{\partial \gamma_8^0} + \frac{1}{2} \sum_{r_i, r_j} \frac{\partial^2 H_0}{\partial J_{r_i} \partial J_{r_j}} \frac{\partial S_1}{\partial \gamma_{r_i}^0} \frac{\partial S_1}{\partial \gamma_{r_j}^0}
\]

\[
+ H_2(J, \gamma^0) = \mathcal{W}_2(J).
\]

It follows from (2.42) that \(S_1\) is independent of the \(\gamma_a^0\)'s. The dependence of \(S_1\) on the \(\gamma_r^0\)'s is found by solving the following equation:

\(^{13}\)Poincaré, op cit., chap. 19.
\[
\frac{1}{2} \sum \frac{\delta^2 H_0}{\delta J_{r_1} \delta J_{r_j}} \frac{\delta S_1}{\delta \gamma_{r_1}^0} \frac{\delta S_1}{\delta \gamma_{r_j}^0} + \langle H_1 \rangle = W_2(J). \quad (2.43)
\]

This equation is obtained by averaging (2.42) over the \( \gamma_{r_1}^0 \)'s. An equation of the same type as (2.43) will have to be solved in order to determine \( S_2, S_3, \ldots \) etc. Equation (2.43) cannot be integrated for arbitrary values of the \( J_r \)'s. This means that, in general, degenerate systems do not remain multiply periodic when perturbed.

\[\text{Note:}\]

\[\text{For a discussion of when (2.43) can be integrated see Poincaré, \textit{op cit.}, chp. 19.}\]
CHAPTER 3

THE BOGOliUBOV METHOD IN ONE DIMENSION

In this chapter we shall present a one dimensional version of Bogoliubov's perturbation theory for a system with a single rapid phase. We shall apply this perturbation theory to a class of one dimensional Hamiltonian systems and shall discuss its connection with classical perturbation theory.

3.1 FORMAL PRESENTATION

We consider a dynamical system (not necessarily a Hamiltonian system) whose state is characterized by an amplitude $x$ and an angle $\gamma$, and whose temporal development is governed by the system of autonomous differential equations

\begin{align}
\dot{x} &= \epsilon A(\gamma, x), \\
\dot{\gamma} &= \omega(x) + \epsilon B(\gamma, x),
\end{align}

where $\epsilon$ is a small parameter and $A$ and $B$ are periodic functions of $\gamma$ with period $2\pi$.

The formal perturbation theory consists of making the following change of variables:

\begin{align}
x &= y + \sum_{n=1}^{\infty} \epsilon^n x(n)(\phi, y), \\
\gamma &= \phi + \sum_{n=1}^{\infty} \epsilon^n \gamma(n)(\phi, y),
\end{align}

\textsuperscript{15} For the original presentation see Bogoliubov and Mitropolsky, \textit{op cit.}, p. 412 et seq.

\textsuperscript{16} The ideas of this chapter are generalized in Appendices A and B to deal with systems having several degrees of freedom.
where the \( F^{(n)} \)'s and \( G^{(n)} \)'s are required to be periodic functions of \( \phi \) with period \( 2\pi \). In order to obtain definite single valued expressions for the \( F^{(n)} \)'s and \( G^{(n)} \)'s, Bogoliubov requires that they shall contain no zero harmonics in \( \phi \). The new variables \( y \) and \( \phi \) are assumed to be governed by the following equations:

\[
\begin{align*}
\dot{y} &= \sum_{n=1}^{\infty} e_n^{n_b}(n)(y), \\
\dot{\phi} &= \omega(y) + \sum_{n=1}^{\infty} e_n^{n_b}(n)(y).
\end{align*}
\tag{3.3a, 3.3b}
\]

The right hand sides of (3.3) are required to be independent of the angle variable \( \phi \). The object of the perturbation theory is to determine the \( F^{(n)} \)'s, \( G^{(n)} \)'s, \( a^{(n)} \)'s and \( b^{(n)} \)'s.

The physical meaning of the transformation (3.2) lies in the decomposition of the true motion described by \( x \) and \( y \) into an average motion described by \( y \) and \( \phi \) and rapid fluctuations about this average which are described by the functions \( F^{(n)} \) and \( G^{(n)} \). In order that this decomposition shall be valid the motion must be characterized by two very different time scales. One of these characteristic times, say \( \tau_1 \), measures a typical period of the rapid motion. It is clear from (3.1b) that

\[
\tau_1 = O(2\pi/\omega).
\tag{3.4}
\]

The second characteristic time, say \( \tau_2 \), measures the interval over which the amplitude of oscillation undergoes an appreciable change. In order that the Bogoliubov method shall provide a good description of the motion, the times \( \tau_1 \) and \( \tau_2 \) must be such that

\[
\tau_2 \gg \tau_1.
\tag{3.5}
\]

The characteristic time \( \tau_2 \) is clearly related to the parameter \( \epsilon \) since \( \tau_2 \to \infty \) and \( \epsilon \to 0 \). The precise connection between \( \tau_2 \) and \( \epsilon \) depends on the specific analytical form of equations (3.1).

It is important that the reader is aware of the asymptotic separation (asymptotic as \( \epsilon \to 0 \)) of the fast and slow time scales. An awareness of this separation is crucial to an understanding and justification of the methods.
which are developed in this chapter and in the remaining chapters.

The details of the perturbation calculation are straightforward. We first differentiate (3.2) with respect to time to obtain

\[ \dot{x} = \dot{y} + \sum_{n=1}^{\infty} \varepsilon^n \left[ \frac{\partial p(n)}{\partial \phi} \dot{\phi} + \frac{\partial p(n)}{\partial y} \dot{y} \right], \quad (3.6a) \]

\[ \dot{y} = \dot{\phi} + \sum_{n=1}^{\infty} \varepsilon^n \left[ \frac{\partial q(n)}{\partial \phi} \dot{\phi} + \frac{\partial q(n)}{\partial y} \dot{y} \right]. \quad (3.6b) \]

Next we substitute the right hand sides of (3.3) into (3.6) to obtain

\[ \dot{x} = \sum_{n=1}^{\infty} \varepsilon^n a(n) + \sum_{n=1}^{\infty} \varepsilon^n \left[ \frac{\partial p(n)}{\partial \phi} (\omega + \sum_{m=1}^{\infty} \varepsilon^m b(m)) \right. \]

\[ \left. + \frac{\partial p(n)}{\partial y} \sum_{m=1}^{\infty} \varepsilon^m a(m) \right], \quad (3.7a) \]

\[ \dot{y} = \omega(y) + \sum_{n=1}^{\infty} \varepsilon^n b(n) + \sum_{n=1}^{\infty} \varepsilon^n \left[ \frac{\partial q(n)}{\partial \phi} (\omega + \sum_{m=1}^{\infty} \varepsilon^m b(m)) \right. \]

\[ \left. + \frac{\partial q(n)}{\partial y} \sum_{n=1}^{\infty} \varepsilon^m a(m) \right]. \quad (3.7b) \]

We now write the original equations (3.1) in terms of the new variables and then expand the functions A, \( \omega \) and B in multiple Taylor series about the arguments \( \phi \) and \( y \). The result of this expansion is a formal infinite series in powers of \( \varepsilon \) for \( \dot{x} \) and \( \dot{y} \). Upon equating the coefficient of \( \varepsilon^n \) from these formal series with the corresponding coefficient of equations (3.7) we obtain an infinite set of coupled differential equations. These equations recursively determine the functions \( p(n), q(n), a(n) \) and \( b(n) \).

The first order functions are found from the equations

\[ a^{(1)}(y) + \omega(y) \frac{\partial p^{(1)}}{\partial \phi} = A(\phi, y), \quad (3.8a) \]
\[ b^{(1)}(y) + \omega(y) \frac{\partial g^{(1)}}{\partial \phi} = f^{(1)} \frac{\partial \omega}{\partial y} + B(\phi, y). \quad (3.8b) \]

Similarly the second order functions are found from the equations

\[ s^{(2)} + \omega \frac{\partial f^{(2)}}{\partial \phi} = g^{(1)} \frac{\partial A}{\partial \phi} + f^{(1)} \frac{\partial A}{\partial y} \]

\[- b^{(1)} \frac{\partial f^{(1)}}{\partial \phi} - a^{(1)} \frac{\partial f^{(1)}}{\partial y}, \quad (3.9a)\]

\[ b^{(2)} + \omega \frac{\partial g^{(2)}}{\partial \phi} = \frac{1}{2} \left[ f^{(1)} \right] \frac{\partial^2 \omega}{\partial y^2} + f^{(2)} \frac{\partial \omega}{\partial y} \]

\[ + g^{(1)} \frac{\partial B}{\partial \phi} + f^{(1)} \frac{\partial B}{\partial y} \]

\[- b^{(1)} \frac{\partial g^{(1)}}{\partial \phi} - a^{(1)} \frac{\partial g^{(1)}}{\partial y}. \quad (3.9b)\]

In general, the equations which determine the n'th order functions will have the form

\[ s^{(n)}(y) + \omega(y) \frac{\partial f^{(n)}}{\partial y} = x^{(n)}(\phi, y), \quad (3.10a) \]

\[ b^{(n)}(y) + \omega(y) \frac{\partial g^{(n)}}{\partial y} = y^{(n)}(\phi, y), \quad (3.10b) \]

where the functions \( x^{(n)}(\phi) \) and \( y^{(n)}(\phi) \) are determined by the solutions of the lower order equations.

The solutions of (3.10) are

\[ a^{(n)}(y) = \frac{1}{2\pi} \int_{0}^{2\pi} x^{(n)}(\phi, y) \, d\phi, \quad (3.11a) \]

\[ f^{(n)}(\phi, y) = \frac{1}{\omega(y)} \int \left[ x^{(n)}(\phi, y) - s^{(n)}(y) \right] \, d\phi, \quad (3.11b) \]
\[ b^{(n)}(y) = \frac{1}{2\pi} \int_0^{2\pi} y^{(n)}(\phi, y) \, d\phi , \quad (3.11c) \]

\[ g^{(n)}(\phi, y) = \frac{1}{\omega(y)} \int \left[ y^{(n)}(\phi, y) - b^{(n)}(y) \right] \, d\phi . \quad (3.11d) \]

As required by Bogoliubov, the functions \( F^{(n)} \) and \( G^{(n)} \) have been determined so as to contain no zero harmonic in \( \phi \).

In order to obtain the \( n \)'th order approximation we must determine all the functions under the summation signs in the following equations

\[ x = y + \sum_{n=1}^{N} \epsilon^n F^{(n)} , \quad (3.12a) \]

\[ y = \phi + \sum_{n=1}^{N} \epsilon^n G^{(n)} . \quad (3.12b) \]

The \( N \)'th order solution is not obtained, however, until we solve the following system of differential equations:

\[ \dot{y} = \sum_{n=1}^{N} \epsilon^n a^{(n)}(y) , \quad (3.13a) \]

\[ \dot{\phi} = \omega(y) + \sum_{n=1}^{N} \epsilon^n b^{(n)}(y) . \quad (3.13b) \]

Thus we have reduced the problem from one of solving the original system of equations to one of solving the system (3.13). We see that the Bogoliubov method does not automatically produce a solution. Rather it gives us an approximately equivalent system of equations in a reduced number of variables. The significance of the reduced system (3.13) lies in the absence of the rapidly varying phase. The reduced system depends only on the slowly varying amplitude and because of this, is usually much simpler to discuss than is the original system (3.1).

We have completed our formal discussion of the one dimensional Bogoliubov method. In order to enhance our understanding of this method we shall now apply it to the special case of an anharmonic oscillator. The Hamiltonian
H for such an oscillator is

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2 + \epsilon a q^3 \]  

(3.14)

where \( p \) is the canonical momentum, \( q \) is its conjugate coordinate and \( \epsilon \) is a small parameter. We introduce a new canonical momentum \( J \) and a new coordinate \( \gamma \) via the transformation

\[ q = \left( \frac{2J}{m \omega_0} \right)^{1/2} \sin \gamma, \]  

(3.15a)

\[ p = (2m \omega_0 J)^{1/2} \cos \gamma. \]  

(3.15b)

The Hamiltonian \( H \), when written in terms of \( J \) and \( \gamma \) is

\[ H = J \omega_0 + \epsilon a \left( \frac{2J}{m \omega_0} \right)^{3/2} \sin^3 \gamma. \]  

(3.16)

Hamilton's equations of motion are found from (3.16) to be

\[ \dot{J} = -3 \epsilon a \left( \frac{2J}{m \omega_0} \right)^{3/2} \sin^2 \gamma \cos \gamma, \]  

(3.17a)

\[ \dot{\gamma} = \omega_0 + \frac{3\epsilon a}{m \omega_0} \left( \frac{2J}{m \omega_0} \right)^{1/2} \sin^3 \gamma. \]  

(3.17b)

Equations (3.17) are of the same type as equations (3.1). We can, therefore, apply the Bogoliubov method. To do so we introduce new variables \( K \) and \( \phi \) as follows

\[ J = K + \sum_{n=1}^{\infty} \epsilon^n F^n (\phi, K), \]  

(3.18a)

\[ \gamma = \phi + \sum_{n=1}^{\infty} \epsilon^n G^n (\phi, K), \]  

(3.18b)

such that
\[ K = \sum_{n=1}^{\infty} \varepsilon^n a_n(K) \]  
\[ \dot{\varphi} = \omega_0 + \sum_{n=1}^{\infty} \varepsilon^n b_n(K). \]

We now substitute expressions (3.18) and (3.19) into (3.17), and equate equal powers of \( \varepsilon \). The first order equations are

\[ a^{(1)}(K) + \omega_0 \frac{\partial F^{(1)}}{\partial \varphi} = -3a \left( \frac{2K}{m_0} \right)^{3/2} \sin^2 \varphi \cos \varphi \]  
\[ b^{(1)}(K) + \omega_0 \frac{\partial G^{(1)}}{\partial \varphi} = 3a \left( \frac{2K}{m_0} \right)^{1/2} \sin^3 \varphi. \]

The right hand sides of (3.20) have no zero harmonics in \( \varphi \). Therefore, we must have

\[ a^{(1)}(K) = b^{(1)}(K) = 0. \]

It follows from (3.20) and (3.21) that

\[ F^{(1)}(\varphi, K) = -\frac{a}{\omega_0} \left( \frac{2K}{m_0} \right)^{3/2} \sin^3 \varphi, \]  
\[ G^{(1)}(\varphi, K) = -\frac{a}{m_0^2} \left( \frac{2K}{m_0} \right)^{1/2} \cos \varphi (\sin^2 \varphi + 2). \]

It should be noticed that \( F^{(1)} \) and \( G^{(1)} \) have been chosen to have no zero harmonics.

We can immediately write down the second order equations by appealing to equations (3.9a,b). We find that

\[ a^{(2)} + \omega_0 \frac{\partial F^{(2)}}{\partial \varphi} = -\frac{a}{m_0} \left( \frac{2K}{m_0} \right)^{1/2} \cos \varphi (\sin^2 \varphi + 2) \left[ -3a \left( \frac{2K}{m_0} \right)^{3/2} (-\sin^3 \varphi \right. \\

+ 2 \sin \varphi \cos^2 \varphi \right] \\

- a \left( \frac{2K}{m_0} \right)^{3/2} \sin^3 \varphi \left[ -\frac{9a}{m_0^2} \left( \frac{2K}{m_0} \right)^{1/2} \sin \varphi \cos \varphi \right]. \]  

(3.23a)
\[ b^{(2)} + \omega_o \frac{\partial g^{(2)}}{\partial \phi} = - \frac{6a^2K}{(\omega_o)^3} \left[ 3\sin^4 \phi \cos^2 \phi + 6\cos^2 \phi \sin^2 \phi \right. + \sin^6 \phi \right]. \tag{3.23b} \]

The right hand side of (3.23a) contains no zero harmonic. We conclude then that
\[ a^{(2)}(k) = 0, \tag{3.24} \]
and that
\[ F^{(2)}(\phi, K) = - \frac{3}{8\omega_o} \left( \frac{2K}{\omega_o} \right)^2 \left( \cos 4 \phi + 4 \cos 2 \phi \right). \tag{3.25} \]

Similarly we find from (3.23b) that
\[ b^{(2)} = - \frac{15}{2} \frac{a^2K}{(\omega_o)^3} \tag{3.26} \]
and
\[ g^{(2)} = - \frac{3}{8\omega_o} \left( \frac{2K}{\omega_o} \right)^3 \left[ - \frac{9}{2} \sin 2 \phi - 3 \sin 4 \phi + \frac{1}{6} \sin 6 \phi \right]. \tag{3.27} \]

The expressions found above, when substituted into (3.18) and (3.19) give
\[ J = K - \epsilon \frac{a}{\omega_o} \left( \frac{2K}{\omega_o} \right)^{3/2} \sin^3 \phi \tag{3.28a} \]
\[ - \frac{3}{2} \frac{a^2K}{(\omega_o)^3} \left( \frac{2K}{\omega_o} \right)^2 \cos 4 \phi + 4 \cos 2 \phi, \]
\[ + 0 \left( \epsilon^3 \right), \]
\[ \gamma = \phi - \epsilon \frac{a}{(\omega_o)^2} \left( \frac{2K}{\omega_o} \right)^{1/2} \cos \phi (\sin^2 \phi + 2) \tag{3.28b} \]
\[ - \frac{3}{2} \frac{a^2K}{(\omega_o)^3} \left[ - \frac{9}{2} \sin 2 \phi - 3 \sin 4 \phi + \frac{1}{6} \sin 6 \phi \right. + \left. 0 \left( \epsilon^3 \right) \right], \]
and

\[ \dot{\mathbf{k}} = 0(\epsilon^3), \quad (3.29a) \]

\[ \phi = \omega_0 - \epsilon^2 \frac{15 a^2 K}{2(m\omega_0)^3} + O(\epsilon^3). \quad (3.29b) \]

We find from (3.29a,b) that, through second order,

\[ K = \text{constant}, \quad (3.30a) \]

\[ \phi = \left[ \omega_0 - \epsilon^3 \frac{15 a^2 K}{2(m\omega_0)^3} \right] t + \text{constant}. \quad (3.30b) \]

When (3.30a,b) are substituted into (3.28a,b), we obtain the time dependence of \( J \) and \( \gamma \) through second order. In the next section we shall demonstrate that \( \dot{K} = 0 \) to all orders in the perturbation theory. This means that the perturbed oscillator remains periodic to all orders in the perturbation theory, the constant frequency of oscillation being given by (3.19b).

3.2 A CLASS OF HAMILTONIAN SYSTEMS

We consider a system whose unperturbed state is described by the action and angle variables \( J^0 \) and \( \gamma^0 \) and whose actual motion is derivable from a Hamiltonian of the form

\[ H = H_0(J^0) + \sum_{n=1}^{R} \epsilon^n H_n(J^0,\gamma^0), \quad (3.31) \]

where \( \epsilon \) is a small parameter, \( R \) is an upper limit, and where the functions \( H_n \) are periodic in \( \gamma^0 \) with period \( 2\pi \).

Hamilton's equations of motion are found from (3.31) to be

\[ \dot{J} = - \sum_{n=1}^{R} \epsilon^n \frac{\partial H_n}{\partial \gamma^0}, \quad (3.32a) \]
\[ \gamma^o = \omega_o + \sum_{n=1}^{R} \epsilon^n \frac{\partial H_n}{\partial j^o} , \quad (3.32b) \]

where

\[ \omega_o = \frac{\partial H_o}{\partial j^o} . \quad (3.33) \]

Equations (3.32) are a special case of equations (3.1). We can, therefore, apply the Bogoliubov method. In order to do so we make the change of variables

\[ j^o = K + \epsilon F(\phi, K) , \quad (3.34a) \]
\[ \gamma^o = \phi + \epsilon G(\phi, K) , \quad (3.34b) \]

such that

\[ \dot{K} = \epsilon A(K) , \quad (3.35a) \]
\[ \dot{\phi} = \omega_o(K) + \epsilon B(K) \quad (3.35b) \]
\[ = \omega(K) . \]

The functions F, G, A, and \( \omega \) are formal infinite series and are of order \( \epsilon^o \).

The Bogoliubov transformation (3.34) is constructed such that the functions F and G are periodic in \( \phi \) with period 2\( \pi \). This means that the Hamiltonian (3.31), when expressed in terms of the new variables K and \( \phi \), will be a periodic function of \( \phi \) with period 2\( \pi \). This fact allows us to prove the following theorem:

Theorem 3.1. A system which can be described by the Hamiltonian (3.1) is such that the variable K is constant to all orders in the perturbation theory.

Proof. The Hamiltonian \( H = h(\phi, K) \) is a constant of the motion; its time derivative must vanish. Thus

\[ \dot{h} = \dot{K} \frac{\partial h}{\partial K} + \dot{\phi} \frac{\partial h}{\partial \phi} \]
\[ = \epsilon A(K) \frac{\partial h}{\partial K} + \omega(K) \frac{\partial h}{\partial \phi} \]
\[ = 0 . \]

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It follows from (3.36) that

$$\frac{\partial h}{\partial \phi} = -\epsilon \frac{A(K)}{\omega(K)} \frac{\partial h}{\partial K} .$$

(3.37)

Let us now suppose that $h(\phi, K)$ is known to be independent of $\phi$ through order $\epsilon^n$. Thus

$$h = h_0(K) + \epsilon^{n+1} h_1(\phi, K) .$$

(3.38)

Upon substituting (3.38) into (3.37) we find that

$$\frac{\partial h}{\partial \phi} = -\epsilon \frac{A(K)}{\omega(K)} \left[ \frac{\partial h_0(K)}{\partial K} + \epsilon^{n+1} \frac{\partial h_1(\phi, K)}{\partial K} \right] .$$

(3.39)

The left hand side of (3.39) has no zero harmonic. This means that

$$\frac{A(K)}{\omega(K)} \frac{\partial h_0(K)}{\partial K} = 0$$

(3.40)

through order $\epsilon^{n+1}$. This implies that

$$\frac{\partial h}{\partial \phi} = 0(\epsilon^{n+2}) .$$

(3.41)

Thus if $h(\phi, K)$ is independent of $\phi$ through order $\epsilon^n$ then it is independent of $\phi$ through order $\epsilon^{n+1}$.

It is clear from (3.31) and (3.34) that $h(\phi, K)$ is independent of $\phi$ through order $\epsilon^0$. Hence, by mathematical induction, we conclude that $h(\phi, K)$ is independent of $\phi$ of all orders in the perturbation theory. This means that the right hand side of (3.37) must vanish to all orders. Since $\partial h/\partial K$ must not vanish, it follows that

$$A(K) = 0$$

(3.42)

to all orders in the perturbation theory. The theorem is therefore proved.

The perturbation formalism is such that $\phi$ depends only on $K$. Thus, according to Theorem 3.1, $\phi$ is a linear function of time. Since $\gamma^0$ increases by $2\pi$, and since $J^0$ undergoes fluctuations about constant $K$, it
follows that the perturbed system is periodic in time. We may state this as a corollary to Theorem 3.1.

Corollary 3.1. A system described by the Hamiltonian (3.31) is periodic in time to all orders in the perturbation theory.

3.3 THE CONNECTION WITH THE CLASSICAL METHOD

Since the perturbed system, when discussed by the Bogoliubov method, remains periodic in time we might expect that there is a connection between the Bogoliubov method and classical perturbation theory. The simplest way to determine if a connection exists is to assume that we have completely determined the motion using both methods.

The classical solution has the form

\[ J^o = J + \varepsilon \frac{\partial S(\gamma^o, J)}{\partial \gamma} \]
\[ \gamma = \gamma^o + \varepsilon \frac{\partial S(\gamma^o, J)}{\partial J} \]  

(3.43a)  

(3.43b)

where \( S(\gamma^o, J) \) is a formal infinite series in powers of \( \varepsilon \). The variables \( J \) and \( \gamma \) are the true action and angle variables for the system described by (3.31).

Having found the formal infinite series (3.34) and (3.43) we can eliminate \( \gamma^o \) and \( J^o \) to find

\[ J = J(K, \phi) \]
\[ \gamma = \gamma(K, \phi) \]  

(3.44a)  

(3.44b)

where \( J(K, \phi) \) and \( \gamma(K, \phi) \) are formal infinite series in powers of \( \varepsilon \). It follows from (3.44a) that

\[ \dot{J} = K \frac{\partial J}{\partial K} + \phi \frac{\partial J}{\partial \phi} \].  

(3.45)

Since \( J \) is a true action variable, \( \dot{J} \) vanishes. According to Theorem 3.1, \( K \) vanishes. We conclude from (3.45) that \( J \) is independent of \( \phi \) to all orders.
It is clear from (3.43) and (3.44) that

$$\gamma = \phi + \epsilon R(K, \phi),$$  \hspace{1cm} (3.46)

where $R(K, \phi)$ is a formal infinite series and is periodic in $\phi$. The perturbation theories are such that

$$\dot{\gamma} = \omega_1(J(K)),$$  \hspace{1cm} (3.47a)
$$\dot{\phi} = \omega(K).$$  \hspace{1cm} (3.47b)

Thus

$$\dot{\epsilon R} = \omega_1(J(K)) - \omega(K),$$  \hspace{1cm} (3.48)

and also

$$\dot{\epsilon R} = \epsilon \omega(K) \frac{\partial R}{\partial \phi} + \epsilon K \frac{\partial R}{\partial K}.$$  \hspace{1cm} (3.49)

Upon using Theorem 3.1 and equations (3.48) and (3.49) we find that

$$\epsilon \frac{\partial R}{\partial \phi} = \frac{\omega_1(J(K)) - \omega(K)}{\omega(K)}.$$  \hspace{1cm} (3.50)

The right hand side of (3.50) is independent of $\phi$, yet $R(K, \phi)$ must be a periodic function of $\phi$. Both of these conditions can be met only if $R(K, \phi)$ is independent of $\phi$. This means that

$$\omega_1(J(K)) = \omega(K)$$  \hspace{1cm} (3.51)

or

$$\dot{\gamma} = \dot{\phi}$$  \hspace{1cm} (3.52)

to all orders in $\epsilon$.

We now ask whether $J=K$? To answer this question we shall evaluate

$$\int_0^{2\pi} J^\circ \, d\gamma^\circ,$$  \hspace{1cm} (3.53)
first in the classical scheme, and then in the Bogoliubov scheme. We shall
then equate the results.

In the classical scheme we have

\[ \int_0^{2\pi} J^0 \, d\gamma^0 = \int_0^{2\pi} \left[ J + \epsilon \frac{\partial}{\partial \gamma^0} \right] \, d\gamma^0 = 2\pi J. \tag{3.54} \]

In the Bogoliubov scheme we have

\[ \int_0^{2\pi} J^0 \, d\gamma^0 = \int_0^{2\pi} (K + \epsilon F(\phi, K)) \frac{\partial \gamma^0}{\partial \phi} \, d\phi = \int_0^{2\pi} (K + \epsilon F(\phi, K)) \left(1 + \epsilon \frac{\partial G(K, \phi)}{\partial \phi}\right) \, d\phi. \tag{3.55} \]

If we assume with Bogoliubov that \( F(\phi, K) \) contains no zero harmonic in \( \phi \) then

\[ \int_0^{2\pi} J^0 \, d\gamma^0 = 2\pi K + \epsilon^2 \int_0^{2\pi} F(K, \phi) \frac{\partial G(K, \phi)}{\partial \phi} \, d\phi. \tag{3.56} \]

The integral in (3.56) does not vanish. In order to see this we simply construct the first order contributions to \( F \) and \( \partial G/\partial \phi \). These are found to be

\[ F^{(1)} = \frac{1}{\omega_0(K)} \left[ H_1(K, \phi) - \frac{1}{2\pi} \int_0^{2\pi} H_1(K, \phi) \, d\phi \right], \tag{3.57a} \]

\[ \frac{\partial G^{(1)}}{\partial \phi} = \frac{1}{\omega_0(K)} \frac{\partial}{\partial K} \left[ H_1(K, \phi) - \frac{1}{2\pi} \int_0^{2\pi} H_1(K, \phi) \, d\phi \right]. \tag{3.57b} \]

If we denote the phase average of \( H_1 \) by \( \langle H_1 \rangle \), we find, to lowest order, that

\[ \int_0^{2\pi} F(K, \phi) \frac{\partial G(K, \phi)}{\partial \phi} \, d\phi = \frac{1}{2\omega_0^2} \frac{\partial}{\partial K} \int_0^{2\pi} \left[ H_1 - \langle H_1 \rangle \right]^2 \, d\phi. \tag{3.58} \]
The right hand side of (3.58) in general does not vanish. Thus we have shown that \( J \neq K \).

It is a simple matter to show that the variables \( \phi \) and \( K \) are not canonical. To do this we form the Poisson bracket of the canonical variables \( J \) and \( \gamma \) with respect to \( \phi \) and \( K \). We find that

\[
\{ \gamma, J \} = \frac{\partial \gamma}{\partial \phi} \frac{\partial J}{\partial K} - \frac{\partial \gamma}{\partial K} \frac{\partial J}{\partial \phi} = \frac{\partial J}{\partial K} \neq 1.
\]

Hence \( \phi \) and \( K \) are not canonical.

According to (3.59), the variables \( \phi \) and \( K \) will be canonical if and only if

\[
J = K + C,
\]

where \( C \) is a constant independent of \( K \). We have shown above that, when \( F(\phi, K) \) is required to have no zero harmonic in \( \phi \), then (3.60) is not satisfied. We shall now show how the Bogoliubov scheme must be modified so that (3.60) holds, hence so that \( \phi \) and \( K \) are canonical. This is accomplished by removing the restriction that \( F(\phi, K) \) and \( G(\phi, K) \) shall contain no zero harmonic in \( \phi \). According to (3.55) the equality (3.60) will be satisfied only if

\[
\int_0^{2\pi} \left[ \phi_K \frac{\partial G}{\partial \phi} + \phi_F + \phi^2 F \frac{\partial G}{\partial \phi} \right] d\phi = 0.
\]

The functions \( \phi_F \) and \( \phi_G \) are formal infinite series of the form

\[
\phi_F = \sum_{n=1}^{\infty} \phi_{F_n}(\phi, K),
\]

\[
\phi_G = \sum_{n=1}^{\infty} \phi_{G_n}(\phi, K).
\]

If we substitute (3.62) into (3.61) and then equate each term of the resulting power series to zero, we obtain following sequence of equations:
\begin{align}
\int_0^{2\pi} \left[ K \frac{\partial g^{(1)}}{\partial \phi} + F^{(1)} \right] d\phi &= 0 , \\
\int_0^{2\pi} \left[ K \frac{\partial g^{(2)}}{\partial \phi} + F^{(2)} + F^{(1)} \frac{\partial g^{(1)}}{\partial \phi} \right] d\phi &= 0 , \\
\int_0^{2\pi} \left[ K \frac{\partial g^{(n)}}{\partial \phi} + F^{(n)} + F^{(1)} \frac{\partial g^{(n-1)}}{\partial \phi} + \cdots + F^{(n-1)} \frac{\partial g^{(1)}}{\partial \phi} \right] d\phi &= 0.
\end{align}

The first term in each integrand of (3.63) contributes nothing to the integral. The sequence (3.63) can be satisfied to all orders. For example, \( F^{(1)} \) must have no zero harmonic. \( F^{(2)} \) must have the zero harmonic

\[- \int_0^{2\pi} F^{(1)} \frac{\partial g^{(1)}}{\partial \phi} d\phi ,\]

and so forth. The zero harmonic of \( G(\phi, K) \) does not enter the sequence (3.63); it is, therefore, arbitrary and may be set equal to zero.

In conclusion, we have demonstrated that the Bogoliubov method, when applied to the one dimensional Hamiltonian (3.31), is equivalent to classical perturbation theory. By satisfying the conditions expressed in (3.63) we can modify the Bogoliubov method so that it is identical with classical perturbation theory.
CHAPTER 4

PERTURBATION THEORY FOR SYSTEMS HAVING TWO RAPID PHASES

In this chapter we shall develop techniques for dealing with two dimensional systems which have two rapid phases. As in classical perturbation theory, it will be necessary to distinguish between degenerate and non-degenerate systems. After formally presenting the methods, we shall apply them to a class of Hamiltonian systems.

4.1 NON-DEGENERATE PERTURBATION THEORY

We consider a dynamical system (not necessarily a Hamiltonian system) whose configuration is characterized by the angles $\gamma_1$ and $\gamma_2$ and the amplitudes $x_1$ and $x_2$, and whose temporal behavior is governed by the following system of autonomous differential equations:

$$
\dot{x}_i = \epsilon A_i(\gamma, x) \quad (i = 1, 2), \quad (4.1a)
$$

$$
\dot{\gamma}_i = \omega_i(x) + \epsilon B_i(\gamma, x) \quad (i = 1, 2), \quad (4.1b)
$$

where $\epsilon$ is a small parameter and the $A_i$'s and $B_i$'s are periodic functions of $\gamma_1$ and $\gamma_2$ with period $2\pi$.

An obvious way to treat the system of equations (4.1) is to extend the Bogoliubov method to the case of two phases. In order to do this we introduce new variables $y_1$, $y_2$, $\varphi_1$, and $\varphi_2$ as follows:

$$
x_i = y_i + \sum_{n=1}^{\infty} \epsilon^n F_i^{(n)}(\varphi, y) \quad (i = 1, 2), \quad (4.2a)
$$

$$
\gamma_i = \varphi_i + \sum_{n=1}^{\infty} \epsilon^n G_i^{(n)}(\varphi, y) \quad (i = 1, 2), \quad (4.2b)
$$

where the $F_i^{(n)}$'s and $G_i^{(n)}$'s are required to be periodic functions of $\varphi_1$ and $\varphi_2$ with period $2\pi$. In addition to the transformation (4.2) we require that

$^{17}$A general discussion of systems having several rapid phases will be found in Appendix C.
\[
\dot{y}_1 = \sum_{n=1}^{\infty} \epsilon^n a_1^{(n)}(y) \quad (i = 1,2), \quad (4.3a)
\]

\[
\dot{\phi}_1 = \omega_1(y) + \sum_{n=1}^{\infty} \epsilon^n b_1^{(n)}(y) \quad (i = 1,2). \quad (4.3b)
\]

The right hand sides of (4.3a,b) are to be independent of \(\phi_1\) and \(\phi_2\). The object of the perturbation theory is to determine the functions \(F_1^{(n)}\), \(G_1^{(n)}\), \(a_1^{(n)}\) and \(b_1^{(n)}\). The method of determining these functions is essentially the same as that used in the Bogoliubov theory.

The perturbation procedure described above is a straightforward generalization of the Bogoliubov method. Like the Bogoliubov method it rests upon the asymptotic separation of the fast and slow time scales. The new method is, however, subjected to constraints which are not present in the Bogoliubov scheme. In order to illustrate these constraints we now consider two similar bodies each with moment of inertia \(I\) about their common axis of rotation. Let their positions be defined by the angles \(\gamma_1\) and \(\gamma_2\), and denote their respective angular momenta by \(J_1\) and \(J_2\). If the bodies do not interact then the Hamiltonian \(H_0\) which describes the system is

\[
H_0 = \frac{1}{2I}(J_1^2 + J_2^2). \quad (4.4)
\]

We now introduce an energy of interaction \(H_1\) which is proportional to the \(\cos(\gamma_1 - \gamma_2)\). Thus we may write

\[
H_1 = \epsilon(1 - \cos(\gamma_1 - \gamma_2)), \quad (4.5)
\]

where \(\epsilon\) measures the strength of the interaction. The total Hamiltonian \(H\) is

\[
H = H_0 + H_1. \quad (4.6)
\]

Hamilton's equations of motion are found from (4.6) to be

\[
\dot{J}_1 = -\epsilon \sin(\gamma_1 - \gamma_2), \quad (4.7a)
\]

\[
\dot{J}_2 = \epsilon \sin(\gamma_1 - \gamma_2), \quad (4.7b)
\]
\[ \dot{\gamma}_1 = \omega_1, \quad (4.7c) \]
\[ \dot{\gamma}_2 = \omega_2, \quad (4.7d) \]

where
\[ \omega_i = J_i/I \quad (i = 1, 2). \quad (4.8) \]

We now perform first order perturbation theory\(^{18}\) by introducing new variables $K_1, K_2, \phi_1$ and $\phi_2$ as follows:

\[ J_i = K_i + \epsilon F_i(\phi, K), \quad (4.9a) \]
\[ \gamma_i = \phi_i + \epsilon G_i(\phi, K), \quad (4.9b) \]

such that
\[ \dot{K}_1 = \epsilon a_1(K), \quad (4.10a) \]
\[ \dot{\phi}_1 = \omega_1(K) + \epsilon b_1(K). \quad (4.10b) \]

when we substitute (4.9) and (4.10) into (4.7) we find that

\[ \epsilon a_1(K) + \omega_1 \frac{\partial F_1}{\partial \phi_1} + \omega_2 \frac{\partial F_1}{\partial \phi_2} = -\epsilon \sin(\phi_1 - \phi_2) + O(\epsilon^2) \quad (4.11a) \]
\[ \epsilon a_2(K) + \omega_1 \frac{\partial F_2}{\partial \phi_1} + \omega_2 \frac{\partial F_2}{\partial \phi_2} = \epsilon \sin(\phi_1 - \phi_2) + O(\epsilon^2) \quad (4.11b) \]
\[ \epsilon b_1(K) + \omega_1 \frac{\partial G_1}{\partial \phi_1} + \omega_2 \frac{\partial G_1}{\partial \phi_2} = 0(\epsilon^2), \quad (4.11c) \]
\[ \epsilon b_2(K) + \omega_1 \frac{\partial G_2}{\partial \phi_1} + \omega_2 \frac{\partial G_2}{\partial \phi_2} = 0(\epsilon^2). \quad (4.11d) \]

The right hand sides of (4.11) contain no zero harmonics. It follows then, from (4.10), that

\(^{18}\) The system (4.7) is integrable. This, however, does not concern us here.
\[ a_1 = a_2 = b_1 = b_2 = 0, \quad (4.12) \]

and consequently that

\[ F_1 = \frac{\cos(\phi_1 - \phi_2)}{\omega_1 - \omega_2}, \quad (4.13a) \]

\[ F_2 = -\frac{\cos(\phi_1 - \phi_2)}{\omega_1 - \omega_2}, \quad (4.13b) \]

and

\[ G_1 = G_2 = 0. \quad (4.13c) \]

Upon substituting (4.13a, b) into (4.9a, b) we find that

\[ J_1 = K_1 + \epsilon \frac{\cos(\phi_1 - \phi_2)}{\omega_1 - \omega_2} + O(\epsilon^2), \quad (4.14a) \]

\[ J_2 = K_2 - \epsilon \frac{\cos(\phi_1 - \phi_2)}{\omega_1 - \omega_2} + O(\epsilon^2). \quad (4.14b) \]

The solution (4.14) is legitimate provided the following inequality does not hold

\[ \omega_1 - \omega_2 \leq 0(\epsilon). \quad (4.15) \]

When (4.15) holds the perturbation solution (4.14) becomes meaningless because of the small divisors. When condition (4.15) holds the system is said to be degenerate. For this reason the perturbation theory presented in this section is called non-degenerate perturbation theory. This means that it is valid only in the absence of small divisors.\(^{19}\) For most problems small divisors will occur at some order of non-degenerate perturbation theory. This means that non-degenerate perturbation theory is almost never valid to all orders.\(^{20}\)

4.2 DEGENERATE PERTURBATION THEORY

The small divisors of the previous section have occurred because we failed to make provisions for internal resonance. When phase relationships

\(^{19}\)For a more complete discussion of small divisors see Appendix C, sec. C.1.

\(^{20}\)The validity of non-degenerate perturbation theory is discussed in Appendix E.
of the type implied by (4.15) exist, it becomes fairly easy for the two
degrees of freedom to exchange energy. These energy exchanges give rise
to appreciable changes in the amplitudes of oscillation. The small di-
visors are simply a manifestation of this fact. It is desirable to have
such resonant changes in amplitude occur through the average motion rather
than through the rapid motion. This means that we must include in the equa-
tions which govern the time development of the average motion those terms
which give rise to small divisors. We, therefore, formulate degenerate
perturbation theory as follows: First we introduce new variables \( y_1, y_2, \phi_1 \text{ and } \phi_2 \) such that

\[
x_i = y_i + \sum_{n=1}^{\infty} e^{nF_i(n)}(\phi, y) \quad (i=1,2), \tag{4.16a}
\]

\[
\gamma_i = \phi_i + \sum_{n=1}^{\infty} e^{nG_i(n)}(\phi, y) \quad (i=1,2), \tag{4.16b}
\]

where the \( F_i(n) \)'s and \( G_i(n) \)'s are periodic functions of \( \phi_1 \text{ and } \phi_2 \) with period
\( 2\pi \). Next we require that

\[
\dot{y}_i = \sum_{n=1}^{\infty} e^{nA_i(n)}(\theta(n), y) \quad (i=1,2) \tag{4.17a}
\]

\[
\dot{\phi}_i = \omega_i(y) + \sum_{n=1}^{\infty} e^{nB_i(n)}(\theta(n), y) \quad (i=1,2), \tag{4.17b}
\]

where the angles \( \theta(n) \) represent those terms which give rise to \( n \)-th order
resonant behavior. In other words, the \( \theta(n) \) are those angles which would
give rise to small divisors in the \( n \)'th order of non-degenerate perturbation
theory. The object of the perturbation theory is to construct the functions
\( F_i(n), G_i(n), A_i(n) \text{ and } B_i(n) \).

In order to illustrate degenerate perturbation theory let us reconsider
the coupled rotators which we discussed in section 4.1. The first order
equations of degenerate perturbation theory are identical in form with the
system (4.11). We must, however, now include in \( a_1, a_2, b_1 \text{ and } b_2 \) not just
the zero harmonics but also those terms which give rise to small divisors.
Thus we find that

\[
a_1 = -\sin(\phi_1 - \phi_2), \tag{4.18a}
\]

\[
a_2 = \sin(\phi_1 - \phi_2), \tag{4.18b}
\]
and
\[ b_1 = b_2 = 0 . \]  
\[(4.19)\]

Consequently
\[ F_1 = F_2 = G_1 = G_2 = 0 . \]  
\[(4.20)\]

Upon substituting (4.18a,b) and (4.19) into (4.17) we find that
\[
\begin{align*}
\dot{\phi}_1 & = -\epsilon \sin(\phi_1 - \phi_2) , & (4.21a) \\
\dot{\phi}_2 & = \epsilon \sin(\phi_1 - \phi_2) , & (4.21b) \\
\dot{\phi}_1 & = \omega_1(k) , & (4.21c) \\
\dot{\phi}_2 & = \omega_2(k) . & (4.21d)
\end{align*}
\]

These equations are identical with the original equations \((4.7)\). In this case degenerate perturbation theory does not simplify the problem. It tells us that we must solve the exact problem. In most cases degenerate perturbation theory will considerably simplify the problem at hand. We consider such a case in Chapter 6.

The process of putting the resonant terms into the average motion (hence into the \(a_{1(n)}\)'s and \(b_1^{(n)}\)'s) can be carried out at each order of perturbation theory. This means that degenerate perturbation theory can be performed so that small divisors never occur in any order of perturbation theory. This fact is of the utmost importance when we discuss the asymptotic convergence of degenerate perturbation theory in Appendix E.

4.3 APPLICATION OF NON-DEGENERATE PERTURBATION THEORY TO A CLASS OF HAMILTONIAN SYSTEMS

In this section we shall apply non-degenerate perturbation theory to a two dimensional Hamiltonian system which is described by the angles \(\gamma_1\) and \(\gamma_2\). Let the momenta conjugate to \(\gamma_1\) and \(\gamma_2\) be \(J_1\) and \(J_2\) respectively. We shall assume the system's Hamiltonian is of the form
\[
\mathcal{H} = \mathcal{H}_0(J) + \epsilon \mathcal{H}_1(J,\gamma) ,
\]
\[(4.22)\]
where $H_1$ is a periodic function of $\gamma_1$ and $\gamma_2$ with period $2\pi$, and $\epsilon$ is a small parameter. Hamilton's equations of motion are found from (4.22) to be

\begin{align}
\dot{J}_1 &= -\epsilon \frac{\partial H_1}{\partial \gamma_1}, \\
\dot{\gamma}_1 &= \omega_1^0(J) + \epsilon \frac{\partial H_1}{\partial J_1},
\end{align}

(4.23a)

(4.23b)

where

\begin{equation}
\omega_1^0 = \frac{\partial H_0}{\partial J_1}.
\end{equation}

(4.24)

Equations (4.23) are a special case of equations (4.1) and can, therefore, be treated by non-degenerate perturbation theory. In order to do so we make the following change of variables:

\begin{align}
J_1 &= K_1 + \epsilon F_1(\phi, K), \\
\gamma_1 &= \phi_1 + \epsilon G_1(\phi, K),
\end{align}

(4.25a)

(4.25b)

where $F_1$ and $G_1$ are formal infinite series in powers of $\epsilon$ and are periodic in $\phi_1$ and $\phi_2$ with period $2\pi$. We further require that

\begin{align}
\dot{K}_1 &= \epsilon a_1(K), \\
\dot{\phi}_1 &= \omega_1^0(K) + \epsilon b_1(K), \\
&= \omega_1(K),
\end{align}

(4.26a)

(4.26b)

where $a_1$ and $b_1$ are formal infinite series and are independent of $\phi_1$ and $\phi_2$.

The Hamiltonian (4.22) is required to be a periodic function of $\gamma_1$ and $\gamma_2$. The functions $F_1$ and $G_1$ are periodic functions of $\phi_1$ and $\phi_2$. We conclude that the Hamiltonian $H$, when expressed in terms of the new variables, is a periodic function of $\phi_1$ and $\phi_2$. This fact allows us to prove the following theorem:

Theorem 4.1. The Hamiltonian (4.22), when expressed in terms of $K_1$, $K_2$, $\phi_1$ and $\phi_2$, is independent of $\phi_1$ and $\phi_2$ to all orders of non-degenerate perturbation theory.
Proof. The Hamiltonian is a constant of the motion; its time derivative must vanish. Thus
\[ \dot{H} = \omega_1 \frac{\partial H}{\partial \phi_1} + \omega_2 \frac{\partial H}{\partial \phi_2} + \epsilon a_1 \frac{\partial H}{\partial k_1} + \epsilon a_2 \frac{\partial H}{\partial k_2} = 0. \] (4.27)

It follows from (4.27) that
\[ \omega_1(K) \frac{\partial H}{\partial \phi_1} + \omega_2(K) \frac{\partial H}{\partial \phi_2} = \left[ -\epsilon \int a_1(K) \frac{\partial H}{\partial k_1} + a_2(K) \frac{\partial H}{\partial k_2} \right]. \] (4.28)

Let us now suppose that \( H \) is known to be independent of \( \phi_1 \) and \( \phi_2 \) through order \( \epsilon^n \). We may, therefore, write
\[ H = H^0(K) + \epsilon^{n+1} H^1(K, \phi). \] (4.29)

When we substitute (4.29) into the right hand side of (4.28) we find that
\[ \omega_1 \frac{\partial H}{\partial \phi_1} + \omega_2 \frac{\partial H}{\partial \phi_2} = -\epsilon \left[ a_1(K) \frac{\partial H^0(K)}{\partial k_1} + a_2(K) \frac{\partial H^0(K)}{\partial k_2} \right] \]
\[ -\epsilon^{n+2} \left[ a_1(K) \frac{\partial H^1(K, \phi)}{\partial k_1} + a_2(K) \frac{\partial H^1(K, \phi)}{\partial k_2} \right]. \] (4.30)

Since the left hand side of (4.30) contains no zero harmonic, the first bracket on the right hand side must vanish through order \( \epsilon^{n+1} \). This means that
\[ \omega_1 \frac{\partial H}{\partial \phi_1} + \omega_2 \frac{\partial H}{\partial \phi_2} = 0(\epsilon^{n+2}). \] (4.31)

It follows from (4.31) that \( H(K, \phi) \) is independent of \( \phi_1 \) and \( \phi_2 \) through order \( \epsilon^{n+1} \).

It is clear from (4.22) and (4.25) that \( H(K, \phi) \) is independent of \( \phi_1 \) and \( \phi_2 \) through order \( \epsilon^0 \). It follows by mathematical induction that \( H(K, \phi) \) is independent of \( \phi_1 \) and \( \phi_2 \) to all orders. The theorem is therefore proved.

It can be shown$^{21}$ that the transformation (4.25) can be made so that \( K_i \) and \( \phi_i \) (\( i=1,2 \)) are canonically conjugate variables. When this is done $^{21}$See Appendix C, sec. C.3.
it follows from Theorem 4.1 that $K_1$ and $K_2$ are constants of the motion to all orders in non-degenerate perturbation theory. Since $\dot{\theta}_1$ and $\dot{\theta}_2$ depend only upon $K_1$ and $K_2$, it follows that $\phi_1$ and $\phi_2$ are linear functions of time. We conclude then that the solutions of equations (4.23) are automatically rendered to all orders in non-degenerate perturbation theory by requiring that the transformation (4.25) be canonical to all orders. When (4.25) is canonical conditions (A) and (B) of section 2.2 are satisfied. Thus $K_1$ and $\phi_1$ ($i=1,2$) are the classical action and angle variables, and the system described by (4.22) is multiply periodic to all orders in non-degenerate perturbation theory. It can be shown that canonical non-degenerate perturbation theory is equivalent to classical non-degenerate perturbation theory.\textsuperscript{22}

It must be emphasized that the fact that we have been able to prove results which are true to all orders in the perturbation theory does not mean that the perturbation theory is valid to all orders. Non-degenerate perturbation theory is valid only until small divisors occur. If the first small divisor occurs at order $\epsilon^n$, then the results of this section are valid through order $\epsilon^{n-1}$. That this is so is demonstrated in Appendix E.

4.4 \textit{APPLICATION OF DEGENERATE PERTURBATION THEORY TO HAMILTONIAN SYSTEMS}

In the case where the system discussed in the previous section is undergoing internal resonance we must use degenerate perturbation theory. To do so we make the following change of variables:

\begin{equation}
J_i = K_i + \epsilon P_i (\phi, K) \quad (i=1,2), \tag{4.32a}
\end{equation}

\begin{equation}
\gamma_i = \phi_i + \epsilon G_i (\phi, K) \quad (i=1,2). \tag{4.32b}
\end{equation}

We must include the resonant behavior in the average motion. Hence we require that

\begin{equation}
\dot{K}_i = \epsilon a_i (\theta, K) \quad (i=1,2), \tag{4.33a}
\end{equation}

\begin{equation}
\dot{\phi}_i = \omega_1^0 (K) + \epsilon b_i (\theta, K) \quad (i=1,2), \tag{4.33b}
\end{equation}

\begin{equation}
= \omega (\theta, K), \tag{4.33c}
\end{equation}

\textsuperscript{22}See Appendix C, sec. C.3.
where $\theta$ represents the resonant angles which are encountered in the course of the perturbation calculation. This vector $\theta$ will have $m$ components of the form

$$\theta_i = p_{i1}\phi_1 + p_{i2}\phi_2 \quad (i=1,2,\ldots,m),$$

(4.34)

where the $p_{ij}$'s are integers. There are an infinite number of ways in which $p_{11}$ and $p_{12}$ can be selected so that $\theta_1$ is slowly varying. For example, if $\phi_1 - \phi_2$ is slowly varying then $100\phi_1 - 101\phi_2$, $1000\phi_1 - 1001\phi_2$ may also be slowly varying. When perturbation theory is carried to all orders we will, therefore, usually encounter an infinite number of $\theta_i$'s (only two of which are independent). However, when perturbation theory is carried out to a finite order there will be only a finite number of $\theta_i$'s and, indeed, through some sufficiently low order, say $\epsilon^n$, there will be only one $\theta_1$ which we shall take to be

$$\theta = p\phi_1 + q\phi_2$$

(4.35)

Here $p$ and $q$ are assumed to have no common divisor (other than unity). We now assume that the resonance is adequately described by the single angle (4.35). In other words, we are going to be satisfied with carrying out perturbation theory through order $\epsilon^n$. The effect of neglecting the higher order resonances is discussed in Appendix E. It turns out that the solution which we obtain is valid to within order $\epsilon^n$ for times of order $(|\nabla \omega(y)| + \epsilon)^{-1}$.

In what follows we shall prove results to all orders. It must be remembered, however, that we are dealing with only a single resonant angle $\theta$. This means that our results are, in fact, valid only to some finite order.

Let us describe the system in terms of the resonant angle $\theta = p\phi_1 + q\phi_2$ and the angle $\phi_2$. We have that

$$\phi_1 = \frac{1}{p}(\theta - q\phi_2).$$

(4.36)

The perturbation theory is such that the Hamiltonian (4.22) is a periodic function of $\phi_1$ and $\phi_2$. It is clear from (4.36) that the Hamiltonian is also a periodic function of $\theta$ and $\phi_2$. This fact allows us to prove the following theorem:
Theorem 4.2. The Hamiltonian (4.22), when expressed in terms of $K_1$, $K_2$, $\theta$ and $\varphi_2$, is independent of $\varphi_2$ to all orders in degenerate perturbation theory.

Proof. Since the Hamiltonian is a constant of the motion its time derivative must vanish. Thus

$$
\dot{H} = \omega_1 \frac{\partial H}{\partial \varphi_1} + \omega_2 \frac{\partial H}{\partial \varphi_2} + \varepsilon \left[ a_1 \frac{\partial H}{\partial K_1} + a_2 \frac{\partial H}{\partial K_2} \right]
$$

$$
= 0.
$$

(4.37)

It follows from (4.37) that

$$
\omega_1 \frac{\partial H}{\partial \varphi_1} + \omega_2 \frac{\partial H}{\partial \varphi_2} = -\varepsilon \left\{ a_1 \frac{\partial H}{\partial K_1} + a_2 \frac{\partial H}{\partial K_2} \right\}.
$$

(4.38)

From (4.36) we find that

$$
\partial / \partial \varphi_1 + p\partial / \partial \theta, \quad \partial / \partial \varphi_2 + q\partial / \partial \theta + \partial / \partial \varphi_2.
$$

(4.39a,b)

Upon substituting (4.39) into (4.38) we find that

$$
(p\omega_1 + \omega_2) \frac{\partial H}{\partial \theta} + \omega_2 \frac{\partial H}{\partial \varphi_2} = -\varepsilon \left\{ a_1 \frac{\partial H}{\partial K_1} + a_2 \frac{\partial H}{\partial K_2} \right\}.
$$

(4.40)

We know the following facts: first $a_1$ and $a_2$ depend on the angles only through $\theta$, second, $\omega_1 = \omega_1(K)$ and $\omega_2 = \omega_2(K)$ through order $\varepsilon^0$. Let us now suppose that the Hamiltonian is known to be independent of $\varphi_2$ through order $\varepsilon^n$. This means that the right hand side of (4.40) is independent of $\varphi_2$ through order $\varepsilon^{n+1}$. The left hand side must also be independent of $\varphi_2$ through order $\varepsilon^{n+1}$. Now let the $n+1$st contribution to $H$ be $H^{n+1}$. The function $H^{n+1}$ must be a periodic function of $\theta$ and $\varphi_2$. Therefore it may be written in the form

$$
H^{n+1} = \sum_{r,s} H_{r,s}^{n+1}(K)e^{it(2\pi r \theta + s\varphi_2)},
$$

(4.41)

where $r$ and $s$ are integers and $T$ is determined by (4.36). Since $H(H, \theta, \varphi_2)$ is assumed to be independent of $\varphi_2$ through order $\varepsilon^n$, the left hand side of (4.40) could depend upon $\varphi_2$ only in order $\varepsilon^{n+1}$ (or higher order). This $\varphi_2$ dependence could come only from the term
\[
\sum_{r,s} H^{n+1}_{r,s}(K)(r(p_1^0 + q_2^0) + s\omega_2^0) e^{iT(r\theta + s\phi_2^0)}.
\]  

(4.42)

This term must be independent of \(\phi_2^0\). This will be so only if \(s = 0\) or if

\[
 r(p_1^0 + q_2^0) + s\omega_2^0 = 0
\]  

(4.43)

for all times. The equality (4.43) tells us that there is a pure commensurability between the frequencies other than the near commensurability \(p_1^0 + q_2^0\) which we have already assumed. In other words the system is not adequately described by a single resonance condition which is in violation of our initial assumption. Hence we exclude the case described by (4.43). This means that

\[
 s = 0,
\]  

(4.44)

which means that \(H^{n+1}(K,\theta,\phi_2^0)\) is independent of \(\phi_2^0\). Thus the Hamiltonian \(H(K,\theta,\phi_2^0)\), if independent of \(\phi_2^0\) through order \(e^n\) is also independent of \(\phi_2^0\) through order \(e^{n+1}\). It is clear that the Hamiltonian (4.22) is independent of \(\phi_2^0\) through order \(e^0\). If follows, therefore, by mathematical induction that it is independent of \(\phi_2^0\) through all orders in degenerate perturbation theory. The theorem is proved.

Suppose now that we are able to make degenerate perturbation theory canonical with \(K_i\) and \(\phi_i^0\) (i=1,2) as canonically conjugate variables. We could introduce \(\theta\) and \(\phi_2^0\) as new canonical variables. The momenta conjugate to \(\theta\) and \(\phi_2^0\) are \(k_1 = K_1/p\) and \(k_2 = K_2 - qK_1^0/p\) respectively. According to Theorem 4.2, the Hamiltonian \(H(K,\theta,\phi_2^0)\) is independent of \(\phi_2^0\) to all orders. Thus for canonical degenerate perturbation theory we have that

\[
 k_2 = K_2 - qK_1^0/p
\]  

(4.45)

is a constant of the motion to all orders. This integral of the motion, of course, reduces the number of variables by one and thereby reduces the complexity of the problem.

Unfortunately, in the general case, degenerate perturbation theory cannot be made canonical to all orders.\(^2\) We are, therefore, only able to state the following conditional theorem:

\(^2\) For a discussion of why this is so see Appendix C, sec. C.4.
Theorem 4.3. If degenerate perturbation theory can be made canonical through order \( \epsilon^n \), then the quantity \( k_2 = K_2 - qK_1/p \) is constant through order \( \epsilon^n \).

The frequencies of oscillation \( \phi_1 \) and \( \phi_2 \) usually depend upon time in degenerate perturbation theory. This means that the motion obtained by degenerate perturbation theory is not multiply periodic in the general case. Since classical perturbation theory seeks only the multiply periodic solutions, we can conclude immediately that degenerate perturbation theory and classical degenerate perturbation theory are not equivalent schemes. Of course some of the solutions generated by degenerate perturbation theory may be multiply periodic. It would be interesting to determine if these multiply periodic solutions are the same as those generated by classical perturbation theory. Unfortunately this appears to be a very difficult task.
CHAPTER 5

ASYMPTOTIC INVARIANCE

The concept of adiabatic invariance is well known to physicists. An adiabatic invariant is, however, a special case of a more general class of invariants which we shall call asymptotic invariants. In this chapter we shall be primarily interested in methods of constructing asymptotic invariants.

5.1 DEFINITION OF AN ASYMPTOTIC INVARIANT

We consider a system into whose description a small parameter $\epsilon$ enters in a natural fashion. The system is supposed to have a well known behavior when $\epsilon=0$. The parameter $\epsilon$ may, for example, measure the weakness of a perturbation or the slow variation of a parameter on which the system depends. The important idea is that the asymptotic behavior of the system is known in the limit $\epsilon \rightarrow 0$.

Definition. A quantity $K$ is an $n$'th order asymptotic invariant if

$$\dot{K} = \epsilon^n M_n,$$  \hspace{1cm} (5.1)

where $M_n$ is of order $\epsilon^n$.

The quantity $M_n$ will, in general, depend upon $n$ and upon the coordinates and momenta which describe the state of the system, and upon the time. The importance of $K$ derives from its time derivative's vanishing as the $n$'th power of $\epsilon$. If $\epsilon$ is small and $n \geq 1$, then $K$ varies slowly and may for many purposes, be assumed to be constant.

5.2 THE HIERARCHY OF ADIABATIC INVARIANTS

Consider a system having one degree of freedom which is such that the Hamiltonian which describes it is a slowly varying function of time. Such a Hamiltonian may be written as

$$H = H_0(q,p,\epsilon t).$$ \hspace{1cm} (5.2)
We shall assume that the Hamiltonian $H_0$ is an infinitely differentiable function of its arguments, and that the trajectories $H_0 = \text{constant}$ ($t$ being fixed) form a one parameter family of closed curves in the q-p plane. A typical trajectory is sketched in Fig. 1.

![Diagram](image)

Figure 1. A typical trajectory in the q-p plane.

Let us introduce the slow time $T$ according to the equation

$$T = \epsilon t.$$  \hfill (5.3)

In what follows we shall perform a series of canonical transformations in which $T$ is held fixed. We begin this series of transformations by introducing the new momentum

$$J_1 = \oint_{H_0} p \, dq,$$  \hfill (5.4)

where $\oint_{H_0}$ indicates an integration along the complete trajectory $H_0 = \text{constant}$ ($t$ being fixed). This momentum and its conjugate coordinate $\gamma_1$ are obtained from the variables q and p through the generating function $S_0(q, J_1, T)$ which is found by solving the following Hamilton-Jacobi equation:

$$H_0(q, \partial S_0/\partial q, T) = W_0(J_1, T).$$  \hfill (5.5)

The variables $J_1$ and $\gamma_1$ are, of course, analogous to action and angle variables.
The time development of $J_1$ and $\gamma_1$ is governed by the following Hamiltonian:

$$H_1 = W_0(J_1, T) + \epsilon \frac{\partial S_0}{\partial T}. \quad (5.6)$$

It is known from the theory of action and angle variables that $q$ will be a periodic function of $\gamma_1$. Thus the generator $S_0$ will be a periodic function of $\gamma_1$.

It is clear from (5.6) that $J_1$ is a first order asymptotic invariant. We next introduce the momentum

$$J_2 = \oint_{H_1} J_1 \, d\gamma_1 \quad (5.7)$$

and its conjugate coordinate $\gamma_2$. These variables are related to $J_1$ and $\gamma_1$ by the generator $S_1(\gamma_1, J_2, T)$ which is found by solving the following Hamilton-Jacobi equation:

$$H_1(\gamma_1, \partial S_1/\partial \gamma_1, T) = W_1(J_2, T). \quad (5.8)$$

The trajectories $H_1 = \text{constant}$ ($T$ being fixed) in the $J_1-\gamma_1$ plane are similar to that sketched in Fig. 2.

![Figure 2. Trajectory $H_1 = \text{constant}$ in the $J_1-\gamma_1$ plane.](image-url)
Since the fluctuations in $J_1$ are of order $\epsilon$ it follows that

$$J_2 = J_1 + 0(\epsilon) \quad (5.9)$$

This means that the generating function $S_1$ will have the form

$$S_1 = J_2 \gamma_1 + \epsilon S_1^0(\gamma_1, J_2, T) \quad (5.10)$$

The Hamiltonian which describes $J_2$ and $\gamma_2$ is, therefore,

$$H_2 = W_1(J_2, T) + \epsilon^2 \frac{\partial S_1^0}{\partial T} \quad (5.11)$$

We now introduce new variables $J_3$ and $\gamma_3$ through the generating function

$$S_2 = J_3 \gamma_2 + \epsilon^2 S_2^0(\gamma_2, J_3, T) \quad (5.12)$$

where $S_2^0$ is found by solving the following Hamilton-Jacobi equation:

$$H_2(\gamma_2, \frac{\partial S_2}{\partial \gamma_2}, T) = W_2(J_3, T) \quad (5.13)$$

The variables $J_3$ and $\gamma_3$ are described by the Hamiltonian

$$H_3 = W_2(J_3, T) + \epsilon^3 \frac{\partial S_2^0}{\partial T} \quad (5.14)$$

This process can be continued to any desired order. The Hamiltonian which describes the $n$'th order variables $J_n$ and $\gamma_n$ will have the form

$$H_n = W_{n-1}(J_n, T) + \epsilon^n \frac{\partial S_{n-1}^0}{\partial T} \quad (5.15)$$

The time derivative of $J_n$ is found from Hamilton's equations of motion to be

$$\dot{J}_n = -\epsilon^n \frac{\partial}{\partial \gamma_n} \left( \frac{\partial S_{n-1}^0}{\partial T} \right) \quad (5.16)$$
Equation (5.16) is of the same type as (5.1), hence $J_n$ is an $n$'th order asymptotic invariant. Since the system undergoes an adiabatic (slow) change, $J_n$ is more commonly called an $n$'th order adiabatic invariant.

The complete set of $J_n$'s constitutes what we shall call the hierarchy of adiabatic invariants. The first construction of a hierarchy of adiabatic invariants was given by Chandrasekhar. Chandrasekhar considered Newton's equation for a harmonic oscillator with a slowly varying frequency. By making a sequence of changes in both the dependent and the independent variables he was able to demonstrate the existence of a hierarchy of invariants. The hierarchy which we have presented is mathematically equivalent to one which was presented by Gardner. It is, however, easier to obtain than is Gardner's hierarchy. Gardner considered the same system which we have considered. Rather than introducing a sequence of action and angle variables $J_n$ and $\gamma_n$, he introduces a sequence of Cartesian variables $p_n$ and $q_n$. Whereas our $n$'th order Hamiltonian is

$$H_n = W_{n-1}(J_n, T) + \varepsilon^n \frac{\partial S_{n-1}}{\partial T}, \quad (5.17)$$

Gardner's $n$'th order Hamiltonian is

$$H_n = W_{n-1}(p_n^2 + q_n^2, T) + \varepsilon^n \frac{\partial F_{n-1}}{\partial T}, \quad (5.18)$$

where $F_{n-1}$ is the generator of the transformation from $p_{n-1}$, $q_{n-1}$ to $p_n$, $q_n$. It follows from (5.18) that

$$\frac{d(p_n^2 + q_n^2)}{dt} = 0(\varepsilon^n). \quad (5.19)$$

Hence $p_n^2 + q_n^2$ is an $n$'th order adiabatic invariant. Gardner's $p_n^2 + q_n^2$ is proportional to our $J_n$.

---


5.3 CONSTRUCTION OF THE HIERARCHY OF ADIABATIC INVARIANTS BY PERTURBATION THEORY

The systems to which the methods of the preceding section are applicable are nearly periodic in a single phase. It is, therefore, reasonable, to investigate the properties of these systems using the Bogoliubov method. In order to do this we first express the Hamiltonian in the form

\[ H = H_0(J,T) + \epsilon H_1(J,\gamma,T), \quad (5.20) \]

where \( T \) is the slow time and where \( H_1 \) is a periodic function of the coordinate \( \gamma \) with period \( 2\pi \). Hamilton's equations of motion are

\[ \dot{J} = -\epsilon \frac{\delta H_1}{\delta \gamma}, \quad (5.21a) \]

\[ \dot{\gamma} = \omega_0(J,T) + \epsilon \frac{\delta H_1}{\delta J}, \quad (5.21b) \]

where

\[ \omega_0 = \frac{\delta H_0}{\delta J}. \quad (5.22) \]

In order to perform perturbation theory we introduce new variables \( K \) and \( \phi \) as follows:

\[ J = K + \epsilon F(K,\phi,T), \quad (5.23a) \]

\[ \gamma = \phi + \epsilon G(K,\phi,T), \quad (5.23b) \]

where \( F \) and \( G \) are formal infinite series in powers of \( \epsilon \) and are periodic functions of \( \phi \) with period \( 2\pi \). We further require that

\[ \dot{K} = \epsilon a(K,T), \quad (5.24a) \]

\[ \dot{\phi} = \omega_0(K,T) + \epsilon b(K,T), \quad (5.24b) \]

where \( a(K,T) \) and \( b(K,T) \) are formal infinite series in powers of \( \epsilon \).

We now require that \( K \) and \( \phi \) be canonical variables. In other words we determine \( F \) and \( G \) such that \( K \) and \( \phi \) are canonical to all orders. This re-
quirement plus the fact that \( H(K,\phi,T) \) will be a periodic function of \( \phi \) allow us to prove the following theorem:

**Theorem 5.1.** The Hamiltonian (5.20), when expressed in terms of the canonical variables \( K \) and \( \phi \), will be independent of \( \phi \) to all orders in the perturbation theory.

**Proof.** Since \( K \) and \( \phi \) are canonical we have

\[
\frac{\partial H}{\partial K} \frac{\partial}{\partial \phi} = \phi \frac{\partial H}{\partial \phi}, \tag{5.25}
\]

and

\[
\frac{\partial H}{\partial K} \frac{\partial}{\partial \phi} = -K \frac{\partial H}{\partial K}. \tag{5.26}
\]

Thus

\[
\frac{\partial}{\partial \phi} = -K \frac{\partial H}{\partial K}, \tag{5.27}
\]

or

\[
\frac{\partial H}{\partial \phi} = -\epsilon \frac{a(K,T)}{\omega(K,T)} \frac{\partial}{\partial K} H(K,\phi,T). \tag{5.28}
\]

It is clear from (5.28) that if \( H(K,\phi,T) \) is independent of \( \phi \) through order \( \epsilon^n \) then it is independent of \( \phi \) through order \( \epsilon^{n+1} \). Since the Hamiltonian (5.20) is certainly independent of \( \phi \) through order \( \epsilon^0 \) it follows by mathematical induction that \( H(K,\phi,T) \) is independent of \( \phi \) to all orders in the perturbation theory. The theorem is proved.

It follows from Theorem 5.1 that \( K \) is constant to all orders in the perturbation theory. Thus we can construct a quantity which is constant to any desired order by performing canonical perturbation theory to that order. In other words canonical perturbation theory generates a hierarchy of adiabatic invariants.

### 5.4 THE TIME DEPENDENT OSCILLATOR

As an illustration of the ideas of sections 5.2 and 5.3 we shall now discuss a one dimensional oscillator whose frequency varies slowly with time. We shall derive the first few members of the hierarchy of adiabatic invariants, first by a successive transformation to action and angle variables and then by perturbation theory.
In the q-p representation the Hamiltonian which describes the oscillator has the form

$$H_0 = \frac{1}{2} p^2 + \frac{1}{2} \omega^2(T)q^2.$$  \hspace{1cm} (5.29)

The mass of the oscillator is taken to be unity. Following the procedure outlined in section 5.2 we introduce the new momentum

$$J_1 = \frac{1}{2\pi} \oint_{\mathcal{H}_0} pdq.$$ \hspace{1cm} (5.30)

If we define \( W_0 \) by the equation

$$H_0 = W_0(J_1, T),$$ \hspace{1cm} (5.31)

then we find from (5.29) that

$$p = (2W_0 - \omega^2 q^2)^{1/2}.$$ \hspace{1cm} (5.32)

Upon setting

$$q = (2W_0/\omega^2)^{1/2} \sin \theta$$ \hspace{1cm} (5.33)

we find that

$$p = (2W_0)^{1/2} \cos \theta.$$ \hspace{1cm} (5.34)

Thus

$$J_1 = (2W_0/2\pi\omega) \int_0^{2\pi} \cos^2 \theta \, d\theta$$ \hspace{1cm} (5.35)

$$= W_0/\omega.$$ \hspace{1cm} (5.35)

The generating function \( S_0 \) is found from the differential equation

$$p = \frac{\partial S_0}{\partial q} = (2J_1 \omega - \omega^2 q^2)^{1/2}.$$ \hspace{1cm} (5.36)
This equation has the solution
\[
S_o = \frac{1}{2} \left\{ q(2J_1 \omega - \omega^2 q^2) \frac{1}{2} + 2J_1 \sin^{-1}(q^2 \omega/2J_1) \right\}^{1/2}.
\]
(5.37)

The angle variable \( \gamma_1 \) conjugate to \( J_1 \) is found from (5.37) to be
\[
\gamma_1 = \frac{\partial S_o}{\partial J_1} = \sin^{-1}(q^2 \omega/2J_1)^{1/2}.
\]
(5.38)

The Hamiltonian in the \( J_1 - \gamma_1 \) representation is
\[
H_1 = W_0(J_1, T) + \epsilon \frac{\partial S_o}{\partial T}
\]
= \( J_1 \omega(T) + \epsilon(\omega/2\omega)J_1 \sin 2\gamma_1 \),
where
\[
\omega' = \frac{\partial \omega}{\partial T}.
\]
(5.40)

The second order invariant \( J_2 \) is defined by the following integral:
\[
J_2 = \frac{1}{2\pi} \int_0^{2\pi} J_1 \, d\gamma_1.
\]
(5.41)

We find from (5.39) that
\[
J_1 = \frac{W_1}{\omega + \epsilon(\omega'/2\omega)\sin 2\gamma_1},
\]
(5.42)

where
\[
H_1 = W_1(J_2, T).
\]
(5.43)

The integral (5.41) can be evaluated to give
\[
J_2 = \frac{W_1}{\omega [1-(\epsilon\omega'/2\omega)^2]^{1/2}}.
\]
(5.44)
or

\[ W_1 = J_2 \omega (1 - (\omega'/2\omega^2)^2)^{1/2} \]  \hspace{1cm} (5.45)

The generating function \( S_1 \) is found by solving the following differential equation:

\[ J_1 = \frac{\partial S_1}{\partial \gamma_1} = \frac{W_1}{\omega \left[ 1 + (\omega'/2\omega^2) \sin 2\gamma_1 \right]} . \]  \hspace{1cm} (5.46)

This equation has the solution

\[ S_1 = \frac{W_1}{\omega \left[ 1 - (\omega'/2\omega^2)^2 \right]^{1/2}} \tan^{-1} \left\{ \frac{\tan \gamma_1 + \omega'/2\omega}{\left[ 1 - (\omega'/2\omega^2)^2 \right]^{1/2}} \right\} . \]  \hspace{1cm} (5.47)

By making use of (5.44) we can express \( S_1 \) as

\[ S_1 = J_2 \tan^{-1} \left\{ \frac{\tan \gamma_1 + \omega'/2\omega}{\left[ 1 - (\omega'/2\omega^2)^2 \right]^{1/2}} \right\} . \]  \hspace{1cm} (5.48)

It should be noted that \( S_1 \) has the form

\[ S_1 = J_2 \gamma_1 + O(\epsilon) \]  \hspace{1cm} (5.49)

as was expected.

The variable \( \gamma_2 \) conjugate to \( J_2 \) is

\[ \gamma_2 = \frac{\partial S_1}{\partial J_2} = \tan^{-1} \left\{ \frac{\tan \gamma_1 + \omega'/2\omega}{\left[ 1 - (\omega'/2\omega^2)^2 \right]^{1/2}} \right\} . \]  \hspace{1cm} (5.50)

The Hamiltonian in the \( J_2 - \gamma_2 \) representation is

\[ H_2 = W_1(J_2, T) + \epsilon^2 \frac{\partial S_1}{\partial T} \]  \hspace{1cm} (5.51)
\[ J_2 \left\{ \right. \\
= \frac{\omega(1 - (\omega'/2\omega)^2)^{1/2}}{1 + \tan^2 \gamma_2} + \\
+ \varepsilon^2 \frac{\partial}{\partial T} \left( \frac{\omega'/2\omega^2}{1 - (\omega'/2\omega^2)^2} \right)^{1/2} \\
+ \varepsilon \frac{(\omega'/2\omega^2) \tan \gamma_2}{1 - (\omega'/2\omega^2)^2} \left\} \right. \\
\]

The third order invariant \( J_3 \) is defined by the following integral:

\[ J_3 = \frac{1}{2\pi} \int_{0}^{2\pi} J_2 d\gamma_2. \] (5.52)

Upon setting \( H_2 = W_2(J_3, T) \) we find from (5.51) that

\[ J_3 = \frac{W_2}{2\pi \omega \left[ 1 - (\omega'/2\omega^2)^2 \right]^{1/2}} \int_{0}^{2\pi} \frac{\sec^2 \gamma_2 d\gamma_2}{\tan^2 \gamma_2 + b \tan \gamma_2 + a}. \] (5.53)

\[ = \frac{2W_2}{\omega \left[ 1 - (\omega'/2\omega^2)^2 \right]^{1/2} \left( 4a - b^2 \right)^{1/2}} \]

where

\[ a = 1 + \varepsilon^2 \frac{\partial}{\partial T} \left( \frac{\omega'/2\omega^2}{\omega \left[ 1 - (\omega'/2\omega^2)^2 \right]} \right), \] (5.54a)

\[ b = \varepsilon^3 \frac{(\omega'/2\omega^2) \frac{\partial}{\partial T} (\omega'/2\omega^2)}{\left[ 1 - (\omega'/2\omega^2)^2 \right]^{3/2}}. \] (5.54b)

This process may be continued to produce the higher order invariants \( J_4, J_5, \) etc. It is clear that the process becomes increasingly tedious with each succeeding order.

We shall now construct the invariants by perturbation theory. We begin by writing the equations of motion in the \( J_1 - \gamma_1 \) representation. These are found from (5.41) to be

\[ J_1 = -\varepsilon (\omega'/\omega) J_1 \cos 2\gamma_1, \] (5.55a)
\[ \gamma_1 = \omega(T) + \epsilon(\omega'/2\omega)\sin 2\gamma_1. \]  
\hspace{1cm} (5.55b)

According to the remarks made in section 5.3 we should seek a solution in the form

\[ J_1 = K + \sum_{n=1}^{\infty} \epsilon F_n(K, \phi, T), \]  
\hspace{1cm} (5.56a)

\[ \gamma_1 = \phi + \sum_{n=1}^{\infty} \epsilon G_n(K, \phi, T), \]  
\hspace{1cm} (5.56b)

such that

\[ \dot{K} = \sum_{n=1}^{\infty} \epsilon A_n(K, T), \]  
\hspace{1cm} (5.57a)

\[ \dot{\phi} = \omega(T) + \sum_{n=1}^{\infty} \epsilon B_n(K, T). \]  
\hspace{1cm} (5.57b)

These equations, when substituted into (5.55), produce an infinite set of coupled differential equations. The first order members of this set are as follows:

\[ A^{(1)} + \omega \frac{\partial F^{(1)}}{\partial \phi} = - (\omega'/\omega)K\cos 2\phi, \]  
\hspace{1cm} (5.58a)

\[ B^{(1)} + \omega \frac{\partial G^{(1)}}{\partial \phi} = (\omega'/2\omega)\sin 2\phi. \]  
\hspace{1cm} (5.58b)

These equations have the solutions

\[ A^{(1)} = B^{(1)} = 0, \]  
\hspace{1cm} (5.59a,b)

\[ F^{(1)} = -(\omega'/2\omega^2)K\sin 2\phi, \]  
\hspace{1cm} (5.60a)

\[ G^{(1)} = -(\omega'/4\omega^2)\cos 2\phi. \]  
\hspace{1cm} (5.60b)

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The transformation as determined by (5.60) is canonical through first order. This means that \( K \) must be constant through first order, or \( K \) is a second order invariant. We shall now compare this first order expression for \( K \) with the second order invariant \( J_2 \). We have that

\[
J_1 = K - \epsilon (\omega'/2\omega^2)K \sin 2\phi + O(\epsilon^2), \tag{5.61a}
\]

\[
\gamma_1 = \phi - \epsilon (\omega'/4\omega^2) \cos 2\phi + O(\epsilon^2). \tag{5.61b}
\]

Upon inverting (5.61a) we find that

\[
K = J_1 + \epsilon (\omega'/2\omega^2)J_1 \sin 2\gamma_1 + O(\epsilon^2). \tag{5.62}
\]

When we compare (5.62) with (5.44) we see that

\[
K = J_2 + O(\epsilon^2). \tag{5.63}
\]

Thus through first order the invariant \( K \) constructed by perturbation theory is the same as the invariant \( J_2 \) constructed by the successive transformation to action and angle variables.

The second order perturbation equations are

\[
A^{(2)} + \omega \frac{\partial A^{(2)}}{\partial \phi} - \frac{\partial}{\partial T} \left( \frac{\omega'}{2\omega^2} \right) K \sin 2\phi = 0, \tag{5.64a}
\]

\[
B^{(2)} = - \left( \frac{\omega'}{4\omega} \right) \left( \frac{\omega'}{2\omega^2} \right), \tag{5.65b}
\]

\[
F^{(2)} = - \frac{1}{2\omega} \frac{\partial}{\partial T} \left( \frac{\omega'}{2\omega^2} \right) K \cos 2\phi + \frac{\omega'}{4\omega^2} K, \tag{5.66a}
\]

\[
G^{(2)} = \frac{1}{4\omega} \frac{\partial}{\partial T} \left( \frac{\omega'}{2\omega^2} \right) \sin 2\phi - \frac{\omega'}{16\omega^2} \sin 4\phi. \tag{5.66b}
\]

The zero harmonic of \( F^{(2)} \) has been chosen so that the transformation (5.56) is canonical through second order. This means that the variable \( K \) must be a constant through second order, hence a third order invariant. Thus we shall compare \( K \) with the third order invariant \( J_3 \). In order to do this we first express \( K \) in terms of \( J_1 \) and \( \gamma_1 \). When we do this we find that

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\[ K = J_1 \left\{ 1 + \varepsilon (\omega'/2\omega^2) \sin 2\gamma_1 + \varepsilon^2 \left[ \frac{1}{2\omega} \frac{\partial}{\partial T} \left( \frac{\omega^1}{2\omega^2} \right) \cos 2\gamma_1 + \frac{\omega^1}{4\omega^2} \right] \right\} . \] (5.67)

The variable \( J_3 \) is given by (5.53) as

\[ J_3 = \frac{W_2}{\omega [1-(\omega'/2\omega^2)]^{1/2} (a-b/4)^{1/2}} . \] (5.68)

If we expand the denominator and retain terms through second order we find that

\[ J_3 = (W_2/\omega) \left\{ 1 - \varepsilon^2 (1/2\omega) \frac{\partial}{\partial T} \left( \frac{\omega^1}{2\omega^2} \right) + \varepsilon^2 (\omega'/2\omega^2)^2 \right\} . \] (5.69)

Upon expanding \( W_2/\omega \) and retaining terms through second order we find that

\[ W_2/\omega = J_1 \left\{ 1 + \varepsilon (\omega'/2\omega^2) \sin 2\gamma_1 + \varepsilon^2 \frac{1}{2\omega} \frac{\partial}{\partial T} \left( \frac{\omega^1}{2\omega^2} \right) (1 + \cos 2\gamma_1) \right\} . \] (5.70)

It follows from (5.69) and (5.70) that

\[ J_3 = J_1 \left\{ 1 + \varepsilon (\omega'/2\omega^2) \sin 2\gamma_1 + \varepsilon^2 \left[ \frac{1}{2\omega} \frac{\partial}{\partial T} \left( \frac{\omega^1}{2\omega^2} \right) \cos 2\gamma_1 + \frac{1}{2} \left( \frac{\omega^1}{2\omega^2} \right)^2 \right] \right\} + O(\varepsilon^3) . \] (5.71)

When we compare (5.71) with (5.67) we see that

\[ K = J_2 + O(\varepsilon^3) . \] (5.72)

The perturbation theory, when carried to higher orders, will produce the higher order invariants. As with the iteration method, the perturbation method increases in complexity with each succeeding order. In practice, however, the perturbation method is simpler than the iteration method. Even in simple example considered above the integrals encountered when applying the
iteration method were more complicated than those encountered in applying the perturbation method.

5.5 AN EXACTLY SOLVABLE EXAMPLE

There is an interesting case in which the motion of the adiabatic oscillator can be expressed exactly in terms of elementary functions. We begin our discussion of this case by writing down the equations of motion in the \( J_1-\gamma_1 \) representation. These equations are

\[
\dot{J}_1 = -\varepsilon(\omega'/\omega)J_1 \cos 2\gamma_1
\]

\[
\dot{\gamma}_1 = \omega(T) + \varepsilon(\omega'/2\omega)\sin 2\gamma_1.
\]  

We find from (5.73) that

\[
\frac{dJ_1}{J_1} = -\frac{\varepsilon(\omega'/2\omega^2)\cos 2\gamma_1}{1+\varepsilon(\omega'/2\omega^2)\sin 2\gamma_1} \, d\gamma_1.
\]  

Upon integrating equation (5.74) we find that

\[
\log J_1 = -\int \frac{\varepsilon(\omega'/\omega^2)\cos 2\gamma_1}{1+\varepsilon(\omega'/2\omega^2)\sin 2\gamma_1} \, d\gamma_1
\]

\[+ \log M,\]

where \( M \) is a constant.

Let us now consider the special case where

\[
\omega(T) = \frac{\omega_0}{1 - 2d \omega_0 t},
\]

\( \omega_0 \) and \( d \) being constants. This time dependence is such that

\[
\varepsilon \omega'/2\omega^2 = \frac{\omega'}{2\omega} = d.
\]

Upon substituting (5.77) into (5.75) we find that
\[ J_1 = \frac{M}{1 + d \sin 2\gamma_1}. \]  

(5.78)

The time dependence of \( \gamma_1 \) can be found by rewriting (5.73) in the following way:

\[ \frac{d\gamma_1}{1 + d \sin 2\gamma_1} = \omega dt. \]  

(5.79)

It follows from (5.79) that

\[ \gamma_1 = \tan^{-1}\left\{ (1 - d^2)^{1/2} \tan \left[ (1 - d^2)^{1/2} (f \omega dt + C) \right] - d \right\}, \]  

(5.80)

where \( C \) is an integration constant.

We can now compare the perturbation solution of section 5.4 with the exact solution represented by (5.78) and (5.80). Through second order the angle \( \phi \) is governed by the following equation:

\[ \dot{\phi} = \omega (1 - \frac{1}{2} d^2) + O(\epsilon^3). \]  

(5.81)

Thus

\[ \phi = (1 - \frac{1}{2} d^2) (f \omega dt + D) + O(\epsilon^3), \]  

(5.82)

where \( D \) is an integration constant.

According to (5.60) and (5.66) \( \gamma_1 \) and \( \phi \) are related by the following equation:

\[ \gamma_1 = \phi - \frac{1}{2} d \cos 2\phi - \frac{1}{8} d^2 \sin 4\phi + O(\epsilon^3). \]  

(5.83)

Upon substituting (5.82) into (5.83) we find that

\[ \gamma_1 = (1 - \frac{1}{2} d^2) (f \omega dt + D) - \frac{1}{2} d \cos 2(f \omega dt + D) \]  

\[ - \frac{1}{8} d^2 \sin 4(f \omega dt + D) + O(\epsilon^3). \]  

(5.84)
We now expand the exact solution (5.80) and retain terms through second order. The result is

$$\gamma_1 = (1 - \frac{1}{2} d^2)(J\omega dt + C) - \frac{1}{2} d\cos 2(J\omega dt + C)$$

$$\quad - \frac{1}{2} d - \frac{1}{2} d^2 \sin 2(J\omega dt + C) - \frac{1}{8} d^2 \sin 4(J\omega dt + C) + O(\epsilon^3).$$

(5.85)

If we choose the integration constant D to be

$$D = C - \frac{1}{2} d + O(\epsilon^3),$$

(5.86)

then the perturbation solution is identical with the exact solution through second order.

We must now compare the expressions for $J_1$. The perturbation solution is found from (5.60) and (5.66) to be

$$J_1 = K(1 - d\sin 2\phi + \frac{1}{2} d^2) + O(\epsilon^3).$$

(5.87)

If we expand the exact solution (5.78) we find that

$$J_1 = M(1 - d\sin 2\gamma_1 + d^2 \sin^2 2\gamma_1) + O(\epsilon^3).$$

(5.88)

By making use of (5.83) we can rewrite (5.88) in the form

$$J_1 = M(1 - d\sin 2\phi + d^2) + O(\epsilon^3).$$

(5.89)

We can make (5.87) and (5.89) identical through order $\epsilon^2$ by choosing

$$M = K(1 - \frac{1}{2} d^2) + O(\epsilon^3).$$

(5.90)

We conclude that, for the special case considered here, the perturbation solution is identical with the exact solution at least through order $\epsilon^2$.  

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Another interesting property of this special oscillator becomes apparent when one attempts to construct the hierarchy of invariants. According to (5.44) the second order invariant \( J_2 \) is

\[
J_2 = \frac{W_1}{\omega (1 - d^2)^{1/2}}.
\]  

(5.91)

We find from (5.42) that

\[
W_1 = J_1 \omega (1 + d \sin 2\gamma_1).
\]  

(5.92)

Since \( J_1 \) is given by (5.78) we conclude that

\[
J_2 = M(1 - d^2)^{-1/2}.
\]  

(5.93)

The second order invariant \( J_2 \) is, therefore, a constant of the motion. The higher order invariants \( J_3, J_4, \) etc., are identical to \( J_2 \), hence the hierarchy terminates with \( J_2 \).

The special example which we have been considering is, of course, not physically realizable for arbitrary time intervals. This is because \( \omega(t) \) is infinite when \( t = 1/2\omega_0 d \).

5.6 ASYMPTOTIC INVARIANTS OF HARMONICALLY DRIVEN SYSTEMS

Quite often we are led to discuss the motion of an oscillatory system which is driven by a weak harmonic force. In this section we shall attempt to find asymptotic invariants for such systems.

We suppose that the system can be described by the following Hamiltonian:

\[
H = H_0(J) + \epsilon H_1(J, \gamma, \omega t),
\]  

(5.94)

where \( J \) is the momentum conjugate to the coordinate \( \gamma \), and where \( H_1 \) is a periodic function of \( \gamma \) and of \( \omega t \). Hamilton's equations of motion are

\[
\dot{J} = -\epsilon \frac{\partial H_1}{\partial \gamma},
\]  

(5.95a)
\[
\dot{\gamma} = -\omega_o(J) + \varepsilon \frac{\partial H_1}{\partial J}, \\
\omega_o(J) = \frac{\partial H_0}{\partial J},
\]

where

We now assume that the system is non-resonant. This means that we can use non-degenerate perturbation theory. We shall require the perturbation theory to be canonical. Thus we introduce new canonical variables \( K \) and \( \phi \) as follows:

\[
J = K + \varepsilon F(K, \phi, \omega t),
\]

\[
\gamma = \phi + \varepsilon G(K, \phi, \omega t),
\]

where \( F \) and \( G \) are formal infinite series in powers of \( \varepsilon \) and are periodic functions of \( \phi \) and of \( \omega t \). We further require that

\[
\dot{K} = \varepsilon a(K),
\]

\[
\dot{\phi} = \omega_o(K) + \varepsilon b(K),
\]

where \( a(K) \) and \( b(K) \) are formal infinite series in powers of \( \varepsilon \).

The Hamiltonian, when expressed in terms of \( K \) and \( \phi \), will be a periodic function of \( \phi \). This allows us to prove the following theorem:

**Theorem 5.2.** The Hamiltonian (5.94), when expressed in terms of the canonical variables \( K \) and \( \phi \), will be independent of \( \phi \) to all orders of non-degenerate perturbation theory.

The proof of this theorem is identical with that of Theorem 5.1. Theorem 5.2 implies that the canonical variable \( K \) is constant to all orders of non-degenerate perturbation theory. In other words \( K \) can be made asymptotically invariant to any desired order.

The ideas presented above can be made more concrete by considering a simple harmonic oscillator which is subjected to a weak harmonic driving force. The Hamiltonian \( H(p, q) \) for this system has the form

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\[ H(p,q) = \frac{1}{2} p^2 + \frac{1}{2} \omega_0 q^2 - \epsilon q \cos \omega t , \]

where \( \epsilon \) measures the strength of the driving force. We begin our discussion by introducing new variables \( J \) and \( \gamma \) through the following canonical transformation:

\[ q = (2J/\omega_0)^{1/2} \cos \gamma , \tag{5.100a} \]
\[ p = (2\omega_0 J)^{1/2} \sin \gamma . \tag{5.100b} \]

The Hamiltonian, when written in terms of \( J \) and \( \gamma \), is

\[ H = J\omega_0 - \epsilon (2J/\omega_0)^{1/2} \cos \gamma \cos \omega t . \tag{5.101} \]

Hamilton's equations of motion are found from (5.101) to be

\[ \dot{J} = -\epsilon (2J/\omega_0)^{1/2} \sin \gamma \cos \omega t , \tag{5.102a} \]
\[ \dot{\gamma} = \omega_0 - \epsilon (2\omega_0 J)^{-1/2} \cos \gamma \cos \omega t . \tag{5.102b} \]

We next perform non-degenerate perturbation theory by introducing new canonical variables \( K \) and \( \phi \) as follows:

\[ J = K + \epsilon F^1(K,\phi,\omega t) + \epsilon^2 F^2(K,\phi,\omega t) + \ldots , \tag{5.103a} \]
\[ \gamma = \phi + \epsilon G^1(K,\phi,\omega t) + \epsilon^2 G^2(K,\phi,\omega t) + \ldots , \tag{5.103b} \]

such that

\[ K = \epsilon a^{(1)}(K) + \epsilon^2 a^{(2)}(K) + \ldots , \tag{5.104a} \]
\[ \phi = \omega_0 + \epsilon b^{(1)}(K) + \epsilon^2 b^{(2)}(K) + \ldots . \tag{5.104b} \]
When we substitute (5.103) and (5.104) into (5.102) we find that

\[ \epsilon a^{(1)}(K) + \epsilon^2 a^{(2)}(K) + \epsilon^2 a^{(1)} \frac{\partial F^{(1)}}{\partial K} + \epsilon^2 \frac{\partial F^{(1)}}{\partial \phi} b^{(1)} + \epsilon \omega \frac{\partial F^{(1)}}{\partial \phi} + \frac{\epsilon \omega}{\omega_0} \frac{\partial F^{(2)}}{\partial \phi} + \frac{\epsilon^2 \omega}{\omega_0} \frac{\partial F^{(2)}}{\partial \phi} + \frac{\epsilon^2 \omega}{\omega_0} \frac{\partial F^{(2)}}{\partial \phi} = -\frac{\epsilon}{\omega_0} \frac{(2K)^{1/2}}{G^{(1)}} \cos \phi \cos \omega t - \frac{\epsilon^2}{(2\omega_0 K)^{3/2}} \frac{F^{(1)}}{\omega_0^{3/2}} \cos \phi \cos \omega t \]

+ \mathcal{O}(\epsilon^3), \quad (5.105)

and

\[ \epsilon b^{(1)}(K) + \epsilon^2 b^{(2)}(K) + \frac{\epsilon \omega}{\omega_0} \frac{\partial G^{(1)}}{\partial \phi} + \frac{\epsilon \omega}{\omega_0} \frac{\partial G^{(1)}}{\partial \phi} + \frac{\epsilon^2}{\omega_0} \frac{\partial G^{(1)}}{\partial \phi} = -\frac{\epsilon}{(2\omega_0 K)^{1/2}} \cos \phi \cos \omega t \]

\[ + \frac{\epsilon}{(2\omega_0 K)^{1/2}} G^{(1)} \sin \phi \cos \omega t + \frac{\epsilon^2}{(2\omega_0 K)^{3/2}} \frac{w_0 F^{(1)}}{\omega_0^{3/2}} \cos \phi \cos \omega t + \mathcal{O}(\epsilon^3). \quad (5.106)\]

We equate powers of \( \epsilon \) and thereby find the first order equations to be

\[ a^{(1)}(K) + \omega \frac{\partial F^{(1)}}{\partial \phi} + \omega \frac{\partial F^{(1)}}{\partial \omega} = -\frac{(2K/\omega)^{1/2}}{G^{(1)}} \sin \phi \cos \omega t, \quad (5.107a)\]

\[ b^{(1)}(K) + \omega \frac{\partial G^{(1)}}{\partial \phi} + \omega \frac{\partial G^{(1)}}{\partial \omega} = -\frac{1}{(2\omega_0 K)^{1/2}} \cos \phi \cos \omega t. \quad (5.107b)\]

Since we are using non-degenerate perturbation theory, and since the right hand sides of (5.107) contain no zero harmonics we conclude that

\[ a^{(1)} = b^{(1)} = 0 \quad (5.108)\]

and

\[ F^{(1)} = \frac{1}{2} \left( \frac{2K}{\omega_0} \right)^{1/2} \left\{ \frac{\cos(\phi + \omega t)}{\omega_0 + \omega} + \frac{\cos(\phi - \omega t)}{\omega_0 - \omega} \right\}, \quad (5.109a)\]

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\[ G^{(1)} = -\frac{1}{2(2\omega_0 K)^{1/2}} \left\{ \frac{\sin(\phi + \omega t)}{\omega + \omega_0} + \frac{\sin(\phi - \omega t)}{\omega - \omega_0} \right\} \]  

This choice of \( F^{(1)} \) and \( G^{(1)} \) is canonical. We can use these first order solutions to write the second order equations as follows:

\[ a^{(2)}(K) + \omega \frac{\partial a^{(2)}}{\partial \phi} + \omega \frac{\partial a^{(2)}}{\partial \omega t} = \]  

\[ = -\frac{\omega}{2\omega_0(\omega^2 - \omega_0^2)} \sin 2\omega t, \]  

\[ b^{(2)}(K) + \omega \frac{\partial b^{(2)}}{\partial \phi} + \omega \frac{\partial b^{(2)}}{\partial \omega t} = \]  

\[ = \frac{1}{8\omega_0 K} \left\{ \frac{\cos 2(\phi + \omega t)}{\omega + \omega_0} + \frac{\cos 2(\phi - \omega t)}{\omega - \omega_0} + \frac{2\omega_0}{\omega^2 - \omega_0^2} \cos 2\phi \right\}. \]

There are no zero harmonics in either (5.110a) or (5.110b). We conclude then that

\[ a^{(2)}(K) = b^{(2)}(K) = 0 \]  

(5.111)

and that

\[ F^{(2)}(K) = \frac{1}{4\omega_0(\omega^2 - \omega_0^2)} \cos 2\omega t, \]  

(5.112a)

\[ G^{(2)} = \frac{1}{16\omega_0 K} \left\{ \frac{\sin 2(\phi + \omega t)}{(\omega + \omega_0)^2} + \frac{\sin 2(\phi - \omega t)}{(\omega - \omega_0)^2} \right\} \]  

\[ + \frac{2}{\omega^2 - \omega_0^2} \sin 2\phi \right\}. \]  

(5.112b)

It is straightforward to verify that this choice of \( F^{(2)} \) and \( G^{(2)} \) is canonical. We have shown above that, in the context of non-degenerate perturbation theory,

\[ J = K + \varepsilon \frac{1}{2} \left\{ \frac{\cos(\phi + \omega t)}{\omega + \omega_0} + \frac{\cos(\phi - \omega t)}{\omega - \omega_0} \right\} \]  

(5.113a)
\[ + \varepsilon^2 \frac{\cos 2\omega t}{4\omega_0 (\omega_0^2 - \omega^2)} + o(\varepsilon^3) , \]

\[ \gamma = \phi \frac{\varepsilon}{2(2\omega_0 K)^{1/2}} \left\{ \frac{\sin(\phi + \omega t)}{\omega_0 + \omega} + \frac{\sin(\phi - \omega t)}{\omega_0 - \omega} \right\} \]

\[ + \frac{\varepsilon^2}{16\omega_0 K} \left\{ \frac{\sin 2(\phi + \omega t)}{(\omega_0 + \omega)^2} + \frac{\sin 2(\phi - \omega t)}{(\omega_0 - \omega)^2} \right\} \]

\[ - \frac{2}{\omega_0^2 - \omega^2} \sin 2 \phi \right\} + o(\varepsilon^3) , \quad (5.113b) \]

where

\[ k = 0 (\varepsilon^3) , \quad (5.114a) \]

\[ \phi = \omega_0 + 0 (\varepsilon^3) . \quad (5.114b) \]

It is clear from (5.114a) that \( K \) is a third order asymptotic invariant. We could continue this process and make \( K \) constant to any desired order. It should be noted that, since \( \phi = \omega_0 t + \text{constant} \) and \( K = \text{constant} \), equation (5.113a,b) represent the complete solution of the problem through second order.

Suppose now that

\[ \omega_0 - \omega \leq 0 (\varepsilon) . \quad (5.115) \]

In this case the solution given by equations (5.113a,b) is meaningless because of the small divisors. The problem is degenerate and must be treated by degenerate perturbation theory. We shall sketch the degenerate solution of the problem requiring that \(\omega t\) is a canonical variable with \( r \) its conjugate momentum; \( r \) will, of course, be a measure of the total energy of the system. The equation of motion for \( r \) is found from (5.101) to be

\[ r = -\frac{\partial H}{\partial \omega t} = -\varepsilon \left( \frac{2J}{\omega_0} \right)^{1/2} \cos \gamma \sin \omega t . \quad (5.116) \]

In order to perform degenerate perturbation theory we let
\[ J = K + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + \ldots \]  
\[ \gamma = \phi + \epsilon G^{(1)} + \epsilon^2 G^{(2)} + \ldots \]  
\[ r = s + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \ldots \]  
(5.117a)  
(5.117b)  
(5.117c)

where the \( F^{(n)} \)'s, \( G^{(n)} \)'s and \( I^{(n)} \)'s are periodic functions of \( \phi \) and \( \omega t \). We further let

\[ \dot{K} = \epsilon a^{(1)}(K, \theta^{(1)}) + \epsilon^2 a^{(2)}(K, \theta^{(2)}) + \ldots, \]  
(5.118a)

\[ \dot{s} = \epsilon c^{(1)}(K, \theta^{(1)}) + \epsilon^2 c^{(2)}(K, \theta^{(2)}) + \ldots. \]  
(5.118b)

\[ \dot{\phi} = \omega_0 + \epsilon b^{(1)}(K, \theta^{(1)}) + \epsilon^2 b^{(2)}(K, \theta^{(2)}) + \ldots, \]  
(5.118c)

where \( \theta^{(1)} \) and \( \theta^{(2)} \) represent the slowly varying angles encountered in first and second order respectively.

When we substitute equations (5.117a) through (5.118c) into equations (5.102a,b) we again obtain equations (5.105) and (5.106) plus the following equation

\[ \epsilon c^{(1)} + \epsilon^2 c^{(2)} + \epsilon \omega_0 \frac{I^{(1)}}{\partial \phi} + \epsilon \omega \frac{\delta I^{(1)}}{\partial \omega t} + \epsilon \omega \frac{\delta I^{(1)}}{\partial K} + \epsilon \omega \frac{\delta I^{(1)}}{\partial \phi} = \]  
\[ + \epsilon^2 \omega_0 \frac{\delta I^{(2)}}{\partial \phi} + \epsilon^2 \omega \frac{\delta I^{(2)}}{\partial \omega t} = -\epsilon(2K/\omega_0)^{1/2} \cos \phi \sin \omega t, \]

\[ -\epsilon^2 F^{(1)}(2\omega_0 K)^{-1/2} \cos \phi \sin \omega t + \epsilon^2 (2K/\omega_0)^{1/2} G^{(1)}(\sin \phi + \omega t) \sin \phi \sin \omega t \]

\[ + O(\epsilon^3). \]  
(5.119)

We now extract the first order terms from the above equations to find that

\[ a^{(1)} + \omega_0 \frac{\delta p^{(1)}}{\partial \phi} + \omega \frac{\delta p^{(1)}}{\partial \omega t} = -\frac{1}{2} (2K/\omega_0)^{1/2} (\sin(\phi + \omega t) + \sin(\phi - \omega t)). \]  
(5.120a)
\[ b'(1) + \omega_o \frac{\partial g'(1)}{\partial \phi} + \omega \frac{\partial g'(1)}{\partial \omega t} = -\frac{1}{2} (2\omega_o K)^{-1/2} (\cos(\phi + \omega t) + \cos(\phi - \omega t)) , \]

\[ (5.120) \]

\[ c'(1) + \omega_o \frac{\partial I'(1)}{\partial \phi} + \omega \frac{\partial I'(1)}{\partial \omega t} = -\frac{1}{2} (2K/\omega_o)^{1/2} (\sin(\phi + \omega t) - \sin(\phi - \omega t)) . \]

\[ (5.120c) \]

We must gather the slowly varying harmonics into the average motion. Hence we choose

\[ a'(1) = -\frac{1}{2} (2K/\omega_o)^{1/2} \sin(\phi - \omega t) , \]

\[ (5.121a) \]

\[ b'(1) = -\frac{1}{2} (2\omega_o K)^{-1/2} \cos(\phi - \omega t) , \]

\[ (5.121b) \]

\[ c'(1) = \frac{1}{2} (2K/\omega_o)^{1/2} \sin(\phi - \omega t) . \]

\[ (5.121c) \]

Equations (5.120a,b,c) can now be integrated to give

\[ F(1) = \frac{1}{2} (2K/\omega_o)^{1/2} \frac{\cos(\phi + \omega t)}{\omega_o + \omega} , \]

\[ (5.122a) \]

\[ G(1) = -\frac{1}{2 (2\omega_o K)^{1/2}} \frac{\sin(\phi + \omega t)}{\omega_o + \omega} , \]

\[ (5.122b) \]

\[ I(1) = \frac{1}{2} (2K/\omega_o)^{1/2} \frac{\cos(\phi + \omega t)}{\omega_o + \omega} , \]

\[ (5.122c) \]

It is easily verified that the above choice of \( F(1) \), \( G(1) \), and \( I(1) \) is canonical.

The substitution of the first order functions \( a'(1) \), \( b'(1) \) etc., into (5.105), (5.106) and (5.119) results in the following equations for the second order motion

\[ a''(2) + \omega_o \frac{\partial F''(2)}{\partial \phi} + \omega \frac{\partial F''(2)}{\partial \omega t} = 0 , \]

\[ (5.123a) \]

\[ b''(2) + \omega_o \frac{\partial G''(2)}{\partial \phi} + \omega \frac{\partial G''(2)}{\partial \omega t} = \frac{\cos 2(\phi + \omega t)}{8\omega_o K(\omega_o + \omega)} , \]

\[ (5.123b) \]
\[ c^{(2)} + \omega_0 \frac{\partial I^{(2)}}{\partial \phi} + \omega \frac{\partial I^{(2)}}{\partial t} = -\frac{\sin 2\omega t}{2\omega_0 (\omega_0 + \omega)}. \]  

Since there are no zero harmonics or slowly varying harmonics on the right hand sides of (5.123) we conclude that

\[ s^{(2)} = b^{(2)} = c^{(2)} = 0 \]  

and

\[ f^{(2)} = 0, \]  

\[ g^{(2)} = \frac{\sin 2(\phi + \omega t)}{16\omega_0 K(\omega_0 + \omega)^2}, \]  

\[ I^{(2)} = \frac{\cos 2\omega t}{4\omega_0 (\omega_0 + \omega)}. \]  

We have now shown that

\[ J = K + \epsilon \frac{1}{2} (2K/\omega_0)^{1/2} \frac{\cos(\phi + \omega t)}{\omega_0 + \omega} + O(\epsilon^3), \]  

\[ r = s + \epsilon \frac{1}{2} (2K/\omega_0)^{1/2} \frac{\cos(\phi + \omega t)}{\omega_0 + \omega} + \epsilon^2 \]  

\[ \frac{\cos 2\omega t}{4\omega_0 (\omega_0 + \omega)} + O(\epsilon^3), \]  

\[ \gamma = \phi - \frac{\epsilon}{2(2K/\omega_0)^{1/2}} \frac{\sin(\phi + \omega t)}{\omega_0 + \omega} + \frac{\epsilon^2}{16\omega_0 K(\omega_0 + \omega)^2} \sin 2(\phi + \omega t) + O(\epsilon^3), \]  

with

\[ \dot{K} = -\frac{\epsilon}{2} (2K/\omega_0)^{1/2} \sin(\phi - \omega t) + O(\epsilon^3), \]  

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\[
\dot{s} = \frac{\varepsilon}{2} (2K/\omega_o)^{1/2} \sin(\phi - \omega t) + O(\varepsilon^3), \quad (5.127b)
\]
\[
\dot{\phi} = \omega_o - \frac{\varepsilon}{2} (2\omega_o K)^{-1/2} \cos(\phi - \omega t) + O(\varepsilon^3). \quad (5.127c)
\]

It is clear from (5.127a,b) that
\[
\dot{K} + \dot{s} = O(\varepsilon^3). \quad (5.128)
\]

It follows that $K + s$ is a third order asymptotic invariant. We shall discuss the physical significance of the variable $s$ (or $r$) shortly. Let us first introduce a new variable
\[
\theta = \phi - \omega t. \quad (5.129)
\]

From (5.127a,c) we find that
\[
\dot{\theta} = \omega_o - \omega - \frac{\varepsilon}{2} (2\omega_o K)^{-1/2} \cos \theta. \quad (5.130b)
\]
\[
\dot{\theta} = -\frac{\varepsilon}{2} (2K/\omega_o)^{1/2} \sin \theta, \quad (5.130a)
\]

Upon dividing (5.130b) by (5.130a) we find
\[
\frac{\sin \theta \, d\theta}{dK} = -\frac{2(\omega_o - \omega)}{\varepsilon (2K/\omega_o)^{1/2}} + \frac{\cos \theta}{2K}. \quad (5.131)
\]

The general integral of (5.131) is
\[
\cos \theta = \left(\frac{2\omega_o}{\varepsilon}\right)^{1/2} \Delta K^{1/2} + dK^{-1/2}, \quad (5.132)
\]

where $d$ is a constant and $\Delta = \omega_o - \omega$. If we now assume that $\Delta \neq 0$ then equation (5.130a) becomes, after a simple calculation,
\[
\dot{K} = -\Delta (-K^2 - bK - c)^{1/2}, \quad (5.133)
\]
where
\[ b = \frac{2\epsilon}{(2\omega_0)^{1/2}\Delta} - \frac{\epsilon^2}{2\omega_0 \Delta^2}, \quad (5.134) \]
\[ c = \frac{d^2 \epsilon^2}{2\omega_0 \Delta^2}. \quad (5.135) \]

Equation (5.133) is easily integrated to give
\[ K = -\frac{b}{2} - \frac{(b^2 - 4c)^{1/2}}{2} \sin \Delta (t - t_0), \quad (5.136) \]

where \( t_0 \) is an integration constant. It is clear from (5.136) that the square 2K/\( \omega_0 \) of the amplitude of oscillation varies in a periodic fashion with period \( 2\pi/\Delta \). Both the period and the amplitude of oscillation increase without bound as \( \Delta \to 0 \).

In the case where \( \Delta = 0 \) the solution (5.136) is not valid. For this special case equation (5.132) becomes
\[ \cos \theta = dK^{-1/2}. \quad (5.137) \]

A simple calculation gives
\[ \dot{K} = -\epsilon(2\omega_0)^{-1/2}(K - d^2)^{1/2}, \quad (5.138) \]

which is easily integrated to give
\[ K = d + \frac{\epsilon^2}{8\omega_0} (t - t_0)^2, \quad (5.139) \]

where \( t_0 \) is an integration constant. Thus under precise resonance conditions (i.e., \( \Delta = 0 \)) the square of the amplitude of oscillation increases quadratically with time.

We are now in a position to say something about the average energy of the system. The Hamiltonian \( H' \) which is appropriate to the canonical variables \( J, \gamma, r \) and \( \omega t \) is
\[ H' = H + \omega r, \quad (5.140) \]

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where $H$ is the system energy (5.101). We have constructed $r$ such that $H'$ is a constant of the motion. Thus

$$r = \frac{1}{\omega} (H' - H)$$  \hspace{1cm} (5.141)$$

is a measure of the system's energy. The average energy is therefore determined by the average value of $r$, hence by $s$. We have seen above that $K + s$ is a third order asymptotic invariant. Hence

$$s = -K + O(\epsilon^3).$$  \hspace{1cm} (5.142)$$

We conclude immediately that, off resonance, the average energy undergoes a slow periodic change with period $2\pi/\omega - \omega$. On resonance the average energy increases quadratically with time.

The example which we have discussed above is very simple, and the results which we have obtained are better obtained by other methods. The example has, however, illustrated the more important ideas of asymptotic invariance and of degenerate perturbation theory. The techniques which were developed above are directly applicable to many practical problems. We shall consider one such application in the next chapter. The ideas presented in the present chapter have been generalized in Appendix D to deal with systems having several rapid phases.
CHAPTER 6

THE MOTION OF A CHARGED PARTICLE IN A
TRANSVERSE HYDROMAGNETIC WAVE

In this chapter we shall study the interaction between a charged particle and a constant magnetic field on which is superimposed a weak, transverse, linearly polarized hydromagnetic wave. This interaction has played an important role in recent discussions of the stability of protons in the inner Van Allen belt.\(^{26}\) The problem is of interest here, however, because it can be discussed by the perturbation methods which we have presented in the previous chapters. In the course of deriving the perturbation solution we shall obtain the exact solution for the motion of a charged particle in a circularly polarized wave. We shall conclude the chapter by relating the results which we obtain to those obtained in previously published work.

6.1 THE EQUATIONS OF MOTION

We begin by considering the motion of charged particle which moves in a uniform magnetic field

\[
\overline{B}_o = (0,0,B_o). \quad (6.1)
\]

This field can be described by the vector potential

\[
\overline{A}_o = (B_o y, 0, 0) \quad (6.2)
\]

The non-relativistic Hamiltonian \(H_o\) which corresponds to this choice of vector potential is

\[
H_o = \left( \frac{1}{2m} \right) (\overline{\mathbf{p}} - \frac{e}{c} \overline{A}_o)^2 \quad (6.3)
\]

\[ = \frac{1}{2m} \left( p_x + m\omega_0 y \right)^2 + p_y^2 + p_z^2 \],

where

\[ \omega_0 = \frac{eB_0}{mc} . \] (6.4)

In order to prepare the system for perturbation theory we introduce action and angle variables. This is accomplished through the following canonical transformation:

\[ x = r - (2J/m\omega_0)^{1/2} \cos \gamma , \quad p_x = p_r \] (6.5a)

\[ y = - (1/m\omega_0)p_x + (2J/m\omega_0)^{1/2} \sin \gamma , \quad p_y = (2m\omega_0 J)^{1/2} \cos \gamma , \] (6.5b)

\[ z = z , \quad p_z = p_z . \] (6.5c)

The Hamiltonian, when expressed in terms of the new variables is

\[ H_0 = J\omega_0 + \frac{1}{2m} p_z^2 . \] (6.6)

The variables \( r \) and \( p_r \) are both cyclic, hence both are constant. Since \( J \) is also constant and \( \gamma \) is a linear function of time it is clear from (6.5) that \( r \) and \( p_r \) locate the center about which the particle gyrates.

We now introduce a plane hydromagnetic wave of the form

\[ \vec{E}_1 = (B_1 \sin kz, 0, 0) , \] (6.7a)

\[ \vec{B}_1 = (0, 0, 0) , \] (6.7b)

where \( k \) is the wave number. This choice of electric and magnetic fields implies that we are working in a coordinate system which moves with the wave. The wave can be described by the vector potential.

\[ \vec{A}_1 = (0, B_1/k \cos kz, 0) . \] (6.8)
This vector potential contributes the following two terms to the Hamiltonian:

$$H_1 = -\left(\frac{\omega_1}{k}\right)\left(2m_0J\right)^2 \cos \gamma \cos kz,$$

$$H_2 = \left(m_0 \frac{\omega_1}{2k^2}\right)\cos^2 kz,$$  \hspace{1cm} (6.9a)  \hspace{1cm} (6.9b)

where

$$\omega_1 = \frac{eB_1}{mc}. \hspace{1cm} (6.10)$$

Let us define

$$\epsilon = \frac{\omega_1}{\omega_0} \hspace{1cm} (6.11)$$

and introduce a new independent variable $\tau$ as follows

$$\tau = \omega_0 t. \hspace{1cm} (6.12)$$

The Hamiltonian $h$ which is appropriate to the new independent variable is

$$h = (H_0 + H_1 + H_2) \frac{dt}{d\tau} \hspace{1cm} (6.13)$$

$$= J + \left(\frac{1}{2}\frac{m_0}{\omega_0}\right) p_z^2 - \left(\frac{\epsilon}{k}\right)\left(2m_0J\right)^2 \cos \gamma \cos kz$$

$$+ \left(\epsilon^2 m_0 / 2k^2\right) \cos^2 kz.$$ 

Hamilton's equations of motion are found from (6.13) to be

$$J' = -\left(\frac{\epsilon}{k}\right)\left(2m_0J\right)^2 \sin \gamma \cos kz, \hspace{1cm} (6.14a)$$

$$p_z' = -\epsilon\left(2m_0J\right)^2 \cos \gamma \sin kz \hspace{1cm} (6.14b)$$

$$+ \left(\epsilon^2 m_0 / k\right) \cos kz \sin kz,$$

$$kz' = kp_z / m_0. \hspace{1cm} (6.14c)$$
\[ \gamma' = 1 - \left(\frac{\varepsilon}{\lambda}\right) \left(\frac{\omega_0}{2\lambda}\right)^2 \cos \gamma \cos kz, \quad (6.14d) \]

where, for example,

\[ J' = \frac{dJ}{d\tau}. \quad (6.15) \]

It will become clear as we proceed that the most interesting motion occurs when

\[ \gamma' - kz' = O(\varepsilon). \quad (6.16) \]

When this condition is satisfied the system is said to be accidentally degenerate. Thus, if \( \varepsilon \) is sufficiently small, we may attempt to solve the equations of motion by degenerate perturbation theory.

Following the formalism which was presented in Chapter 4 we introduce the average variables \( U, V, \phi, K \) according to the equations

\[ kz = ku + \sum_{n=1}^{\infty} \varepsilon^n D(n)(U, V, K, \phi), \quad (6.17a) \]

\[ p_z = V + \sum_{n=1}^{\infty} \varepsilon^n E(n)(U, V, K, \phi), \quad (6.17b) \]

\[ J = K + \sum_{n=1}^{\infty} \varepsilon^n F(n)(U, V, K, \phi), \quad (6.17c) \]

\[ \gamma = \phi + \sum_{n=1}^{\infty} \varepsilon^n G(n)(U, V, K, \phi). \quad (6.17d) \]

The average motion will be governed by the equations

\[ kU' = kV/\omega_0 + \sum_{n=1}^{\infty} \varepsilon^n a(n)(V, K, \theta), \quad (6.18a) \]

\[ V' = \sum_{n=1}^{\infty} \varepsilon^n b(n)(V, K, \theta), \quad (6.18b) \]

\[ K' = \sum_{n=1}^{\infty} \varepsilon^n A(n)(V, K, \theta), \quad (6.18c) \]
\[ \phi' = 1 + \sum_{n=1}^{\infty} \epsilon^n B(n)(V,K,\Theta), \quad (6.18d) \]

where \( \Theta \) represents the slowly varying angles which are encountered in the course of the perturbation calculation.

When we substitute (6.17) and (6.18) into (6.14) we obtain an infinite set of coupled differential equations. The first order members of this set of equations are\(^{27}\)

\[ A^{(1)} + \left( kV/mw_o \right) \partial_{kU} F^{(1)} + \partial F^{(1)} = -(1/2k)(2mw_oK)^{1/2} \left[ \sin(\phi - kU) + \sin(\phi) \right], \quad (6.19a) \]

\[ b^{(1)} + \left( kV/mw_o \right) \partial_{kU} E^{(1)} + \partial E^{(1)} = - (1/2) (2mw_oK)^{1/2} \left[ \sin(\phi + kU) - \sin(\phi - kU) \right], \quad (6.19b) \]

\[ a^{(1)} + \left( kV/mw_o \right) \partial_{kU} D^{(1)} = \left( k/mw_o \right) E^{(1)}, \quad (6.19c) \]

\[ B^{(1)} + \left( kV/mw_o \right) \partial_{kU} G^{(1)} + \partial G^{(1)} = -(1/2k)(mw_o/2K)^{1/2} \left[ \cos(\phi + kU) + \cos(\phi - kU) \right]. \quad (6.19d) \]

Since we wish solutions which are valid when the difference angle

\[ \Theta = \phi - kU \quad (6.20) \]

is slowly varying, we must absorb the \( \Theta \) dependence into the average motion. Therefore, we choose

\(^{27}\)Because of the large amount of partial differentiation encountered in the remainder of the dissertation we shall often use \( \partial_z \) as a shorthand notation for \( \partial/\partial z \). We make the convention that \( \partial_z xy = y \partial_z x \) while \( \partial_z (xy) = \partial(xy)/\partial z \).
\[ A^{(1)} = -(1/2k)(2\omega_o K)^2 \sin \theta, \quad B^{(1)} = -(1/2k)(2\omega_o/2K)^2 \cos \theta, \] (6.21a, b)

\[ a^{(1)} = 0, \quad b^{(1)} = (1/2)(2\omega_o K)^2 \sin \theta. \] (6.21c, d)

With this choice for \( A^{(1)}, B^{(1)}, a^{(1)}, \) and \( b^{(1)} \) we find that

\[ F^{(1)} = (1/2k\omega_2)(2\omega_o K)^2 \cos(\phi + kU), \] (6.22a)

\[ E^{(1)} = (1/2\omega_2^2)(2\omega_o K)^2 \cos(\phi + kU), \] (6.22b)

\[ D^{(1)} = (k/2\omega_2^2)(2K/\omega_o)^2 \sin(\phi + kU), \] (6.22c)

\[ G^{(1)} = -(1/2k\omega_2)(\omega_o/2K)^2 \sin(\phi + kU), \] (6.22d)

where

\[ \omega_2 = 1 + kV/\omega_o. \] (6.23)

It is straightforward to show that the transformation determined by (6.22) is canonical through first order.

The second order members of the hierarchy of perturbation equations are:

\[ A^{(2)} + \frac{kV}{\omega_o} \partial_{kU} F^{(2)} + \partial_\phi F^{(2)} + A^{(1)} \partial_K F^{(1)} + B^{(1)} \partial_\phi F^{(1)} + b^{(1)} \partial_Y F^{(1)} \]

\[ = -(1/k)(\omega_o/2K)^2 F^{(1)} \sin\phi \cos kU - (1/k)(2\omega_o K)^2 G^{(1)} \cos\phi \cos kU \]

\[ + (1/k)(2\omega_o K)^2 D^{(1)} \sin\phi \sin kU, \] (6.24a)

\[ b^{(2)} + \frac{kV}{\omega_o} \partial_{kU} G^{(2)} + \partial_\phi G^{(2)} + A^{(1)} \partial_K G^{(1)} + B^{(1)} \partial_\phi G^{(1)} + b^{(1)} \partial_Y G^{(1)} \]

\[ = -(\omega_o/2K)^2 F^{(1)} \cos\phi \sin kU + (2\omega_o K)^2 G^{(1)} \sin\phi \sin kU \]

\[ - (2\omega_o K)D^{(1)} \cos\phi \cos kU + (\omega_o/k) \cos kU \sin kU, \] (6.24b)
\[ a^{(2)} + \frac{kV}{m_0} \partial_{kU} b^{(2)} + \partial_{\phi} D^{(2)} + A^{(1)} \partial_{kU} D^{(1)} + B^{(1)} \partial_{\phi} D^{(1)} + b^{(1)} \partial_{\phi} D^{(1)} = \left(\frac{k}{m_0}\right)_E^{(2)}, \quad (6.24c) \]
\[ B^{(2)} + \frac{kV}{m_0} \partial_{kU} D^{(2)} + A^{(1)} \partial_{kU} D^{(1)} + B^{(1)} \partial_{\phi} D^{(1)} + b^{(1)} \partial_{\phi} D^{(1)} + \partial_{\phi} G^{(2)} = \frac{3}{2} \left[ \frac{1}{m_0} \right] (m_0/2K)^2 F^{(1)} \cos \phi \cos kU + \left(\frac{1}{m_0} \right) (m_0/2K)^2 \left[ \frac{1}{2} \right] \sin \phi \cos kU \]
\[ + \left(\frac{1}{m_0} \right) (m_0/2K)^2 \left[ \frac{1}{2} \right] \cos \phi \sin kU. \quad (6.24d) \]

When the first order solutions are substituted into (6.24) we obtain a set of equations which determine the second order motion. It is found that the solutions of this set of equations can be chosen so that the transformation (6.22) is canonical through second order. These canonical solutions are:

\[ A^{(2)} = 0, \quad B^{(2)} = \frac{1}{8m_0^2}, \quad (6.25a, b) \]
\[ a^{(2)} = -(k^2K/4m_0\omega_2^2) + \frac{1}{8m_0^2}, \quad b^{(2)} = 0 \quad (6.25c, d) \]

and

\[ F^{(2)} = \left[ -\left(\frac{k^4}{4m_0^2} \right) \cos 2\phi - \left(\frac{1}{2m_0} \right) \cos 2(\phi + kU) \right] \quad (6.26a) \]
\[ + \frac{m_0}{8k^2\omega_2^2} - K^2/4m_0^2, \]
\[ c^{(2)} = \left[ \left(\frac{m_0}{16k^2\omega_2^2} - \frac{1}{16m_0^2} \right) \sin 2(\phi + kU) \right] \quad (6.26b) \]
\[ + \left(\frac{1}{8m_0^2} \right) \sin 2\phi - \left(\frac{1}{8m_0^2} \right)(\omega_2 - 1)) \sin 2kU, \]
\[ d^{(2)} = \left[ \left(\frac{1}{8m_0^2} \right)(\omega_2 - 1)) + \left(\frac{k^4}{4m_0^2} \right)(\omega_2^3 - 1)) \right] \quad (2.26c) \]
\[ + \left(\frac{1}{8m_0^2} \right)(\omega_2 - 1)) + \left(\frac{k^4}{8m_0^2} \right)(\omega_2^3 - 1)) - \left(\frac{1}{8m_0^2} \right)(\omega_2 - 1)^2 \right] \sin 2kU. \]
\[ - \left(\frac{k^4}{4m_0^2} \right) \sin 2\phi + \left(\frac{k^4}{16m_0^4} \right) \sin 2(\phi + kU), \]

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\[ E^{(2)} = \left[ \frac{m_0}{4k_2}(\omega_2 - 1) \right] + \left( \frac{k_2}{4k_2^2}(\omega_2 - 1) \right) \cos 2kU + \left( \frac{k_2}{8k_2^2} \right) \cos 2(\phi + kU) \]

\[ - \left( \frac{k_2}{4k_2^2} \right) + \frac{m_0}{8k_2^2} \cdot \]

Since the transformation has been made canonical through second order it follows from Corollary C.2 of Appendix C that we must be able to construct a quantity \( I \) which is a third order asymptotic invariant. This quantity is to be derived from the generating function (C.84) which, in this case, has the form

\[ F = P(\phi = kU) + I\phi. \quad (6.27) \]

Thus

\[ K = \partial_{\phi} F = P + I, \quad (6.28a) \]

\[ V = \partial_{U} F = - kP. \quad (6.28b) \]

It follows from (6.28) that

\[ I = K + V/k. \quad (6.29) \]

We can verify that \( I \) as defined by (6.29) is indeed constant through second order. To do this we simply write the equations for \( K \) and \( V \) through second order. These are found from (6.21) and (6.25) to be

\[ K' = -(\epsilon/2k)(2m_0k)^{\frac{1}{2}} \sin \theta + O(\epsilon^3), \quad (6.30a) \]

\[ V' = (\epsilon/2)(2m_0k)^{\frac{1}{2}} \sin \theta + O(\epsilon^3). \quad (6.30b) \]

It follows from (6.29) and (6.30) that

\[ I' = O(\epsilon^3). \quad (6.31) \]
Equations (6.29) and (6.31) reduce by one the number of differential equations which we must solve in order to present a second order solution.

The complete set of equations which govern the average motion is

\[ K' = -(\varepsilon/2k)(2m_0K)^2 \sin \theta + O(\varepsilon^3), \]  
\[ V' = (\varepsilon/2)(2m_0K)^2 \sin \theta + O(\varepsilon^3), \]  
\[ kU' = (kV/m_0) + \varepsilon^2 \left[ (1/8\omega_c^2) - (k^2K/4m_0\omega_c^2) \right] + O(\varepsilon^3), \]  
\[ \phi' = 1 - (\varepsilon/2k)(m_0/2K) \cos \theta + (\varepsilon^2/8\omega_c^2) + O(\varepsilon^3). \]  

If, in these equations, we replace \( \varepsilon \) by \( 2\varepsilon \) and retain terms only through order \( \varepsilon \) we will obtain equations (F.9) of Appendix F. Equations (F.9) describe the complete motion of a charged particle in a circularly polarized hydromagnetic wave. In what follows we shall present the complete first order solution of (6.32). If in this first order solution we replace \( \varepsilon \) by \( 2\varepsilon \) we will obtain the exact solution for the motion of a charged particle which moves in a circularly polarized hydromagnetic wave.

In order to facilitate the description of the average motion we introduce the variables \( a \) and \( b \) which are defined by the equations

\[ a = R \cos \theta, \]  
\[ b = R \sin \theta, \]  
where

\[ R = k(2K/m_0)^{1/2}. \]

The quantity \( R \) measures the ratio of the Larmor radius to the hydromagnetic wavelength. The equations of motion in the \( a-b \) representation are

\[ a' = - \left[ 1 - C + (R_c^2/2) + \frac{1}{8} \varepsilon^2 R_c^2 (1 + C - R_c^2/2)^{-3} \right] b, \]  
\[ b' = \left[ 1 - C + (R_c^2/2) + \frac{1}{8} \varepsilon^2 R_c^2 (1 + C - R_c^2/2) \right] a - \varepsilon/2, \]
where
\[ C = k^2 l/mw_0. \quad (6.36) \]

These equations give rise to the differential form
\[
\left[ 1 - C + (R^2/2) + \epsilon^2 R^2 (2 + 2C - R^2)^{-3} \right] bdb + \\
+ \left[ 1 - C + (R^2/2) + \epsilon^2 R^2 (2 + 2C - R^2)^{-3} \right] a - \epsilon/2 \right] da = 0. \quad (6.37)
\]

This is an exact differential whose integral is
\[
R^4 + 4(1 - C)R^2 + \epsilon^2 \left[ (1 + C)(1 + C - R^2/2)^{-2} - 2(1 + C - R^2/2)^{-3} \right] - 4\epsilon a = M, \quad (6.38)
\]

where M is a constant.

Equation (6.38) expresses the conservation of energy through second order. To see that this is so we express the Hamiltonian (6.13) in terms of the average variables, retaining terms through second order. We find that\(^{28}\)
\[
h = K + (V^2/2m\omega_0) - (\epsilon/2k)(2m\omega_0k)^2 \cos \theta + \frac{1}{8\omega_2^2} \left( \frac{m\omega_0/k^2 + K + V/k}{4k\omega_2} \right). \quad (6.39)
\]

By making use of equation (6.29) we are able to rewrite (6.39) as
\[
8k^2 h/m\omega_0 = 4R^2 + R^4 - 4C R^2 - 4\epsilon R \cos \theta \quad (6.40)
\]
\[
+ \epsilon^2 \left[ (1 + C)\omega_2^2 - 2\omega_2^{-1} + 2 \right].
\]

\(^{28}\) The Hamiltonian (6.39) depends on the angle variables \( \phi \) and \( kU \) only through their difference \( \theta \). This is in agreement with Theorem 4.2.
By definition

\[ \omega_2 = 1 + \left( \frac{kV}{m_0} \right) = 1 + C - \left( \frac{R^2}{2} \right). \]  \hspace{1cm} (6.41)

Therefore equation (6.38) is identical with (6.40) provided we choose

\[ M = \left( \frac{8k^2h}{m_0} \right) - 4C^2 - 2\epsilon^2. \]  \hspace{1cm} (6.42)

The equivalence with the conservation of energy is therefore proved.

The important aspects of the motion are most easily illustrated by plotting equation (6.40) in the a-b plane. Before doing this it is worth while to anticipate the nature of the trajectories by examining the singular points\(^{29}\) of (6.35). If we neglect second order effects we find that

\[ a' = - (1 - C + R^2/2)b, \]  \hspace{1cm} (6.43a)

\[ b' = (1 - C + R^2/2)a - \epsilon/2. \]  \hspace{1cm} (6.43b)

The singular points of (6.43) are given by

\[ b = 0, \]  \hspace{1cm} (6.44a)

\[ a^3 + 2(1 - C)a - \epsilon = 0. \]  \hspace{1cm} (6.44b)

If \( \epsilon \) is sufficiently small the cubic equation will have three real roots. These roots, which we shall call \( a_1, a_2, a_3 \), are approximately as follows:

\[ a_1 = 2(-2(1 - C)/3)^{1/2}(3/4)^{1/2} - 5/6) + 0(\epsilon^2) > 0, \]  \hspace{1cm} (6.45a)

\[ a_2 = 2(-2(1 - C)/3)^{1/2}(3/4)^{1/2} - 5/6) + 0(\epsilon^2) < 0, \]  \hspace{1cm} (6.45b)

\(^{29}\) A singular point is one where the ratio \( da/db \) is indeterminant, hence one where \( a' \) and \( b' \) vanish simultaneously. Singular points are significant because they represent positions of equilibrium.

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\[ a_3 = 2(-2(1 - c)/3)\frac{1}{2}(c/3) + 0(e^2) < 0, \]  
(6.45c)

where

\[ \delta = - (c/2)(-2(1 - c)/3)^{-\frac{3}{2}}. \]  
(6.46)

The nature of these singular points can be determined by examining the behavior of the motion in their vicinity. To do this we let

\[ a = a_1 + x, \]  
(6.47a)

\[ b = y, \]  
(6.47b)

where \( a_1 \) is one of the singular points and \( x \) and \( y \) represent small displacements from this singular point. Upon substituting (6.47) into (6.43) and retaining terms through first order in \( x \) and \( y \) we find that

\[ x' = -(1-C + a_1^2/2)y, \]  
(6.48a)

\[ y' = (1 - C + 3a_1^2/2)x. \]  
(6.48b)

We seek solutions of the form

\[ x = x_0 e^{x \tau}, \]  
(6.49a)

\[ y = y_0 e^{y \tau}. \]  
(6.49b)

These solutions are valid if

\[ \lambda = \pm \left[ -(1 - C + a_1^2/2)(1 - C + 3a_1^2/2) \right]^{1/2}. \]  
(6.50)

The values of \( a_1 \) as expressed by (6.45) may be substituted into (6.50). The results of this substitution are summarized in Table 1.
<table>
<thead>
<tr>
<th>Singular Points</th>
<th>$\lambda$</th>
<th>Nature of Singular Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>imaginary</td>
<td>Center (stable)</td>
</tr>
<tr>
<td>$a_2$</td>
<td>real</td>
<td>Saddle point (unstable)</td>
</tr>
<tr>
<td>$a_3$</td>
<td>imaginary</td>
<td>Center (stable)</td>
</tr>
</tbody>
</table>

Table 1. Classification of singular points

Table 1. shows that the trajectories must close about the point $a_3$ which is close to the origin, and they must also close about the point $a_1$ which lies away from the origin on the positive $a$ axis. According to (6.38), for large values of $R$, the trajectories have the form

$$R^4 = \text{constant} \quad (6.51)$$

and are therefore circles centered about the origin. The manner in which these requirements are satisfied is shown in Figure 3. Here several trajectories are plotted for a particular physical situation. The parameters such as the energy, background field strength, etc., have been assigned values which are appropriate to a proton which is moving in the Van Allen belt at a distance of two earth radii. The value of $\epsilon$ which was used in Figure 3 was chosen to be about ten times larger than what one would expect at two earth radii. This larger value was used to facilitate the plotting of the trajectories.

### 6.2 A QUALITATIVE DISCUSSION OF THE TRAJECTORIES

The trajectories of Figure 3 consist of a family of closed curves. This implies that $K$, and consequently $V$, is a periodic function of time; we shall obtain the period in the next section. The trajectories can be divided into two groups; those which are centered about the point $a_3$, and those which are centered about the point $a_1$. The motion corresponding to the first group is non-synchronous since the difference angle $\theta$ increases without bound. The motion corresponding to the second group is synchronous since, for these trajectories, the angle $\theta$ oscillates between well-defined limits. The synchronous and non-synchronous regions are separated by the trajectory, called a separatrix, which passes through the unstable point $a_2$.

---

$^{30}$Dragt, op cit., p. 1647.
Figure 3. Trajectories in the a-b plane for a 50 mev proton. 
k = 2x10^-8 cm^-1, \omega_o = 370 rad/sec, \epsilon = .01.
The largest fluctuations in $K$ occur on trajectories which pass close to the separatrix. As one moves away from the separatrix into the non-synchronous regions the trajectories rapidly become circles centered about the origin. As one moves away from the separatrix into the synchronous region the fluctuations in $K$ and $\theta$ become smaller until, at the point $a_1$, they vanish. The separatrix, therefore, determines the range of values of $K$ and $\theta$ for which maximum resonance occurs.

The simplest motion occurs at the singular points. For example, at the point $a_1$ the equations determining the average motion have the form

$$K' = 0, \quad V' = 0, \quad ku' = n_V, \quad \phi' = n_K, \quad (6.52)$$

where $n_V$ and $n_K$ are constants. The solutions of the original equations have the form

$$J = K + \varepsilon F(K,V,kU,\phi), \quad (6.53a)$$
$$p_z = V + \varepsilon E(K,V,kU,\phi), \quad (6.53b)$$

where $F$ and $E$ are periodic functions of $\phi$ and $kU$. According to (6.52) $\phi$ and $kU$ are linear functions of time. The motion at the point $a_1$ is, therefore, doubly periodic in time. A similar result holds for the point $a_2$; this point is, however, unstable.

In general, the quantities $J$ and $p_z$ oscillate in time about some average value. The precise time dependence of the oscillations is complicated. The amplitudes are, however, bounded, being greatest in the resonance region and rapidly decreasing outside this region.

6.3 THE TIME DEPENDENCE OF THE AVERAGE MOTION

In order to find the explicit time dependence of $K$ and an expression for the period $T$ we introduce a new variable $S$ according to the equation

$$S = R^2 \quad (6.54)$$
$$= \left(2k^2/m_0\right)K.$$ 

It follows from (6.30) that

$$S' = - \varepsilon R \sin \theta. \quad (6.55)$$
If we agree to neglect second order effects then we find from (6.40) that

$$e R \cos \theta = \left(R^4/4\right) + (1 - C) R^2 + C^2 - 2k^2\hbar/m_\omega. \quad (6.56)$$

We may, therefore, write

$$S' = -f(S), \quad ((6.57)$$

where

$$(f(S))^2 = e^2 S - \left[(S^2/4) + (1 - C) S + C^2 - 2k^2\hbar/m_\omega\right]^2. \quad (6.58)$$

If we make the definitions

$$L = 1 - C, \quad (6.59a)$$
$$N = C^2 - 2k^2\hbar/m_\omega, \quad (6.59b)$$

then we find that

$$S' = -\frac{1}{4} \left[-S^4 - 8LS^3 - 16\left(\frac{N}{2} + L^2\right)S^2 - 16(2LN - e^2)S - 16N^2\right]^{1/2}. \quad (6.60)$$

If we denote the roots of the polynomial $(4f(S))^2$ by $S_1$, $S_2$, $S_3$, $S_4$, then

$$S' = -\frac{1}{4} \left[(S_1 - S)(S - S_2)(S - S_3)(S - S_4)\right]^{1/2}. \quad (6.61)$$

It is necessary that we consider separately the solutions of (6.61) in the synchronous and non-synchronous regions.

A. Synchronous Region

In the synchronous region there is only one trajectory corresponding to each value of the constant $C$. This means that there are two values of $R^2$ for which $\sin \theta$ vanishes. This implies that the polynomial $(4f(S))^2$ has only two real roots which we shall take to be $S_1$, $S_2$ with $S_1 \geq S_2$. The roots $S_3$, $S_4$ must be complex conjugates and may be written as

$$S_3 = m + in, \quad (6.62a)$$

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\[ S_4 = m - \text{ in.} \quad \text{(6.62b)} \]

Equation (6.61) can be expressed in differential form as

\[
\frac{dS}{\left[(S_1 - S)(S - S_2)(S - S_2)(S - S_4)\right]^{1/2}} = -\frac{d\tau}{4}. \quad \text{(6.63)}
\]

Upon integrating (6.63) we find that\(^{31}\)

\[
-\frac{1}{4}(\tau - \tau_0) = \text{gcn}^{-1}(\cos \phi, \kappa), \quad \text{(6.64)}
\]

where \(\text{cn}(z)\) is the Jacobi elliptic function,

\[
g = (AB)^{-1/2}, \quad \text{(6.65a)}
\]

\[
\cos \phi = \frac{(S_1 - S)B - (S - S_2)A}{(S_1 - S)B + (S - S_2)A}, \quad \text{(6.65b)}
\]

\[
\kappa^2 = \frac{(S_1 - S_2)^2 - (A - B)^2}{4AB}, \quad \text{(6.65c)}
\]

\[
A^2 = (S_1 - m)^2 + n^2, \quad \text{(6.66a)}
\]

\[
B^2 = (S_2 - m)^2 + n^2, \quad \text{(6.66b)}
\]

It follows from (6.64) that

\[
\text{cn} \left[ \frac{1}{4g} (\tau - \tau_0) \right] = \frac{(S_1 - S)B - (S - S_2)A}{(S_1 - S)B + (S - S_2)A}. \quad \text{(6.67)}
\]

If at \(\tau = 0, S = S_0\) then the constant \(\tau_0\) is determined by substituting the values \(\tau = 0\) and \(S = S_0\) into (6.67).

From (6.67) we find that

\[ S(\tau) = \frac{S_1 B + S_2 A + (S_2 A - S_1 B) \text{cn} \left( \frac{\tau - \tau_0}{4g} \right)}{A + B + (A - B) \text{cn} \left( \frac{\tau - \tau_0}{4g} \right)} \]  \hspace{1cm} (6.68)

The time dependence of \( K \) is obtained by multiplying both sides of (6.68) by \( \frac{m \omega_0}{2k^2} \).

The function \( \text{cn}(x) \) is periodic in \( x \) with period \( 4\Lambda \), \( \Lambda \) being the complete elliptic integral of the first kind with modulus \( \kappa \) given by (6.65c). If follows that the period of oscillation (in seconds) is given by

\[ T = 16\Lambda (\omega_0^2 AB)^{-1/2} \] \hspace{1cm} (6.69)

The time dependence of \( V \) is found from equation (6.29).

B. Non-synchronous Region

In the non-synchronous regions there are two trajectories corresponding to each value of the constant \( C \). There are, therefore, four values of \( R^2 \) for which \( \sin \theta \) vanishes. This means that the polynomial \( (4f(S))^2 \) has four real roots which we shall arrange such that \( S_1 \geq S_2 \geq S_3 \geq S_4 \).

Let us consider first the case where

\[ S_1 \geq S \geq S_2 \geq S_3 \geq S_4 \] \hspace{1cm} (6.70)

We find from (6.61) that\(^\text{32}\)

\[ -\frac{1}{4}(\tau - \tau_0) = \text{gsn}^{-1} (\sin \phi, \kappa), \] \hspace{1cm} (6.71)

where \( \text{sn}(z) \) is the Jacobi elliptic function,

\[ g = 2 \left[ (S_1 - S_3)(S_2 - S_4) \right]^{-1/2}, \] \hspace{1cm} (6.72a)

\[ \sin \phi = \left\{ \left( \frac{(S_1 - S_3)(S - S_2)}{(S_1 - S_2)(S - S_3)} \right) \right\}^{1/2}, \] \hspace{1cm} (6.72b)

\(^{32}\)Byrd and Friedman, op cit., Eq. (256.00), p. 120.
\[ \kappa^2 = \frac{(S_1 - S_2)(S_3 - S_4)}{(S_1 - S_3)(S_2 - S_4)}. \] (6.72c)

It follows from (6.71) that

\[ \text{sn} \left[ \frac{1}{4g}(\tau - \tau_0) \right] = \left\{ \frac{(S_1 - S_2)(S - S_2)}{(S_1 - S_2)(S - S_3)} \right\}^{1/2}, \] (6.73)

or

\[ S(\tau) = \frac{S_2 (S_1 - S_3) - S_3 (S_1 - S_2) \text{sn}^2 \left[ (\tau - \tau_0)/4g \right]}{S_1 - S_3 - (S_1 - S_2) \text{sn}^2 \left[ (\tau - \tau_0)/4g \right]}. \] (6.74)

The time dependence of \( K \) is obtained by multiplying both sides of (6.74) by \( m \omega_0 / 2k^2 \).

The function \( \text{sn}^2 x \) is periodic in \( x \) with period 2 \( \Lambda \), \( \Lambda \) being the complete elliptic integral of the first kind with modulus \( k \) given by (6.72c). It follows that the period of oscillation (in seconds) is given by

\[ T = \frac{32 \Lambda}{\omega_0^2 (S_1 - S_2)(S_2 - S_4)} \left[ \frac{\omega_0^2 (S_1 - S_2)(S_2 - S_4)}{\omega_0^2 (S_1 - S_2)(S_2 - S_4)} \right]^{-1/2}. \] (6.75)

We must now consider the remaining case where

\[ S_1 > S_2 > S_3 > S_4. \] (6.76)

For this case we find that\(^{33}\)

\[ S(\tau) = \frac{S_4 (S_1 - S_3) + S_1 (S_3 - S_4) \text{sn}^2 \left[ (\tau - \tau_0)/4g \right]}{S_1 - S_3 + (S_3 - S_4) \text{sn}^2 \left[ (\tau - \tau_0)/4g \right]}. \] (6.77)

The time dependence of \( K \) is obtained by multiplying both sides of (6.77) by \( m \omega_0 / 2k^2 \). The time dependence of \( V \) is found from (6.29). The period of oscillation is given by (6.75).

The time dependence of the difference angle \( \theta \) is found from (6.56) to be given by

\(^{33}\) Byrd and Friedman, op. cit., Eq. (252.00), p. 103.
\[ \theta(\tau) = \cos^{-1}\left[ \frac{3}{4\epsilon} + \epsilon^{-1}(1 - \epsilon)S_{\theta}^{\frac{1}{2}} + \epsilon^{-1}\left( c^2 - skh/m_o \right)S_{\theta}^{\frac{1}{2}} \right] \] \quad (6.78)

The particular function \( S(\tau) \) to be used in (6.78) depends upon the trajectory, hence upon the constant \( C \).

With the exact solutions for \( K, V, \theta \) which we have obtained we can, in principle, write down the explicit time dependence of the variables \( \phi \) and \( kU \). This allows us to write down the explicit time dependence, through first order, of the original variables \( J, p_z, \gamma, \) and \( k_z \). We shall not present these expressions since they are complicated and add little to our understanding of the problem.

As was mentioned earlier, the average solutions which we have obtained constitute the exact solution for the motion of a charged particle which moves in a circularly polarized hydromagnetic wave. It is only necessary that, in the above solutions, we replace \( \epsilon \) by \( 2\epsilon \) to obtain the correct expressions for the circularly polarized case. These expressions are presented in Appendix F.

6.4 BEHAVIOR UNDER MAXIMUM RESONANCE CONDITIONS

In this section we shall obtain an estimate for the amplitude of oscillation of \( K \) and for the period of oscillation in the case where the particle enters the wave nearly on resonance. Exact resonance occurs when the particle traverses a single hydromagnetic wavelength in one Larmor period. This exact resonance is nearly fulfilled when

\[ \frac{kV}{\omega_o} = 1. \] \quad (6.79)

If we substitute condition (6.79) into the Hamiltonian (6.40) and neglect the first order term we find that

\[ \frac{k^2K}{\omega_o} = \frac{k^2h}{\omega_o} - 1/2. \] \quad (6.80)

Conditions (6.79) and (6.80) allow us to obtain a value for the constant \( C \) which corresponds to near resonance. The appropriate value of \( C \) is found to be

\[ C = \frac{k^2I}{\omega_o} = \frac{k^2h}{\omega_o} + 1/2 \] \quad (6.81)

\[ = a + 1/2, \]
where
\[
d = \frac{k^2 \hbar}{m \omega_0}.
\] (6.82)

We now substitute the value of \( \alpha \) given by (6.81) into the polynomial \((4f(S))^2\) to find that
\[
(4f(S))^2 = - \left[ S^4 + (4 - 8d)S^3 + (6 - 24d + 24d^2)S^2 \right. \\
+ (4 - 24d + 4d^2 - 32d^3 - 16d^4)S \\
+ \left. (1 - 8d + 24d^2 - 32d^3 + 16d^4) \right].
\] (6.83)

We must obtain the roots of \((4f(S))^2\). In order to do this we introduce a new variable \( x \) defined by the equation
\[
S = x - (1 - 2d).
\] (6.84)

Upon substituting (6.84) into (6.83) we find that
\[
(4f(S))^2 = - \left[ x^4 - (4\epsilon)^2 x + (1 - 2d)(4\epsilon)^2 \right].
\] (6.85)

The roots of (6.85) are found by solving the equation
\[
x^4 - (4\epsilon)^2 x + (1 - 2d)(4\epsilon)^2 = 0,
\] (6.86)

which, for any \( u \) is equivalent to
\[
(x^2 + u/2)^2 - \left\{ u x^2 + (4\epsilon)^2 x + \left[ (u^2/4) - (1 - 2d)(4\epsilon)^2 \right] \right\} = 0.
\] (6.87)

We choose \( u \) such that the left hand side of (6.87) is the difference of two squares. Thus
\[
(4\epsilon)^4 = 4u \left[ (u^2/4) - (1 - 2d)(4\epsilon)^2 \right],
\] (6.88)
or
\[ u^3 - 4(1 - 2d)(4\epsilon)^2u - (4\epsilon)^4 = 0. \] (6.89)

If we denote a root of (6.89) by \( u_o \) then we may rewrite the quartic (6.86) as
\[ \left( x^2 + \frac{1}{2} u_o x + 16\epsilon^2(4u_o)^{-1} + u_o/2 \right) \left( x^2 - \frac{1}{2} u_o x - (4\epsilon)^2(4u_o)^{-1} + u_o/2 \right) = 0 \] (6.90)

The roots of (6.90) are given by
\[ x_{1,2} = \frac{1}{2} u_o^{1/2} + \frac{1}{2} \left[ -u_o + 4(4\epsilon)^2(4u_o)^{-1/2} \right]^{1/2}, \] (6.91)
\[ x_{3,4} = \frac{1}{2} u_o^{1/2} + \frac{1}{2} \left[ -u_o - 4(4\epsilon)^2(4u_o)^{-1/2} \right]^{1/2}. \] (6.92)

A root \( u_o \) of the cubic is given by
\[ u_o = \left\{ \left(4\epsilon\right)^{4/2} + \left[\left(4\epsilon\right)^{8/4} + \left(-4(1 - 2d)\right)^3(4\epsilon)^6/27\right]^{1/2} \right\}^{1/2}. \] (6.93)

Upon expanding the right hand side of (6.93) and retaining only the lowest non-vanishing power of \( \epsilon \) we find that
\[ u_o = 4\epsilon^2/(2d - 1) + O(\epsilon^3). \] (6.94)

When we substitute (6.94) into (6.91) and (6.92) and make use of (6.84) we find the roots of the polynomial \((4f(S))^2\) to be
\[ S_{1,2} = (2d - 1) \pm 2(2d - 1)^{1/4} \epsilon^2 + O(\epsilon), \] (6.95)
\[ S_{3,4} = (2d - 1) \pm 12(2d - 1)^{1/4} \epsilon^2 + O(\epsilon). \] (6.96)

Since two roots are imaginary it follows that the trajectory corresponding to condition (6.81) lies in the synchronous region.
The expression \(2d - 1\) can be presented in terms of easily understood physical quantities. By definition

\[
d = \frac{k^2 h}{m \omega_0}
\]

\[
= \frac{k^2 H}{m \omega_0},
\]

where \(H\) is the energy. Thus

\[
2d - 1 = \left(\frac{2k^2}{m \omega_0^2}\right)(H - \frac{m \omega_0^2}{2k^2}) \tag{6.98}
\]

\[
= \left(\frac{2k^2}{m \omega_0^2}\right)\frac{m}{2} v_{\perp 0} + \frac{1}{2m} p_{z 0}^2 z_0 + O(\epsilon) - \frac{m \omega_0^2}{2k^2},
\]

where \(v_{\perp 0}\) is the initial transverse velocity and \(p_{z 0}\) the initial longitudinal momentum. According to (6.79).

\[
p_{z 0} \approx \frac{m \omega_0}{k}. \tag{6.99}
\]

Upon substituting (6.99) into (6.98) we find that

\[
2d - 1 = k^2 \frac{v_{\perp 0}^2}{\omega_0^2} + O(\epsilon). \tag{6.100}
\]

The roots of the polynomial \((k f(S))^2\) are, therefore, given by

\[
S_{1,2} = \left(\frac{k^2 v_{\perp 0}^2}{\omega_0^2}\right) + 2(\epsilon v_{\perp 0} / \omega_0)^{1/2} + O(\epsilon), \tag{6.101}
\]

\[
S_{3,4} = \left(\frac{k^2 v_{\perp 0}^2}{\omega_0^2}\right) + 12(\epsilon v_{\perp 0} / \omega_0)^{1/2} + O(\epsilon). \tag{6.102}
\]

We find from (6.101) that the total fluctuation in \(K\) is given by

\[
\Delta K = \left(\frac{m \omega_0}{2k^2}\right)(S_1 - S_2) \tag{6.103}
\]

\[
= 2(\epsilon m^2 \omega_0 v_{\perp 0} / k^3)^{1/2}.
\]
The period is given by (6.69) as

\[ T = 16 \Lambda (\omega_o^2 \Delta B)^{-1/2} \quad . \] \hfill (6.104)

We find from (6.66) that

\[ A^2 = B^2 = 8(2d - 1)^{1/2} \, \epsilon + O(\epsilon^2) \] \hfill (6.105)

\[ = 8\epsilon (k v_o / \omega_o) + O(\epsilon^2). \]

The modulus of the complete elliptic integral \( \Lambda \) is found from (6.65c) to be

\[ \kappa^2 = \frac{16(2d - 1)^{1/2} \epsilon}{32(2d - 1)^{1/2} \epsilon} \] \hfill (6.106)

\[ = 1/2. \]

The period of the average motion is, therefore, given by

\[ T = 16 \Lambda (8\epsilon k v_o \gamma_o)^{-1/2} + O(\epsilon). \] \hfill (6.107)

From the above expressions we can obtain the approximate fluctuations of \( J \) and the approximate period for the true motion of a particle which moves in a circularly polarized wave under resonance conditions. To do this we simply replace \( \epsilon \) by \( 2\epsilon \). Thus, for the circularly polarized case,

\[ \Delta J = 2(2\epsilon m^2 w_o \gamma_o / k^3)^{1/2} + O(\epsilon) \] \hfill (6.108)

and

\[ T = 4 \Lambda (\epsilon k w_o \gamma_o)^{-1/2} + O(\epsilon), \] \hfill (6.109)

where \( \Lambda \) is the complete elliptic integral of the first kind with modulus

\[ \kappa = 2^{-1/2} . \] \hfill (6.110)
6.5 RELATION OF THE SOLUTIONS TO PREVIOUSLY PUBLISHED RESULTS

In a coordinate system which is at rest the linearly polarized hydromagnetic wave has the form

\[
\overline{E}_l = (B_l \sin(\omega t - kz'), 0,0), \quad (6.111a)
\]

\[
\overline{E}_l = (0, -E_l \sin(\omega t - kz'),0), \quad (6.111b)
\]

where \(\omega\) is the frequency of the wave and the prime on \(z\) indicates the rest system. In the rest system the circularly polarized wave has the form

\[
\overline{E}_l = (B_l \sin(\omega t - kz'), B_l \cos(\omega t - kz'),0), \quad (6.112a)
\]

\[
\overline{E}_l = (E_l \cos(\omega t - kz'),-E_l \sin(\omega t - kz'), 0). \quad (6.112b)
\]

The equations of motion in these fields are given by

\[
\dot{p} = e(\overline{E}_l + \overline{v} \times \overline{B}/c + \overline{v} \overline{E}_l/c). \quad (6.113)
\]

The electric and magnetic fields are related by the equation

\[
\overrightarrow{k} \times \overrightarrow{E}_l = \omega \overrightarrow{E}_l/c, \quad (6.114)
\]

where

\[
\overrightarrow{k} = (0,0,k). \quad (6.115)
\]

Upon substituting (6.114) into (6.113) we find that

\[
\dot{p}_z = (ek/\omega) \overrightarrow{v} \cdot \overrightarrow{E}_l \quad (6.116)
\]

The rate of change of the particle's energy \(E\) is given by

\[
\dot{E} = e \overrightarrow{v} \cdot \overrightarrow{E}_l. \quad (6.117)
\]
It follows that

\[ \mathcal{E} = \omega \mathbf{P}_z'/k. \]  \hspace{1cm} (6.118)

This last equation may be integrated to give the result that

\[ \mathcal{E}(t) = (P_z(t) - P_z' - \omega/k + \mathcal{E}_o. \]  \hspace{1cm} (6.119)

The momentum \( P_z' \) is related to the momentum \( P_z \) of the system which moves with the wave by the equation

\[ P_z' = P_z + \omega/\mathbf{k}. \]  \hspace{1cm} (6.120)

In order to obtain (6.120) we have neglected relativistic effects and used the fact that

\[ \omega/\mathbf{k} = v_{hm}. \]  \hspace{1cm} (6.121)

where \( v_{hm} \) is the velocity of the hydromagnetic wave. Upon substituting (6.120) into (6.119) we find that

\[ \mathcal{E}(t) = \mathcal{E}_o + \frac{mv_{hm}^2}{\mathbf{k}} - v_{hm}P_z' + v_{hm}P_z(t). \]  \hspace{1cm} (6.122)

We have previously shown that, for a linearly polarized wave, the momentum \( P_z \) is a nearly periodic function of time with period given by either (6.69) or (6.75) depending on whether the motion occurs in the synchronous region or in the nonsynchronous region. Thus, for a linearly polarized wave, \( \mathcal{E}(t) \) is a nearly periodic function of time with period given by (6.69) or (6.75).

In the case of a circularly polarized wave the momentum \( P_z \) has been shown to be a strictly periodic function of time with period found from (F.17) or F.18, depending on whether the motion occurs in the synchronous region or in the non-synchronous region. Thus, for the circularly polarized case, \( \mathcal{E}(t) \) is a strictly periodic function of time.

Roberts and Buchsbaum have considered, from the point of view of special relativity, the problem of finding \( \mathcal{E}(t) \) for the case of a circularly polarized
electromagnetic wave.\textsuperscript{34} They find that \( \mathcal{E}(t) \) is strictly periodic in time. A hydromagnetic wave may be viewed as an electromagnetic wave which moves in a medium having a large index of refraction. We may, therefore, compare the results which we obtained with those obtained by Roberts and Buchsbaum.

For the case where a particle, whose velocity is much less than \( c \),\textsuperscript{35} enters the wave initially on resonance Roberts and Buchsbaum give an approximate expression for the period \( T \). This expression is subject to the conditions that the wave is weak compared to \( B_0 \). They find that\textsuperscript{36}

\[
T = 8 \Lambda \left[ \left| \Omega \right| l - n^2 \left( \frac{E_1}{B_0} \right)^{2/3} \right]^{-1}, \tag{6.123}
\]

where \( n \) is the index of refraction,

\[
\Omega = -eB_0 \left( 1 - \frac{v^2}{c^2} \right)^{1/2} / mc, \tag{6.124}
\]

and \( \Lambda \) is the complete elliptic integral of the first kind with modulus \( \kappa = \). (.067)\(^{1/2} \). If we use the fact that, for a hydromagnetic wave, \( n \gg 1 \) we may rewrite (6.123) as

\[
T = 8 \Lambda \left[ \left| \Omega \right| (nE_1/B_0)^{2/3} \right]^{-1} \tag{6.125}
\]

\[
= 8 \kappa \left[ \left| \Omega \right| \epsilon^{2/3} \right]^{-1} .
\]

The period given by (6.125) does not have even the same functional dependence on \( \epsilon \) as the correct non-relativistic period which is given by (6.109).

In order to remove this disagreement we shall now consider the "high energy" period which Roberts and Buchsbaum present. This period is valid for a weak wave and under initial resonance conditions and has the form\textsuperscript{37}

\textsuperscript{34}C. S. Roberts and S. J. Buchsbaum, Motion of a Charged Particle in a Constant Magnetic Field and a Transverse Electromagnetic Wave Propagating Along the Field, Phys. Rev. \textbf{135}, p. 381, 1961.

\textsuperscript{35}c is the velocity of light in vacuum.

\textsuperscript{36}Roberts and Buchsbaum, op. cit., Eq. (3.23), p. 385.

\textsuperscript{37}Roberts and Buchsbaum, op. cit. Eq. (3.38), p. 386.
\[ T = 4\Lambda \left[ \omega |t_0| (v_{\perp}/c) (E_1/E_0) |n^2 - 1| \right]^{-1/2}, \quad (6.126) \]

where \( \Lambda \) is the complete elliptic integral of the first kind with modulus \( \kappa \) given by

\[ \kappa^2 = (1/2)(1 + \sin \theta_0), \quad (6.127) \]

\( \theta_0 \) being the initial phase difference. In the non-relativistic limit \( |\theta_0| \ll \omega_0 \).

If we write (6.126) in the non-relativistic limit and under the conditions that \( n \gg 1 \) we find that

\[ T = 4 \Lambda \left[ \omega_0 (n v_{\perp} / c) \epsilon \right]^{-1/2}. \quad (6.128) \]

The index of refraction is defined by the relation

\[ n = c/v_{hm}. \quad (6.129) \]

The wave number \( k \) relates the frequency \( \omega \) and the velocity \( v_{hm} \) through the equation

\[ k = \omega / v_{hm}. \quad (6.130) \]

When we substitute (6.129) and (6.130) into (6.128) we find that

\[ T = 4 \Lambda (\epsilon \omega_0 v_{\perp} \epsilon) ^{-1/2}. \quad (6.131) \]

The period which is given by (6.131) is identical, to within \( O(\epsilon) \), to the correct non-relativistic period (6.109) when the initial phase difference \( \theta_0 \) is such that

\[ \sin \theta_0 = 0. \]

For the high energy approximation Roberts and Buchsbaum find, in the case
where \( \theta_o = 0 \), that the total fluctuation in the energy is given by\(^\text{38}\)

\[
\Delta E = 2 \xi_0 \left[ 2 \left( |\Omega_0|/\omega \right) (v_{\perp 0}/c)(E_1/B_0)/(n^2 - 1) \right]^{1/2}, \tag{6.132}
\]

\( \xi_0 \) being the initial relativistic energy \( m_0 c^2 \). If we assume \( n \gg 1 \) then (6.132) becomes

\[
\Delta E = 2 \xi_0 \left[ 2 \left( |\Omega_0|/\omega \right) (v_{\perp 0}/n^3 c) \xi_0 \right]^{1/2}. \tag{6.133}
\]

In the non-relativistic limit

\[
\xi_0 = m c^2 + m v_0^2 / 2, \tag{6.134}
\]

\( m \) being the rest mass; the term \( m v_0^2 / 2 \) can be dropped from (6.134) since \( m c^2 \gg m v_0^2 \). We now take the non-relativistic limit of (6.133) to find that

\[
\Delta E = 2 m \left[ 2 \left( \omega_0 / \omega \right) (v_{\perp 0} / n^3) \xi_0 \right]^{1/2}. \tag{6.135}
\]

Equations (6.129) and (6.130) can be used to write (6.135) as

\[
\Delta E = 2 m v_{hm} \left[ \left( 2 \omega_0 v_{\perp 0} / k \right) \right]^{1/2}. \tag{6.136}
\]

Let us now consider the \( \Delta z \) which the exact nonrelativistic theory predicts. According to (6.122)

\[
\Delta z = v_{hm} \Delta p_z, \tag{6.137}
\]

while from (F.11)

\[
\Delta p_z = k \Delta J. \tag{6.138}
\]

The quantity \( \Delta J \) given by (6.108), when substituted into (6.138), gives

\(^{38}\) Roberts and Buchsbaum, op.cit. Eq. (3.25), p. 385.
\[ \Delta E = 2 \text{mv} \left[ \frac{2\varepsilon_0 \nu}{4} / k \right] \frac{1}{\xi} + O(\varepsilon). \]

This expression is identical, to within \( O(\varepsilon) \), with (6.136). We conclude that it is the non-relativistic limit of Roberts' and Buchsbaum's "high energy" approximation which yields the non-relativistic motion. Their so called "low energy" approximation is not the non-relativistic limit.\(^{39}\)

Wentzel and Dragt, in separate papers, have discussed the change in a particle's orbital magnetic moment as the particle moves through a hydromagnetic wave. It is a simple matter to show that the magnetic moment \( \mu \) is proportional to the action variable \( J \); the precise connection between the two is given by

\[ \mu = eJ/mc. \tag{6,140} \]

This equality permits us to relate the results which we have obtained to those which were obtained by the authors mentioned above.

In his paper\(^{40}\) Wentzel considers the motion of a charged particle in a transverse, linearly polarized hydromagnetic wave. He employs a perturbation method to construct solutions of the equations of motion. He concludes, among other things, that there is a resonant decrease in the magnetic moment when the particle traverses one half of a hydromagnetic wavelength per Larmor period. This is in complete disagreement with the results which we have obtained. We can be sure that Wentzel's conclusion is incorrect since the exact solution for the motion in a circularly polarized wave shows no resonance where Wentzel finds one. Wentzel's results are incorrect because the perturbation method which he used was too restrictive to properly deal with the problem.\(^{41}\)

Dragt\(^{42}\) considers the motion of a charged particle in a dipole field. He assumes that a weak hydromagnetic wave is propagating along the field lines. Without presenting an explicit form for the wave, he quite properly concludes that a resonant change in the magnetic moment is possible in the case where the particle traverses one hydromagnetic wave per Larmor period. Having reached this conclusion he then embarks upon a course of dangerous speculation. For example, without examining the equations of motion in detail he concludes that

\(^{39}\) Their limiting process is legitimate. It is, however, not the non-relativistic limit but a mathematical limit with no apparent physical meaning.

\(^{40}\) Wentzel, \textit{op cit.}

\(^{41}\) A proper critique of Wentzel's perturbation theory is too lengthy to give here. Briefly, the fault of his method lies in the appearance of apparent (i.e. not true) secular terms.

\(^{42}\) Dragt, \textit{op cit.}

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the magnetic moment can undergo secular changes. Our studies do not support this conclusion. It is clear that the magnetic moment will fluctuate; the fluctuations are, however, of rather small amplitude, the amplitude varying as $\epsilon^{1/2}$.

Both Wentzel and Dragt appeal to the idea of a secular change in the magnetic moment in order to explain the removal of protons from the inner Van Allen belt. A hydromagnetic wave is assumed to cause the secular change in the magnetic moment. Our studies imply that a mono-frequency hydromagnetic wave will not be an effective mechanism for removing protons from the inner Van Allen belt.
APPENDIX A

FORMAL PRESENTATION OF THE BOGOLIUBOV METHOD

In this appendix we shall present a slightly revised version of Bogoliubov's perturbation theory for a system with a single rapid phase.\(^3\)

We consider a dynamical system whose state is characterized by the angle variable \(\gamma\) and the \(r\) variables \(x_1, x_2, \ldots, x_r\), and whose temporal behavior is governed by the system of autonomous differential equations

\[
\dot{x}_k = \sum_{n=1}^{R} \epsilon^n A_k^{(n)}(\gamma, x), \quad (A.1a)
\]

\[
\dot{\gamma} = \omega(x) + \sum_{n=1}^{R} \epsilon^n B^{(n)}(\gamma, x) \quad (A.1b)
\]

where \(\epsilon\) is a small parameter, \(A_k^{(n)}\) and \(B^{(n)}\) are periodic functions of \(\gamma\) with period \(2\pi\), and \(R\) is an upper limit.

The formal perturbation theory consists of making the following change of variables:

\[
x_k = y_k + \sum_{n=1}^{\infty} \epsilon^n F_k^{(n)}(\phi, y), \quad (A.2a)
\]

\[
\gamma = \phi + \sum_{n=1}^{\infty} \epsilon^n G^{(n)}(\phi, y), \quad (A.2b)
\]

where the \(F_k^{(n)}\)'s and the \(G^{(n)}\)'s are periodic functions of \(\phi\) with period \(2\pi\).

The new variables \(y_k\) and \(\phi\) are to be governed by the following equations:

\[
\dot{y}_k = \sum_{n=1}^{\infty} \epsilon^n A_k^{(n)}(y), \quad (A.3a)
\]

\(^3\)For the original presentation see Bogoliubov and Mitropolsky, *op cit.*, p. 412 et seq.
\[ \dot{\phi} = \omega(y) + \sum_{n=1}^{\infty} \epsilon^n b(n)(y). \quad (A.3b) \]

The right hand sides of (A.3) are independent of the angle variable \( \phi \).

The physical meaning of the transformation (A.2) lies in the decomposition of the true motion described by the variables \( x_k \) and \( \gamma \) into an average motion described by \( y_k \) and \( \phi \) and rapid fluctuations about this average which are described by the functions \( F_k(n) \) and \( G(n) \).

The functions \( F_k(n) \) and \( G(n) \) are uniquely determined only after some restrictions are placed upon the transformation (A.2). In order to obtain definite single-valued expressions for \( F_k(n) \)'s and \( G(n) \)'s Bogoliubov requires that they shall contain no zero harmonics in the angle variable \( \phi \).

The details of the perturbation calculation are straightforward. We first differentiate (A.2) with respect to time to obtain

\[ \dot{x}_k = \dot{y}_k + \sum_{n=1}^{\infty} \epsilon^n (\partial_{\phi} F_k(n) \phi + \sum_{i=1}^{r} \partial_{y_i} F_k(n) \dot{y}_i), \quad (A.4a) \]

\[ \dot{\gamma} = \dot{\phi} + \sum_{n=1}^{\infty} \epsilon^n \left[ \partial_{\phi} G(n) \phi + \sum_{i=1}^{r} \partial_{y_i} G(n) \dot{y}_i \right]. \quad (A.4b) \]

Next we substitute the right hand sides of (A.3) into (A.4) to obtain

\[ \dot{x}_k = \sum_{n=1}^{\infty} \epsilon^n a_k(n) + \sum_{n=1}^{\infty} \epsilon^n \left[ \partial_{\phi} F_k(n) (\omega + \sum_{m} \epsilon^m b(m)) + \sum_{i=1}^{r} \partial_{y_i} F_k(n) \sum_{m} \epsilon^m a_i(m) \right] \quad (A.5a) \]

\[ \dot{\gamma} = \omega(y) + \sum_{n=1}^{\infty} \epsilon^n b(n) + \sum_{n=1}^{\infty} \epsilon^n \left[ \partial_{\phi} G(n) (\omega + \sum_{m} \epsilon^m b(m)) + \sum_{i=1}^{r} \partial_{y_i} G(n) \sum_{m} \epsilon^m a_i(m) \right]. \quad (A.5b) \]

We now write the original equations (A.1) in terms of the new variables and then expand the functions \( A_k(n) \), \( \omega \) and \( B(n) \) in multiple Taylor series about the arguments \( \phi \) and \( y_k \). The result of this expansion is a formal infinite series in powers of \( \epsilon \) for \( x_k \) and \( \gamma \). Upon equating the coefficient of \( \epsilon^n \) from these formal series with the corresponding coefficient of equations (A.5) we obtain an infinite set of coupled differential equations. These equations recursively determine the functions \( F_k(n) \), \( G(n) \), \( a_k(n) \) and \( b(n) \).
The first order functions are found from the equations

\begin{align}
a_k^{(1)}(y) + \omega(y) \partial_{\phi} F_k^{(1)} &= A_k^{(1)}(\phi, y), \quad (A.6a) \\
b_k^{(1)}(y) + \omega(y) \partial_{\phi} G_k^{(1)} &= \sum_{i=1}^{r} \partial_{y_i} \omega F_i^{(1)} + B^{(1)}(\phi, y). \quad (A.6b)
\end{align}

Similarly the second order functions are found from the equations

\begin{align}
a_k^{(2)} + \omega \partial_{\phi} F_k^{(2)} &= A_k^{(2)} + \partial_{\phi} A_k^{(1)} G_k^{(1)} + \sum_{i=1}^{r} \partial_{y_i} A_k^{(1)} F_i^{(1)} \quad (A.7a) \\
&- \partial_{\phi} F_k^{(1)} b_k^{(1)} - \sum_{i=1}^{r} \partial_{y_i} F_k^{(1)} a_k^{(1)}, \\
b_k^{(2)} + \omega \partial_{\phi} G_k^{(2)} &= B_k^{(2)} + \sum_{i,j} \frac{1}{2} \partial_{y_i} \partial_{y_j} \omega F_i^{(1)} F_j^{(1)} \quad (A.7b) \\
&+ \sum_{i} \partial_{y_i} \omega F_i^{(2)} + \partial_{\phi} B_k^{(1)} G_k^{(1)} + \sum_{i} \partial_{y_i} B_k^{(1)} F_i^{(1)} \\
&- \partial_{\phi} G_k^{(1)} b_k^{(1)} - \sum_{i} \partial_{y_i} G_k^{(1)} a_k^{(1)}.
\end{align}

In general the equations which determine the \(n^{th}\) order functions will have the form

\begin{align}
a_k^{(n)}(y) + \omega(y) \partial_{\phi} F_k^{(n)} &= x_k^{(n)}(\phi, y), \quad (A.8a) \\
b_k^{(n)}(y) + \omega(y) \partial_{\phi} G_k^{(n)} &= y_k^{(n)}(\phi, y), \quad (A.8b)
\end{align}

where the functions \(x_k^{(n)}\) and \(y_k^{(n)}\) are determined by the solutions of the lower order equations.
The solutions of (A.8) are

\[ a_k^{(n)}(y) = \frac{1}{2\pi} \int_0^{2\pi} x_k^{(n)}(\phi, y) d\phi, \quad (A.9a) \]

\[ f_k^{(n)}(\phi, y) = \frac{1}{\omega(y)} \int_0^{2\pi} [x_k^{(n)}(\phi, y) - a_k^{(n)}(y)] d\phi, \quad (A.9b) \]

\[ b^{(n)}(y) = \frac{1}{2\pi} \int_0^{2\pi} y^{(n)}(\phi, y) d\phi, \quad (A.9c) \]

\[ g^{(n)}(\phi, y) = \frac{1}{\omega(y)} \int_0^{2\pi} [y^{(n)}(\phi, y) - b^{(n)}(y)] d\phi. \quad (A.9d) \]

The functions \( f_k^{(n)} \) and \( g^{(n)} \) have been determined so as to contain no zero harmonic in \( \phi \).

In order to obtain the \( N' \)th order approximation we must determine all the functions under the summation signs in the following equations:

\[ x_k = y_k + \sum_{n=1}^{N} \epsilon^n f_k^{(n)}, \quad (A.10a) \]

\[ \gamma = \phi + \sum_{n=1}^{N} \epsilon^n g^{(n)}. \quad (A.10b) \]

The \( N' \)th order solution is not obtained, however, until we solve the following system of differential equations:

\[ \dot{y}_k = \sum_{n=1}^{N} \epsilon^n a_k^{(n)}(y), \quad (A.11a) \]

\[ \dot{\phi} = \omega(y) + \sum_{n=1}^{N} \epsilon^n b^{(n)}(y), \quad (A.11b) \]

where the functions \( a_k^{(n)} \) and \( b^{(n)} \) are known. Thus we have reduced the problem from one of solving the original system of equations to one of solving the sys-
tem (A.11). We see then that the Bogoliubov method does not automatically produce a solution. Rather, it produces an approximately equivalent system of equations in a reduced number of variables. The significance of the reduced system lies in the absence of the rapidly varying phase. This independence of the phase usually makes the reduced system much easier to discuss than the original system.
APPENDIX B

APPLICATION OF THE BOGOLIUBOV METHOD TO HAMILTONIAN SYSTEMS HAVING SEVERAL DEGREES OF FREEDOM

In this appendix we shall apply the Bogoliubov method to a class of Hamiltonian systems which has several degrees of freedom. We shall demonstrate that the Bogoliubov transformation can be made canonical to all orders. This will permit us to construct a quantity which is constant to all orders in the perturbation theory.

B.1 THE EQUATIONS OF MOTION

Let the system have $N$ degrees of freedom and be described by the Hamiltonian $H = H(q,p)$. We assume that there exists a canonical transformation to new variables $u_i (i = -N, \ldots, 1, \ldots, N)$ such that the Hamiltonian separates into the form

$$H = H_0(u_{-N}) + \sum_{n=1}^{R} \epsilon^n H_n(u),$$

(B.1)

where $\epsilon$ is a small parameter, $R$ is an upper limit, and where the $H_n$'s are periodic functions of $u_N$. The variables $u_i$ are chosen such that the momentum conjugate to the coordinate $u_{i}$ ($i > 0$) is $u_{-i}$.

Hamilton's equations of motion are found from (B.1) to be

$$\dot{u}_i = sgn i \sum_{n=1}^{R} \epsilon^n \partial_{u_{-i}} H_n \quad (i \neq N),$$

(B.2a)

$$\dot{u}_N = \omega_0(u_{-N}) + \sum_{n=1}^{R} \epsilon^n \partial_{u_{-N}} H_n,$$

(B.2b)

where

$$\omega_0(u_{-N}) = \partial_{u_{-N}} H_0(u_{-N}),$$

(B.3)

and where $sgni$ is the sign of $i$.  

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The Bogoliubov method can be applied to the system of equations (B.2) since the $H_n$'s are periodic functions of $u_{N}$. Before we proceed with the discussion of (B.2) we should mention another type of system to which the Bogoliubov method is applicable. Suppose that, when written in terms of the canonical variables $u_{1}$, the Hamiltonian has the form

$$H = H_{0}(u_{-N},  \hat{u}) + \sum_{n=1}^{R} \varepsilon^{n}H_{n}(u),$$  \hspace{1cm} (B.4)

where the $H_n$'s are periodic functions of $u_N$ and where the caret over $u$ excludes the indices $N$ and $-N$. The equations of motion which we obtain from (B.4) have a form to which the Bogoliubov method is applicable. The results which we obtain from a discussion of a system which is described by (B.1) will also hold for a system which is described by (B.4). Hamiltonians of the form (B.4) arise, for example, when one discusses the motion of a charged particle which moves in a slowly varying electromagnetic field.

In order to apply the Bogoliubov method to (B.2) we make the following change of variables:

$$u_{i} = v_{i} + \varepsilon F_{i}(v) \hspace{1cm} (i = -N, \ldots, N),$$  \hspace{1cm} (B.5)

where the $F_i$'s are formal infinite series in powers of $\varepsilon$ and are periodic functions of $v_N$. We require the average motion to be governed by the following equations:

$$\dot{v}_{i} = \varepsilon A_{i}(u_{-N}, \hat{u}) \hspace{1cm} (i \neq N),$$  \hspace{1cm} (B.6a)

$$\dot{v}_{N} = \omega_{0}(u_{-N}) + \varepsilon B(u_{-N}, \hat{u})$$  \hspace{1cm} (B.6b)

$$= \omega(u_{-N}, \hat{u}),$$

where the $A_i$'s and $\omega$ are formal infinite series in powers of $\varepsilon$.

The Bogoliubov transformation (B.5) is such that the Hamiltonian (B.1), when expressed in terms of the new variables, is a periodic function of $v_N$. This fact permits us to prove the following theorem:

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44 We shall use this convention throughout this appendix.
Theorem B.1. The Hamiltonian \( (B.1) \), when expressed in terms of the \( v \)'s, is independent of \( v_N \) to all orders in the perturbation theory.

Proof. Since the Hamiltonian \( H = h(v) \) is autonomous its time derivative must vanish. Thus

\[
\dot{h} = \sum_{j=-N}^{N-1} v_j \dot{v}_j h + v_N \dot{v}_N h
\]

\( = 0. \) \hfill (B.7)

If follows from (B.6) and (B.7) that

\[
\partial_{v_N} h = -\varepsilon^{n-1} (v_{-N}, \dot{v}) \sum_{j=-N}^{N-1} A_j (v_{-N}, \dot{v}) \partial_{v_j} h. \]

Equation (B.8) is of the same type as (3.37). Thus if \( h(v) \) is independent of \( v_N \) through order \( \varepsilon^n \) then it is independent of \( v_N \) through order \( \varepsilon^{n+1} \). It is clear from (B.1) and (B.5) that \( h \) is independent of \( v_N \) through order \( \varepsilon^0 \). Hence, by mathematical induction, \( h(v) \) is independent of \( v_N \) to all orders in the perturbation theory. The theorem is therefore proved.

It is clear that Theorem B.1 is valid for any function \( h(v) \) which is periodic in \( v_N \) and which satisfies an equation similar to (B.8). More precisely let \( M(v) \) be a periodic function of \( v_N \) and be independent of \( v_N \) through order \( \varepsilon^0 \). For such a function we have the following theorem:

Theorem B.2. If a function \( M(v) \) satisfies the differential equation

\[
\partial_{v_N} M = P(v_{-N}, \dot{v}) + \varepsilon \text{IM}, \]

where \( P \) is independent of \( v_N \) and \( L \) is some linear operator which is independent of \( v_N \), then \( M(v) \) is independent of \( v_N \) to all orders in the perturbation theory.

Theorem B.1 has an interesting corollary which we may state as follows:

Corollary B.1. If the transformation (B.5) can be made canonical with \( v_1 \) and \( v_{-1} \) as conjugate variables then \( v_N \) will be constant to all orders in the perturbation theory.

In the next section we shall prove that the transformation (B.5) can be made canonical to all orders in the perturbation theory. Thus, because of
Corollary B.1, the number of differential equations which must be solved is reduced. The fact that we are able to construct a quantity which is constant to an arbitrarily high power of $\epsilon$ may also have some practical applications. One possible application lies in the design of controlled fusion devices where the first order constancy of the magnetic moment is often employed. For some devices it may be necessary to employ a quantity which is constant to second or even higher order; Corollary B.1 allows us to construct such a quantity.

**B.2 CANONICAL PERTURBATION THEORY**

The formalism of the Bogoliubov method places certain restrictions on the Bogoliubov variables. In this section we shall demonstrate that these restrictions are not incompatible with the conditions that the variables be canonical. We shall show that we can satisfy the appropriate Poisson bracket relations and we shall examine to what extent the canonical transformation is arbitrary.

If the Bogoliubov variables are to be canonical then they must satisfy the following Poisson bracket relations to all orders in the perturbation theory:

$$ (v_i, v_j) = \text{sgn} \delta_{i,-j}, \quad (B.10) $$

where the brackets are formed with respect to any complete set of canonical variables.

We have observed in Appendix A that the Bogoliubov formalism uniquely determines the $v_N$ dependence of the $n'$th order functions in the transformation (assuming the lower order functions are known). The zero harmonics of the $n'$th order functions are, however, arbitrary. Suppose that we have determined the transformation functions so that the transformation is canonical through order $c^{n-1}$. These transformation functions will uniquely determine the $v_N$ dependence of the $n'$th order transformation functions. There is, therefore, no hope of satisfying conditions (B.10) unless the choice of canonical lower order functions implies that the $n'$th order contributions to the left hand sides of (B.10) are independent of $v_N$. If this is so then we may select the arbitrary zero harmonics of the $n'$th order functions so that conditions (B.10) are satisfied through $n'$th order.

We shall now prove that, no matter how we choose the zero harmonics, the left hand sides of (B.10) are independent of $v_N$ to all orders in the perturbation theory. In order to do this we construct a $(2N-1) \times (2N-1)$ dimensional matrix $L$ whose $i-j$'th component $L_{i,j}$ is

$$ L_{i,j} = (v_i, v_j) \quad (i,j \neq N). \quad (B.11) $$
We also construct a \((2N-1)\times(2N-1)\) dimensional matrix \(M\) whose \(i,j\)'th component \(M_{ij}\) is

\[
M_{ij} = \partial_{v_i} A_j \quad (i, j \neq N) \tag{B.12}
\]

The rows and columns of these matrices are labeled as follows: \(i, j = N-1, N-2, \ldots, 1\). The time derivative of \(L_{ij}\) is

\[
\dot{L}_{ij} = (\dot{v}_i, v_j) + (v_i, \dot{v}_j) \tag{B.13}
\]

\[
= (\varepsilon A_i, v_j) + (v_i, \varepsilon A_j).
\]

We can rewrite (B.13) in the form \(^{45}\)

\[
\dot{L}_{ij} = \varepsilon \sum_{m=N-1}^{-N} \partial_{v_m} A_i(v_m, v_j) + \varepsilon \sum_{m=N-1}^{-N} \partial_{v_m} A_j(v_i, v_m). \tag{B.14}
\]

It is clear from (B.14) that

\[
\sim \quad L = \varepsilon M L + \varepsilon LM, \tag{B.15}
\]

where \(\sim\) is the transposed of the matrix \(M\). It must also be true that

\[
\dot{L} = \omega \partial_{v_N} L + \varepsilon \sum_{i=-N}^{-1} A_i \partial_{v_i} L \tag{B.16}
\]

It follows from (B.15) and (B.16) that

\[
\partial_{v_N} L = \varepsilon \omega^{-1} \left\{ \tilde{M} L + LM - \sum_{i=-N}^{-1} A_i \partial_{v_i} L \right\}. \tag{B.17}
\]

^{45}Goldstein, op cit., Eq. (8.50), p. 254.
The Bogoliubov transformation is the identity transformation to lowest order. The matrix $L$ is, therefore, independent of $v_N$ through order $\epsilon^0$. Since $L$ must be a periodic function of $v_N$ and since equation (B.17) is of the same type as equation (B.9) we conclude that $L$ is independent of $v_N$ to all orders in the perturbation theory.

We must now consider those brackets which involve the angle variable $v_N$. In order to do this we construct the $(2N-1)\times 1$ dimensional matrix $R$ whose $i$'th component $R_{i1}$ is

$$R_{i1} = (v_i, v_N) \quad (i \neq N).$$  \hfill (B.18)

We also construct the $(2N-1)\times 1$ dimensional matrix $S$ whose $j$'th component $S_{j1}$ is

$$S_{j1} = \partial_{v_j} \omega \quad (j \neq 1).$$  \hfill (B.19)

The time derivative of $R_{i1}$ is

$$\dot{R}_{i1} = (\dot{v}_i, v_N) + (v_i, \dot{v}_N)$$  \hfill (B.20)

$$= (\epsilon A_i, v_N) + (v_i, \omega).$$

We can rewrite (B.20) in the form

$$\dot{R}_{i1} = \epsilon \sum_{m=N-1}^{-N} (v_m, v_N) \partial_{v_m} A_i + \sum_{m=N-1}^{-N} (v_i, v_m) \partial_{v_m} \omega.$$  \hfill (B.21)

It follows from (B.21) that

$$\ddot{R} = \epsilon M R + LR.$$  \hfill (B.22)

We must also have

$$\dot{R} = \omega \partial_{v_N} R + \epsilon \sum_{i=N-1}^{-N} A_i \partial_{v_i} R.$$  \hfill (B.23)
Thus

\[ \dot{v}_N^R = L_0 + \varepsilon \left\{ M R - \sum_{i=N-1}^{-N} A_i \partial v_i \right\} . \quad (B.24) \]

The matrix \( L_0 \) is independent of \( v_N \) to all orders. The matrix \( R \) must be a periodic function of \( v_N \). Since \( R \) is independent of \( v_N \) through order \( \varepsilon^0 \), and since (B.24) is of the same type as (B.9) we conclude that \( R \) is independent of \( v_N \) to all orders in the perturbation theory.

We have demonstrated above that, no matter how the zero harmonics are selected, the left hand sides of (B.10) are independent of \( v_N \) to all orders in the perturbation theory. This allows us to select the arbitrary zero harmonics so that conditions (B.10) are satisfied order by order. It is, however, inconvenient to work with (B.10) since, in order to do so, we must invert the Bogoliubov transformation. To see how we can avoid having to do this let us construct the \( 2N \times 2N \) dimensional matrix \( P \) whose \( i-j' \)th component \( P_{ij} \) is

\[ P_{ij} = (v_i, v_j) (i, j = -N, \ldots, N) . \quad (B.25) \]

The matrix \( P \) is independent of \( v_N \) to all orders, therefore the inverse \( P^{-1} \) of \( P \) is also independent of \( v_N \) to all orders. It is a well known fact\(^{46}\) that \( P^{-1} \) is the negative of the matrix of the corresponding Lagrange brackets. We can, therefore, express the conditions that the transformation be canonical in terms of the Lagrange brackets and thereby avoid having to invert the transformation. A still more compact way to obtain the conditions to be placed on the zero harmonics is to evaluate the Poisson brackets of the original variables in terms of the Bogoliubov variables. This approach is legitimate since the inverse of a contact transformation is itself a contact transformation.

We shall use the latter of the above methods and, therefore, shall work with the following Poisson bracket relations:

\[ (u_i, u_j) = (v_i, v_j) + (\varepsilon F_i, v_j) + (v_i, \varepsilon F_j) + (\varepsilon F_i, \varepsilon F_j) , \quad (B.26) \]

where the brackets are formed with respect to the complete set of Bogoliubov variables.

When we say that the transformation is canonical through order \( \varepsilon^n \) we mean that

\(^{46}\) Goldstein, op cit., p. 252.
\[
\left( \sum_{m=1}^{n} \epsilon_{F_i}^m (v_j), v_i \right) + \left( v_i, \sum_{m=1}^{n} \epsilon_{F_j}^m (m) \right) + \left( \sum_{m=1}^{n-1} \epsilon_{F_i}^m (m) , \sum_{m=1}^{n-1} \epsilon_{F_j}^m (m) \right) = 0 ,
\]

(B.27)

where the prime on the final bracket excludes all terms of order greater than \( \epsilon^n \). Suppose that we know that the transformation is canonical through order \( \epsilon^{n-1} \), what is the requirement that it is canonical through order \( \epsilon^n \)? In order to answer this question we rewrite (B.27) in the following way:

\[
\left( \epsilon_{F_i}^{(n)} v_j \right) + \left( v_i, \epsilon_{F_j}^{(n)} \right) + \sum_{r+s=n} \left( \epsilon_{F_i}^r v_j, \epsilon_{F_j}^s \right) + \left\{ \left( \sum_{m=1}^{n-1} \epsilon_{F_i}^m (m) v_j \right) + \left( v_i, \sum_{m=1}^{n-1} \epsilon_{F_j}^m (m) \right) + \left( \sum_{m=1}^{n-2} \epsilon_{F_i}^m (m) , \sum_{m=1}^{n-2} \epsilon_{F_j}^m (m) \right) \right\} = 0.
\]

(B.28)

Since the expression in curly brackets vanishes by assumption, equation (B.28) reduces to

\[
\left( \epsilon_{F_i}^{(n)} v_j \right) + \left( v_i, \epsilon_{F_j}^{(n)} \right) + \sum_{r+s=n} \left( \epsilon_{F_i}^r v_j, \epsilon_{F_j}^s \right) = 0 .
\]

(B.29)

The summation in (B.29) depends only upon the lower order functions and is, therefore, known.

If we denote the zero harmonic of \( \epsilon_{F_i}^{(n)} \) by \( f_i^{(n)} \) and denote the average over \( v_N \) by the symbol \( <> \), then we may rewrite (B.29) as

\[
\text{sgn} - j \partial_{v_j} f_i^{(n)} + \text{sgn} \partial_{v_i} f_j^{(n)} = - \sum_{r+s=n} \left( \epsilon_{F_i}^r v_j, \epsilon_{F_j}^s \right) .
\]

(B.30)

In order to obtain (B.30) we have used the fact that\(^7\)

\[
\left( v_i, \epsilon_{F_j}^{(n)} \right) = \text{sgn} \partial_{v_i} f_j^{(n)} .
\]

(B.31)

The functions \( f_i^{(n)} \) must satisfy equations (B.30) order by order beginning with \( n=1 \) if the transformation is to be canonical.

The equations which determine the function \( f_{-N}^{(n)} \) are the easiest members of (B.30) to discuss. These equations are

\(^7\)Goldstein, op cit., Eq. (8.51), p. 254.

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\[ \text{sgn} \cdot \partial_{v_j} f_{-N}^{(n)} = - \sum_{r+s=n} \left( f_{-N}^{(r)} , f_{j}^{(s)} \right) >. \] (B.32)

They are integrable if and only if

\[ \partial_{v_i} \partial_{v_j} f_{-N}^{(n)} = \partial_{v_j} \partial_{v_i} f_{-N}^{(n)} \quad (i, j \neq N) \] (B.33)

It is interesting to verify that these conditions are satisfied since it illustrates nicely the self-consistency of the perturbation theory. We find from (B.32) that

\[ \partial_{v_i} \partial_{v_j} f_{-N}^{(n)} = - \text{sgn} j \sum_{r+s=n} \left( \partial_{v_i} F_{-N}^{(r)} , F_{-j}^{(s)} \right) + \left( F_{-N}^{(r)} , \partial_{v_i} F_{-j}^{(s)} \right) >. \] (B.34)

and

\[ \partial_{v_j} \partial_{v_i} f_{-N}^{(n)} = - \text{sgn} i \sum_{r+s=n} \left[ \left( \partial_{v_j} F_{-N}^{(r)} , F_{-i}^{(s)} \right) + \left( F_{-N}^{(r)} , \partial_{v_j} F_{-i}^{(s)} \right) \right] >. \] (B.35)

Thus

\[ \text{sgn} i \text{sgn} j \left( \partial_{v_i} \partial_{v_j} f_{-N}^{(n)} - \partial_{v_j} \partial_{v_i} f_{-N}^{(n)} \right) = \] (B.36)

\[ = \sum_{r+s=n} \left\{ \left( F_{-N}^{(r)} , -\text{sgn} i \partial_{v_i} F_{-j}^{(s)} + \text{sgn} j \partial_{v_j} F_{-i}^{(s)} \right) + \left( -\text{sgn} i \partial_{v_i} F_{-N}^{(r)} , F_{j}^{(s)} \right) + \left( \text{sgn} j \partial_{v_j} F_{-N}^{(r)} , F_{-i}^{(s)} \right) \right\} >. \]

Upon using the fact that the lower order functions must also satisfy (B.30), we find that (B.36) can be rewritten as

\[ \text{sgn} i \text{sgn} j \left( \partial_{v_i} \partial_{v_j} f_{-N}^{(n)} - \partial_{v_j} \partial_{v_i} f_{-N}^{(n)} \right) = \] (B.37)

\[ = \sum_{r+s=n} \left\{ \left( F_{-N}^{(r)} , - \sum_{p+q=r} \left( F_{-i}^{(p)} , F_{-j}^{(q)} \right) + \sum_{p+q=s} \left( F_{-i}^{(p)} , F_{-N}^{(q)} \right) + \partial_{v_i} F_{-N}^{(r)} \right) + \left( F_{-i}^{(s)} , \sum_{p+q=r} \left( F_{-N}^{(p)} , F_{-j}^{(q)} \right) - \partial_{v_j} F_{-N}^{(r)} \right) \right\} >. \]
If we make use of the Jacobi identity

\[ (u, (v,w)) + (v, (w,u)) + (w, (u,v)) = 0 \]  \hspace{1cm} (B.38)

and the fact that the summation indices are such that

\[ p + q + r = n, \]  \hspace{1cm} (B.39a)
\[ p + q + s = n. \]  \hspace{1cm} (B.39b)

then equations (B.37) reduce to

\[
\text{sgn}_i \text{sgn}_j (\partial_{v_i} \partial_{v_j} f_{-N}^{(n)} - \partial_{v_i} \partial_{v_j} f_{-N}^{(n)}) = \\
= \sum_{r+s=n} \left\{ (F_{i}^{(r)}, F_{-i}^{(s)}) + (F_{-i}^{(r)}, F_{i}^{(s)}) \right\} \\
= \sum_{r+s=n} \partial_{v_i} (F_{-i}^{(r)}, F_{i}^{(s)}) > \\
= 0.
\]  \hspace{1cm} (B.40)

We conclude that the function \( f_{-N}^{(n)} \) is uniquely determined, apart from an additive constant, by the lower order functions. We shall see that the functions \( f_{-N}^{(m)}(i \neq -N) \) are not unique. Thus there will be many ways of specifying the lower order functions and, as a result, many possible functions \( f_{-N}^{(n)} \). In the one dimensional case, however, the functions \( f_{-N}^{(n)} \) will be uniquely determined to all orders. This is in agreement with the results of Chapter 3 where it was demonstrated that \( K \) could be canonical only if it were equal to the classical action variable (plus an arbitrary constant).

Let us now assume that the function \( f_{N}^{(n)} \) is known. We can integrate (B.30) to obtain

\[
f_{i}^{(n)} = \int \partial_{v_i} f_{N}^{(n)} dv_{-N} \text{sgn} + \int \sum_{r+s=n} (F_{i}^{(r)}, F_{N}^{(s)}) dv_{-N} \\
+ d_{i}(\wedge) \quad (i \neq N,-N) \]  \hspace{1cm} (B.41)
where the $d_i$'s are yet to be determined. When we substitute the right hand side of (B.41) into (B.30) we observe that the dependence on the function $f^{(n)}_N$ vanishes identically. The function $f^{(n)}_N$ is, therefore, arbitrary and may, for convenience, be set equal to zero. Once the function $f^{(n)}_N$ is chosen it follows from (B.41) that the $v_N$ dependence of the $f^{(n)}_i$'s ($i \neq N,-N$) is uniquely determined.

We can verify that the solutions (B.41) are correct by substituting them into (B.30). When we do this we find that

\[
\text{sgn}-j \partial_{v_j} d_i^{(n)} + \text{sgn} \partial_{v_{-i}} d_j^{(n)} < \sum_{r+s=n} \left\{ (\partial_{v_j} F_i^{(r)} , F_i^{(s)}) + (F_i^{(r)}, \partial_{v_j} F_i^{(s)}) \right\} dv_N + \text{sgn} \left< \sum_{r+s=n} \left\{ (\partial_{v_{-i}} F_j^{(r)} , F_j^{(s)}) + (F_j^{(r)}, \partial_{v_{-i}} F_j^{(s)}) \right\} dv_N = - < \sum_{r+s=n} (F_i^{(r)}, F_j^{(s)}) > .
\]  

(B.42)

Using the fact that the lower order functions satisfy (B.30) we can rewrite (B.42) in the form

\[
\text{sgn}-j \partial_{v_j} d_i^{(n)} + \text{sgn} \partial_{v_{-i}} d_j^{(n)} + j < \sum_{r+s=n} \left\{ (- \sum_{p+q=r} (F_i^{(p)}, F_j^{(q)}), F_j^{(s)}) + (F_i^{(r)}, \sum_{p+q=r} (F_{-i}^{(p)}, F_{-j}^{(q)})) \right\} dv_N = - < \sum_{r+s=n} (F_i^{(r)}, F_j^{(s)}) >
\]  

(B.43)

Upon using the Jacobi identity we find that

\[
\text{sgn}-j \partial_{v_j} d_i^{(n)} + \text{sgn} \partial_{v_{-i}} d_j^{(n)} = < \sum_{r+s=n} (F_i^{(r)}, F_j^{(s)}) >
\]

(B.44)

\[- j < < \sum_{r+s=n} \left\{ (\partial_{v_{-N}} F_i^{(r)} , F_i^{(s)}) + (F_i^{(r)}, \partial_{v_{-N}} F_i^{(s)}) \right\} > dv_N
\]

or

\[
\text{sgn}-j \partial_{v_j} d_i^{(n)} + \text{sgn} \partial_{v_{-i}} d_j^{(n)} = < \sum_{r+s=n} (F_i^{(r)}, F_j^{(s)}) > ,
\]  

(B.45)

where the bar indicates that part of the function which is independent of $v_N$. Thus equations (B.41) do give the correct $v_N$ dependence.
The $d_1^{(n)}$'s are found by solving the system of equations (B.45). This system does not have a unique solution since a solution of the homogeneous system is given by

$$d_1^{(n)} = \text{sgn} \, \delta_{\nu \lambda} \, U(\hat{\nu}),$$  \hspace{1cm} (B.46)

where $U(\hat{\nu})$ is any function of $\hat{\nu}$.

We can now summarize the properties of the canonical Bogoliubov transformation. We shall assume that the transformation has been made canonical through order $\epsilon^{n-1}$. The zero harmonic $f_0^{(n)}$ is uniquely determined. The zero harmonic $f_{-N}^{(n)}$ is completely arbitrary. The zero harmonics $f_1^{(n)}(i \neq N, -N)$ have their $\nu_{-N}$ dependence uniquely determined once the function $f_N$ is chosen. The $\nu$ dependence of the $f_1^{(n)}$'s $(i \neq N, -N)$ is determined by equations (B.45). These equations have an infinite number of solutions.

### B.3 SYSTEMS HAVING SEVERAL PERIODICITIES

According to Corollary B.1 we can apply the Bogoliubov method to a nearly periodic system and thereby eliminate one of the degrees of freedom. Suppose that the remaining degrees of freedom possess a periodicity or near periodicity. We can then re-apply the Bogoliubov method to eliminate another degree of freedom. This process can be continued until all the periodicities or near periodicities are removed.

As an illustration of these ideas let us consider a system of $N$ degrees of freedom which is multiply periodic in the unperturbed state. We shall assume that the unperturbed frequencies $\omega_1^0, \ldots, \omega_N^0$ satisfy the following approximate equalities:

$$\frac{\omega_i^0}{\omega_1^0} \approx \epsilon, \ldots, \frac{\omega_N^0}{\omega_{N-1}^0} \approx \epsilon. $$  \hspace{1cm} (B.47)

The unperturbed state is described by the action and angle variables $J_1, \gamma_1 (i = 1, \ldots, N)$ and by the Hamiltonian

$$H = H_0(J_1) + \epsilon H_1(J_1, J_2) + \ldots + \epsilon^{N-1} H_{N-1}(J_1, J_2, \ldots, J_N).$$  \hspace{1cm} (B.48)
We shall assume that the perturbation contributes the following terms to the Hamiltonian

\[
\sum_{n=N}^{R} \epsilon^n H_n(J, \gamma) . \quad (B.49)
\]

The equations of motion are

\[
\dot{J}_1 = - \sum_{n=N}^{R} \epsilon^n \partial_{J} H_n , \quad (B.50a)
\]

\[
\dot{\gamma}_j = \sum_{n=3,1}^{R} \epsilon^n \partial_{J} H_n , \quad (j \neq 1) \quad (B.50b)
\]

\[
\dot{\gamma}_1 = \omega_1 ^{(1)}(J_1) + \sum_{n=1}^{R} \epsilon^n \partial_{J} H_n , \quad (B.50c)
\]

Equations (B.50) are suitable for the application of the Bogoliubov method since the \( H_n \)'s are periodic functions of the \( \gamma \)'s. As usual we make a canonical transformation

\[
J_1 = K_1^{(1)} + \epsilon F_1^{(1)}(K, \vartheta) , \quad (B.51a)
\]

\[
\gamma_1 = \varphi_1^{(1)} + \epsilon G_1^{(1)}(K, \vartheta) , \quad (B.51b)
\]

where \( K_1^{(1)} \) and \( \varphi_1^{(1)} \) are conjugate variables. The Hamiltonian when written in terms of the new variables, takes the form

\[
H = H_0(K_1^{(1)}, \varphi_1^{(1)}) + \epsilon h_1(K_1^{(1)}, K_2^{(1)}, \ldots) + \epsilon^{N-1} h_{N-1}(K_1^{(1)}, \ldots, K_N^{(1)})
\]

\[
+ \sum_{n=N}^{\infty} \epsilon^n h_n(K^{(1)}, \vartheta^{(1)}). \quad (B.52)
\]

This Hamiltonian is independent of \( \vartheta_1^{(1)} \). The equations of motion for the new variables have the form

\[
\dot{K}_1^{(1)} = - \sum_{n=N}^{\infty} \epsilon^n \partial_{K_1^{(1)}} h_n \quad (i=2, \ldots, N) \quad (B.53a)
\]
\[ \dot{\phi}_j = \sum_{n=j-1}^{\infty} \varepsilon^n \phi_{K_j} (1)^n (j = 3, 4, \ldots, N), \]  \hspace{1cm} (B.53b)

\[ \dot{\phi}_2 = \varepsilon \phi_{K_2} (1)^2 \sum_{n=2}^{\infty} \varepsilon^n \phi_{K_2} (1)^n. \]  \hspace{1cm} (B.53c)

Equations (B.53) are in a form to which the Bogoliubov method can be applied. Again we make a canonical transformation

\[ K_1^{(1)} = K_1^{(2)} + \varepsilon F_1^{(2)} (K_1^{(2)}, \phi_1^{(2)}), \]  \hspace{1cm} (B.54a)

\[ \phi_1^{(1)} = \phi_1^{(2)} + \varepsilon G_1^{(2)} (K_1^{(2)}, \phi_1^{(2)}), \]  \hspace{1cm} (B.54b)

where \( i = 2, 3, \ldots, N \). The Hamiltonian, when expressed in terms of these new variables will be independent of both \( \phi_1^{(1)} \) and \( \phi_2^{(2)} \). Thus both \( K_1^{(1)} \) and \( K_2^{(2)} \) constant. By applying the Bogoliubov method \( N \) times we can introduce \( N \) constants \( K_1^{(i)} \) such that the Hamiltonian depends only on these constants. The conjugate angle variables \( \phi_1^{(i)} \) will be linear functions of time. We conclude that the perturbed system remains multiply periodic in time to all orders in the perturbation theory. The variables \( K_1^{(1)} \) and \( \phi_1^{(1)} \) satisfy conditions (A) and (B) of section 2.2 and are, therefore, the classical action and angle variables.

Another interesting case occurs when the perturbation itself introduces the second periodicity or near periodicity. According to Corollary B.1 there will be a quantity which is constant to all orders in \( \varepsilon \) associated with this second near periodicity. This situation arises, for example, when a charged particle is trapped by a slowly varying magnetic field. Here we have two near periodicities: the rapid cyclotron motion, and the slower drift of the particle between mirror points. The lowest order contribution to the constant associated with the drift between mirror points is the so-called longitudinal invariant of Chew, Goldberger and Low.\(^{48}\)

APPENDIX C

PERTURBATION THEORY FOR SYSTEMS HAVING
SEVERAL RAPID PHASES

In this appendix we shall develop techniques for dealing with systems having several phases. As in the classical theory it will be necessary to distinguish between degenerate systems and non-degenerate systems. After formally presenting the methods we shall apply them to a class of Hamiltonian systems.

C.1 FORMAL PRESENTATION OF NON-DEGENERATE PERTURBATION THEORY

We consider a dynamical system whose state is characterized by the $s$ angle variables $\gamma_1, \ldots, \gamma_s$ and the $r$ variables $x_1, \ldots, x_r$, and whose temporal behavior is governed by the following system of autonomous differential equations:

\[
\dot{x}_k = \sum_{n=1}^{R} \varepsilon^{n_{A_k}(n)}(\gamma, x), \quad (C.1a)
\]

\[
\dot{\gamma}_1 = \omega_1(x) + \sum_{n=1}^{R} \varepsilon^{n_{B_1}(n)}(\gamma, x), \quad (C.1b)
\]

where $\varepsilon$ is a small parameter and the $A_k^{(n)}$'s and $B_1^{(n)}$'s are periodic functions of the angle variables $\gamma_j$ with period $2\pi$.

The formal perturbation theory consists of making the following change of variables:

\[
x_k = y_k + \sum_{n=1}^{\infty} \varepsilon^{n_{F_k}(n)}(\emptyset, y), \quad (C.2a)
\]

\[
\gamma_1 = \phi_1 + \sum_{n=1}^{\infty} \varepsilon^{n_{G_1}(n)}(\emptyset, y) \quad (C.2b)
\]

such that

\[
\dot{y}_k = \sum_{n=1}^{\infty} \varepsilon^{n_{a_k}(n)}(y), \quad (C.3a)
\]

\[
\dot{\phi}_1 = \omega_1(y) + \sum_{n=1}^{\infty} \varepsilon^{n_{b_1}(n)}(y). \quad (C.3b)
\]
The $F_k^{(n)}$'s and $G_1^{(n)}$'s are to be periodic functions of the $\phi_j$'s with period $2\pi$.

This perturbation theory is the obvious generalization of the Bogoliubov method. The physical meaning of the transformation lies in the decomposition of the true motion into an average motion described by $y$ and $\phi$ and fluctuations about this average described by the $F_k^{(n)}$'s and the $G_1^{(n)}$'s. As with the Bogoliubov method, the physical meaning of the transformation lies in the decomposition of the true motion into an average motion described by $y$ and $\phi$ and fluctuations about this average described by the $F_k^{(n)}$'s and the $G_1^{(n)}$'s. As with the Bogoliubov method the new perturbation formalism determines only the $\phi$ dependence of the $F_k^{(n)}$'s and $G_1^{(n)}$'s. The perturbation calculation is analogous to that of the Bogoliubov method.

In order to appreciate the limitations of non-degenerate perturbation theory let us consider the $n$'th order perturbation equation. In general, this equation will have the form

$$a_k^{(n)}(y) + \sum_{i=1}^{s} \omega_i \partial^{(n)}_i F_k = \sum_p x_p^{(n)}(y) e^{i(p_1\phi_1 + \ldots + p_s\phi_s)}.$$  \hspace{1cm} (C.4)

The right hand side of (C.4) is a known, finite Fourier series. Non-degenerate perturbation theory consists of gathering the zero harmonic of the right hand side into the function $a_k^{(n)}(y)$ and integrating the remaining equation. In so doing we find that

$$F_k^{(n)} = -i \sum_p \frac{x_p^{(n)}(y)}{p_1\omega_1 + \ldots + p_s\omega_s} e^{i(p_1\phi_1 + \ldots + p_s\phi_s)},$$  \hspace{1cm} (C.5)

where the prime on $x_p^{(n)}$ excludes the zero harmonic. We can also add an arbitrary function of $y$ to the right hand side of (C.5). It is clear that non-degenerate perturbation theory fails when the right hand side of (C.5) includes denominators of the form

$$p_1\omega_1 + \ldots + p_s\omega_s \leq O(\epsilon).$$  \hspace{1cm} (C.6)

If the first small divisor of the form (C.6) occurs at order $\epsilon^n$ then non-degenerate perturbation theory is valid through order $\epsilon^{n-1}$. The $n$'th order terms must, however, be obtained by degenerate perturbation theory.

C.2 FORMAL PRESENTATION OF DEGENERATE PERTURBATION THEORY

The small divisors of the previous section have occurred because we failed to make provisions for internal resonance. When phase relationships of the
type implied by (C.6) exist, it becomes fairly easy for the various degrees of freedom to exchange energy. These exchanges give rise to appreciable changes in the amplitudes of oscillation. The small divisors are simply a manifestation of this fact. It is clear that such resonant changes in amplitude should occur through the average motion and not through the fluctuating terms. In order to accomplish this we formulate degenerate perturbation theory as follows:

\[ x_k = y_k + \sum_{n=1}^{\infty} \epsilon_F^{(n)}(\phi, y), \quad (C.7a) \]

\[ y_1 = \phi_1 + \sum_{n=1}^{\infty} \epsilon_G^{(n)}(\phi, y) \quad (C.7b) \]

such that

\[ \dot{y}_k = \sum_{n=1}^{\infty} a_k^{(n)}(y, \theta(n)), \quad (C.8a) \]

\[ \dot{\phi}_1 = \omega_1(y) + \sum_{n=1}^{\infty} b_1^{(n)}(y, \theta(n)) \quad (C.8b) \]

The \( F_k \)'s and \( G_i \)'s are to be periodic functions of the \( \phi_i \)'s with period \( 2\pi \). The functions \( a_k^{(n)} \) and \( b_1^{(n)} \) are to contain all the zero harmonics and the slowly varying harmonics of the \( n \)'th order perturbation equation. These slowly varying harmonics are represented by the angles \( \theta(n) \). The functions \( a_k^{(n)} \) and \( b_1^{(n)} \) will be periodic functions of the \( \phi(n) \)'s.

Degenerate perturbation theory is the most general of the techniques which we have discussed; it includes the Bogoliubov method and non-degenerate perturbation theory as special cases.

C.3 APPLICATION OF NON-DEGENERATE PERTURBATION THEORY TO HAMILTONIAN SYSTEMS

In this section we shall apply non-degenerate perturbation theory to a Hamiltonian system which is nearly periodic in several angle variables. Let these angle variables be designated by \( u_i \) (\( i = r+1, \ldots, N \)) and their conjugate momenta by \( u_{-i} \) (\( i = r+1, \ldots, N \)). A complete specification of the system will require the \( r \) additional coordinates \( u_j \) (\( j = 1, \ldots, r \)) and their conjugate momenta \( u_{-j} \) (\( j = 1, \ldots, r \)). We shall assume that the systems Hamiltonian is of the form

\[ H = H_0(u_{-r+1}, \ldots, u_N) + \sum_{n=1}^{R} \epsilon_H^n(u), \quad (C.9) \]
where the $H_n$'s are periodic functions of the $u_i$'s ($i = r+1, \ldots, N$).

The equations of motion are

\[
\dot{u}_k = \text{sgn } k \sum_{n=1}^{R} \epsilon H_{u_k} H_n \quad (k = -N, \ldots, r),
\]

\[
\dot{u}_j = \omega_j^0 \sum_{n=1}^{R} \epsilon H_{u_j} H_n \quad (j = r+1, \ldots, N),
\]

where

\[
\omega_j^0 = \partial_{u_j} H_0 \quad (j = r+1, \ldots, N).
\]

Equations (C.10) are a special case of equations (C.1) and can, therefore, be treated by non-degenerate perturbation theory. In order to do so we make the following changes of variables:

\[
u_p = u_p^0 + \epsilon F_p(v) \quad (p = -N, \ldots, N)
\]

such that

\[
\dot{v}_k = \epsilon \partial_v \{v_{-N}, \ldots, v_r\} \quad (k = -N, \ldots, r)
\]

\[
\dot{v}_j = \omega_j^0(v) + \epsilon b_j(v_{-N}, \ldots, v_r) \quad (j = r+1, \ldots, N)
\]

\[= \omega_j(v_{-N}, \ldots, v_r),\]

where $F_p$, $a_k$, and $\omega_j$ are formal infinite series in powers of $\epsilon$ with the $F_p$'s periodic functions of the $v_j$'s ($j = r+1, \ldots, N$).

The transformation (C.12) is such that the Hamiltonian, when expressed in terms of the $v$'s, is a periodic function of the $v_j$'s ($j = r+1, \ldots, N$). This fact permits us to prove the following theorem:

**Theorem C.1.** The Hamiltonian (C.9), when expressed in terms of the $v$'s, is independent of the $v_j$'s ($j = r+1, \ldots, N$) to all orders in the perturbation theory.

**Proof.** Since the Hamiltonian $H = h(v)$ is autonomous its time derivative must vanish. Thus
\[ h = \sum_{k=-N}^{r} \dot{v}_k \partial v_k h + \sum_{j=r+1}^{N} \dot{v}_j \partial v_j h \]  
\[ = \varepsilon \sum_{k=-N}^{r} a_k \partial v_k h + \sum_{j=r+1}^{N} \omega_j \partial v_j h \]  
\[ = 0 . \]  

If follows from (C.14) that
\[ \sum_{j=r+1}^{N} \omega_j \partial v_j h = -\varepsilon \sum_{k=-N}^{r} a_k \partial v_k h . \]  

Let us now suppose that \( h(v) \) is known to be independent of the \( v_j \)'s (\( j = r+1, \ldots, N \)) through order \( \varepsilon^n \). Thus we may write
\[ h = h_0(v_{-N}, \ldots, v_r) + \varepsilon^{n+1} h_1(v) . \]

Upon substituting (C.16) into the right hand side of (C.15) we find
\[ \sum_{j=r+1}^{N} \omega_j \partial v_j h = -\varepsilon \sum_{k=-N}^{r} a_k (\partial v_k h_0 + \varepsilon^{n+1} \partial v_k h_1) . \]

The left hand side of (C.17) contains no zero harmonic in the \( v_j \)'s, hence the right hand side must contain no zero harmonic. This means that
\[ \sum_{k=-N}^{r} a_k \partial v_k h_0 = 0 , \]  
\[ \sum_{j=r+1}^{N} \omega_j \partial v_j h = 0(\varepsilon^{n+2}) . \]  

Since there is assumed to be no commensurabilities among the \( \omega_j \)'s, it follows from (C.19) that \( h(v) \) is independent of the \( v_j \)'s (\( j = r+1, \ldots, N \)) through order \( \varepsilon^{n+1} \).
It is clear from (C.9) and (C.12) that \( h(v) \) is independent of the \( v_j \)'s (\( j = r+1, \ldots, N \)) through order \( \epsilon^0 \). It follows by mathematical induction that \( h(v) \) is independent of the \( v_j \)'s (\( j = r+1, \ldots, N \)) to all orders in the perturbation theory. The theorem is therefore proved.

Clearly Theorem C.1 is not restricted to Hamiltonian functions. To be more precise let \( M(v) \) be a periodic function of the \( v_j \)'s (\( j = r+1, \ldots, N \)) and be independent of the \( v_j \)'s (\( j = r+1, \ldots, N \)) through order \( \epsilon^0 \). For such a function we have the following theorem:

**Theorem C.2.** If the function \( M(v) \) satisfies the following differential equation

\[
\sum_{j=r+1}^{N} \omega_j \partial_{v_j} M = F(v_{-N}, \ldots, v_r) + \epsilon L M
\]  

(C.20)

where \( P \) is independent of the \( v_j \)'s and \( L \) is some linear operator which is independent of the \( v_j \)'s, then \( M(v) \) is independent of the \( v_j \)'s (\( j = r+1, \ldots, N \)) to all orders in non-degenerate perturbation theory.

We can state the following corollary to Theorem C.1.

**Corollary C.1.** If the transformation (C.12) can be made canonical to all orders with \( v_1 \) and \( v_{-1} \) conjugate variables, then the \( v_k \)'s (\( k = -N, \ldots, -r-1 \)) will be constant to all orders in non-degenerate perturbation theory.

If the transformation is to be canonical then we must satisfy the following Poisson bracket relations to all orders in the perturbation theory:

\[
(v_i, v_j) = \text{sgn } i \delta_{i,-j} (i,j = -N, \ldots, N),
\]  

(C.21)

where the brackets are formed with respect to any complete set of canonical variables.

We have observed that the perturbation formalism allows us to add an arbitrary zero harmonic to the \( n \)'th order perturbation function \( F^{(n)}_p \). We shall now show that, no matter how we choose these zero harmonics, the left hand sides of (C.21) will be independent of the angle variables \( v_j (j = r+1, \ldots, N) \) to all orders in non-degenerate perturbation theory. In order to do this we construct the \( (N+r)x(N+r) \) dimensional matrix \( L \) whose \( i-k \)'th component \( L_{ik} \) is

\[
L_{ik} = (v_i, v_k) \ (i,k \neq r+1, \ldots, N).
\]  

(C.22)
We also construct the \((N+r)\times(N+r)\) dimensional matrix \(M\) whose \(i-k\)'th component \(M_{ik}\) is

\[
M_{ik} = \partial_{v_i} a_k (i,k \neq r+1, \ldots, N). \tag{C.23}
\]

The time derivative of the \(i-k\)'th component of \(L\) is

\[
\dot{L}_{ik} = (\dot{v}_i, v_k) + (v_i, \dot{v}_k) = (\epsilon a_i, v_k) + (v_i, \epsilon a_k). \tag{C.24}
\]

Equation (C.24) may be rewritten as\(^{9}\)

\[
\dot{L}_{ik} = \epsilon \sum_{m=-N}^{N} (v_m, v_k) \partial_{v_m} a_i + \epsilon \sum_{m=-N}^{N} (v_i, v_m) \partial_{v_m} a_k. \tag{C.25}
\]

It follows from (C.25) that

\[
\dot{L} = \tilde{M} L + \epsilon LM, \tag{C.26}
\]

where \(\tilde{M}\) is the transposed of the matrix \(M\). It must also be true that

\[
\dot{L} = \sum_{j=r+1}^{N} \omega_j \partial_{v_j} L + \epsilon \sum_{k=-N}^{N} a_k \partial_{v_k} L. \tag{C.27}
\]

We find from (C.26) and (C.27) that

\[
\sum_{j=r+1}^{N} \omega_j \partial_{v_j} L = \tilde{M} L + \epsilon LM - \epsilon \sum_{k=-N}^{N} a_k \partial_{v_k} L. \tag{C.28}
\]

The matrix \(L\) must be a periodic function of the angle variables \(v_j\); it is also independent of the \(v_j\)'s through order \(\epsilon^0\) since the transformation (C.12) is the identity transformation through order \(\epsilon^0\). Since (C.28) is the same type of equation as (C.20) it follows from Theorem C.2 that \(L\) is independent of the \(v_j\)'s \((j = r+1, \ldots, N)\) to all order in non-degenerate perturbation theory.

We must now investigate those brackets which involve the angle variables. To do this we construct the \((N+r)\times l\) dimensional matrix \(R^{(j)}\) whose \(i^{th}\) component \(R^{(j)}_{il}\) is

\[
R^{(j)}_{il} = (v_i, v_j) (i \neq r+1, \ldots, N) (j = r+1, \ldots, N) . \quad (C.29)
\]

We also construct the \((N+r)\times l\) dimensional matrix \(S^{(j)}\) whose \(i^{th}\) component \(S^{(j)}_{il}\) is

\[
S^{(j)}_{il} = \partial_{v_i} \omega_j (i \neq r+1, \ldots, N) (j = r+1, \ldots, N) . \quad (C.30)
\]

The time derivative of \(R^{(j)}\) is

\[
\dot{R}^{(j)}_{il} = (\dot{v}_i, v_j) + (v_i, \dot{v}_j)
\]

\[
= (e a_i , v_j) + (v_i, \omega_j)
\]

\[
= e \sum_{k=-N}^{r} (v_k, v_j) \partial_{v_i} a_k + \sum_{k=-N}^{r} (v_i, v_k) \partial_{v_i} \omega_j .
\]

It follows from \((C.31)\) that

\[
\dot{R}^{(j)} = e \tilde{N} R^{(j)} . \quad (C.32)
\]

It must also be true that

\[
\dot{R}^{(m)} = \sum_{j=r+1}^{N} \omega_j \partial_{v_j} R^{(m)} + e \sum_{k=-N}^{r} a_k \partial_{v_k} R^{(m)} . \quad (C.33)
\]

It follows from \((C.32)\) and \((C.33)\) that

\[
\sum_{j=r+1}^{N} \omega_j \partial_{v_j} R^{(m)} = e \tilde{M} R^{(m)} - e \sum_{k=-N}^{r} a_k \partial_{v_k} R^{(m)} . \quad (C.34)
\]

The matrix \(R^{(m)}\) is periodic in the angle variables and is independent of the angle variables through order \(e^0\). Thus it follows from Theorem C.2 that \(R^{(m)}\) is independent of the \(v_j\)'s \((j = r+1, \ldots, N)\) to all orders in the perturbation theory.
We must now consider the remaining case of the brackets \((v_i, v_j)\) \((i, j = r+1, \ldots, N)\). We have

\[
\frac{d}{dt} (v_i, v_j) = (\dot{v}_i, v_j) + (v_i, \dot{v}_j) \quad \text{ (C.35)}
\]

\[
= (\omega_i, v_j) + (v_i, \omega_j)
\]

\[
= \sum_{k=-N}^{r} (v_k, v_j) \partial_v \omega_i + \sum_{k=-N}^{r} (v_i, v_k) \partial_v \omega_j.
\]

Thus we see that

\[
\frac{d}{dt} (v_i, v_j) = \tilde{S}(\rho) \delta_s(1) - \tilde{S}(\rho) R_i(1). \quad \text{ (C.36)}
\]

It must also be true that

\[
\frac{d}{dt} (v_i, v_j) = \sum_{m=r+1}^{N} \omega_m \partial_v (v_i, v_j) + \epsilon \sum_{k=-N}^{r} a_k \partial_v (v_i, v_j). \quad \text{ (C.37)}
\]

We find from (C.36) and (C.37) that

\[
\sum_{m=r+1}^{N} \omega_m \partial_v (v_i, v_j) = \tilde{S}(\rho) \delta_s(1) - \tilde{S}(\rho) R_i(1) - \epsilon \sum_{k=-N}^{r} a_k \partial_v (v_i, v_j). \quad \text{ (C.38)}
\]

Equation (C.38) is of the same type as (C.20). Thus it follows from Theorem C.2 that \((v_i, v_j)\) \((i, j = r+1, \ldots, N)\) is independent of the \(v_m\)'s \((m = r+1, \ldots, N)\) to all orders in non-degenerate perturbation theory.
We have shown above that the left hand sides of (C.21) are independent of the \( v_j \)'s \((j = r+1, \ldots, N)\) to all orders in non-degenerate perturbation theory. We may, therefore choose the arbitrary zero harmonics so as to satisfy the equalities (C.21) order by order. It is, however, inconvenient to work with (C.21) since, in order to do so, we must invert the transformation (C.12). It is simpler to work with the Poisson brackets of the original variables with respect to the variables \( v \). This is legitimate since the inverse of a contact transformation is itself a contact transformation. We shall, therefore, work with the following Poisson bracket relations:

\[
(u_i, u_j) = (v_i, v_j) + (eF_i, v_j) + (v_i, eF_j) + (eF_i, eF_j),
\]

(C.39)

where \( i, j = -N, \ldots, N \) and where the brackets are formed with respect to the \( v \)'s.

The functions \( eF_i \) have the form

\[
eF_i = \sum_{n=1}^{\infty} e^F_i^{(n)}(v).
\]

(C.40)

If the transformation is to be canonical then the zero harmonics \( f_i^{(n)}(v_{-N}, \ldots, v_r) \) of the functions \( F_i^{(n)} \) must satisfy the following equations:

\[
\text{sgn} \sum_{-j} \partial_{-j} f_i^{(n)} + \text{sgn} \sum_{i} \partial_{i} f_j^{(n)} = -\langle\langle \sum_{p+q=n} F_i^{(p)}, F_j^{(q)} \rangle\rangle,
\]

(C.41)

where the symbol \( \langle\langle \rangle\rangle \) indicates the average over all the phases \( v_j (j = r+1, \ldots, N) \). In order to obtain (C.41) we have used the fact that

\[
(v_i, F) = \text{sgn} \sum_{-i} \partial_{v_{-i}} F.
\]

(C.42)

We shall now verify that the system of equations (C.41) is integrable. Let us first discuss the equations which determine the \( f_j^{(n)} \)'s \((j = r+1, \ldots, N)\). These equations are

\[
\text{sgn} \sum_{-i} \partial_{v_{-i}} f_k^{(n)} = -\langle\langle \sum_{p+q=n} F_i^{(p)}, F_k^{(q)} \rangle\rangle,
\]

(C.43)

where \( k = -N, \ldots, -r-1 \), and \( i = -N, \ldots, r \). The system (C.43) is integrable if and
only if
\[ \partial_{v_1} \partial_{v_j} f_k^{(n)} = \partial_{v_j} \partial_{v_1} f_k^{(n)}. \] (C.44)

From (C.43) we find that
\[ \partial_{v_1} \partial_{v_j} f_k^{(n)} = \text{sgn } j \sum_{p+q=n} \left[ (\partial_{v_i} F_{-j}^{(p)}, F_k^{(q)}) + (F_{-j}^{(p)}, \partial_{v_i} F_k^{(q)}) \right] \] (C.45)

and
\[ \partial_{v_j} \partial_{v_1} f_k^{(n)} = \text{sgn } i \sum_{p+q=n} \left[ (\partial_{v_j} F_{-i}^{(p)}, F_k^{(q)}) + (F_{-i}^{(p)}, \partial_{v_j} F_k^{(q)}) \right] . \] (C.46)

Upon subtracting (C.46) from (C.45) and multiplying the result by (sgn-i) (sgn-j) we find that
\[ (\text{sgn-i}) (\text{sgn-j}) (\partial_{v_1} \partial_{v_j} - \partial_{v_j} \partial_{v_1}) f_k^{(n)} = \] (C.47)
\[ = \sum_{p+q=n} \left[ (\text{sgn-i} \partial_{v_i} F_{-j}^{(p)}, F_k^{(q)}) + (F_{-j}^{(p)}, \text{sgn-i} \partial_{v_i} F_k^{(q)}) - (\text{sgn-j} \partial_{v_j} F_{-i}^{(p)}, F_k^{(q)}) - (F_{-i}^{(p)}, \text{sgn-j} \partial_{v_j} F_k^{(q)}) \right] . \]

We now use that fact that
\[ \text{sgn-j} \partial_{v_j} F_{-i}^{(p)} + \text{sgn i} \partial_{v_i} F_{-j}^{(p)} = - \sum_{s+b=p} (F_s, F_b) \] (C.48)

to rewrite (C.47) as
\[ (\text{sgn-i})(\text{sgn-j})(\partial_{v_1} \partial_{v_j} - \partial_{v_j} \partial_{v_1}) f_k^{(n)} = \] (C.49)
\[
- \ll \sum_{p+q=n} \left[ \left( \sum_{a+b=p} \left( F_{-i}^a, F_{j}^b \right) - \text{sgn}^{-j} \partial_{v_i} F_{-1}^p, F_k^q \right) \right. \\
+ \left( F_{-j}^p, \sum_{a+b=q} \left( F_{-i}^a, F_k^b \right) - \text{sgn}^{-k} \partial_{v_{-1}} F_{-i}^q \right) \\
- \left( \sum_{a+b=p} \left( F_{-j}^a, F_{-i}^b \right) - \text{sgn}^{i} \partial_{v_j} F_{-j}^p, F_k^q \right) \left. \right] \right] >> .
\]

A simple rearrangement of terms in (C.49) gives

\[
(\text{sgn}^{-i})(\text{sgn}^{-j})(\partial_{v_i} \partial_{v_j} - \partial_{v_j} \partial_{v_i}) f_{M}^{(n)} = \quad (C.50)
\]

\[
= - \ll \sum_{p+q=n} \left[ \left( \partial_{v_{-1}} F_{-1}^p, F_{-j}^q \right) + \left( F_{-1}^p, \partial_{v_{-1}} F_{-j}^q \right) \right] >> .
\]

The right hand side of (C.50) vanishes, hence we conclude that (C.44) is satisfied. It follows from this that the \(f_{M}^{(n)}\)'s (\(k=-N_{-1},..,-r-1\)) are uniquely determined, apart from an additive constant, by the lower order functions.

We now consider the equations which determine the remaining \(f_{i}^{(n)}\)'s (\(i=-r,..,N\)). These equations are

\[
\text{sgn}^{-j} \partial_{v_{-j}} f_{i}^{(n)} + \text{sgn}^{i} \partial_{v_{-1}} f_{j}^{(n)} = - a_{ij}^{(n)} \quad (C.51)
\]

where \(i,j=-r,..,N\) and where

\[
\epsilon_{ij}^{(n)} = - \ll \sum_{p+q=n} \left( F_{i}^p, F_{j}^q \right) >> .
\]

Let us now take

\[
f_{M}^{(n)} = \text{sgn} M \partial_{v_{-M}} U_{M}^{(n)}, \quad (C.53)
\]
where \( M \) is any one of \(-r, \ldots, N\), and where \( u^{(n)} \) is any continuous function of the \( v_m \)'s \((m=-N, \ldots, r)\). With this choice we find from (C.51) that

\[
f^{(n)}_j = \text{sgn } j \partial_{v_{-j}} u^{(n)} + \text{sgn } M \int a_{M_j}^{(n)} dv_M + d^{(n)}_j,
\]

where \( j \neq -M \) and where \( d^{(n)}_j \) is a function independent of \( v_M \). Upon substituting (C.54) into (C.51) we find that

\[
\text{sgn } j \text{sgn } M \int \partial_{v_{-j}} s_{M_i}^{(n)} dv_M + \text{sgn } j \partial_{v_{-j}} d^{(n)}_{i_j}
\]

\[
+ \text{sgn } i \text{sgn } M \int \partial_{v_{-i}} s_{M_j}^{(n)} dv_M + \text{sgn } i \partial_{v_{-i}} d^{(n)}_{j_i} = -a^{(n)}_{i_j}.
\]

We use the definition of \( s_{M_j}^{(n)} \) in order to rewrite (C.55) as follows:

\[
\text{sgn } M \int \left[ \sum_{p+q=n} \text{sgn } j \left\{ (\partial_{v_{-j}} F_{M}^{(p)}, F_{i}^{(q)}) + (F_{M}^{(p)}, \partial_{v_{-j}} F_{i}^{(q)}) \right\} \right] dv_M
\]

\[
+ \text{sgn } i \left\{ (\partial_{v_{-i}} F_{M}^{(p)}, F_{j}^{(q)}) + (F_{M}^{(p)}, \partial_{v_{-i}} F_{j}^{(q)}) \right\} dv_M
\]

\[
+ \text{sgn } j \partial_{v_{-j}} d^{(n)}_{i_j} + \text{sgn } i \partial_{v_{-i}} d^{(n)}_{j_i} = -a^{(n)}_{i_j}.
\]

We next use the fact that the \( F_{i}^{(p)} \)'s \((p < n)\) satisfy equations (C.48) in order to rewrite (C.56) as follows:

\[
\text{sgn } j \partial_{v_{-j}} d^{(n)}_{i_j} + \text{sgn } i \partial_{v_{-i}} d^{(n)}_{j_i} =
\]

\[
- \sum_{p+q=n} \left( F_{i}^{(p)}, F_{j}^{(q)} \right) + \int \sum_{p+q=n} \left[ (\partial_{v_{-M}} F_{i}^{(p)}, F_{j}^{(q)}) + (F_{i}^{(p)}, \partial_{v_{-M}} F_{j}^{(q)}) \right] dv_M.
\]

Hence
where the bar indicates that part of the function which is independent of ν_{-M}.

The system (C.58) is of the same type as (C.51). It has, however, one less variable, namely ν_{-M}. By repeating the above process we can eliminate another variable. A continuation of this process will eventually yield the complete solution for the Ψ_{j}^{(n)}'s (i=-r,...,N). We conclude that the system (C.41) is integrable.

Let us now suppose that r = 0. In other words the system is assumed to be completely described by the N angle variables ν_{i} and their conjugate momenta ν_{-i}. The Hamiltonian will depend only upon the ν_{-i}'s (i = 1,..., N). If we make the transformation canonical by satisfying equations (C.41) to all orders, then the ν_{-i}'s will be constants and the ν_{i}'s will be linear functions of time. The perturbed system, therefore, remains multiply periodic to all orders in the perturbation theory. It is a simple matter to show that, for this system, non-degenerate perturbation theory is identical with non-degenerate classical perturbation theory. In order to do this we first write down the classical solution. This has the form

\[ u_{i} = J_{i} + \epsilon \partial_{i} S(u_{k}, J), \quad (C.59a) \]

\[ \gamma_{i} = u_{i} + \epsilon \partial_{i} S(u_{k}, J), \quad (C.59b) \]

where J_{i} and γ_{i} are the true action and angle variables.

We can now express the J's and γ's in terms of the v's. In so doing we obtain the following relations:

\[ J_{i} = v_{-i} + \epsilon G(v_{-N},...,v_{-1}), \quad (C.60a) \]

\[ \gamma_{i} = v_{i} + \epsilon R_{i}(v). \quad (C.60b) \]

Equation (C.60a) holds because the J_{i} are constants. The classical method is such that

\[ \gamma_{i} = o_{i}^{(1)}(J) \quad (C.61) \]
while non-degenerate perturbation theory gives

\[ \dot{v}_i = \omega_1(v_{-N}, \ldots, v_{-1}) . \]  
(C.62)

Thus the time derivative of \( R_i \) is given by

\[ \dot{eR}_i = \omega^{(1)}_1(J(v_{-N}, \ldots, v_{-1})) - \omega_1(v_{-N}, \ldots, v_{-1}) \]  
(C.63)

and by

\[ \dot{eR}_i = \sum_{j=1}^{N} \omega_j \frac{\partial}{\partial v_j} R_i . \]  
(C.64)

From (C.63) and (C.64) we find that

\[ \sum_{j=1}^{N} \omega_j \frac{\partial}{\partial v_j} R_i = \omega^{(1)}_1 - \omega_1 . \]  
(C.65)

The right hand side of (C.65) is independent of the \( v_j \)’s (\( j = 1, \ldots, N \)). \( R_i \) must, however, be a periodic function of the \( v_j \)’s. Hence we conclude from (C.65) that \( R_i \) is independent of the \( v_j \)’s and that

\[ \omega^{(1)}_1 = \omega_1 . \]  
(C.66)

Equation (C.60b) becomes

\[ \gamma_i = v_i + eR_i(v_{-N}, \ldots, v_{-1}) . \]  
(C.67)

We now form the following Poisson brackets:

\[ (\gamma_i, \gamma_j) = \sum_{m=1}^{N} \left\{ \frac{\partial}{\partial v_m} \gamma_i \frac{\partial}{\partial v_{-m}} J_k - \frac{\partial}{\partial v_{-m}} \gamma_i \frac{\partial}{\partial v_m} J_k \right\} \]  
(C.68)

\[ = \delta_{im} \frac{\partial}{\partial v_{-m}} J_k , \]

where use has been made of equations (C.60a) and (C.67). It follows from (C.68) that the \( v_i \)'s and \( v_{-i} \)'s can be canonical only if
\[ J_1 = v_{-1} + C \]  \hspace{1cm} (C.69)

where \( C \) is a constant independent of the \( v_{-i} \)'s. We have demonstrated above that when the transformation (C.12) is canonical, the \( v_{-i} \)'s are uniquely determined (apart from an additive constant). Thus according to (C.69) non-degenerate perturbation theory leads to results identical with those of classical perturbation theory.

It must be emphasized that non-degenerate perturbation theory and classical perturbation theory are not the same. They do, however, lead to identical results when applied to a system to which the classical method is applicable. It must also be emphasized that the fact that we have been able to prove results which are true to all orders in the perturbation theory does not imply that the perturbation theory is valid to all orders. Non-degenerate perturbation is valid only until small divisors occur. If the first small divisor occurs at order \( \varepsilon^n \) then the results of this section are valid through order \( \varepsilon^{n-1} \). A more complete discussion of the asymptotic convergence of non-degenerate perturbation theory will be found in Appendix E.

### C.4 APPLICATION OF DEGENERATE PERTURBATION THEORY TO HAMILTONIAN SYSTEMS

In the case where the system is undergoing internal resonance we must use degenerate perturbation theory. Thus we make the change of variables

\[ u_i = v_i + \varepsilon F_i(v) \]  \hspace{1cm} (C.70)

such that

\[ \dot{v}_k = \varepsilon e_k(v_{-N}, \ldots, v_r, \theta) \hspace{1cm} (k = -N, \ldots, r), \]  \hspace{1cm} (C.71a)

\[ \dot{v}_j = \omega_j^0 + \varepsilon e_j(v_{-N}, \ldots, v_r, \theta) \hspace{1cm} (j = r+1, \ldots, N). \]  \hspace{1cm} (C.71b)

\[ = \omega_j(v_{-N}, \ldots, v_r, \theta) \]

The vector \( \theta \) will have the \( m \) components

\[ \Theta_i = p_{i1}v_{r+1} + \ldots + p_{iN-r}v_N \hspace{1cm} (i = i, \ldots, m), \]  \hspace{1cm} (C.72)

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where the $p_{ij}$'s are integers. Let us now assume that the resonance can be adequately described by the restriction

$$m < N - r.$$ \hfill (C.73)

In general this restriction will not be satisfied. In fact $m$ will usually be infinite. In practice, however, condition (C.73) will hold to a sufficiently high order.\(^{50}\) We will proceed as if condition (C.73) were true to all orders. The results which we obtain will be valid to as high an order as (C.73) can be maintained.

Let us introduce in addition to the angles $\Theta_i$ the angles

$$\mu_j = \nu_j \quad (j = r+m+1, \ldots, N). \hfill (C.74)$$

The angles $\nu_k$ ($k = r+1, \ldots, N$) can be expressed as linear combinations of the angles $\Theta_i$ and $\mu_j$. The perturbation theory is such that the Hamiltonian will be a periodic function of the $\nu_k$'s. It will also be a periodic function of the $\Theta_i$'s and the $\mu_j$'s. This fact allows us to prove the following theorem:

**Theorem C.3.** The Hamiltonian, when expressed in terms of the $\nu_k$'s $(k = -N, \ldots, r)$ and the $\Theta_i$'s and $\mu_j$'s, will be independent of the $\mu_j$'s to all orders in degenerate perturbation theory.

**Proof.** Since the Hamiltonian is autonomous its time derivative must vanish. Thus

$$\dot{h} = \dot{h}(v) = \sum_{j=r+1}^{N} \omega_j \partial_{\nu_j} h + \epsilon \sum_{k=-N}^{r} a_k \partial_{\nu_k} h = 0 \hfill (C.75)$$

It follows from (C.75) that

$$\sum_{j=r+1}^{N} \omega_j \partial_{\nu_j} h = -\epsilon \sum_{k=-N}^{r} a_k \partial_{\nu_k} h. \hfill (C.76)$$

The partial derivatives in (C.76), when written in terms of the new variables are

\(^{50}\) For a discussion of the two dimensional case see Chapter 4.
\[ \partial v_j = \sum_i p_{ij-r} \partial \Theta_i \quad (j=r+1, \ldots, r+m) \quad (C.77) \]

and

\[ \partial v_k = \partial \mu_k + \sum_i p_{ik-r} \partial \Theta_i \quad (k = r+m+1, \ldots, N) \quad (C.78) \]

If we now write the Hamiltonian in terms of the \( \Theta_i \)'s and \( \mu_j \)'s and make use of equations (C.77) and (C.78) then equation (C.76) becomes

\[ \sum_j \sum_i \omega_j p_{ij-r} \partial \Theta_i \quad L + \sum_{k=r+m+1}^N \omega_k \partial \mu_k \quad (C.79) \]

\[ \sum_i \omega_k p_{ik-r} \partial \Theta_i \quad L = -\epsilon \sum_{k=-N}^r a_k \partial v_k L, \]

where

\[ H = L(v_{-N}, \ldots, v_r, \Theta, \mu). \quad (C.80) \]

The frequencies \( \omega_j \) will be such that \( \omega_j = \omega_j^0(v_{-N}, \ldots, v_r) \) to lowest order.

Let us suppose that the Hamiltonian is known to be independent of \( \mu \) through order \( \epsilon^R \). This means that the right hand side of (C.79) will be independent of \( \mu \) through order \( \epsilon^{n+1} \). The left hand side of (C.79) must also be independent of \( \mu \) through order \( \epsilon^{n+1} \). Let the \( n+1 \)'st contribution to \( L \) be \( L^{n+1} \). The function \( L^{n+1} \) must be a periodic function of the \( \Theta_i \)'s and the \( \mu_j \)'s and, therefore, may be written as

\[ L^{n+1} = \sum_s L_s^{n+1} e^{iT(s_p \Theta_p + s_q \mu_q)}, \quad (C.81) \]

where \( T \) is determined by (C.72). Since \( L \) is assumed to be independent of \( \mu \) through order \( \epsilon^R \), the left hand side of (C.79) can depend upon \( \mu \) only in order \( \epsilon^{n+1} \). This \( \mu \) dependence will come from the term

\[ \sum_s L_s^{n+1} \left\{ \sum_i s_i \sum_{j=r+1}^N p_{ij-r} \omega_j^0 + \sum_{k=r+m+1}^N s_k \omega_k^0 \right\} e^{iT(s_p \Theta_p + s_q \mu_q)}. \quad (C.82) \]
We have observed that (C.82) must be independent of $\mu$. This will be true if the $s_q$'s are all zero or if the coefficient

$$\sum_{i=1}^{m} s_i \sum_{j=r+1}^{N} p_{ij} \o_j^0 + \sum_{k=r+m+1}^{N} s_k \o_k^0 = 0 \quad (C.83)$$

for all times. Equation (C.83) implies the existence of commensurabilities other than those listed in equations (C.72). The existence of such commensurabilities violates our initial assumptions. We must therefore exclude the possibility of satisfying equation (C.83). This means that the $s_q$'s must all vanish which implies that $L^{n+1}$ is independent of $\mu$.

It is clear that the Hamiltonian is independent of $\mu$ through order $\varepsilon^0$. Hence by mathematical induction we conclude that the Hamiltonian is independent of $\mu$ to all orders in the perturbation theory. The theorem is therefore proved.

In general the transformation (C.70) can not be made canonical to all orders in degenerate perturbation theory. The reason for this is that the matrix of Poisson brackets (C.22) will, in general, depend upon the $\Theta_1$'s just as the Hamiltonian depends upon the $\Theta_1$'s. Since degenerate perturbation theory uniquely determines the $\Theta_1$ dependence of the $n$'th order transformation functions, we are not able to eliminate the $\Theta$ dependence of the Poisson brackets. We can, however, proceed in a conditional fashion and assume that the transformation can be made canonical through order $\varepsilon^n$. The precise value of $n$ will depend upon the Hamiltonian in question. When we introduce the variables $I_p$ ($p=1,\ldots,m$) and $K_q$ ($q=r+m+1,\ldots,N$) which are conjugate to the $\Theta_p$'s and $\mu_q$'s and are related to the $v$'s by the generating function

$$F = \sum_{p=1}^{m} \sum_{j=r+1}^{N} I_p p_{pj} \o_j^0 + \sum_{k=r+m+1}^{N} K_k \o_k^0 \quad (C.84)$$

we have the following corollary to Theorem C.3:

**Corollary C.2.** If the transformation (C.70) can be made canonical through order $\varepsilon^n$, then the variables $K_k$ ($k = r+m+1,\ldots,N$) defined by (C.84) will be constant through order $\varepsilon^n$.

If it should happen that the degeneracy conditions

$$p_{11} \o_{r+1}^0 + \ldots + p_{IN-r} \o_N^0 = O(\varepsilon) \quad (i = 1,\ldots,m) \quad (C.85)$$

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remain valid for all values of the \( u_i \)'s (\( i=-N,\ldots,r \)), then the problem can be treated by non-degenerate perturbation theory. In order to do this we simply express the original problem in terms of the \( u_k \)'s (\( k=-r,\ldots,r \)) and the variables \( J_i \) and their conjugate coordinates \( \gamma_i \) (\( i=1,\ldots,N-r \)). These latter variables are related to the \( u \)'s by the generating function

\[
F = \sum_{i=1}^{m} \sum_{j=r+1}^{N} J_{ij} u_{i-j} + \sum_{k=r+1}^{N} u_k J_{k-r} . \tag{C.86}
\]

The equations of motion for these new variables are in a form to which non-degenerate perturbation theory may be applied. A system of this type is said to be intrinsically degenerate.

The solutions obtained by applying degenerate perturbation theory are usually not multiply periodic. Since classical perturbation theory seeks only multiply periodic solutions, we can conclude immediately that degenerate perturbation theory and classical degenerate perturbation theory are not equivalent schemes.
APPENDIX D

GENERAL ASYMPTOTIC INVARIANTS

In this appendix we shall formally extend the results of Chapter 5 to systems which have several degrees of freedom.

D.1 CONSTRUCTION OF THE HIERARCHY OF GENERAL ADIABATIC INVARIANTS BY PERTURBATION THEORY

In order to obtain the most general results we shall consider the system of section C.3 which has several rapid phases. The Hamiltonian (C.9) will now depend explicitly upon the slow time $T = \epsilon t$. Thus (C.9) becomes

$$ H = H_0(u_{-N}, \ldots, u_{-r-1}, T) + \sum_{n=1}^{R} \epsilon^n H_n(u, T). \quad (D.1) $$

The zero'th order term of (D.1) may also be a slowly varying function of the $u_k$'s ($k = -r, \ldots, r$). The results which we obtain by considering such a system are the same as those obtain by discussing (D.1).

We now introduce $t = \frac{T}{\epsilon}$ as a new coordinate and a quantity $s$ as its conjugate momentum. The autonomous Hamiltonian $h$ which is appropriate to the new canonical variables $\frac{T}{\epsilon}$ and $s$ is

$$ h = H_0 + \sum_{n=1}^{R} \epsilon^n H_n + s. \quad (D.2) $$

The equations of motion are

$$ \dot{u}_k = \text{sgn} k \sum_{n=1}^{R} \epsilon^n \partial_{u_k} H_n \quad (k = -N, \ldots, r), \quad (D.3a) $$

$$ \dot{s} = -\epsilon \partial_t H_0 - \epsilon \sum_{n=1}^{R} \epsilon^n \partial_s H_n, \quad (D.3b) $$

$$ \dot{T} = \epsilon, \quad (D.3c) $$

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\[
   u_j = \omega_j^0 + \sum_{n=1}^{R_n} \epsilon^n \partial_{u_j} H_n \quad (j = r+1, \ldots, N),
   \tag{D.3d}
\]

where
\[
   \omega_j^0 = \partial_{u_j} H_0 \quad (j = r+1, \ldots, N).
   \tag{D.4}
\]

We can apply non-degenerate perturbation theory to the system (D.3) by making a change of variables

\[
   u_i = v_i + \epsilon F_i(v, T, s) \quad (i = -N, \ldots, N)
   \tag{D.5a}
\]

\[
   T = \bar{T} + \epsilon D(v, T, s),
   \tag{D.5b}
\]

\[
   s = \bar{s} + \epsilon E(v, T, s),
   \tag{D.5c}
\]

such that

\[
   \dot{v}_k = \epsilon a_k(v_{-N}, \ldots, v_r, T, s) \quad (k = -N, \ldots, r),
   \tag{D.6a}
\]

\[
   \dot{\bar{T}} = \epsilon c(v_{-N}, \ldots, v_r, \bar{T}, \bar{s})
   \tag{D.6b}
\]

\[
   \dot{\bar{s}} = \epsilon d(v_{-N}, \ldots, v_r, \bar{T}, \bar{s}),
   \tag{D.6c}
\]

\[
   \dot{v}_j = \omega_j^0(v_{-N}, \ldots, v_{r-1}, T) + \epsilon b_j(v_{-N}, \ldots, v_r, \bar{T}, \bar{s})(j = r+1, \ldots, N).
   \tag{D.6d}
\]

Let us indicate the Poisson bracket with respect to all the \(v\)'s by ( ) and the Poisson bracket with respect to all the variables (\(v\)'s and \(s\) and \(\bar{T}/\epsilon\)) by [ ] . With these conventions the requirement that the transformation (D.5) be canonical is that the following equations be satisfied:

\[
   \left[ u_i, u_j \right] = (v_i, v_j) + (\epsilon F_i, v_j) + (v_i, \epsilon F_j) + \epsilon^2 \partial_{T_i} F_i \partial_{T_j} F_i
   \tag{D.7a}
\]

\[
   - \epsilon^2 \partial_{T_i} F_i \partial_{T_j} F_j = \text{sgn } i \delta_{i-j},
\]

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\[
\begin{align*}
\begin{bmatrix} T/\epsilon, u_i \end{bmatrix} &= (D,v_i) + [D,\epsilon F_i] + \epsilon \partial_{S} F_i = 0 \quad (D.7b) \\
[s,u] &= \text{sgn} -i \partial_{v_{-1}} \epsilon E + \left[ \epsilon \partial_{T} F_i, \epsilon F_i \right] - \epsilon^2 \partial_{T} F_i = 0, \quad (D.7c) \\
\begin{bmatrix} T/\epsilon, s \end{bmatrix} &= \left[ T/\epsilon, s \right] + \partial_{T} D + \partial_{S} \epsilon E + [D,\epsilon E] = 1. \quad (D.7d)
\end{align*}
\]

It is clear from the equations of motion that \( T = \overline{T} \) to all orders. Therefore, it follows from equation (D.7b) that

\[
\partial_{S} F_i = 0 \quad (D.8)
\]

to all orders. Similarly from (D.7d) we find that

\[
\partial_{S} E = 0 \quad (D.9)
\]

to all orders. Thus the transformation (D.5) becomes

\[
\begin{align*}
u_i &= v_i + \epsilon F_i(v,T), \quad (D.10a) \\
T &= \overline{T}, \quad (D.10b) \\
s &= \overline{s} + \epsilon E(v,T). \quad (D.10c)
\end{align*}
\]

The conditions that this transformation be canonical are

\[
\begin{align*}
(\epsilon F_i, v_j) + (v_i, \epsilon F_j) + (\epsilon F_i, \epsilon F_j) &= 0, \quad (D.11a) \\
\text{sgn} -i \partial_{v_{-1}} \epsilon E - \epsilon^2 \partial_{T} F_i + (\epsilon E, \epsilon F_i) &= 0. \quad (D.11b)
\end{align*}
\]

According to Theorem C.1 the Hamiltonian D.2 will be independent of the \( v_j \)'s (\( j = r+1, \ldots, N \)) to all orders. Thus if we make the transformation canonical to all orders the variables \( v_{-j} \) (\( j = r+1, \ldots, N \)) will be constant to all orders.
We now observe that the equations of motion for the \( u \)'s are independent of \( s \). This means that we need not introduce \( s \). We may work directly with the Hamiltonian (D.11) if we require that

\[
  u_i = v_i + \varepsilon F_i(v,T) \tag{D.12}
\]

such that

\[
  \dot{v}_k = \varepsilon a_k(v_{-N}, \ldots, v_r, T) (k = -N, \ldots, r) \tag{D.13a}
\]

\[
  \dot{v}_j = \omega_j(v_{-N}, \ldots, v_{r-1}, T) + \varepsilon b_j(v_{-N}, \ldots, v_r, T) (j = r+1, \ldots, N) \tag{D.13b}
\]

and

\[
  \dot{u}_i = \dot{v}_i + \varepsilon \sum_{k=-N}^{N} \dot{v}_k \partial_{v_k} F_i + \varepsilon \partial_{T} F_i \tag{D.14}
\]

Equation (D.14) differs from the equations which we obtained in the earlier chapters by the appearance of the term \( \varepsilon \partial_{T} F_i \).

If we perform non-degenerate perturbation theory according to equations (D.12) through (D.14), and if we determine the arbitrary zero harmonics of the \( F_i \)'s so as to satisfy equations (D.11a), then the quantities \( v_{-j} (j = r+1, \ldots, N) \) will be constant to any desired order in \( \varepsilon \). The \( v_{-j} \)'s constitute what we shall call the hierarchy of adiabatic invariants for non-degenerate systems.

Clearly we can treat degenerate systems containing an adiabatic time dependence in a fashion analogous to our treatment of non-degenerate systems. To do so we require:

\[
  u_i = v_i + \varepsilon F_i(v,T) (i = -N, \ldots, N), \tag{D.15}
\]

such that

\[
  \dot{v}_k = \varepsilon a_k(v_{-N}, \ldots, v_r, \Theta, T) (K = -N, \ldots, r), \tag{D.16a}
\]

\[
  \dot{v}_j = \omega_j(v_{-N}, \ldots, v_r, \Theta, T) (j = r+1, \ldots, N) \tag{D.16b}
\]

and

\[
  \dot{u}_i = \dot{v}_i + \varepsilon \sum_{n=-N}^{N} \dot{v}_n \partial_{v_n} F_i + \varepsilon \partial_{T} F_i . \tag{D.17}
\]
We can introduce n'th order adiabatic invariants as described in Theorem C.3 and Corollary C.2.

D.2 ASYMPTOTIC INVARIANTS OF HARMONICALLY DRIVEN SYSTEMS

Quite often we are led to discuss the behavior of an oscillatory system which is driven by a weak harmonic force. In this section we shall attempt to find asymptotic invariants for such systems.

We shall assume that the Hamiltonian which describes the system has the following form:

\[ H = H_0(u_{N}, \ldots, u_{r-1}) + \sum_{n=1}^{R} \epsilon^n H_n(u, \omega t), \tag{D.18} \]

where the \( u_i \)'s have the usual meaning and the \( H_n \)'s are periodic functions of the \( u_j \)'s (\( j = r+1, \ldots, N \)) and of \( \omega t \).

In order to apply the general theorems which were established in Appendix C we must convert (D.18) into an autonomous Hamiltonian. This can be accomplished by introducing \( \gamma = \omega t \) as a new coordinate and a quantity \( s \) as its conjugate momentum. The Hamiltonian \( h \) which is appropriate to these new variables is

\[ h = H_0 + \sum_{n=1}^{R} \epsilon^n H_n + \omega s. \tag{D.19} \]

The equations of motion are

\[ \dot{u}_k = \text{sgn} k \sum_{n=1}^{R} \epsilon^n \frac{\partial}{\partial u_{-k}} H_n \quad (k = -N, \ldots, r), \tag{D.20a} \]

\[ \dot{s} = -\sum_{n=1}^{R} \epsilon^n \frac{\partial}{\partial \gamma} H_n, \tag{D.20b} \]

\[ \dot{\gamma} = \omega, \tag{D.20c} \]

\[ \dot{u}_j = \omega_j^0 + \sum_{n=1}^{R} \epsilon^n \frac{\partial}{\partial u_{-j}} H_n \quad (j = r+1, \ldots, N). \tag{D.20d} \]

We shall assume that

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\[ \epsilon \ll \frac{\omega}{\omega_0} \ll \epsilon^{-1} \quad \text{(D.21)} \]

Let us suppose that the system is non-degenerate. If this is so then the proper perturbation theory consists of making a change of variables

\[ u_i = v_i + \epsilon F_i(v, \bar{\gamma}, \bar{s}), \quad (D.22a) \]

\[ s = \bar{s} + \epsilon D(v, \bar{\gamma}, \bar{s}), \quad (D.22b) \]

\[ \gamma = \bar{\gamma} + \epsilon E(v, \bar{\gamma}, \bar{s}) \quad (D.22c) \]

such that

\[ \dot{v}_k = \epsilon a_k(v_{-N}, \ldots, v_r, \bar{s}) \quad (k = -N, \ldots, r), \quad (D.23a) \]

\[ \dot{s} = \epsilon c(v_{-N}, \ldots, v_r, \bar{s}) \]

\[ \dot{\gamma} = \omega + \epsilon d(v_{-N}, \ldots, v_r, \bar{s}) \quad (D.23c) \]

\[ \dot{v}_j = \omega_j^0 + \epsilon b_j(v_{-N}, \ldots, v_r, \bar{s}) \quad (j = r+1, \ldots, N) \quad (D.23d) \]

We shall denote the Poisson bracket with respect to the \( v \)'s by \( \{ \, \} \) and the Poisson bracket with respect to all the variables (\( v \)'s and \( \gamma \) and \( s \)) by \( [ \, \, ] \). If (D.22) is to be canonical then we must satisfy the following Poisson bracket relations:

\[ [u_i, u_j] = (v_i, v_j) + (\epsilon F_i, v_j) + (v_i, \epsilon F_j) + (\epsilon F_i, \epsilon F_j) \quad (D.24a) \]

\[ + \epsilon^2 \partial_{\gamma} F_i \partial_{\gamma} F_j - \epsilon^2 \partial_s F_i \partial_s F_j = \text{sgn } i \delta_{i, -j} \]

\[ [\gamma, u_i] = \text{sgn } i \partial_{\gamma} v_i \epsilon E + \epsilon \partial_s F_i + [\epsilon E, \epsilon F_i] = 0 \quad (D.24b) \]

\[ [u_i, s] = \text{sgn } i \partial_{\gamma} v_i \epsilon D + \partial_{\gamma} \epsilon F_i + [\epsilon F_i, \epsilon D] = 0 \quad (D.24c) \]

\[ [\gamma, s] = 1 + \partial_{s} \epsilon D + \partial_{\gamma} \epsilon E + [\epsilon E, \epsilon D] = 1 \quad (D.24d) \]
It is clear from the equations of motion that $\gamma = \overline{\gamma}$ to all orders. This means that $E = 0$ to all orders. It follows, therefore, from (D.24b) that

$$\partial_{E}F_{1} = 0 \tag{D.25}$$

to all orders. Similarly it follows from (D.24d) that

$$\partial_{E}D = 0 \tag{D.26}$$

to all orders. Thus the transformation (D.22) becomes

$$u_{i} = v_{i} + \epsilon F_{1}(v, \gamma) \tag{D.27a}$$

$$s = \overline{s} + \epsilon D(v, \gamma) \tag{D.27b}$$

$$\gamma = \overline{\gamma} \tag{D.27d}$$

The requirement that (D.27) be canonical is

$$(u_{i}, u_{j}) = (v_{i}, v_{j}) + (\epsilon F_{1}, v_{j}) + (v_{i}, \epsilon F_{j}) + (\epsilon F_{1}, \epsilon F_{j}) = \text{sgn} \delta_{i,j} \tag{D.28a}$$

$$\text{sgn} \epsilon \frac{\partial s}{\partial v_{i}} D + \text{sgn} \frac{\partial \gamma}{\partial v_{i}} F_{1} + (\epsilon F_{1}, \epsilon D) = 0 \tag{D.28b}$$

According to Corollary C.1 the Hamiltonian (D.19) will be independent of the $v_{j}$'s ($j = r+1, \ldots, N$) to all orders in the perturbation theory. Thus if (D.27) is canonical to all orders, then the variables $v_{-j}$ ($j = r+1, \ldots, N$) will be constant to all orders. In addition the Hamiltonian will be independent of $\gamma$ to all orders. Thus the variable $\overline{s}$ will be constant to all orders.

The equations of motion for the $u$'s do not involve $s$. We may, therefore, avoid introducing $s$ if we formulate the perturbation theory as follows:

$$u_{i} = v_{i} + \epsilon F_{1}(v, \gamma) \tag{D.29}$$
such that
\[ \dot{\nu}_k = \epsilon a_k(\nu_{-N}, \ldots, \nu_r)(k = -N, \ldots, r), \quad (D.30a) \]
\[ \dot{\nu}_j = \omega_j^0 + \epsilon b_j(\nu_{-N}, \ldots, \nu_r)(j = r+1, \ldots, N) \quad (D.30b) \]
and
\[ \dot{u}_1 = \dot{\nu}_1 + \epsilon \sum_{k=-N}^N \dot{\nu}_k \partial_{\nu_k} F_1 + \epsilon \omega \partial_{\gamma} F_1. \quad (D.31) \]
The variable \( \gamma \) is to be treated as a dynamical variable. The \( F_1 \)'s are, of course, required to be periodic functions of \( \gamma \).

If we perform non-degenerate perturbation theory according to equations (D.29), through (D.31) and if we make the transformation (D.29) canonical to all orders by satisfying equations (D.28a) then the \( \nu_j \)'s (\( j = r+1, \ldots, N \)) will be constant to all orders. These \( \nu_j \)'s form a hierarchy of non-degenerate asymptotic invariants.

Clearly the above ideas can be extended to degenerate systems. In order to do so we formulate degenerate perturbation theory as follows:
\[ u_1 = v_1 + \epsilon F_1(v, \gamma) \quad (D.32) \]
such that
\[ \dot{\nu}_k = \epsilon a_k(v_{-N}, \ldots, v_r, \theta)(k = -N, \ldots, r), \quad (D.33a) \]
\[ \dot{\nu}_j = \omega_j^0 + \epsilon b_j(v_{-N}, \ldots, v_r, \theta)(j = r+1, \ldots, N), \quad (D.33b) \]
and
\[ \dot{u}_1 = \dot{\nu}_1 + \epsilon \sum_{k=-N}^N \dot{\nu}_k \partial_{\nu_k} F_1 + \epsilon \omega \partial_{\gamma} F_1. \quad (D.34) \]
The quantity \( \gamma \) is to be treated as a dynamical variable. The \( F_1 \)'s are to be periodic functions of \( \gamma \). The vector \( \theta \) is to contain all those phase relationships which give rise to resonance (both internal resonance and external resonance). We can introduce \( n \)'th order asymptotic invariants as described in Theorem C.3 and Corollary C.2.
APPENDIX E

A DISCUSSION OF THE ASYMPTOTIC CONVERGENCE OF DEGENERATE PERTURBATION THEORY

We wish to investigate the extent to which the degenerate perturbation theory of Appendix C provides an accurate description of the true motion. The original equations of motion are

\[ \dot{x}_i = \epsilon A_i (x, \gamma), \quad (i = 1, \ldots, r) \]  
\[ \dot{\gamma}_j = \omega_j (x) + \epsilon B_j (x, \gamma), \quad (j = 1, \ldots, s), \]  

where \( A_i \) and \( B_j \) are periodic functions of the \( \gamma \)'s. We always assume that the right hand sides of (E.1a,b) are infinitely differentiable functions of their arguments. We introduce new variables \( Y_i \) and \( \Phi_j \) such that

\[ x_i = Y_i + \sum_{n=1}^{\infty} \epsilon^n P_i^{(n)}(Y, \Phi), \]  
\[ = X_i (Y, \Phi, \epsilon) \]  
\[ \gamma_j = \Phi_j + \sum_{n=1}^{\infty} \epsilon^n \phi_j^{(n)}(Y, \Phi), \]  
\[ = \Gamma_j (Y, \Phi, \epsilon). \]  

At this point we assume that we have used degenerate perturbation theory to all orders. This means that the right hand sides of (E.2a,b) contain no small divisors to all orders. Since the right hand sides of (E.1a,b) are infinitely differentiable functions it follows that \( X_i(Y, \Phi) \) and \( \Gamma_j(Y, \Phi) \) are also infinitely differentiable functions of their arguments. We now shall demonstrate that the transformation (E.2a,b) has a unique inverse. To do this we write

\textsuperscript{52}The reader should go no further until he has satisfied himself that this statement is true.
\[ Y_i = Y_i(x, \gamma, \epsilon), \quad (E.3a) \]
\[ \phi_j = \phi_j(x, \gamma, \epsilon). \quad (E.3b) \]

We can certainly expand the right hand sides of (E.3a, b) in formal Taylor series about \( \epsilon = 0 \). Thus

\[ Y_i = Y_i(x, \gamma, 0) + \epsilon \frac{\partial Y_i}{\partial \epsilon} \bigg|_{\epsilon=0} (x, \gamma, \epsilon) + \frac{\epsilon^2 \partial^2 Y_i}{\partial \epsilon^2} \bigg|_{\epsilon=0} (x, \gamma, \epsilon) + \ldots, \quad (E.4a) \]
\[ \phi_j = \phi_j(x, \gamma, 0) + \epsilon \frac{\partial \phi_j}{\partial \epsilon} \bigg|_{\epsilon=0} (x, \gamma, \epsilon) + \frac{\epsilon^2 \partial^2 \phi_j}{\partial \epsilon^2} \bigg|_{\epsilon=0} (x, \gamma, \epsilon) + \ldots. \quad (E.4b) \]

Equations (E.4a, b) can be written as

\[ Y_i = Y_i(x, \gamma, 0) + \sum_{n=1}^{\infty} \epsilon^n f_i^{(n)}(x, \gamma), \quad (E.5a) \]
\[ = Y_i(x, \gamma, 0) + \epsilon f_i(x, \gamma) \]
\[ \phi_j = \phi_j(x, \gamma, 0) + \sum_{n=1}^{\infty} \epsilon^n g_j^{(n)}(x, \gamma). \quad (E.5b) \]
\[ = \phi_j(x, \gamma, 0) + \epsilon g_j(x, \gamma) \]

It is clear from (E.2a, b) that

\[ Y_i(x, \gamma, 0) = x_i, \quad (E.6a) \]
\[ \phi_j(x, \gamma, 0) = \gamma, \quad (E.6b) \]

We now substitute (E.5a, b) into (E.2a, b) to find that

\[ \sum_{n=1}^{\infty} \epsilon^n f_i^{(n)}(x, \gamma) + \sum_{n=1}^{\infty} \epsilon^n f_i^{(n)}(x + \epsilon f(x, \gamma), + \epsilon g(x, \gamma)) = 0, \quad (E.7a) \]
\[ \sum_{n=1}^{\infty} \epsilon^n g_j^{(n)}(x, \gamma) + \sum_{n=1}^{\infty} \epsilon^n g_j^{(n)}(x + \epsilon f(x, \gamma), + \epsilon g(x, \gamma)) = 0. \quad (E.7b) \]
We expand the known functions $F_i^{(n)}$ and $G_j^{(n)}$ in formal Taylor series about the arguments $x$ and $\gamma$. Since (E.7a,b) are to be satisfied for arbitrary $\epsilon$ the coefficient of each power of $\epsilon$ must vanish. Thus a knowledge of the $f_i^{(m)}$'s and $g_j^{(m)}$'s ($m = 1 \ldots, n-1$) uniquely determines the next functions $f_i^{(n)}$, $g_j^{(n)}$. Since we know from (E.7a,b) that

\begin{align}
  f_i^{(1)}(x, \gamma) &= -F_i^{(1)}(x, \gamma), \\
  g_j^{(1)}(x, \gamma) &= -G_j^{(1)}(x, \gamma),
\end{align}

(E.8a)

(E.8b)

it follows by induction that the $f_i^{(n)}$'s and $g_j^{(n)}$'s are uniquely determined to all orders.

The time development of the $Y_i$'s and $\Phi_j$'s are governed by the following equations:

\begin{align}
  \dot{Y}_i &= \sum_{n=1}^{\infty} \epsilon^n a_i^{(n)}(y, \Phi), \\
  \Phi_j &= \omega_j(y) + \sum_{n=1}^{\infty} \epsilon^n b_j^{(n)}(y, \Phi).
\end{align}

(E.9a)

(E.9b)

Since the variables $x_i$, $\gamma_1$ and $Y_1$, $\Phi_1$ are uniquely related it follows that a solution of equations (E.9a,b), subject to the proper initial conditions, is a formal solution of the original problem. However, there is no reason to believe that the perturbation theory will be convergent. This means that the formal solution may very well be meaningless. We shall now verify that the perturbation theory does provide an asymptotically correct solution.

In perturbation theory we solve the following system of differential equations:

\begin{align}
  \dot{y}_i &= \sum_{n=1}^{N} \epsilon^{n_a}(y, \Phi), \\
  \dot{\Phi}_j &= \omega_j(y) + \sum_{n=1}^{N} \epsilon^{n_b}(y, \Phi).
\end{align}

(E.10a)

(E.10b)
where $N$ is some finite integer. We subject equations (E.10a,b) to the following initial conditions:

\[
y_i(0) = x_i^0 + \sum_{n=1}^{N} \epsilon n_i(n) (x^0, y^0), \tag{E.11a}
\]

\[
\phi_j(0) = \gamma_j^0 + \sum_{n=1}^{N} \epsilon n_j(n) (x^0, y^0), \tag{E.11b}
\]

where $x_i^0$ and $\gamma_j^0$ are the initial values of the original variables $x_i$ and $\gamma_j$. Let us now introduce the functions

\[
y_i(N) = x_i + \sum_{n=1}^{N} \epsilon n_i(n) (x, y), \tag{E.12a}
\]

\[
\phi_j(N) = \gamma_j + \sum_{n=1}^{N} \epsilon n_j(n) (x, y). \tag{E.12b}
\]

The functions $y_i(N)$ and $\phi_j(N)$ are true functions of the $x_i$'s and $\gamma_j$'s. When we invert equations (E.12a,b) we find that

\[
x_i = y_i(N) + \sum_{n=1}^{N} \epsilon n_i(n) (y(N), \phi(N)) + O(\epsilon N + 1), \tag{E.13a}
\]

\[
\gamma_j = \phi_j(N) + \sum_{n=1}^{N} \epsilon n_j(n) (y(N), \phi(N)) + O(\epsilon N + 1). \tag{E.13b}
\]

Equation (E.13a), for example, is easily verified by making use of equations (E.12). Thus

\[
x_i = x_i + \sum_{n=1}^{N} \epsilon n_i(n) (x, y) \tag{E.14}
\]
\[
+ \sum_{n=1}^{N} \varepsilon^n f_1^{(n)}(x) + \sum_{n=1}^{N} \varepsilon^n f_1^{(n)}(x, \gamma), \gamma + \sum_{n=1}^{N} \varepsilon^n g^{(n)}(x, \gamma))
+ O(\varepsilon^{N+1}).
\]

The functions \(F_1^{(n)}\) are now expanded about \(x\) and \(\gamma\). It follows from equations (E.7a) that the \(F_1^{(n)}\)'s \((n = 1, \ldots, N)\) will just cancel the corresponding terms in the Taylor series expansion of the \(F_1^{(n)}\)'s. The remainder of the functions \(F_1^{(n)}\) are found from Taylor's theorem with a remainder to be of order \(\varepsilon^{N+1}\). (We assume, of course, that the \(F_1^{(n)}\)'s such that one can use Taylor's theorem and also that they are bounded over the range of interest to us.) If follows then that equations (E.15a,b) are indeed correct.

The functions \(y_i^{(N)}\) and \(\phi_j^{(N)}\) satisfy the following differential equations:

\[
\dot{y}_i^{(N)} = x_i + \sum_{n=1}^{N} \varepsilon^n \left\{ \sum_{k=1}^{r} \frac{\partial f_1^{(n)}}{\partial x_k} \dot{x}_k + \sum_{k=1}^{s} \frac{\partial f_1^{(n)}}{\partial \gamma_k} \dot{\gamma}_k \right\}
\]

\(= \sum_{n=1}^{N} \varepsilon^n s_1^{(n)}(y^{(N)}, \gamma^{(N)}) + O(\varepsilon^{N+1}),
\]

\[
\dot{\phi}_j^{(N)} = \dot{\gamma}_j + \sum_{n=1}^{N} \varepsilon^n \left\{ \sum_{k=1}^{r} \frac{\partial g_1^{(n)}}{\partial x_k} \dot{x}_k + \sum_{k=1}^{s} \frac{\partial g_1^{(n)}}{\partial \gamma_k} \dot{\gamma}_k \right\}
\]

\(= \omega_j(y^{(N)}) + \sum_{n=1}^{N} \varepsilon^n b_j^{(n)}(y^{(N)}, \gamma^{(N)}) + O(\varepsilon^{N+1}).
\]

Equations (E.15) follow from the same type of reasoning which produced equations (E.13).

We now observe that equations (E.15) are very similar to equations (E.10). We also observe that \(y_1^{(N)}, \phi_j^{(N)}\) and \(y_i, \phi_j\) satisfy the same initial conditions. Thus the problem is reduced to comparing the time development of functions which satisfy nearly identical differential equations and the same initial conditions.
This problem is discussed in most textbooks on differential equations. We shall simply state the well known results. Let the vector functions \( x = (x_1, x_2, \ldots, x_T) \) and \( y = (y_1, y_2, \ldots, y_T) \) satisfy the following differential equations:

\[
\begin{align*}
\dot{x} &= f(x), \\
\dot{y} &= f(y) + a,
\end{align*}
\] (E.16a)

with the initial conditions

\[
x(0) = y(0),
\] (E.17)

where \( a \) is assumed small. Further let \( f(x) \) satisfy a Lipschitz condition

\[
|f(x) - f(y)| \leq K|x - y|,
\] (E.18)

where \( K \) is the Lipschitz constant. Under these conditions it is straightforward to show that

\[
|x - y| \leq |a/K| e^{Kt} - |a/K|.
\] (E.19)

We see from (E.19) that \( x \) and \( y \) remain close for times of order \( 1/K \).

When we apply the above ideas to \( y_i^{(N)}, \phi_j^{(N)} \) and \( y_i, \phi_j \) we find that

\[
K = O(|\text{grad } \omega(y)| + \epsilon),
\] (E.20)

\[
|a/K| = O(\epsilon^{N+1}/|\text{grad } \omega(y)| + \epsilon).
\] (E.21)

---

Equation (E.20) follows from (E.10a,b). We conclude that

\[ y_i^{(N)}(t) = y_i(t) + O(\varepsilon^{N+1} / |\text{grad}\ \omega(y)| + \varepsilon), \]  
(E.22a)

\[ \dot{y}_j^{(N)}(t) = \dot{y}_j(t) + O(\varepsilon^{N+1} / |\text{grad}\ \omega(y)| + \varepsilon), \]  
(E.22b)

for times of order \((|\text{grad}\ \omega(y)| + \varepsilon)^{-1}\). When we substitute equations (E.22) into (E.13) we find that

\[ x_i(t) = y_i(t) + \sum_{n=1}^{N} \varepsilon^n F_{i}^{(n)}(y,\phi) \]  
(E.23a)

\[ + O(\varepsilon^{N+1} / |\text{grad}\ \omega(y)| + \varepsilon), \]

\[ y_j(t) = \dot{y}_j(t) = \sum_{n=1}^{N} \varepsilon^n G_j^{(n)}(y,\phi) \]  
(E.23b)

\[ + O(\varepsilon^{N+1} / |\text{grad}\ \omega(y)| + \varepsilon), \]

for times of order \((|\text{grad}\ \omega(y)| + \varepsilon)^{-1}\). In other words N'th order degenerate perturbation theory provides a solution which is correct to within order

\[ (\varepsilon^{N+1} / |\text{grad}\ \omega(y)| + \varepsilon) \]  
for times of order \((|\text{grad}\ \omega(y)| + \varepsilon)^{-1}\).

When the system has only one phase the validity of the perturbation theory is somewhat different from the general case. In this special case equations (E.15a) and (E.10a) have the form

\[ y_i^{(N)} = \varepsilon a_i y^{(N)} \]  
(E.24a)

\[ \dot{y}_i = \varepsilon a_i y + O(\varepsilon^{N+1}). \]  
(E.24b)

From (E.24) we see that the Lipschitz constant K is of order \(\varepsilon\). This means that
\[ y^{(N)}_1(t) = y_1(t) + O(\epsilon^N) \]  

(E.25)

for times of order \( \epsilon^{-1} \). There is, so far, no restriction on \(|\text{grad } \omega(y)|\). We now rewrite equations (E.15b) and (E.10b) for this special case. They have the form

\[ \dot{\rho}^{(N)} = \alpha(y^{(N)}) + \epsilon \beta(y^{(N)}) \]  

(E.26a)

\[ \dot{\rho} = \alpha(y) + \epsilon \beta(y) + O(\epsilon^{N+1}) \]  

(E.26b)

When we substitute (E.25) into (E.26a,b) we find that

\[ \dot{\rho}^{(N)} = \dot{\rho} + |\text{grad } \omega(y)| O(\epsilon^N) + O(\epsilon^{N+1}) \]  

(E.27)

Hence

\[ \dot{\rho}^{(N)}(t) = \dot{\rho}(t) + |\text{grad } \omega(y)| O(\epsilon^{N-1}) + O(\epsilon^N) \]  

(E.28)

for times of order \( \epsilon^{-1} \). It follows from (E.28) that the \( N \)'th order single phase method (hence the Bogoliubov method) produces a solution which is valid only to order \( \epsilon^{N-1} \) for times of order \( \epsilon^{-1} \) unless \(|\text{grad } \omega(y)| = O(\epsilon)|\), in which case the solution is valid to within order \( \epsilon^N \) for times of order \( \epsilon^{-1} \).

A few remarks are now in order concerning the validity of non-degenerate perturbation theory. Suppose that we perform degenerate perturbation theory and that through order \( \epsilon^N \) we encounter no resonant behavior. In this case degenerate perturbation theory is identical to non-degenerate perturbation theory through order \( \epsilon^N \). Thus, for a system which is non-resonant through order \( \epsilon^N \), non-degenerate perturbation theory, when carried out only through order \( \epsilon^N \), provides a solution which is valid to within order \( \epsilon^N \) for times of order \( (|\text{grad } \omega(y)| + \epsilon)^{-1} \). It is important to realize that if we carry non-degenerate perturbation theory to higher orders in which resonance occurs, then not only don't we improve the approximate solution, we actually destroy the asymptotic convergence of the perturbation theory. In other words non-degenerate perturbation theory must not be carried beyond the highest order for which resonance does not occur.
APPENDIX F

THE MOTION OF A CHARGED PARTICLE IN A CIRCULARLY POLARIZED HYDROMAGNETIC WAVE

We shall describe the motion in a coordinate system which moves with the wave. In such a system the wave has the form

\[ \vec{B}_1 = (B_1 \sin kx, B_1 \cos kx, 0). \] (F.1)

This field can be described by the vector potential

\[ \vec{A}_1 = ((B_1/k) \sin kx, (B_1/k) \cos kx, 0). \] (F.2)

The uniform background field \( B_0 \) is described by the following vector potential:

\[ \vec{A}_0 = (-B_0 y, 0, 0). \] (F.3)

The non-relativistic Hamiltonian \( H \) is, therefore, given by

\[ H = \left( \frac{1}{2m} \right) \left[ \left( p_x - \frac{m_0}{(m_1/k) \sin kx} \right)^2 + \left( p_y - \frac{m_0}{(m_1/k) \cos kx} \right)^2 + p_z^2 \right]. \] (F.4)

We shall now change variables according to equations (6.5). In this new representation the Hamiltonian has the form

\[ H = J \omega_0 + \left( \frac{p_z^2}{2m} \right) - \left( \frac{1}{(m_1/k)(2m_0 J)} \right)^2 \cos(\gamma - kx) + \frac{m_1^2}{2k^2}. \] (F.5)

As in Chapter 6 we define \( \epsilon \) as follows

\[ \epsilon = \frac{\omega_1}{\omega_0}, \] (F.6)

and we change the independent variable according to the equation

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\[ \tau = \omega_0 t. \] (F.7)

The Hamiltonian \( h \) which is appropriate to the new independent variable is

\[ h = J + \left( p_z^2 / 2 \omega_0 \right) - \left( \epsilon / k \right)(2 \omega_0 J)^2 \cos(\gamma - kz) + \epsilon^2 \omega_0 / 2k^2 \] (F.8)

The equations of motion are found from (F.8) to be

\[ J' = -(\epsilon / k)(2\omega_0 J)^2 \sin(\gamma - kz), \] (F.9a)

\[ p_z' = \epsilon (2\omega_0 J)^2 \sin(\gamma - kz), \] (F.9b)

\[ k z' = k p_z / \omega_0, \] (F.9c)

\[ \gamma' = 1 - (\epsilon / k)(\omega_0 / 2J)^2 \cos(\gamma - kz). \] (F.9d)

Upon replacing \( \epsilon \) by \( \epsilon / 2 \) equations (F.9) become identical in form with equations (6.32) (through first order). In Chapter 6 we have presented the complete solutions (through first order) of (6.32). We can use these complete solutions to write down the exact solutions of (F.9). In order to simplify this task we make the following definitions:

\[ \Theta = \gamma - kz, \] (F.10a)

\[ R = k(2J/\omega_0)^2, \] (F.10b)

\[ a = R \cos \Theta, \] (F.10c)

\[ b = R \sin \Theta. \] (F.10d)

These variables are analogous to their counterparts of Chapter 6.

The variables \( J \) and \( p_z \) are related by the equation
\[ I = J + \frac{p_z}{k}, \quad \text{(F.11)} \]

where \( I \) is a constant. We define another constant \( C \) by the equation

\[ C = L^2 I / m \nu_0. \quad \text{(F.12)} \]

The constant \( C \) plays the same role as its counterpart of Chapter 6.

We now describe the solutions of (F.9) by copying the appropriate equations from Chapter 6 and, in these equations, replacing \( \varepsilon \) by \( 2\varepsilon \). Thus the singular points of the equations for \( a' \) and \( b' \) are found from (6.44) to be given by

\[ b = 0, \quad \text{(F.13a)} \]

\[ a^2 - 2(1 - 0)a - 2\varepsilon = 0, \quad \text{(F.13b)} \]

When \( \varepsilon \) is sufficiently small the cubic (F.13b) will have three real roots; we shall assume this to be the case. The trajectories in the \( a-b \) plane will resemble those plotted in Figure 3 of Chapter 6. There will be a synchronous region and a non-synchronous region.

We introduce a new variable \( S \) according to the equation

\[ S = R^2, \quad \text{(F.14)} \]

The equation of motion for \( S \) is found from (6.60) to be

\[ S' = -f(S), \quad \text{(F.15)} \]

where

\[ (4f(S))^2 = \left[ 8L^4 + 8LS^3 + 16(M^2 + L^2)S^2 + 16(2LM - 4\varepsilon)S + 16M^2 \right], \quad \text{(F.16)} \]

\( L \) and \( M \) being defined by (6.59). We shall denote the roots of \( (4f(S))^2 \) by \( S_1, S_2, S_3, S_4 \).
In the synchronous region there are only two real roots which we shall take to be \( S_1, S_2 \) with \( S_1 \geq S_2 \). The time dependence of \( S \) is found from (6.68) to be given by

\[
S(\tau) = \frac{S_1 B + AS_2 + (S_2 A - S_1 B) \text{cn} \left[ \frac{(\tau - \tau_o)/4g}{(\tau - \tau_o)/4g} \right]}{A + B + (A - B) \text{cn} \left[ \frac{(\tau - \tau_o)/4g}{(\tau - \tau_o)/4g} \right]}, \tag{F.17}
\]

where \( g \) is given by (6.65c) and \( A \) and \( B \) by (6.66).

In the non-synchronous region the polynomial \((4f(S))^2\) has four real roots which we shall arrange such that \( S_1 \geq S_2 \geq S_3 \geq S_4 \). When \( S_1 \geq S \geq S_2 \geq S_3 \geq S_4 \) then the time dependence of \( S \) is found from (6.75) to be given by

\[
S(\tau) = \frac{S_2(S_1 - S_3) - S_3(S_1 - S_2) \text{sn}^2 \left[ \frac{(\tau - \tau_o)/4g}{(\tau - \tau_o)/4g} \right]}{S_1 - S_3 - (S_1 - S_2) \text{sn}^2 \left[ \frac{(\tau - \tau_o)/4g}{(\tau - \tau_o)/4g} \right]}, \tag{F.18}
\]

where \( g \) is defined by (6.73a). When \( S_1 \geq S_2 \geq S_3 \geq S \geq S_4 \) we find from (6.78) that

\[
S(\tau) = \frac{S_4(S_1 - S_3) + S_3(S_1 - S_4) \text{sn}^2 \left[ \frac{(\tau - \tau_o)/4g}{(\tau - \tau_o)/4g} \right]}{S_1 - S_3 - (S_3 - S_4) \text{sn}^2 \left[ \frac{(\tau - \tau_o)/4g}{(\tau - \tau_o)/4g} \right]}, \tag{F.19}
\]

g being given by (6.73).

The time dependence of \( J \) is found by multiplying the appropriate function \( S(\tau) \) by \( m^2_0/2k^2 \). The time dependence of \( p_z \) is found from (F.11). The time dependence of \( \theta \) is found from (6.79) to be given by

\[
\theta(\tau) = \cos^{-1} \left[ \frac{3}{2} \frac{\mathcal{E}}{\delta \epsilon} + \frac{1}{2} \frac{\mathcal{E}}{\delta \epsilon} (1 - \epsilon) \frac{c^2 - 2k^2 h/mw_0}{2\epsilon} \right]. \tag{F.20}
\]
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