MOTION OF A CHARGED PARTICLE IN A SPATIALLY PERIODIC MAGNETIC FIELD

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Degenerate perturbation theory is employed to discuss the motion of a charged particle in a constant magnetic field on which is superimposed a weak, transverse, spatially periodic magnetic field. A first order solution of the equations of motion is presented. It is shown that the secular motion is periodic in time. The significance of this result with respect to the stability of protons in the inner Van Allen belt is discussed.

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1. INTRODUCTION

In the preceding paper[1] (henceforth cited as I) we have presented a new formulation of classical perturbation theory. There we illustrated the non-degenerate form of this theory by discussing the van der Pol equation. The van der Pol equation has, of course, been adequately discussed by many authors using a variety of techniques. In the present paper, however, we shall discuss a problem which has not been adequately treated in previous publications. Here we shall employ degenerate perturbation theory to discuss the interaction between a charged particle and a constant magnetic field on which is superimposed a weak, transverse, spatially periodic magnetic field.

This interaction has played an important role in recent discussions of the stability of protons in the inner Van Allen belt. For example, Dragt[2] and Wentzel[3] have argued that a resonant interaction between the charged particle and a periodic magnetic field would cause a breakdown of the adiabatic invariance of the particle's orbital magnetic moment. They further argue that such a breakdown of the adiabatic invariance of the magnetic moment would destroy the magnetic trapping effect. This reasoning has led them to assert that a periodic disturbance (produced, for example, by a hydromagnetic wave) on the geomagnetic field is responsible for the removal of protons, which would otherwise be trapped, from the inner Van Allen belt. Of crucial importance to their assertions is the assumption that a weak periodic disturbance can cause a large change in the orbital magnetic moment.

In what follows we shall obtain a complete first order solution of the equations of motion in the case where the periodic field is a sinusoid. We shall find that the secular changes produced by such a field are of bounded variation. In particular, the "average" magnetic moment is a periodic function of time. The relative fluctuation in the "average" magnetic moment depends upon the ratio of the particle's cyclotron radius to the wavelength of the periodic disturbance: The fluctuation is large when the ratio is small and small when the ratio is large. The essential point is that a resonant interaction between the particle and the periodic field is not sufficient to cause large changes in the magnetic moment. This is contrary to the assumptions of Dragt and Wentzel.

Aside from its application to the question of stability of protons in the inner Van Allen belt, the example which we shall discuss is an interesting mathematical exercise. It illustrates quite nicely many of the phenomena which are characteristic of non-linear oscillatory systems. For example, the ideas of secular growth, stability and instability, and synchronous and non-synchronous behavior arise in a very natural way. The example also illustrates that a non-linear resonance is considerably more complicated than a linear resonance.

Our program is as follows: In section 2. we derive Hamilton's equations of motion which describe the interaction between the particle and the field. In section 3. we introduce the appropriate perturbation theory and obtain the differential equations which describe the secular motion. In section 4. we perform a phase plane analysis in order to characterize the secular motion. In section 5. we obtain an explicit solution of the differential equations which describe the secular motion. In section 6. we discuss the behavior of the secular motion under resonance conditions. The final section summarizes the main conclusions of the paper.

2. THE EQUATIONS OF MOTION

In the Cartesian reference frame x, y, z the magnetic field is taken to have the form

$$\underset{\sim}{B} = \left[B_1 \sin kz,\ 0,\ B_0\right]. \tag{2.1}$$

This field can be described by the vector potential

$$A = \left[-B_0 y,\ (B_1/k)\cos kz,\ 0\right]. \tag{2.2}$$

The non-relativistic Hamiltonian H which describes the system is

$$H = (1/2m)\left\{\underset{\sim}{p} - (e/c)\underset{\sim}{A}\right\}^2 \tag{2.3}$$

$$= (1/2m)\left\{\left[p_x + m\omega_0 y\right]^2 + \left[p_y - (m\omega_1/k)\cos kz\right]^2 + p_z^2\right\},$$

where

$$\omega_o = eB_o/mc \ , \quad \omega_1 = eB_1/mc \ . \tag{2.4}$$

In order to prepare the system for perturbation theory we introduce the new canonical momenta J, p_r, P_Z and their conjugate coordinates ψ, r, and Z as follows:

$$x = r - (2J/m\omega_o)^{1/2}\cos\psi \ , \qquad p_x = p_r, \tag{2.5}$$

$$y = -(1/m\omega_o)p_r + (2J/m\omega_o)^{1/2}\sin\psi \ , \quad p_y = (2m\omega_o J)^{1/2}\cos\psi \ ,$$

$$z = Z, \qquad p_z = P_Z.$$

The quantities r, $p_r/m\omega_o$, and Z are the Cartesian coordinates of the guiding center. In the unperturbed state the particle gyrates about this center with angular velocity ω_o in a circle of radius $(2J/m\omega_o)^{1/2}$. We shall measure the time in units of the rotation period. To do this we introduce a new independent variable

$$\tau = \omega_o t. \tag{2.6}$$

It is straightforward to show that the Hamiltonian h which is appropriate to the new variables is

$$h = J + (1/2m\omega_o)P_Z^2 - (\varepsilon/k)(2m\omega_o J)^{1/2}\cos\psi\cos kZ$$

$$+ (\varepsilon^2 m\omega_o/2k^2)\cos^2 kZ, \tag{2.7}$$

where

$$\varepsilon = \omega_1/\omega_o = B_1/B_o. \tag{2.8}$$

Hamilton's equations of motion are found from Eq.(2.7) to be

$$J' = -(\varepsilon/2k)(2m\omega_o J)^{1/2}\left[\sin(\psi + kZ) + \sin(\psi - kZ)\right], \tag{2.9a}$$

$$P_Z' = -(\varepsilon/2)(2m\omega_o J)^{1/2}\left[\sin(\psi + kZ) - \sin(\psi - kZ)\right] \tag{2.9b}$$

$$+ (\varepsilon^2 m\omega_o/k)\cos kZ \sin kZ,$$

$$kZ' = kP_Z/m\omega_o, \tag{2.9c}$$

$$\psi' = 1 - (\varepsilon/2k)(m\omega_o/2J)^{1/2}\left[\cos(\psi + kZ) + \cos(\psi - kZ)\right], \tag{2.9d}$$

where, for example,

$$J' = dJ/d\tau . \tag{2.10}$$

The system of differential equations (2.9) is in the standard form to which the perturbation theory of I is applicable. The parameter of smallness is $\varepsilon = B_1/B_0$. We see from equations (2.9) that the sum angle $\psi + kZ$ contributes only small amplitude, rapid fluctuations to the motion. However, the difference angle $\psi - kZ$ can give rise to secular motion when $\psi' - kZ' = O(\varepsilon)$. The system (2.9) must, therefore, be treated by degenerate perturbation theory. We shall carry out this treatment in the next section.

3. PERTURBATION THEORY

In this section we shall perform first order perturbation theory according to the formalism presented in I. Our object is to separate the rapidly fluctuating motion from the secular motion. In order to do this we introduce new variables U, V, K, and ϕ as follows:

$$kZ = kU + \varepsilon D_1(U,V,K,\phi) + O(\varepsilon^2), \tag{3.1a}$$

$$P_Z = V + \varepsilon E_1(U,V,K,\phi) + O(\varepsilon^2), \tag{3.1b}$$

$$J = K + \varepsilon F_1(U,V,K,\phi) + O(\varepsilon^2), \tag{3.1c}$$

$$\psi = \phi + \varepsilon G_1(U,V,K,\phi) + O(\varepsilon^2), \tag{3.1d}$$

where D_1, E_1, F_1, and G_1 are required to be periodic functions of ϕ and of kU with period 2π. The variables U, V, K, and ϕ are to contain the secular motion and D_1, E_1, F_1, and G_1 are to contain the rapidly fluctuating motion. In order to guarantee that U,V,K, and ϕ represent the secular motion we require that

$$kU' = (kV/m\omega_0) + \varepsilon a_1(U,V,K,\phi) + O(\varepsilon^2), \qquad (3.2a)$$

$$V' = \varepsilon b_1(U,V,K,\phi) + O(\varepsilon^2), \qquad (3.2b)$$

$$K' = \varepsilon A_1(U,V,K,\phi) + O(\varepsilon^2), \qquad (3.2c)$$

$$\phi' = 1 + \varepsilon B_1(U,V,K,\phi) + O(\varepsilon^2). \qquad (3.2d)$$

The functions a_1, b_1, A_1, and B_1 are to contain only those combinations of U and ϕ which can give rise to secular motion. The precise manner in which this choice is made is fully described in I.

If we substitute the ansatz (3.1) and (3.2) into equations (2.9) and retain only terms through first order in ε, then we obtain the following set of equations:

$$A_1 + \hat{O}\, F_1 = -(1/2k)(2m\omega_0 K)^{1/2} \Big[\sin(\phi + kU) + \sin(\phi - kU)\Big], \qquad (3.3a)$$

$$b_1 + \hat{O}\, E_1 = -(1/2)(2m\omega_0 K)^{1/2} \Big[\sin(\phi + kU) - \sin(\phi - kU)\Big], \qquad (3.3b)$$

$$a_1 + \hat{O}\, D_1 = (k/m\omega_o)E_1, \tag{3.3c}$$

$$B_1 + \hat{O}\, G_1 = -(1/2k)(m\omega_o/2K)^{1/2}\left[\cos(\phi + kU) + \cos(\phi - kU)\right], \tag{3.3d}$$

where the operator

$$\hat{O} = (kV/m\omega_o)\partial\, /\partial kU + \partial\, /\partial\phi\ . \tag{3.4}$$

The difference angle $\theta = \phi - kU$ can give rise to secular behavior when it is slowly varying. In order that U, V, K, and ϕ shall contain all of the secular motion we must absorb the θ dependence into the functions A_1, a_1, b_1, and B_1. We, therefore, choose

$$A_1 = -(1/2k)(2m\omega_o K)^{1/2}\sin\theta, \tag{3.5a}$$

$$B_1 = -(1/2k)(m\omega_o/2K)^{1/2}\cos\theta, \tag{3.5b}$$

$$a_1 = 0, \tag{3.5c}$$

$$b_1 = (1/2)(2m\omega_o K)^{1/2}\sin\theta. \tag{3.5d}$$

With this choice of A_1, B_1, a_1, and b_1 we find that

$$D_1 = (k/2\omega_2^2)(2K/m\omega_o)^{1/2}\sin(\phi + kU), \qquad (3.6a)$$

$$E_1 = (1/2\omega_2^2)(2m\omega_o K)^{1/2}\cos(\phi + kU), \qquad (3.6b)$$

$$F_1 = (1/2k\omega_2)(2m\omega_o K)^{1/2}\cos(\phi + kU), \qquad (3.6c)$$

$$G_1 = -(1/2k\omega_2)(m\omega_o/2K)^{1/2}\sin(\phi + kU), \qquad (3.6d)$$

where

$$\omega_2 = 1 + (kV/m\omega_o). \qquad (3.7)$$

When our choice for A_1, B_1, a_1, and b_1 is substituted into equations (3.2) we find that

$$K' = -(\varepsilon/2k)(2m\omega_o K)^{1/2}\sin\theta, \qquad (3.8a)$$

$$V' = (\varepsilon/2)(2m\omega_o K)^{1/2}\sin\theta, \qquad (3.8b)$$

$$\theta' = 1 - (kV/m\omega_o) - (\varepsilon/2k)(m\omega_o/2K)^{1/2}\cos\theta, \qquad (3.8c)$$

where $\theta' = \phi' - kU'$. It follows from equations (3.8a,b) that

$$K + (V/k) = I, \qquad (3.9)$$

where I is a constant. The system of equations (3.8), therefore, reduces to two equations relating the two variables K and θ. In the next section we shall use these equations to obtain some general information concerning K and θ without producing an explicit solution.

4. PHASE PLANE ANALYSIS

We begin this section by introducing the new variables a and b which are defined by the equations

$$a = R\cos\theta\ , \quad b = R\sin\theta, \tag{4.1}$$

where

$$R = k(2K/m\omega_0)^{1/2}. \tag{4.2}$$

The quantity R measures the ratio of the cyclotron radius to the fundamental period of the disturbance. The equations of motion for a and b are found from equations (3.8) to be

$$a' = -(1 - C + R^2/2)b, \tag{4.3a}$$

$$b' = (1 - C + R^2/2)a - \varepsilon/2, \tag{4.3b}$$

where

$$C = k^2 I/m\omega_0 \tag{4.3c}$$

is a constant. These equations give rise to the differential form

$$\left[1 - C + (R^2/2)\right] bdb + \left\{\left[1 - C + (R^2/2)\right]a - (\varepsilon/2)\right\}da = 0. \tag{4.4}$$

This is an exact differential whose integral is

$$R^4 + 4(1 - C)R^2 - 4\varepsilon a = M, \tag{4.5}$$

where M is a constant. Eq. (4.5) expresses the conservation of energy through first order. It follows from the Hamiltonian (2.7) that, to first order in ε,

$$M = (8k^2h/m\omega_o) - 4C^2. \tag{4.6}$$

The important aspects of the motion can be illustrated by plotting Eq. (4.5) in the a-b plane. Before doing this it is useful to examine the points where a' and b' are simultaneously zero. These are the points of equilibrium and are usually termed singular points. It follows from equations (4.3) that the singular points are to be found from the equations

$$b = 0, \tag{4.7a}$$

$$a^3 + 2(1 - C)a - \varepsilon = 0. \tag{4.7b}$$

If ε is sufficiently small the cubic equation will have three real roots; we shall assume this to be the case. These roots, which we shall call a_1, a_2, a_3, are approximately as follows:

$$a_1 = 2\left[-2(1 - C)/3\right]^{1/2}\left[(3/4)^{1/2} - \delta/6\right] + O(\varepsilon^2) > 0, \tag{4.8a}$$

$$a_2 = 2\left[-2(1 - C)/3\right]^{1/2}\left[-(3/4)^{1/2} - \delta/6\right] + O(\varepsilon^2) < 0, \quad (4.8b)$$

$$a_3 = 2\left[-2(1 - C)/3\right]^{1/2}(\delta/3) + O(\varepsilon^2) < 0, \quad (4.8c)$$

where

$$\delta = -(\varepsilon/2)\left[-2(1 - C)/3\right]^{-3/2}. \quad (4.9)$$

The nature of these singular points can be determined by examining the behavior of the motion in their vicinity. In order to do this we let

$$a = a_i + \xi \quad , \quad b = \eta \; , \quad (4.10)$$

where a_i is one of the singular points and ξ and η represent small displacements from this singular point. Upon substituting equations (4.10) into equations (4.3) and retaining terms through first order in ξ and η we find that

$$\xi' = -\left[1 - C + (a_i^2/2)\right]\eta \; , \quad (4.11a)$$

$$\eta' = \left[1 - C + (3a_i^2/2)\right]\xi \; . \quad (4.11b)$$

We seek solutions in the form

$$\xi = \xi_0 e^{\lambda\tau} \quad , \quad \eta = \eta_0 e^{\lambda\tau} \; . \quad (4.12)$$

These solutions are valid if

$$\lambda = \pm\left\{-\left[1 - C + (a_i^2/2)\right]\left[1 - C + (3a_i^2/2)\right]\right\}^{1/2}. \quad (4.13)$$

When the values of a_i as expressed by equations (4.8) are substituted into equations (4.13) we find that we can classify the singular points as to their stability. This classification is given in Table I.

Singular Points	λ	Nature of Singular Point
a_1	**complex**	Center (stable)
a_2	real	Saddle point (unstable)
a_3	**complex**	Center (stable)

Table I. Classification of singular points

Table I. shows that the trajectories must close about the point a_3 and they must also close about the point a_1. Furthermore, it follows from Eq. (4.5) that, for large values of R, the trajectories have the form

$$R^4 = \text{constant} \quad (4.14)$$

and are, therefore, circles centered about the origin. The manner in which these requirements are satisfied is shown in Figure 1. There several trajectories are plotted for a particular physical situation. The parameters such as the

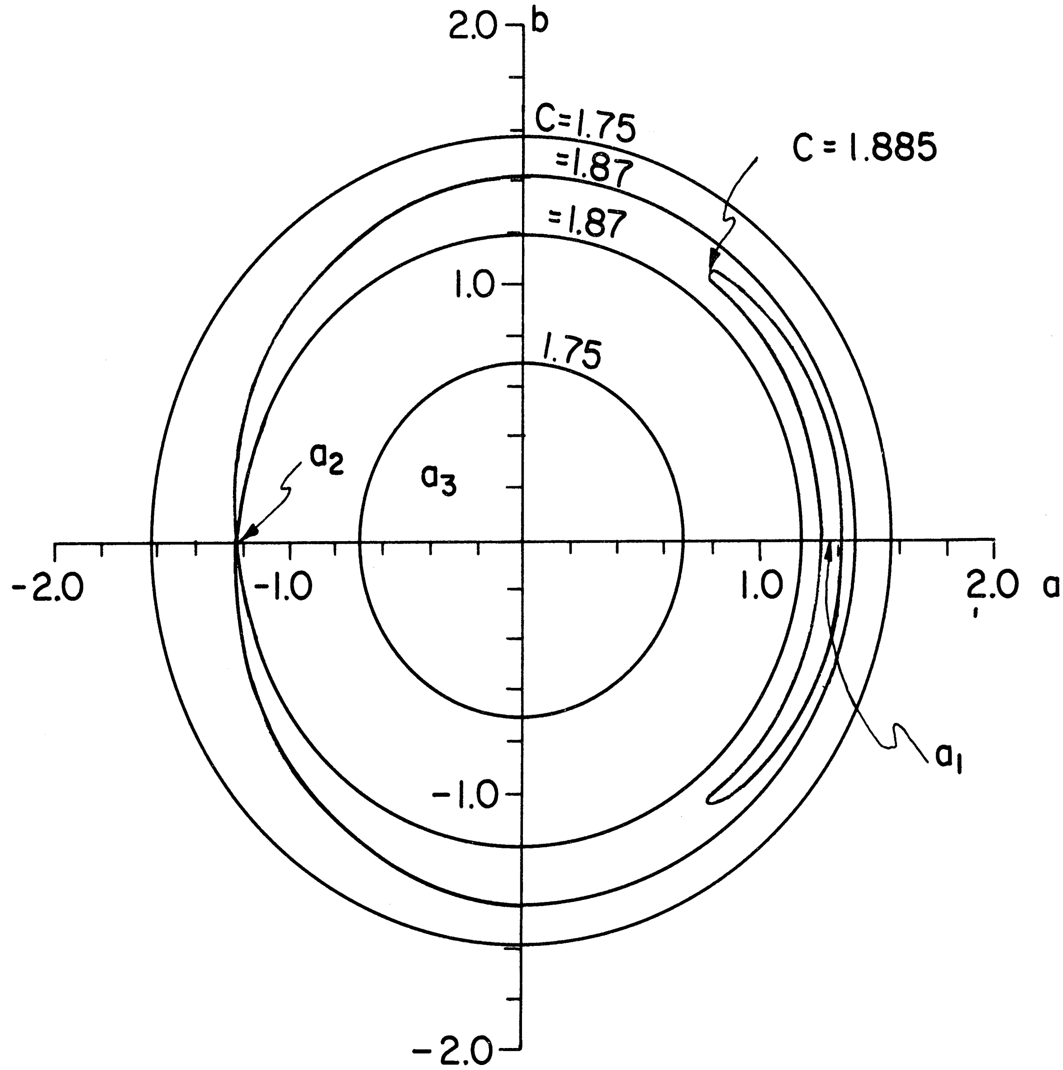

Figure 1. Trajectories of a 50 Mev proton in the a-b plane.

$k = 6.28 \times 10^{-8}$ cm^{-1}, $\omega_o = 370$ rad/sec, $\varepsilon = .01$.

energy, background field strength, etc., have been assigned values which are appropriate to a proton which is moving in the inner Van Allen belt at a distance of two earth radii. However, the value of ε which was used in Figure 1. was chosen to be about ten times larger than what one would expect at two earth radii.[4] This larger value of ε was used to facilitate the plotting of the trajectories.

The trajectories in Figure 1. consist of a family of closed curves. This means that K, and consequently V, is a periodic function of time; we shall obtain the period in the next section. The trajectories can be divided into two groups: those which are centered about the point a_3 and those which are centered about the point a_1. The motion corresponding to the first group is non-synchronous since the difference angle θ increases without bound. The motion corresponding to the second group is synchronous since, for these trajectories, the angle θ oscillates between well defined limits. The synchronous and non-synchronous regions are separated by the trajectory, called a separatrix, which passes through the unstable point a_2.

The largest fluctuations in K occur on trajectories which pass close to the separatrix. As one moves away from the separatrix into the non-synchronous regions the trajectories rapidly become circles centered about the point a_3. As one moves away from the separatrix into the synchronous region the fluctuations in K and θ become smaller until, at the point a_1, they vanish. The separatrix, therefore, determines the range of values of K and θ for which maximum resonance occurs. In the following sections we shall obtain the time dependence of K and we shall estimate the total fluctuation in K under resonance conditions.

5. TIME DEPENDENCE OF THE MOTION

In order to find the explicit time dependence of K and an expression for the period T we introduce a new variable

$$S = R^2 = (2k^2/m\omega_o)K. \tag{5.1}$$

It follows from equations (3.8) that

$$S' = -\varepsilon R \sin \theta . \tag{5.2}$$

The right hand side of Eq. (5.2) can be expressed as a function of S alone by making use of equations (4.5) and (4.6). A straightforward calculation gives

$$S' = -(1/4)\Big[-S^4 - 8LS^3 - (8N + 16L^2)S^2 - 16(2LN - \varepsilon^2)S - 16N^2\Big]^{1/2}, \tag{5.3}$$

where

$$L = 1 - C , \quad N = C^2 - 2Ch/I. \tag{5.4}$$

If we denote the roots of the quartic in the square bracket in (5.3) by S_1, S_2, S_3, and S_4, then Eq. (5.3) becomes

$$S' = -(1/4)\left[(S_1 - S)(S - S_2)(S - S_3)(S - S_4)\right]^{1/2}. \quad (5.5)$$

This is a first order differential equation for $S(\tau)$ which is solvable in terms of elliptic functions. The solutions depend upon the nature of the roots S_1, S_2, S_3, and S_4; we must distinguish the case of two real roots from the case of four real roots.

Case of two Real Roots

It should be clear from Figure 1. that two real roots corresponds to motion in the synchronous region. We shall arrange the real roots S_1 and S_2 such that $S_1 \geq S_2$. The complex roots S_3 and S_4 may be written as

$$S_3 = m + in, \quad S_4 = m - in. \quad (5.6)$$

In this case Eq. (5.5) has the solution[5]

$$S(\tau) = \frac{S_1B + S_2A + (S_2A - S_1B)\mathrm{cn}\left[(\tau - \tau_0)/4g\right]}{A + B + (A - B)\mathrm{cn}\left[(\tau - \tau_0)/4g\right]}, \quad (5.7)$$

where

$$A^2 = (S_1 - m)^2 + n^2, \quad B^2 = (S_2 - m)^2 + n^2, \quad (5.8)$$

and

$$g = (AB)^{-1/2}. \tag{5.9}$$

The function cn(x) is a Jacobi elliptic function. The modulus κ of the elliptic function is given by

$$\kappa^2 = \frac{(S_1 - S_2)^2 - (A - B)^2}{4AB} . \tag{5.10}$$

The constant τ_0 is chosen to satisfy the initial conditions. The function cn(x) is periodic in x with period $4K$, K being the complete elliptic integral of the first kind. It follows that $S(\tau)$, and hence $K(\tau)$, is a periodic function of τ with period T given by

$$T = 16 K(AB)^{-1/2} , \tag{5.11}$$

where the modulus κ of the complete elliptic integral is given by Eq. (5.10).

Case of four Real Roots

Four real roots corresponds to motion in the non-synchronous region. We consider first the case where $S_1 \geq S \geq S_2 \geq S_3 \geq S_4$. In this case the solution of Eq. (5.5) is[6]

$$S(\tau) = \frac{S_2(S_1 - S_3) - S_3(S_1 - S_2)\mathrm{sn}^2\left[(\tau - \tau_0)/4g\right]}{S_1 - S_3 - (S_1 - S_2)\mathrm{sn}^2\left[(\tau - \tau_0)/4g\right]} , \tag{5.12}$$

where

$$g = 2\left[(S_1 - S_3)(S_2 - S_4)\right]^{-1/2}. \qquad (5.13)$$

The modulus κ of the Jacobi elliptic sine function is given by

$$\kappa^2 = \frac{(S_1 - S_2)(S_3 - S_4)}{(S_1 - S_3)(S_2 - S_4)} . \qquad (5.14)$$

The function $sn^2(x)$ is periodic in x with period 2 K, K being the complete elliptic integral of the first kind. It follows that $S(\tau)$, and hence $K(\tau)$, is a periodic function of τ with period

$$T = 32K\left[(S_1 - S_3)(S_2 - S_4)\right]^{-1/2}. \qquad (5.15)$$

In the remaining case where $S_1 \geq S_2 \geq S_3 \geq S \geq S_4$ we find that [7]

$$S(\tau) = \frac{S_4(S_1 - S_3) + S_1(S_3 - S_4)sn^2\left[(\tau - \tau_0)/4g\right]}{S_1 - S_3 + (S_3 - S_4)sn^2\left[(\tau - \tau_0)/4g\right]}, \qquad (5.16)$$

where g is given by Eq. (5.13) and where the modulus κ of the Jacobi elliptic sine function is given by Eq. (5.14). It follows that $S(\tau)$ is a periodic function of τ with period given by Eq. (5.15).

We have now determined the function $S(\tau)$ in the synchronous and non-synchronous regions. A knowledge of the function $S(\tau)$ immediately determines $K(\tau)$, $V(\tau)$, and $\cos\theta(\tau)$. These latter functions allow us to find the explicit time dependence of $\phi(\tau)$ and $kU(\tau)$. We shall not attempt to do this since the resulting expressions would add little to our understanding of the motion. The significant point is that $K(\tau)$ and $V(\tau)$ are strictly periodic functions of τ. In the next section we shall estimate the maximum fluctuation in $K(\tau)$.

6. BEHAVIOR UNDER RESONANCE CONDITIONS

Exact resonance occurs when the particle traverses a single period of the sinusoidal field in one cyclotron period. This exact resonance is nearly fulfilled when

$$kV/m\omega_0 = 1. \tag{6.1}$$

If we substitute condition (6.1) into Eq. (2.7) and neglect the first order term we find that

$$k^2K/m\omega_0 = (k^2h/m\omega_0) - (1/2). \tag{6.2}$$

Equations (4.5), (4.6), (6.1), and (6.2) allow us to find a value of the constant C which corresponds to near resonance. The appropriate value of C is found to be

$$C = d + 1/2, \tag{6.3}$$

where $d = k^2h/m\omega_o$. With this value of C the quartic in Eq. (5.3) becomes

$$S^4 + (4 - 8d)S^3 + (6 - 24d + 24d^2)S^2 \tag{6.4}$$

$$+ (4 - 24d + 48d^2 - 32d^3 - 16\varepsilon^2)S$$

$$+ 1 - 8d + 24d^2 - 32d^3 + 16d^4.$$

The roots of this quartic are approximately as follows :

$$S_{1,2} = (2d - 1) \pm 2(2d - 1)^{1/4}\varepsilon^{1/2} + O(\varepsilon), \tag{6.5}$$

$$S_{3,4} = (2d - 1) \pm i2(2d - 1)^{1/4}\varepsilon^{1/2} + O(\varepsilon). \tag{6.6}$$

Since two roots are complex it follows that the trajectory corresponding to condition (6.3) lies in the synchronous region. Now, by definition,

$$d = k^2h/m\omega_o = k^2H/m\omega_o^2 , \tag{6.7}$$

where H is the energy. Thus

$$2d - 1 = (2k^2/m\omega_o^2)\left[H - (m\omega_o^2/2k^2)\right] \tag{6.8}$$

$$= (2k^2/m\omega_o^2)\left[(mv_{\perp_o}^2/2) + (p_{z_o}^2/2m) - (m\omega_o^2/2k^2) + O(\varepsilon)\right],$$

where $v_{\perp_o}$ is the initial transverse velocity and p_{z_o} is the initial longitudinal momentum. According to Eq. (6.1)

$$p_{z_o} = m\omega_o/k + O(\varepsilon). \tag{6.9}$$

Upon substituting Eq. (6.9) into Eq. (6.8) we find that

$$2d - 1 = k^2 v_{\perp_o}^2/\omega_o^2 + O(\varepsilon). \tag{6.10}$$

We now define the relative fluctuation ΔK in K as follows:

$$\Delta K = \frac{2(K_{max} - K_{min})}{(K_{max} + K_{min})} = \frac{2(S_1 - S_2)}{(S_1 + S_2)}\ . \tag{6.11}$$

Upon making use of equations (6.5), (6.10), and (6.11) we find that

$$\Delta K = 4(kv_{\perp_o}/\omega_o)^{-3/2}\varepsilon^{1/2} + O(\varepsilon). \tag{6.12}$$

If we denote the initial value of K by K_o, then $v_{\perp_o}$ and K_o are related by the expression

$$mv_{\perp_o}^2/2 = K_o\omega_o + O(\varepsilon). \tag{6.13}$$

It follows that

$$kv_{\perp_o}/\omega_o = k(2K_o/m\omega_o)^{1/2} + O(\varepsilon) \tag{6.14}$$

$$= R_o + O(\varepsilon),$$

where R_o is 2π times the ratio of the initial cyclotron radius to the wavelength of the disturbance. Upon substituting Eq. (6.14) into Eq. (6.12) we find that

$$\Delta K = 4R_o^{-3/2}\varepsilon^{1/2} + O(\varepsilon). \qquad (6.15)$$

Thus the relative fluctuation in K under resonance conditions depends not only on ε but also on the ratio of the cyclotron radius to the wavelength of the periodic disturbance. This dependence of the relative change in K upon R_o is not surprising. An increase of the wavelength of the disturbance requires a corresponding increase in the longitudinal particle velocity in order to achieve resonance. Associated with the increase in the longitudinal velocity is a decrease in the **transverse** velocity and hence **a decrease** in absolute value of K. This decrease in absolute value contributes to the increase in the relative change.

The ideas developed above are best illustrated through an example. We first observe from Eq. (6.15) that relative fluctuations of order unity will occur when

$$R_o \approx (16\varepsilon)^{1/3}. \qquad (6.16)$$

Now consider the trajectories plotted in Figure 1. These trajectories correspond to a 50 Mev proton moving in a disturbance whose wavelength is 10^8 cm (1000 km) and to a field strength ratio $\varepsilon = .01$. It follows from Eq. (6.16) that, for $\varepsilon = .01$, relative fluctuations of order unity will occur when

$$R_o = (.16)^{1/3} = .543 \qquad (6.17)$$

lies in the resonance region. However, it is clear from Figure 1. that $R_o = .543$ lies well outside the resonance region. Thus $K(\tau)$ for a 50 Mev proton moving in a periodic disturbance whose wavelength is 1000km undergoes only relatively small fluctuations.

Let us now increase the wavelength of the disturbance to 1.5×10^8 cm (1500 km) and hold the other parameters fixed. The trajectories in the a-b plane for this situation are plotted in Figure 2. Inspection of Figure 2. reveals that $R_o = .543$ lies in the resonance region. Thus, as is evident from Figure 2., $K(\tau)$ for a 50 Mev proton moving in a periodic disturbance whose wavelength is 1500 km can undergo relative fluctuations of order unity. Here we have graphic evidence of the influence of R_o on the relative fluctuation of $K(\tau)$. We shall conclude this section with a few remarks relating our results to previously published work.

The average orbital magnetic moment $\mu(\tau)$ is related to $K(\tau)$ by the equation

$$\mu(\tau) = eK(\tau)\ /mc. \qquad (6.18)$$

Therefore, what has been said above about $K(\tau)$ also holds

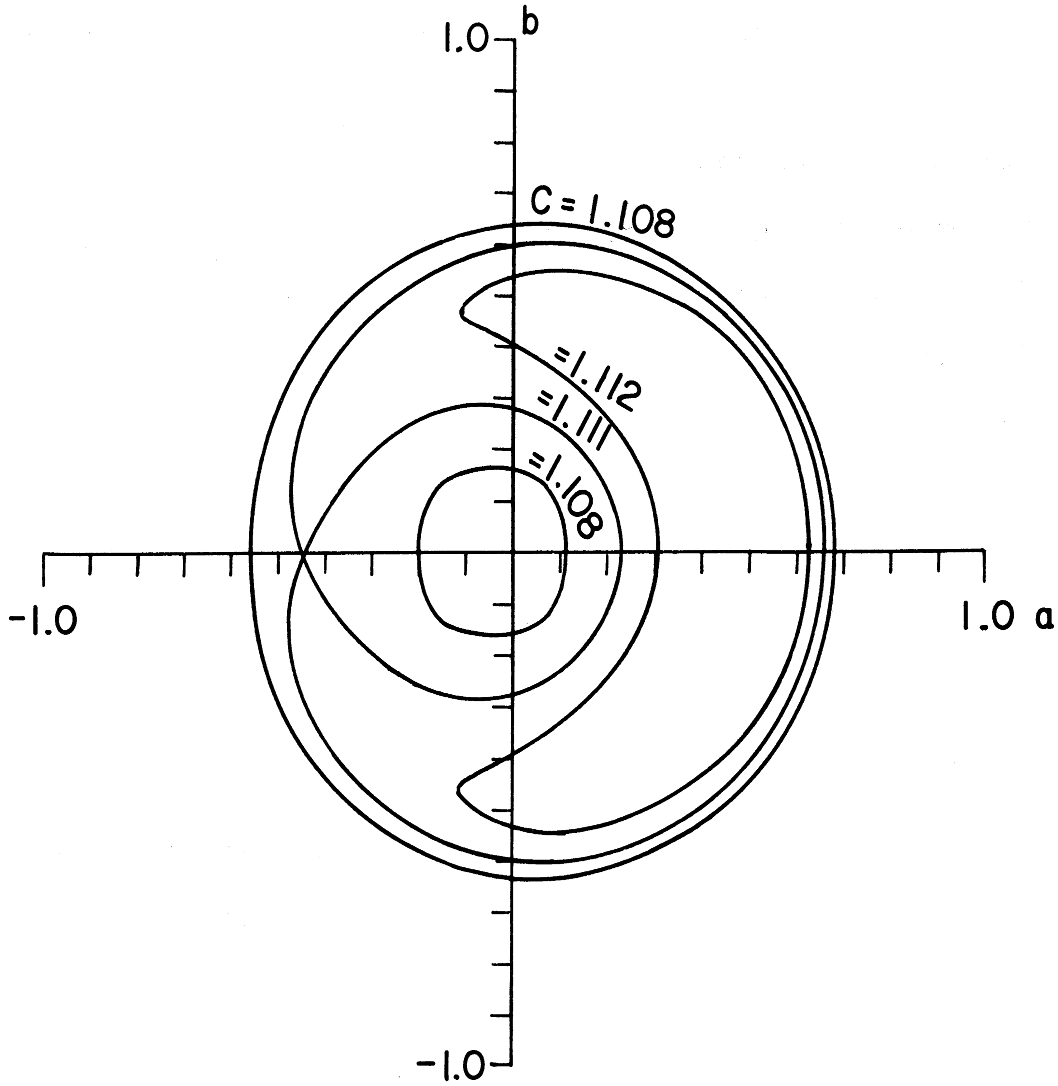

Figure 2. Trajectories of a 50 Mev proton in the a-b plane.
$k = 4.19\text{x}10^{-8}\ \text{cm}^{-1}$, $\omega_o = 370$ rad/sec, $\varepsilon = .01$.

for $\mu(\tau)$. This means that $\mu(\tau)$ is a periodic function of τ. This result was not obtained in either the work of Dragt or that of Wentzel. Furthermore, the relative fluctuation in $\mu(\tau)$ (that is the fluctuation measured with respect to the mean value of μ) depends upon the ratio of the cyclotron radius to the wavelength of the periodic disturbance. This fact was not appreciated by Dragt or Wentzel. For example, Dragt [8] concludes that, when ω_0 = 370rad/sec and $\varepsilon \leq .01$, the orbital magnetic moment of a 160 Mev proton will be significantly effected by a disturbance having a wavelength of 1000 km. However, we have shown in Figure 1. that, under the same conditions, the magnetic moment of a 50 Mev proton is only slightly effected by a 1000 km wave. The resonant cyclotron radius of a 160 Mev proton is greater than that of a 50 Mev proton. It follows, therefore, from Eq. (6.15) that the effect of a 1000 km wave on a 160 Mev proton will be smaller than its effect on a 50 Mev proton. Thus we are forced to conclude that a 1000 km wave will not appreciably effect the orbital magnetic moment of a 160 Mev proton under the conditions stated above.

It is apparent from the above discussion that the interaction between a charged particle and a weak, spatially periodic magnetic field is considerably different from that envisioned by Dragt and Wentzel. This means that their explanations of the origin of the distribution of protons in the inner Van Allen belt can not be considered satisfactory. It does not mean that their work is without merit.

The work of Dragt is especially noteworthy since he makes substantial progress in accounting for the proton distribution in the inner Van Allen belt via heuristic arguments. What is evident from all of this is that the origin of the distribution of protons in the inner Van Allen belt must be carefully re-examined in view of the true nature of the interaction between the protons and spatially periodic magnetic fields. Until this is done the origin of the proton distribution in the inner Van Allen belt must remain an open question.

7. CONCLUSION

We have shown that the perturbation theory presented in I yields a complete first order solution for the motion of a charged particle in a constant magnetic field on which is superimposed a weak, spatially periodic magnetic field. The significant result from a physical viewpoint is the periodic behavior of the secular motion. It is hoped that we have succeeded in showing that this periodic behavior of the secular motion gives rise to serious objections to previous work concerning the stability of protons in the inner Van Allen belt.

Footnotes

* Supported in part by the National Science Foundation.

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1. T. P. Coffey and G. W. Ford, J. Math. Phys. ---.
2. A. J. Dragt, J. Geophys. Research, 66, (1961).
3. D. Wentzel, J. Geophys. Research, 66, (1961)
4. Dragt, op cit., p. 1647.
5. P. F. Byrd and M. D. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists (Springer-Verlag, Berlin, 1954), Eq. (259.00), p. 133.
6. Byrd and Friedman, op cit., Eq. (256.00), p. 120.
7. Byrd and Friedman, op cit., Eq. (252.00), p. 103.
8. Dragt, op cit., p. 1646.