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Personal Reminiscences of the Birth of Algebraic *K*-theory*

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Abstract. These informal reminiscences, presented at the ICTP 2002 Conference on algebraic *K*-theory, recount the trajectory in the author's early research, from work on the Serre Conjecture (on projective modules over polynomial algebras), via ideas from algebraic geometry and topology, to the ideas and constructions that eventually contributed to the founding of algebraic *K*-theory. The solution of the Congruence Subgroup Problem is presented as a pivotal event.

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The organizers, Remi Kuku, Claudio Pedrini, and Max Karoubi, of the ICTP 2002 Conference on Algebraic *K*-theory graciously used the occasion to celebrate my 70th birthday. During the ceremonial session there I offered some impromptu personal reminiscences about how my own research led me into what came to be called algebraic *K*-theory, and about the early development of the subject. The interest shown there led the organizers to request that I record those reminiscences for the conference proceedings. That is the genesis of the still somewhat informal account that follows.

In the late 1950s and early 1960s, a period of great mathematical ferment, there were several interwoven mathematical themes in play from which algebraic *K*-theory gradually took shape.

- Various motifs in algebraic and geometric topology – generalized cohomology theories, fiber bundles, simple homotopy theory.
- The precocious birth of homological algebra, and the development of category theory, in part for an axiomatic treatment of homological algebra. In particular, the paradigm shifting intrusion of homological methods into commutative algebra.
- Grothendieck's radical and visionary re-founding and expansion of algebraic geometry.

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At the University of Chicago (1955–1959), my cohort of graduate students was gently initiated by Kaplansky, with assistance from Dick Swan, into the then current development of homological methods in commutative algebra. This context gave great prominence to projective modules, and raised natural questions about their structure. The best known of these, 'Serre's Conjecture', [14], asked whether (finitely generated) projective modules over polynomial algebras (over a field) are free; it was finally affirmed, over two decades later, independently by Quillen [12] and Suslin [17]. I had been tenaciously attracted by this tantalizingly simple question, apparently congenial to my algebraic training. But my naïve algebraic efforts made little headway.

Two papers transformed my thinking about this, and in ways that helped seed the later development of algebraic K-theory. One was the landmark work of Grothendieck (published in a paper by Borel and Serre [7]) on his generalized Riemann–Roch theorem, in which he introduced the 'Grothendieck group' $K(X)$ of vector bundles on a scheme X. This inspired the creation by Atiyah and Hirzebruch of topological K-theory (see [1]). One of Grothendieck's results, a kind of algebraic homotopy invariance of $K(X)$ for smooth schemes, implied that, for a regular commutative ring A, projective modules over a polynomial extension $A[t]$ are, *stably*, obtained by base change from projective A-modules. This implies that projective modules over polynomial algebras over a field are stably free, an affirmative answer to the 'stabilization' of Serre's Conjecture.

The second influence was a talk of Serre in the Séminaire Dubreil–Pisot [15], in which he showed that 'a large projective module has a free direct summand'. More precisely, if A is a commutative noetherian ring of dimension d, and if P is a projective A-module of rank $r > d$, then P has a direct summand isomorphic to A. This result, unlike anything I had seen in algebra, was best understood when looked at topologically: a vector bundle whose fiber dimension exceeds that of the base has a non-vanishing section. In topology (say with CW complexes) one constructs such a section on successive skeleta, and the obstruction to extension to a cell from its boundary lies, under our dimension assumptions, in a low dimensional homotopy group of a high dimensional sphere, which vanishes.

These simplicial arguments do not directly work in algebra, where the functions behave like analytic functions, in that they are globally determined by their germs, and so cannot be pieced together with things like partitions of unity. Serre invented an alternative approach that worked algebraically. Using the maximal ideal spectrum X, he first produces a section whose zero locus has smaller dimension than X. Then he further reduces the dimension of the zero locus with a second section, sufficiently independent of the first so that the sum does not upset what the first section accomplished. In this way, an induction finally produces the desired nonvanishing section, provided that there is sufficient room to achieve all of the successive linear independence conditions, which is where the assumption that $r > d$ comes in. The above results reduced the Serre Conjecture to a cancellation problem for projective modules of bounded rank.

From where I had started, both of the above contributions to the Serre Conjecture were quite dramatic, because they were strikingly general, and used ideas that were well beyond my algebraic horizon. I took this as a signal from the mathematical spirits: this was a territory where topological thinking was appropriate, and more advanced than the algebraic; I should more seriously and systematically heed topological results and methods. The first benefit of this reorientation allowed me to see Serre's result as analogous to part of a general pattern of stability theorems in topology. It was easy to formulate algebraic analogues of them, and to prove the companion to Serre's result, a cancellation theorem, corresponding to uniqueness up to homotopy of the non-vanishing section when $r > d + 1$ [3].

Encouraged by this, I began to try more systematically to find algebraic translations of the results and constructions of topological K-theory, and to prove the resulting, topologically inspired, conjectures. The fundamental question that emerged was how to construct algebraic analogues of the higher topological K-functors. In topology, this came simply by applying K_0 to successive suspensions; but there was not yet known a sufficiently well behaved algebraic analogue of the suspension to do this. This moment, for me, was the birth of thinking about an 'algebraic K-theory'. Note that this was in the nature of following a general kind of mathematical lead, no longer immediately relevant to the questions about projective modules with which I had started, nor with any *a priori* idea of what mathematical interest it might have. It was more in the nature of an open ended mathematical exploration.

In the absence of adequate ideas for all of the higher algebraic K-functors, I was at least able, in collaboration with Schanuel [2], to fashion a satisfactory definition of K_1, and, in [3], to construct, using relative versions of K_0 and K_1, as much of 'long' exact sequences as made sense. Moreover, Alex Heller, Dick Swan and I were able to prove a K_1-analogue of Grothendieck's homotopy invariance theorem, even generalized to the noncommutative and nonregular case, but with a correction term, coming from unipotent matrices, which vanished for regular rings [5]. The generalization of this result to rings of Laurent polynomials turned out to be of significant interest in topology, for the simple homotopy theory of spaces fibered over a circle. It also had connections with ideas related to Bott periodicity, reflected in the functorial appearance of $K_0(A)$ as a direct summand of $K_1(A[t, t^{-1}])$, and later I used this, by backward iteration, to construct a definition of negative algebraic K-groups [4].

This was, for me, a novel mathematical predicament. While K_1 had been constructed largely as an intellectual exercise, it turned out to be a quite natural and somewhat classical looking algebraic object. Indeed, for the case of an integer group algebra, $\mathbf{Z}\pi$, the quotient $K_1(\mathbf{Z}\pi)/\pm\pi$ (of course not denoted this way) had long ago been introduced by Whitehead, as the vehicle for his simple homotopy invariants, and it was first algebraically studied by his student, G. Higman in 1940 [8]. This work provided some of the basic tools of the subject. But for general rings A, $K_1(A)$, as an instrument for understanding $\mathrm{GL}_n(A)$, could now be seen with an enlarged significance, and with the added perspective of stabilization, which came naturally from topology, but was not part of the traditional algebraic repertoire. Moreover, while the basic questions about K_0 were easily answered for rings like **Z**, the basic questions about (relative) K_1's were not, and they turned out to be essentially a stabilized version of a half century old problem – the so-called 'Congruence Subgroup Problem (CSP)' – about congruence subgroups of unimodular groups over **Z** and, more generally, over integers in number fields. This led to a major, eventually collaborative, research effort.

This story may be of interest to recount in more detail, since it largely stimulated much of the subsequent development of algebraic *K*-theory. Let *A* be the ring of integers in a number field. For an ideal *J* in *A*, put

$$\mathrm{SL}_n(J) = \mathrm{Ker}\left(\mathrm{SL}_n(A) \to \mathrm{SL}_n\left(\frac{A}{J}\right)\right),$$

the congruence subgroup of level *J*. The CSP asks whether every finite index subgroup of $\mathrm{SL}_n(A)$ contains $\mathrm{SL}_n(J)$, for some $J \neq 0$. The group $\mathrm{SL}_n(J)$ contains a subgroup, $E_n(J)$, generated by certain conjugates of 'elementary' matrices, and we introduce the coset space,

$$S_n(J) = \frac{\mathrm{SL}_n(J)}{E_n(J)}.$$

The direct limit of the stabilization maps,

$$\sigma_n : S_n(J) \to S_{n+1}(J),$$

is just the relative group *K*-group, $\mathrm{SK}_1(J)$.

The stability theorems that I proved for K_1 showed that σ_n is surjective for $n \geqslant 2$, and injective for $n \geqslant 3$, hence $S_n(J) \cong \mathrm{SK}_1(J)$ for $n \geqslant 3$. It follows further that

$$\mathrm{SK}_1(J) \cong S_2'(J) = \frac{\mathrm{SL}_2(J)}{(\mathrm{SL}_2(J) \cap E_3(J))}.$$

Moreover, it was easily shown that, for $n \geqslant 3$, the CSP is equivalent to the triviality of $S_n(J)$, hence of $\mathrm{SK}_1(J)$. (It was known classically the CSP fails completely for $n=2$, for example when $A=\mathbf{Z}$. The case $n=2$ was later analyzed in detail by Serre [16].) Thus, using stability results, the CSP was reduced to the calculation of the group $S_2'(J)$ above, essentially a problem in $\mathrm{SL}_2(A)$.

Some of these ideas were being explored also by Mennicke. Independently, he, on the one hand, and Michel Lazard, Serre, and myself, on the other (using profinite completions and group cohomology) solved the CSP affirmatively for the case $A=\mathbf{Z}$. But the case of general A resisted, and it began to exhibit intriguing connections with class field theory, notably Legendre symbols and the explicit reciprocity laws.

Meanwhile, Milnor had become interested in this question, and he made some decisive observations and notational innovations during our collaboration on this while I was at the IAS (1965–1966). Let

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(J).$$

It is easily seen that the element $[\propto]$ of $S_2'(J)$ that it represents depends only on (a, b). Milnor's main observation, from Mennicke's calculations, was the striking fact that this function of (a, b) is *multiplicative* in b (in fact it is *bi-multiplicative* in the two variables). These relations, together with invariance under certain elementary operations, thus presented a certain Abelian group, say $S_2''(J)$, of which $S_2'(J)$ is a quotient. Milnor denoted the defining generators of $S_2''(J)$ by

$$\begin{bmatrix} b \\ a \end{bmatrix}$$

which we came to call 'Mennicke symbols'.

The defining relations among the Mennicke symbols naturally suggested the Legendre symbols,

$$\left(\frac{b}{a}\right)_m$$

with values in the group μ_m of mth roots of unity, assumed to belong to F. These are, by construction, multiplicative in each variable, and depend on b only mod a. But, for the Legendre symbols to define a homomorphism from $S_2''(J)$ to μ_m, we need them to depend on a only mod b – an unnatural looking condition. To get this we would like to use reciprocity, relating the two symbols with a and b reversed. There are two obstacles to this working. One is a correction factor coming from primes dividing m. This obstacle can be avoided if J is sufficiently divisible by m. The other comes from the contribution to the reciprocity law coming from the primes at infinity, of which only the real places are relevant. The only way to escape this obstacle is to assume that there are no real places, i.e. that F is totally imaginary. Under these conditions, and with J sufficiently divisible by m, the Legendre symbol above then defines a surjective homomorphism, $S_2''(J) \to \mu_m$.

When F is not totally imaginary, we were able to use these arguments to show that $S_2''(J) = 0$, and so $\mathrm{SK}_1(J) = 0$ for all J, and we have an affirmative solution of the CSP for $\mathrm{SL}_n(A)$ for all $n \geqslant 3$. (The same arguments apply when A is the affine ring of a smooth curve, in a global field of characteristic $p > 0$.)

There remained two crucial steps to complete the program for totally imaginary fields. One, based on a very opportune paper of Kubota [9], was to sharpen the stability theorems by showing that $S_2''(J) \to S_2'(J)$ is an isomorphism, and so $S_2''(J) \simeq \mathrm{SK}_1(J)$.

The final step, achieved in collaboration with Serre, was to show, for J sufficiently 'deep', and μ_m the full group of roots of unity in F, that $S_2''(J) \to \mu_m$ is an isomorphism, in other words that the Legendre symbol is in fact a universal Mennicke symbol. This involved somewhat delicate number theoretic calculations with Hilbert symbols, and the use of Chebotarev density. What these arguments further show is a new number theoretic result, a kind of uniqueness theorem for the reciprocity laws, viewed as a global relationship between all of the local Hilbert symbols. This result was also obtained, in a different form, by Cal Moore [11].

I found this network of mathematical connections to be quite stunning. A calculation of relative K_1's for integers in global fields was reduced, by stability theorems, to the classical Congruence Subgroup Problem, essentially an algebraic problem about the normal subgroup structure of matrix groups. There seemed (to me) to be no reason *a priori* to expect this to be significantly related to such delicate number theoretic objects as the explicit reciprocity laws of class field theory. The results not only establish such a relationship, but one so intrinsic that one would have had to invent/discover the reciprocity laws in order to solve the CSP, if they had not already been known.

This work stimulated other investigations, for example the CSP for other algebraic groups. In K-theory, Milnor, influenced also by the work of Cal Moore [11], which was in turn dependent on earlier work of Chevalley and Robert Steinberg, was led to propose a general definition of $K_2(A)$, A any ring, as the Schur multiplier of the stable elementary group $E(A) = [\mathrm{GL}(A), \mathrm{GL}(A)]$, generated by elementary matrices. He established fundamental properties of this functor, and carried out the first significant calculations. All of this was elegantly exposed in his Princeton Lecture Notes [10].

By now, the accumulated results and applications of lower K-groups stimulated an intense search for the full blown algebraic K-theory. Many definitions were proposed, and compared. These efforts were given a spectacular synthesis and definitive resolution with the work of Quillen, for which he received the Fields Medal (see [13]). He provided two fundamentally different ways of defining higher algebraic K-groups, one homotopy theoretic, which gave the answer he insisted on for finite fields, the other category theoretic, which brought with it all of the fundamental computational tools with which the study of the lower K-groups had advanced. With this, the gestation of algebraic K-theory was brought to satisfying term, and the field entered a healthy infancy. This 2002 ICTP conference, in particular the spectacular work of Voevodsky and Suslin, amply demonstrates that the field has now also achieved a healthy and vigorous young adulthood.

References

1. Atiyah, M. F.: *K*-theory, In: D. W. Anderson and W. A. Benjamin (eds), Lecture Notes in Math. Springer, New York, 1967.
2. Bass, H. and Schanuel, S.: The homotopy theory of projective modules, *Bull. Am. Math. Soc.* **68** (1962), 425–428.
3. Bass, H.: *K*-theory and stable algebra, *Publ. Math. Inst. Hautes Etudes Sci*. **22** (1964), 5–60.
4. Bass, H.: In: W. A. Benjamin (ed.), *Algebraic K-theory*, New York, 1968.
5. Bass, H., Heller, A. and Swan, R. G.: The Whitehead group of a polynomial extension, *Publ. Math. Inst. Hautes Études Sci.* **22** (1964), 61–79.
6. Bass, H., Milnor, J. and Serre, J.-P.: Solution of the congruence subgroup problem for SL_n $(n \geqslant 3)$ and Sp_{2n} $(n \geqslant 2)$, *Publ. Math. Inst. Hautes Études Sci.* **33** (1967), 59–137.
7. Borel, A. and Serre, J.-P.: Le théoreme de Riemann-Roch, *Bull. Soc. Math. France* **86** (1958), 97–136.
8. Higman, G.: The units of group rings, *Proc. London Math. Soc.* **46** (1940), 231–248.

9. Kubota, T.: Ein arithmetischer Satz über eine Matrizengruppe, *J. Reine Angew. Math.* **222** (1966), 55–57.
10. Milnor, J.: *Introduction to Algebraic K-theory,* Ann. Math. Stud. 72, Princeton University Press, Princeton, NJ, 1971, pp. xiii + 184.
11. Moore, C.: Group extension of p-adic and adelic linear groups, *Publ. Math. Inst. Hautes Études Sci.* **35** (1968), 157–222.
12. Quillen, D.: Projective modules over polynomial rings, *Invent. Math.* **36** (1976), 167–171.
13. Quillen, D.: Higher algebraic *K*-theory, Vol. 1, *Proc. ICM*, Vancouver, BC, 1974, pp. 171–176.
14. Serre, J.-P.: Faisceaux algébriques cohérents, *Ann. Math.* **61** (1955), 197–278.
15. Serre, J.-P.: Modules projectifs et espaces fibrés a fibre vectorielle, Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23, Secrétariat Mathématique, Paris, 1958, 18 p.
16. Serre, J.-P.: Le probleme de groupes de congruence pour SL_2, *Ann. Math.* 92 (1970), 489–527.
17. Suslin, A.: Projective modules over polynomial rings are free, *Dokl. Adad. Nauk SSSR* **229**(5) (1976), 1063–1066 (Russian).