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Absolute Stable Rank and Quadratic Forms over Noncommutative Rings

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Abstract. Given an associative ring A, let $\operatorname{asr}(A)$ denote the absolute stable range of A, as defined in [5]. We prove that $\operatorname{asr}(A) \leqslant 1 + \operatorname{Kdim} A$ if A is a right Noetherian ring, and that $\operatorname{asr}(A) \leqslant 1 + \text{cl-Kdim } A$ if A is an affine PI algebra. Combined with results from [5], this provides a cancellation theorem ('Witt cancellation') for quadratic spaces defined over such a ring A.

Key words. Absolute stable rank, Witt cancellation, quadratic spaces, Noetherian rings, affine PI rings.

0. Introduction

Let A be an associative ring (with identity) and let $\mathcal{M}_A$ denote the set of maximal right ideals of A. Given a right ideal I of A, write

$$J(I) = \bigcap \{M \in \mathcal{M}_A : I \subseteq M\}.$$

In this notation, we set $A = J(A)$; thus the Jacobson radical of A is denoted by $J(0)$. If $I = J(I)$, then I is called a *Jacobson right ideal.* The paper [5] considers the notion of the *absolute stable rank of A*, which is written $\operatorname{asr}(A)$ and defined as follows:

$\operatorname{asr}(A) \leqslant n$ if, given any $a_0, \dots, a_n \in A$, then there exist $t_0, \dots, t_{n-1} \in A$ such that

$$J\left(\sum_0^n a_i A\right) = J\left(\sum_0^{n-1} (a_i + a_n t_i) A\right).$$

One of the main aims of [5] is to obtain bounds for $\operatorname{asr}(A)$ for various classes of noncommutative rings A; in particular, they show that: (i) $\operatorname{asr}(A) \leqslant 1 + d$ whenever A is a module finite algebra over a commutative Noetherian ring R with $\dim(\operatorname{maxspec} R) = d$, and (ii) $\operatorname{asr}(A) = 1$ if A is a semi-local ring (see [5], Theorems 3.1 and 2.4], respectively). The aim of this note is to show that the techniques of [8] can be easily modified to give a simpler proof of these results. Indeed, our proof also works for any right Noetherian ring, and for any affine PI ring (a ring is called affine if it is finitely generated as an algebra over some central subfield and is PI if it satisfies a polynomial identity).

THEOREM A. (i) *If A is a right Noetherian ring, then* $\operatorname{asr}(A) \leqslant 1 + \operatorname{Kdim}(A/J(0))$. (ii) *Let A be a* PI *ring, and assume either that A is finitely generated as an algebra* *over some central, Noetherian subring, or that A is a finitely generated module over some central, J-Noetherian subring. Then* $\operatorname{asr}(A) \leqslant 1 + \dim(\operatorname{maxspec} A)$.

Here Kdim stands for Krull dimension in the sense of Rentschler and Gabriel [6], while dim(maxspec A) is defined as in the commutative case: dim(maxspec A) is the maximum integer n for which there exists a chain of Jacobson, prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$ in A. We prove both parts of Theorem A simultaneously by working with a more general class of rings (which we call strongly right J-Noetherian rings) and a dimension (which we call Kmax) that is bounded above by Kdim A, respectively, by dim(maxspec A), in the two cases mentioned in Theorem A. The definitions of these terms and the elementary results concerned with them are given in Section 1, while the following generalisation of Theorem A is proved in Section 2:

THEOREM B. *If A is a strongly right J-Noetherian ring, then* $\operatorname{asr}(A) \leqslant 1 + \operatorname{Kmax} A$.

We should emphasise, however, that if the reader is satisfied with Theorem A(i), then the results of Section 1 are unnecessary. Indeed, if every occurrence of 'Kmax' in the proof of Theorem B is replaced by 'Kdim', then one obtains a valid proof for Theorem A(i).

Finally, the motivation for considering absolute stable rank is that it provides a version of Witt cancellation for quadratic spaces defined over A. Here the notion of a quadratic space is the very general one of Bak given in [1, p. 255] and rediscovered by Magurn, Van der Kallen, and Vaserstein in [5, §5]. Since it is rather involved, we refer the reader to [1] or [5] for its definition. However, the reader should note that this definition does include all of the quadratic spaces of, for example, Bak [1], Bass [2] and Wall [11]. The *Witt index*, ind(q) of a quadratic space (V, q) is the largest integer r such that (V, q) contains an isomorphic copy of the hyperbolic space $H(A^{(r)})$. The details of this (and other undefined terms) can be found in [5, §§6,7]. Combining Theorem B with [5, Theorem 8.1 and Corollary 8.3] immediately gives:

COROLLARY 1. *Let A be a strongly right J-Noetherian ring (thus, for example, A could be a right Noetherian ring or an affine PI ring). Let (V, q) be a quadratic space defined over A and assume that* $\operatorname{ind}(q) \geqslant \operatorname{Kmax} A + 3$. *Then the orthogonal group $O(q)$ acts transitively on the set of all q-unimodular vectors $v \in V$ with a given length $|v|_q$.*

COROLLARY 2 (Witt Cancellation). *Let A be as in Corollary 1. Suppose that (V, q), (V', q') and (V'', q'') are quadratic spaces defined over A such that* $\operatorname{ind}(q) \geqslant \operatorname{Kmax} A + 2$, *that V'' is a finitely generated, projective right A-module and that q'' is nonsingular. Then any isomorphism*

$$(V, q) \perp (V'', q'') \cong (V', q') \perp (V'', q'')$$

forces $(V, q) \cong (V', q')$.

NOTATION. Throughout this note $\supset$ and $\subset$ will stand for strict inclusions.

1. J-Noetherian Rings and the Dimension Kmax

The aim of this section is to give the definitions and basic properties of strongly J-Noetherian rings and Kmax. Fix a ring A, let $\mathscr{L}$ denote the set of Jacobson right ideals of A, and partially order $\mathscr{L}$ by inclusion. Then Kmax A is defined to be the *deviation* of $\mathscr{L}$, as described, for example, in [4, §6.1.2, p. 174] or [6]. In other words, let $I_1 \supseteq I_2$ be Jacobson right ideals of A. Then $\text{Kmax}\, I_1/I_2 = -1$ if $I_1 = I_2$. Inductively, $\text{Kmax}\, I_1/I_2 = \alpha$ for some ordinal $\alpha \geqslant 0$ if $\text{Kmax}\, I_1/I_2 \nless \alpha$ but, for every infinite descending chain of Jacobson right ideals, $I_1 = K_1 \supseteq K_2 \supseteq \cdots \supseteq I_2$, one has $\text{Kmax}(K_i/K_{i+1}) < \alpha$ for all but finitely many i. Finally, write $\text{Kmax}\, A = \text{Kmax}\, A/J(0)$ (if it exists). By analogy with the commutative case, the ring A is called *right J-Noetherian* if A has ACC (that is, the ascending chain condition) on Jacobson right ideals. Then [4, Proposition 6.1.8, p. 175] ensures that Kmax A is defined for any right J-Noetherian ring A (although it can easily be an infinite ordinal). Finally, a ring A is called *strongly right J-Noetherian* if it is a right J-Noetherian ring such that A/P is a right Goldie ring for every Jacobson, prime ideal P of A. (One may clearly make analogous definitions for modules, but we have no need of that generality.)

While Kmax is defined in terms of Jacobson right ideals, it is frequently useful to extend the definition to arbitrary right ideals of A. Thus, if $I_1 \supseteq I_2$ are right ideals of A, write $\text{Kmax}(I_1/I_2) = \text{Kmax}(J(I_1)/J(I_2))$. Moreover, I_1/I_2 is called *α-critical* if $\text{Kmax}(I_1/I_2) = \alpha$ and, given any right ideal K with $I_1 \supseteq K \supset I_1 \cap J(I_2)$, then

$$(a)\ \text{Kmax}(K/I_2) = \alpha \quad \text{but} \quad (b)\ \text{Kmax}(I_1/K) < \alpha.$$

Finally, I_1/I_2 is called *critical* if it is α-critical for some ordinal α. The elementary properties of Kmax are similar to those of Kdim, as described for example in [4, Chapter 6], and we begin by describing some of these properties.

LEMMA 1.1. *Let $L_1 = J(L_1) \subseteq L_2$ be right ideals of a ring A. If P is an ideal of A such that $L_2 P \subseteq L_1$, then $J(L_2) \cap J(L_1 + P) = L_1$. In particular, $L_2 J(P) \subseteq L_1$.*

Proof. Let $M \in \mathscr{M}_A$ be such that $L_1 \subseteq M$. Then, either $L_2 \subseteq M$ or $L_2 + M = A$. In the latter case, $P = AP = L_2 P + MP \subseteq L_1 + M \subseteq M$. Thus

$$\begin{aligned} L_1 = J(L_1) &= \bigcap \{M \in \mathscr{M}_A : M \supseteq L_2\} \cap \bigcap \{M \in \mathscr{M}_A : M \supseteq L_1 + P\} \\ &= J(L_2) \cap J(L_1 + P); \end{aligned}$$

as required.

LEMMA 1.2. *Let A be any ring for which* Kmax A *is defined. Then*:

(i) *If $I_1 \supseteq I_2 \supseteq I_3$ are right ideals of A, then*

$$\text{Kmax}(I_1/I_3) \geqslant \sup\{\text{Kmax}(I_1/I_2), \text{Kmax}(I_2/I_3)\},$$

with equality if I_2 is an ideal. In particular, equality holds if A is commutative.

(ii) *If $I_1 \supset I_2$ are right ideals of A with* $\text{Kmax}(I_1/I_2) = \alpha \geqslant 0$, *then there exists a right ideal L with $I_1 \supseteq L \supset J(I_2) \cap I_1$ such that L/I_2 is β-critical for some $0 \leqslant \beta \leqslant \alpha$.*

(iii) *If L is in* (ii), *then K/I_2 is β-critical whenever K is a right ideal such that $L \supseteq K \supset J(I_2) \cap I_1$.*

Proof. (i) We may assume that I_1, I_2 and I_3 are Jacobson right ideals of A. Certainly any chain of Jacobson right ideals between I_1 and I_2 (or between I_2 and I_3) can be extended to a chain between I_1 and I_3. This clearly implies the stated inequality (see [4, Proposition 6.1.17, p. 179]).

Assume, now, that I_2 is an ideal. (In this case $J(I_2)$ is also an ideal and so we may again assume that each I_j is Jacobson.) Let K_1 and K_2 be Jacobson right ideals of A with $I_3 \subseteq K_2 \subset K_1 \subseteq I_1$ and suppose that $K_1 \cap I_2 = K_2 \cap I_2$. Then $K_1 I_2 \subseteq K_1 \cap I_2 \subseteq K_2$ and so, by Lemma 1.1, $K_2 = K_1 \cap J(K_2 + I_2)$. Therefore, $K_1 \nsubseteq J(K_2 + I_2)$ and so, certainly, $J(K_2 + I_2) \neq J(K_1 + I_2)$. It now follows easily, as in the proof of [4, Lemma 6.2.4, p. 180], that $\text{Kmax}(I_1/I_3) = \sup\{\text{Kmax}(I_1/I_2), \text{Kmax}(I_2/I_3)\}$.

(ii) This is very similar to the proof of [4, Lemma 6.2.10, p. 182]. First, pick a right ideal K_1 such that $I_1 \supseteq K_1 \supset J(I_2) \cap I_1$ and with $\text{Kmax}(K_1/I_2) = \beta$ as small as possible. Then certainly $\beta = \text{Kmax}(K'/I_2)$ whenever $K_1 \supseteq K' \supset J(I_2) \cap I_1$. Thus, if K_1/I_2 is not β-critical, then there exists $K_1 \supset K_2 \supset J(I_2) \cap I_1$ such that $\text{Kmax}(K_1/K_2) = \beta$. Now repeat the above argument, but with K_1 replaced by K_2. In this way either one finds some $L = K_n \subseteq K_1$ such that L/I_2 is β-critical, or one obtains an infinite descending chain $K_1 \supset K_2 \supset \cdots \supset J(I_2) \cap K_1$ such that $\text{Kmax}(K_i/K_{i+1}) = \beta$ for each i. This second possibility contradicts the fact that $\text{Kmax}(K_1/I_2) = \beta$; as required.

(iii) Let $L \supseteq K \supseteq K' \supset J(I_2) \cap I_1 = J(I_2) \cap L$. By part (a) of the definition of criticality and the fact that L/I_2 is β-critical, certainly $\text{Kmax}(K'/I_2) = \beta = \text{Kmax}(K/I_2)$. Next, by part (b) of the definition of criticality and part (i) of this lemma, $\text{Kmax}(K/K') < \beta$; as required.

The main reason for introducing strongly J-Noetherian rings and Kmax is to allow us to prove the main theorem simultaneously for Noetherian rings and affine PI rings. The next two propositions show that strongly J-Noetherian rings do provide the appropriate class in which to do this.

LEMMA 1.3. *Let A be a ring with* ACC *on Jacobson ideals. If $J(0) = 0$, then A has only finitely many minimal prime ideals.*

Proof. If A is not prime, then there exist nonzero ideals B_1 and B_2 with $B_1 B_2 = 0$. By Lemma 1.1, $J(B_1)J(B_2) = 0$ and so any minimal prime ideal of A is minimal over either $J(B_1)$ or $J(B_2)$. Now apply the obvious J-Noetherian induction.

LEMMA 1.4. *Let F be a finitely generated right module over a semiprime Goldie, PI ring B. Then*:

(a) *Let $L_1 \subset L_2$ be submodules of F such that L_1 is essential in L_2. Then there exists a central, regular element $c \in B$ such that $(L_2)c \subseteq L_1$.*

(b) *Assume, now, that $F = B$ and that $L_1 = J(L_1) \subseteq V_1 \subset V_2 \subseteq L_2$, for some Jacobson right ideals V_1 and V_2. Set $K = J(cB)$. Then $V_1 \supseteq V_2 \cap K$ and $J(V_1 + K) \neq J(V_2 + K)$. In particular,* $\text{Kmax}(L_2 + K/L_1 + K) \geqslant \text{Kmax}\, L_2/L_1$ *(provided that the right hand side exists).*

Proof. (a) Pick a submodule N of F such that $N \cap L_2 = 0$ but $N + L_2$ (and, hence, $N + L_1$) is essential in F. By [7. Theorem 1.7.34, p. 58], B has a semisimple Artinian quotient ring Q, obtained by inverting the central, regular elements of B. Thus $F \otimes_B Q = (L_1 + N) \otimes_B Q$. Moreover, if $F = \Sigma_1^t f_i B$ for some $f_j \in F$, then there exists a central, regular element $c \in B$ such that $f_j c \in L_1 + N$ for each j. Thus $(L_2 + N)c \subseteq Fc \subseteq L_1 + N$ and, hence, $L_2 c \subseteq L_2 \cap (L_1 + N) = L_1$; as required.

(b) Since $V_2 K \subseteq V_1$, Lemma 1.1 implies that $V_1 = V_2 \cap J(V_1 + K)$. This certainly implies that $J(V_1 + K) \neq J(V_2 + K)$, and the inequality for Kmax then follows easily.

PROPOSITION 1.5. *Let A be a* PI *ring and assume that either*:

(a) *A has* ACC *on Jacobson ideals*; *or*

(b) *A is finitely generated as an algebra over a central, Noetherian subring R*; *or*

(c) *A is finitely generated as a module over a J-Noetherian subring R.*

Then A is strongly (*left and right*) *J-Noetherian.*

Remark. It is easy to construct a PI ring A, finitely generated as an algebra over a central, J-Noetherian subring R, such that A is not J-Noetherian. For example, take

$$R = k[x_1, x_2, \ldots]_{(x_1, x_2, \ldots)}$$

and let $A = R[y]$ be the (commutative) polynomial extension of R. (In order to prove this, use the fact that, for any prime factor ring $\bar{R}$ of R, the Jacobson radical of $\bar{R}[y]$ is equal to zero.)

Proof. Since A is PI, every prime factor ring of A is Goldie (see [4, Corollary 13.6.6, p. 465]). Thus it suffices to show that A is right J-Noetherian.

(a) Assume that A has ACC on Jacobson ideals. Then we may assume that $J(0) = 0$ and, by induction, that A/I is J-Noetherian for every non-zero Jacobson ideal I. By Lemma 1.3, A has finitely many minimal prime ideals and hence, by [7, Theorem 1.7.34, p. 58], it is Goldie. Now let $I_1 \subset I_2 \subset \cdots$ be an infinite, strictly ascending chain of Jacobson right ideals of A and set $I = \bigcup I_j$. Since A is right Goldie, it has finite uniform dimension. Thus, there exists an integer k such that I_j is an essential submodule of I_{j+1} for all $j \geqslant k$. In particular, I_k is essential in I. Now apply Lemma 1.4(b), with $L_1 = I_k \subset L_2 = I$. Thus there exists a nonzero, Jacobson ideal K such that

$$J(I_k + K)/K \subset J(I_{k+1} + K)/K \subset \cdots$$

is a strictly ascending chain of Jacobson right ideals of A/K. This contradicts the inductive hypothesis.

(b) By [7, Theorem 4.5.7, p. 220], A has ACC on semiprime ideals and, hence, has ACC on Jacobson ideals. Now apply part (a).

(c) We may assume that the Jacobson radical, $J_A(0)$ of A is zero. Thus, by [4, Corollary 10.2.10, p. 350], $J_R(0) = 0$. Moreover, by J-Noetherian induction, we may assume that A/P is J-Noetherian whenever P is a Jacobson ideal of A for which $P \cap R \neq 0$.

Suppose that there exists an infinite, strictly ascending chain of Jacobson right ideals $I_1 \subset I_2 \subset \cdots$ of A and set $I = \bigcup I_j$. In this case, it is not immediately clear why A should have finite uniform dimension. However, A is a homomorphic image of a finitely generated, free right R-module F and, by Lemma 1.3, F_R *does* have finite uniform dimension. Thus let $I'_1 \subset I'_2 \subset \cdots \subset I' = \bigcup I'_j$ be the inverse image, in F, of the chain $I_1 \subset I_2 \subset \cdots$. Then, for some k, I'_k is essential in I'. Therefore Lemma 1.4(a), with $R = B$, implies that there exists a central, regular element $c \in R$ such that $I'c \subseteq I'_k$. Thus $Ic \subseteq I_k$. As in the proof of part (a), this provides an infinite, strictly ascending chain of Jacobson right ideals of $A/J(AcA)$; contradicting the inductive hypothesis.

PROPOSITION 1.6. (a) *If A is a right Noetherian ring, then* $\operatorname{Kmax} A \leqslant \operatorname{Kdim} A/J(0)$.

(b) *If A is a strongly right J-Noetherian, PI ring, then* $\operatorname{Kmax} A \leq \dim(\operatorname{maxspec} A)$.

Proof. Part (a) is trivial. In order to prove part (b), recall that Kmax does exist for any J-Noetherian ring. Also, since neither side of the inequality is affected by factoring out $J(0)$, we may suppose that A is semiprime. By J-Noetherian induction, we may further assume that $\operatorname{Kmax} A/I \leqslant \dim(\operatorname{maxspec} A/I)$ for any nonzero, Jacobson ideal I. Now, suppose that $\dim(\operatorname{maxspec} A) = n < \operatorname{Kmax} A$. Then there exists an infinite descending chain of Jacobson right ideals $A \supset I_1 \supset I_2 \supset \cdots$ such that $\operatorname{Kmax}(I_i/I_{i+1}) = n$ for each i. Since $J(0) = 0$, [4, Corollary 13.6.9, p. 466] and Lemma 1.3 imply that A is Goldie. In particular, A_A has finite uniform dimension and so, for some $r \geqslant 1$, I_{r+1} is an essential submodule of I_r. Now apply Lemma 1.4(b), with $I_{r+1} = L_1 \subset I_r = L_2$. This provides a Jacobson ideal $K = J(cA)$ such that $\operatorname{Kmax} A/K \geqslant \operatorname{Kmax}(I_r/I_{r+1}) = n$.

Since $J(0) = 0$, it is readily checked that every minimal prime ideal of A is Jacobson. But, by Lemma 1.4, K is contained in no minimal prime ideal of A. Therefore, $\dim(\operatorname{maxspec}(A/K)) < \dim(\operatorname{maxspec} A) = n$. However, by induction and the last paragraph, $\dim(\operatorname{maxspec}(A/K)) = \operatorname{Kmax}(A/K) \geqslant n$; a contradiction.

Remarks. (i) Even for commutative rings, one need not have equality in Proposition 1.6(ii). The obvious example is given by the ring $R = \mathbb{C}[x]_{(x)}[y]$, for which

$$\operatorname{Kmax} R = 1 < 2 = \dim(\operatorname{maxspec} R).$$

To see this, observe that, while (0), (x) and (x, y) are all Jacobson prime ideals of R, one has $0 = \bigcap \{M\}$ for any infinite set of maximal ideals $\{M : x \notin M\}$. This example suggests that, in fact,

$$\operatorname{Kmax} R = \inf\{n : Y = \operatorname{maxspec} R \text{ is a disjoint union of subspaces: } Y = U_1 \cup \cdots \cup U_r \text{ with } \dim U_i \leqslant n \text{ for each } i\}.$$

Finally, observe that, although R is a domain, the module R_R is not critical.

(ii) Propositions 1.5 and 1.6 can easily be generalised to work for rings satisfying Warfield's condition (*). Thus, assume that A is a ring which has ACC on Jacobson ideals, and such that A/P is a right fully bounded, right Goldie ring for each Jacobson, prime ideal P. Then A is strongly right J-Noetherian and $\operatorname{Kmax} A \leqslant \dim(\operatorname{maxspec} A)$. The details are left to the interested reader.

2. The Main Theorem

THEOREM 2.1. *Let A be a strongly right J-Noetherian ring. Then* $\operatorname{asr}(A) \leqslant 1 + \operatorname{Kmax} A$.

Proof. The proof is similar to that used in [8] to prove that the stable range of a right Noetherian ring is bounded above by its Krull dimension. Let $n = \operatorname{Kmax} A$ and $-1 \leqslant s \leqslant n$ an integer. Given Jacobson right ideals $I \subseteq L$, consider the following inductive statement:

$*(s, I, L)$ Suppose that $b_0, \dots, b_s, c \in L$ are such that $\operatorname{Kmax}(cA + I/I) \leqslant s$ and that $L = J\left(I + \sum_0^s b_i A + cA\right)$. Then there exist elements $g_i \in A$ such that $L = J\left(I + \sum_0^s (b_i + cg_i)A\right)$.

The statement of the theorem is just the case $*(n, J(0), L)$, where $L = J(\Sigma_0^{n+1} a_j A)$ and $\{a_0, \dots, a_{n+1}\} = \{b_0, \dots, b_n, c\}$. We will prove $*(s, I, L)$ by a double induction: First, given $-1 \leqslant r < s$, assume that $*(r, I', L')$ holds for any Jacobson right ideals $I' \subseteq L'$ (note that, in the case $r = -1$, there is nothing to prove). Secondly, if I' and L' are Jacobson right ideals such that $I \subset I' \subseteq L' \subseteq L$, then by J-Noetherian induction assume that $*(s, I', L')$ holds (in this case, $*(s, L, L)$ is trivially true).

Thus, assume that $b_0, \dots, c$ are as in $*(s, I, L)$. By Lemma 1.2(ii), we may pick $\lambda \in A$ such that $c\lambda \notin I$ but $c\lambda A + I/I$ is r-critical for some $r \leqslant s$ and the right annihilator, $P = r\text{-}ann(c\lambda A + I/I)$ is as large as possible. Note that Lemma 1.2(iii) implies that P is a prime ideal. Set $I' = J(I + c\lambda A)$. Since $I' \supset I$, the second inductive statement provides elements $g_i \in A$ such that

$$L = J\left(I' + \sum_0^s (b_i + cg_i)A\right) = J\left(I + \sum_0^s (b_i + cg_i)A + c\lambda A\right).$$

In other words, replacing b_i by $b_i + cg_i$ for each i and c by $c\lambda$, we may assume that $cA + I/I$ is r-critical, with a prime annihilator, P.

For each i, write $K_i = \{f \in A : b_i f \in I\}$. Suppose, for some i, that $P \nsubseteq K_i$. Then $I' = J(I + b_i P) \supset I$. Thus, by induction, there exist $g_j \in A$ such that

$$L = J\left(I + b_i P + \sum_{j=0}^{s} (b_j + cg_j)A\right). \tag{1}$$

However, since $cP \subseteq I$, one has $b_iP + I = (b_i + cg_i)P + I$. Combined with Equation (1), this implies that $L = J(I + \Sigma_0^s (b_j + cg_j)A)$; as required.

Alternatively, suppose that there exists i such that $K_i \nsubseteq P$. Then there exists $k \in K_i \backslash P$ and $\mu \in A$ such that $c\mu k \notin I$. But now

$$(b_i + c\mu)A + I \supseteq (b_i + c\mu)kA + I = c\mu kA + I \supset I. \tag{2}$$

Let $I' = J(c\mu kA + I)$. Since $cA + I/I$ is r-critical, Equation (2) implies that $\mathrm{Kmax}(cA + I'/I') < r \leqslant s$. Thus, set

$$L' = J\left(I' + \sum_{\substack{0 \leqslant j \leqslant s \\ j \neq i}} b_jA + cA\right).$$

Then, by induction $*(s-1, I', L')$ may be applied to the elements

$$\{b_j : 0 \leqslant j \leqslant s, j \neq i\} \cup \{c\}$$

to obtain elements $f_j \in A$ such that $L' = J(I' + \Sigma_{j \neq i} (b_j + cf_j)A)$. Thus by Equation (2), again,

$$L = J\left(I + (b_i + c\mu)A + \sum_{j \neq i} (b_j + cf_j)A\right);$$

as required.

It remains to consider one final case; when $K_i = P$ for each $0 \leqslant i \leqslant s$. Set $N = I + b_0A + cA$. Then $P = r\text{-ann}(N/I)$ and so, by Lemma 1.1, $P = J(P)$. Thus, by hypothesis, A/P is right Goldie. Applying Lemma 1.1, again, implies that $I = J(N) \cap J(P + I)$, and hence that

$$(b_0A + J(P + I))/J(P + I) \cong (b_0A + I)/I \cong A/K_0 = A/P. \tag{3}$$

Since A/P is right Goldie, this forces $J(P + I) = P$; that is, $I \subseteq P$. Moreover, Equation (3) now implies that $(b_0A + P)/P$ is an essential submodule of A/P, and hence of $(N + P)/P$. Therefore, as $N \cap P = I$, $(b_0A + I)/I$ is an essential submodule of N/I.

Thus we may choose $x \in (b_0A + I) \cap (cA + I)$ with $x \notin I$. Now repeat the argument given after Equation (2), but applied to the right ideals $I' = J(I + xA)$ and $L' = J(I' + \Sigma_1^s b_iA + cA)$. Then, as before,

$$\mathrm{Kmax}(cA + I'/I') < \mathrm{Kmax}(cA + I/I) \leqslant s.$$

Therefore, $*(s-1, I', L')$ may be applied to find elements $g_1, \ldots, g_s \in A$ such that $L' = J(I' + \Sigma_1^s (b_i + cg_i)A)$. This in turn implies that

$$L = J(L' + b_0A) = J\left(I + b_0A + \sum_1^s (b_i + cg_i)A\right),$$

and therefore completes the proof of the theorem.

Theorem B of the Introduction is just Theorem 2.1. Theorem A is an immediate consequence of this, combined with Propositions 1.5 and 1.6. Another special case of Theorem 2.1 is the Stable Range Theorem. Recall that the *stable range*, sr(A), of a ring A is defined by $\mathrm{sr}(A) \leqslant n$ if, given any $a_0, \ldots, a_n \in A$ such that $A = \Sigma_0^n a_i A$, then $A = \Sigma_0^{n-1} (a_i + a_n \lambda_i)A$ for some $\lambda_i \in A$. Then an immediate consequence of Theorem 2.1 is:

COROLLARY 2.2. *If A is a strongly right J-Noetherian ring, then* $\mathrm{sr}(A) \leqslant 1 + \mathrm{Kmax}\, A$.

For right Noetherian rings (with Kmax replaced by Kdim), this corollary is contained in [8], while for affine PI rings (with Kmax replaced by dim(maxspec)) it is a consequence of [10, Theorem 3.10]. For the numerous applications of the stable range theorem to K-theory and classical groups defined over rings, the reader is referred to [3].

COROLLARY 2.3. *Let I be a right ideal of a strongly right J-Noetherian ring A, and set $n = \mathrm{Kmax}\, A$. Then, there exist $a_0, \ldots, a_n \in I$ such that $J(I) = J(\Sigma_0^n a_j A)$.*

Proof. This is an easy exercise. (For the Noetherian rings this result has also been stated, but not proved, as [9, Proposition 3.9].)

Finally, I should remark that I know almost nothing about the properties of J-Noetherian rings and Kmax. (As sample questions: Are right J-Noetherian rings always strongly right J-Noetherian? Does equality always hold in Lemma 1.2(i)?) However, it is widely recognised that Noetherian rings and affine PI algebras have many properties in common, and so it is possible that strongly J-Noetherian rings will provide a convenient setting in which to prove results about both of these classes of rings.

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