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Ordinary Cokriging Revisited[1]

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This paper sets up the relations between simple cokriging and ordinary cokriging with one or several unbiasedness constraints. Differences between cokriging variants are related to differences between models adopted for the means of primary and secondary variables. Because it is not necessary for the secondary data weights to sum to zero, ordinary cokriging with a single unbiasedness constraint gives a larger weight to the secondary information while reducing the occurrence of negative weights. Also the weights provided by such cokriging systems written in terms of covariances or correlograms are not related linearly, hence the estimates are different. The prediction performances of cokriging estimators are assessed using an environmental dataset that includes concentrations of five heavy metals at 359 locations. Analysis of reestimation scores at 100 test locations shows that kriging and cokriging perform equally when the primary and secondary variables are sampled at the same locations. When the secondary information is available at the estimated location, one gains little by retaining other distant secondary data in the estimation.

KEY WORDS: cokriging, unbiasedness constraints, negative weights, standardization.

INTRODUCTION

Depending on the model adopted for the random function, three kriging variants can be distinguished: simple kriging, ordinary kriging, and kriging with a trend model (universal kriging). Several authors (Matheron, 1970, p. 129; Journel and Rossi, 1989) showed that the latter two algorithms are but simple kriging with the stationary mean replaced by a local mean that is estimated within each search neighborhood. Similar relations exist in the multivariate situation and are developed here for the most frequently used simple and ordinary cokriging. Moreover, the cokriging system for estimating the local primary and secondary means implicitly used in ordinary cokriging is established.

The unbiasedness of the ordinary cokriging estimator is ensured by forcing the primary data weights to sum to one whereas the weights of each secondary variable are constrained to sum to zero. Under these "traditional" constraints

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most of the secondary data weights tend to be small and some of them are necessarily negative with a risk of getting unacceptable results such as negative concentration estimates. To reduce the occurrence of negative weights and avoid limiting artificially the impact of secondary data, Isaaks and Srivastava (1989, p. 416) proposed to use a single constraint that forces all primary and secondary data weights to sum to one. Unbiasedness of the estimator then is ensured by rescaling all secondary variables to the same mean as the primary variable, hence the term "rescaled" or "standardized" ordinary cokriging (Deutsch and Journel, 1992, p. 70). Another advantage of using a single contraint is that the secondary information can be limited to a single datum when the secondary variable is sampled at the estimated point (collocated cokriging; Almeida and Journel, 1994); under the traditional constraints that datum would get, by construction, a zero weight. In this paper, properties of traditional and rescaled ordinary cokriging estimators are compared and the impact of the unbiasedness constraints on the cokriging weights is investigated.

Several authors (Stein and others, 1988; Hevesi, Istok, and Flint, 1992; Asli and Marcotte, 1995) have compared the prediction performances of simple and ordinary kriging or cokriging in the equally and undersampled situations. Using an environmental dataset, that comparison is here extended to rescaled ordinary cokriging and the situation of single collocated secondary data. Two performance criteria are considered: (1) the mean absolute error of prediction of heavy metal concentrations at 100 test locations, and (2) the proportion of test locations that are classified wrongly as safe or contaminated on the basis of cokriging estimates.

SIMPLE COKRIGING

Let $\{z_1(\mathbf{u}_{\alpha_1}), \alpha_1 = 1, \ldots, n_1\}$ be the values of the primary attribute z_1 at n_1 locations $\mathbf{u}_{\alpha_1}$. To alleviate notation consider the situation where there is only one secondary attribute z_2 measured at, possibly different, locations $\mathbf{u}_{\alpha_2}$, $\{z_2(\mathbf{u}_{\alpha_2}), \alpha_2 = 1, \ldots, n_2\}$.

The simple cokriging (SCK) estimator of z_1 at location $\mathbf{u}$ is written:

$$Z^{(1)*}_{SCK}(\mathbf{u}) - m_1 = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda^{SCK}_{\alpha_1}(\mathbf{u})[Z_1(\mathbf{u}_{\alpha_1}) - m_1] + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda^{SCK}_{\alpha_2}(\mathbf{u})[Z_2(\mathbf{u}_{\alpha_2}) - m_2] \quad (1)$$

where $\lambda^{SCK}_{\alpha_1}(\mathbf{u})$ is the weight assigned to the primary datum $z_1(\mathbf{u}_{\alpha_1})$, $\lambda^{SCK}_{\alpha_2}(\mathbf{u})$ is the weight assigned to the secondary datum $z_2(\mathbf{u}_{\alpha_2})$, and m_1 and m_2 are the primary and secondary means assumed known and constant within the study

area $\mathcal{A}$. Typically, only primary and secondary data closest to the location $\mathbf{u}$ being estimated are retained, that is $n_i(\mathbf{u})$ is usually smaller than n_i. Note that the number of data retained and the size of the search neighborhood need not be the same for all attributes.

The simple cokriging estimator (1) can be rewritten as:

$$Z^{(1)*}_{SCK}(\mathbf{u}) = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda^{SCK}_{\alpha_1}(\mathbf{u}) Z_1(\mathbf{u}_{\alpha_1}) + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda^{SCK}_{\alpha_2}(\mathbf{u}) Z_2(\mathbf{u}_{\alpha_2}) + \lambda^{SCK}_{m_1}(\mathbf{u}) m_1 + \lambda^{SCK}_{m_2}(\mathbf{u}) m_2 \tag{2}$$

where $\lambda^{SCK}_{m_1}(\mathbf{u})$ and $\lambda^{SCK}_{m_2}(\mathbf{u})$ are the weights given to the primary and secondary means, respectively:

$$\lambda^{SCK}_{m_1}(u) = 1 - \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda^{SCK}_{\alpha_1}(\mathbf{u})$$

$$\lambda^{SCK}_{m_2}(\mathbf{u}) = - \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda^{SCK}_{\alpha_2}(\mathbf{u})$$

The cokriging weights that minimize the error variance $\sigma^2_E(\mathbf{u}) = \mathrm{Var}\{Z^{(1)*}_{SCK}(\mathbf{u}) - Z_1(\mathbf{u})\}$ are obtained by solving the following system of $(n_1(\mathbf{u}) + n_2(\mathbf{u}))$ linear equations:

$$\begin{cases} \sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{SCK}_{\beta_1}(\mathbf{u}) C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{SCK}_{\beta_2}(\mathbf{u}) C_{12}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_2}) \\ \qquad = C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}), \ \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \\ \sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{SCK}_{\beta_1}(\mathbf{u}) C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{SCK}_{\beta_2}(\mathbf{u}) C_{22}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_2}) \\ \qquad = C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}), \ \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \end{cases} \tag{3}$$

where $C_{ij}(\mathbf{u}_{\alpha_i} - \mathbf{u}_{\beta_j})$ is the cross-covariance between variables Z_i and Z_j at locations $\mathbf{u}_{\alpha_i}$ and $\mathbf{u}_{\beta_j}$.

CORRELOGRAM NOTATION

When the variances of primary and secondary variables differ by several orders of magnitude, there are large differences between the covariance terms of the system (3) with risks of numerical instability when solving it (Isaaks and Srivastava, 1989, p. 416). A solution consists of rescaling the auto and cross-covariance values, e.g., solving the cokriging system (3) in terms of correlo-

grams:

$$\begin{cases} \sum_{\beta_1=1}^{n_1(\mathbf{u})} \nu_{\beta_1}^{SCK}(\mathbf{u})\rho_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \nu_{\beta_2}^{SCK}(\mathbf{u})\rho_{12}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_2}) \\ \qquad = \rho_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}), \ \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \\ \sum_{\beta_1=1}^{n_1(\mathbf{u})} \nu_{\beta_1}^{SCK}(\mathbf{u})\rho_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \nu_{\beta_2}^{SCK}(\mathbf{u})\rho_{22}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_2}) \\ \qquad = \rho_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}), \ \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \end{cases} \tag{4}$$

where the cross-correlogram $\rho_{ij}(\mathbf{h})$ is defined as the ratio $C_{ij}(\mathbf{h})/(\sigma_i \cdot \sigma_j)$ with σ_i^2 being the stationary variance of RF $Z_i(\mathbf{u})$.

Simple cokriging systems written in terms of correlograms or covariances yield two different sets of weights that are linearly related. Indeed, accounting for the definition of the cross correlogram, system (4) becomes:

$$\begin{cases} \sum_{\beta_1=1}^{n_1(\mathbf{u})} \nu_{\beta_1}^{SCK}(\mathbf{u})C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_1})/\sigma_1^2 + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \nu_{\beta_2}^{SCK}(\mathbf{u})C_{12}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_2})/\sigma_1\sigma_2 \\ \qquad = C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u})/\sigma_1^2, \ \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \\ \sum_{\beta_1=1}^{n_1(\mathbf{u})} \nu_{\beta_1}^{SCK}(\mathbf{u})C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_1})/\sigma_1\sigma_2 + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \nu_{\beta_2}^{SCK}(\mathbf{u})C_{22}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_2})/\sigma_2^2 \\ \qquad = C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u})/\sigma_1\sigma_2, \ \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \end{cases}$$

Multiplying the first $n_1(\mathbf{u})$ equations by σ_1^2 and the next $n_2(\mathbf{u})$ equations by $\sigma_1\sigma_2$, one deduces the following relation between the two sets of cokriging weights:

$$\lambda_{\alpha_1}^{SCK}(\mathbf{u}) = \nu_{\alpha_1}^{SCK}(\mathbf{u}) \qquad \alpha_1 = 1, \ldots, n_1(\mathbf{u})$$

$$\lambda_{\alpha_2}^{SCK}(\mathbf{u}) = \frac{\sigma_1}{\sigma_2} \nu_{\alpha_2}^{SCK}(\mathbf{u}) \qquad \alpha_2 = 1, \ldots, n_2(\mathbf{u})$$

Whereas the primary data weights are the same for both systems, the weights of the secondary data are rescaled by the ratio of standard deviations. For example, the larger the variance of the secondary variable relative to the variance of Z_1, the smaller the secondary data weights relative to the primary data weights. The cokriging estimator (1) then is written:

$$Z_{SCK}^{(1)*}(\mathbf{u}) - m_1 = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \nu_{\alpha_1}^{SCK}(\mathbf{u})[Z_1(\mathbf{u}_{\alpha_1}) - m_1] + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \nu_{\alpha_2}^{SCK}(\mathbf{u}) \frac{\sigma_1}{\sigma_2} [Z_2(\mathbf{u}_{\alpha_2}) - m_2]$$

The standardized form of that SCK estimator is obtained by dividing both terms of the expression by σ_1:

$$\frac{Z^{(1)*}_{SCK}(\mathbf{u}) - m_1}{\sigma_1} = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \nu^{SCK}_{\alpha_1}(\mathbf{u}) \left[\frac{Z_1(\mathbf{u}_{\alpha_1}) - m_1}{\sigma_1}\right] + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \nu^{SCK}_{\alpha_2}(\mathbf{u}) \left[\frac{Z_2(\mathbf{u}_{\alpha_2}) - m_2}{\sigma_2}\right] \tag{5}$$

Thus the estimator $Z^{(1)*}_{SCK}(\mathbf{u})$ in the standardized form (5) with weights provided by the cokriging system (4) identifies the estimator (1), as it should.

ORDINARY COKRIGING

For the case of a single secondary attribute z_2, the ordinary cokriging (OCK) estimator of z_1 at $\mathbf{u}$ is:

$$Z^{(1)*}_{OCK}(\mathbf{u}) = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda^{OCK}_{\alpha_1}(\mathbf{u}) Z_1(\mathbf{u}_{\alpha_1}) + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda^{OCK}_{\alpha_2}(\mathbf{u}) Z_2(\mathbf{u}_{\alpha_2}) \tag{6}$$

That estimator is unbiased under the following constraints on the cokriging weights:

$$\sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda^{OCK}_{\alpha_1}(\mathbf{u}) = 1, \qquad \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda^{OCK}_{\alpha_2}(\mathbf{u}) = 0 \tag{7}$$

Minimization of the error variance $\sigma^2_E(\mathbf{u})$ under the two constraints (7) yields the following system of $(n_1(\mathbf{u}) + n_2(\mathbf{u}) + 2)$ linear equations:

$$\begin{cases} \displaystyle\sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{OCK}_{\beta_1}(\mathbf{u}) C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{OCK}_{\beta_2}(\mathbf{u}) C_{12}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_2}) \\ \qquad + \mu^{OCK}_1(\mathbf{u}) = C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}), \qquad \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \\ \displaystyle\sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{OCK}_{\beta_1}(\mathbf{u}) C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{OCK}_{\beta_2}(\mathbf{u}) C_{22}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_2}) \\ \qquad + \mu^{OCK}_2(\mathbf{u}) = C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}), \qquad \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \\ \displaystyle\sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{OCK}_{\beta_1}(\mathbf{u}) = 1 \\ \displaystyle\sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{OCK}_{\beta_2}(\mathbf{u}) = 0 \end{cases} \tag{8}$$

where the two Lagrange parameters $\mu^{OCK}_1(\mathbf{u})$ and $\mu^{OCK}_2(\mathbf{u})$ account for the two unbiasedness constraints (7).

Correlogram Notation

The standardized form of the ordinary cokriging estimator (6) is:

$$\frac{Z^{(1)*}_{OCK}(\mathbf{u}) - m_1}{\sigma_1} = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \nu^{OCK}_{\alpha_1}(\mathbf{u}) \left[\frac{Z_1(\mathbf{u}_{\alpha_1}) - m_1}{\sigma_1}\right] + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \nu^{OCK}_{\alpha_2}(\mathbf{u}) \left[\frac{Z_2(\mathbf{u}_{\alpha_2}) - m_2}{\sigma_2}\right] \quad (9)$$

where the cokriging weights $\nu^{OCK}_{\alpha_i}(\mathbf{u})$ are obtained by solving the ordinary cokriging system (8) expressed in terms of correlograms. Similar to simple cokriging weights, OCK weights of original and standardized variables are linearly related:

$$\lambda^{OCK}_{\alpha_1}(\mathbf{u}) = \nu^{OCK}_{\alpha_1}(\mathbf{u}) \qquad \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \quad (10)$$

$$\lambda^{OCK}_{\alpha_2}(\mathbf{u}) = \frac{\sigma_1}{\sigma_2} \nu^{OCK}_{\alpha_2}(\mathbf{u}) \qquad \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \quad (11)$$

Again, one verifies that both standardized and original forms of ordinary cokriging result in the same cokriging estimate.

Local Reestimation of the Mean

Ordinary cokriging usually is preferred to simple cokriging since it requires neither knowledge nor stationarity of the primary and secondary means over the entire area $\mathcal{A}$. Indeed, one can show that ordinary cokriging with local search neighborhoods amounts to:

(1) reestimating the local primary and secondary means, say $m^{(1)*}_{OCK}(\mathbf{u})$ and $m^{(2)*}_{OCK}(\mathbf{u})$, at each location $\mathbf{u}$ using both primary and secondary data specific to that neighborhood (see Appendix),

(2) then applying the simple cokriging estimator (1) using these local means rather than the stationary means m_1 and m_2:

$$Z^{(1)*}_{OCK}(\mathbf{u}) - m^{(1)*}_{OCK}(\mathbf{u}) = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda^{SCK}_{\alpha_1}(\mathbf{u})[Z_1(\mathbf{u}_{\alpha_1}) - m^{(1)*}_{OCK}(\mathbf{u})] + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda^{SCK}_{\alpha_2}(\mathbf{u})[Z_2(\mathbf{u}_{\alpha_2}) - m^{(2)*}_{OCK}(\mathbf{u})] \quad (12)$$

Accounting for the expression (2) of the simple cokriging estimator, the following relation between the two estimators is deduced:

$$Z^{(1)*}_{OCK}(\mathbf{u}) = Z^{(1)*}_{SCK}(\mathbf{u}) + \lambda^{SCK}_{m_1}(\mathbf{u})\Delta^{OCK}_{m_1}(\mathbf{u}) + \lambda^{SCK}_{m_2}(\mathbf{u})\Delta^{OCK}_{m_2}(\mathbf{u})$$

where: $\Delta^{OCK}_{m_1}(\mathbf{u}) = m^{(1)*}_{OCK}(\mathbf{u}) - m_1, \qquad \Delta^{OCK}_{m_2}(\mathbf{u}) = m^{(2)*}_{OCK}(\mathbf{u}) - m_2$

Differences between simple and ordinary cokriging estimators are caused by departures of estimated primary and secondary local means from the stationary means m_1 and m_2.

Example

The comparison simple versus ordinary cokriging is illustrated using the one-dimensional example of Figure 1. This transect is part of a dataset including the topsoil concentrations of seven heavy metals measured at the same 359 locations (Atteia, Dubois, and Webster, 1994; Webster, Atteia, and Dubois, 1994). The primary variable is here cadmium concentration, whereas nickel is

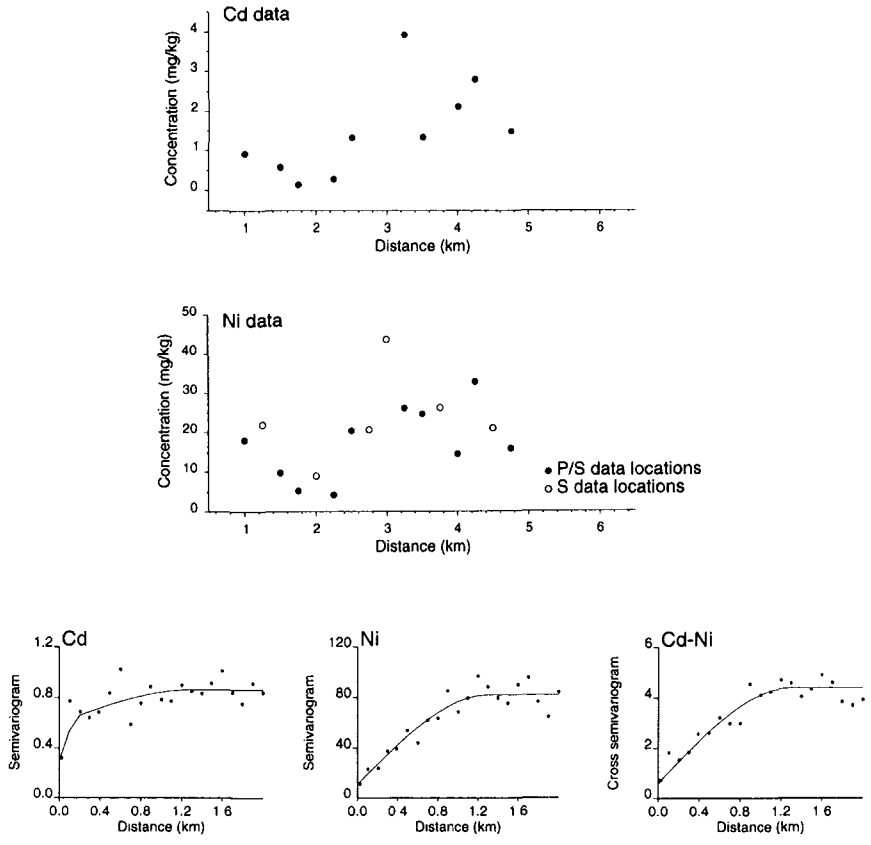

Figure 1. Transect of 10 Cd concentrations and 16 Ni concentrations. Bottom graphs show experimental direct and cross semivariograms inferred from full dataset (n = 259) with linear model of coregionalization fitted.

the secondary variable. The information available for estimating Cd concentrations along the transect consists of:

(1) ten Cd data and 16 Ni data (see Fig. 1, top graphs). Black dots depict locations where both primary and secondary variables are known. Open circles correspond to six locations where only the secondary variable is available.

(2) the linear model of coregionalization shown at the bottom of Figure 1:

$$\gamma_{Cd}(h) = 0.3\ g_0(h) + 0.3\ \text{Sph}(h/200\ \text{m}) + 0.26\ \text{Sph}(h/1.3\ \text{km})$$

$$\gamma_{Ni}(h) = 11\ g_0(h) + 71\ \text{Sph}(h/1.3\ \text{km})$$

$$\gamma_{Cd\text{-}Ni}(h) = 0.6\ g_0(h) + 3.8\ \text{Sph}(h/1.3\ \text{km})$$

where $g_0(h)$ is a nugget effect model, and $\text{Sph}(h/a)$ is an isotropic spherical model of range a.

(3) primary and secondary stationary means identified to the sample means of Cd and Ni data along the transect: $m_1 = 1.49$ mg/kg, $m_2 = 19.6$ mg/kg.

The estimation is performed every 50 m using at each location the five closest primary data and five closest secondary data, that is, $n_1(\mathbf{u}) = n_2(\mathbf{u}) = 5\ \forall\ \mathbf{u}$.

Figure 2 (top graphs) shows the overall means (horizontal dashed line) and local means (solid line) of Cd and Ni that are used implicitly in simple and ordinary cokriging, respectively. The local means were estimated using the procedure described in the Appendix; they both have a staircase shape, each step corresponding to estimates based on the same neighboring primary and secondary data. The profiles of local means follow the general increase in Cd and Ni concentrations along the transect.

Figure 2 (bottom graph) shows the simple (dashed line) and ordinary (solid line) cokriging estimates of Cd concentrations. Note that:

- both interpolators are exact in that they honor Cd data at their locations.
- OCK estimates are smaller than SCK estimates in the left part of the transect where the Cd local mean is smaller than the overall mean.
- OCK estimates are larger than SCK estimates in the right part of the transect where the Cd local mean is larger than the overall mean.

RESCALED ORDINARY COKRIGING

Rescaled or standardized ordinary cokriging (Isaaks and Srivastava, 1989, p. 409–416; Deutsch and Journel, 1992, p. 70) is a variant of ordinary cokriging where the two unbiasedness constraints (7) are replaced by a single constraint

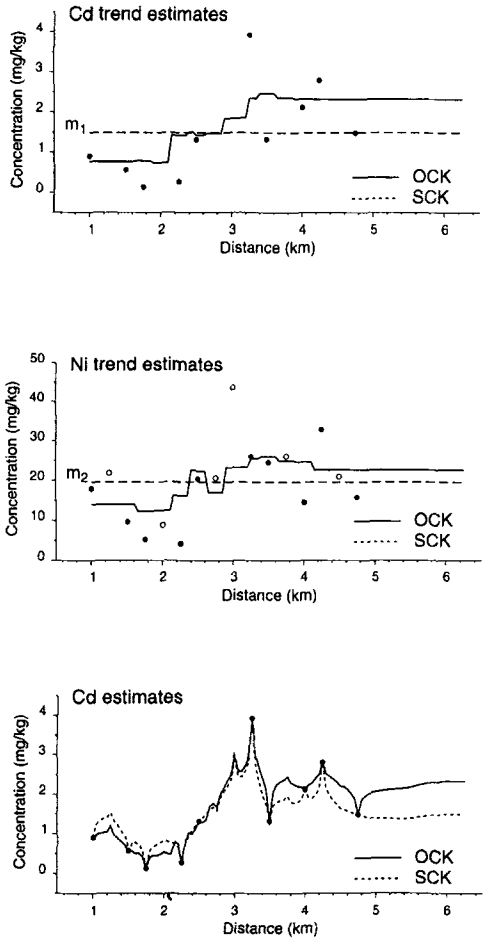

Figure 2. Estimates of Cd and Ni local means, and Cd concentrations using information of Figure 1 and simple cokriging (dashed line) or ordinary cokriging (solid line).

that requires all primary and secondary data weights to sum to one:

$$\sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda_{\alpha_1}^{OCK}(\mathbf{u}) + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda_{\alpha_2}^{OCK}(\mathbf{u}) = 1 \tag{13}$$

By using this single constraint one lessens two shortcomings associated with the traditional constraint that all secondary data weights sum to zero, which are (1) some of the secondary data weights are negative, thereby increasing the risk of getting unacceptable estimates such as negative concentrations, (2) most of the weights $\lambda_{\alpha_2}^{OCK}(\mathbf{u})$ tend to be small, thus reducing the influence of the secondary information.

Under the single constraint (13) the unbiasedness of the ordinary cokriging estimator is ensured by rescaling the secondary variable Z_2 so that its mean is equal to that of the primary variable. The rescaled cokriging (RCK) estimator (6) is written:

$$Z^{(1)*}_{RCK}(\mathbf{u}) = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda^{RCK}_{\alpha_1}(\mathbf{u}) Z_1(\mathbf{u}_{\alpha_1}) + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda^{RCK}_{\alpha_2}(\mathbf{u})[Z_2(\mathbf{u}_{\alpha_2}) - m_2 + m_1] \tag{14}$$

where the means m_1 and m_2 are estimated by the sample means after possible correction for preferential sampling (declustered means). The cokriging weights are obtained by solving an ordinary cokriging system of type (8) with a single unbiasedness constraint:

$$\begin{cases} \sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{RCK}_{\beta_1}(\mathbf{u}) C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{RCK}_{\beta_2}(\mathbf{u}) C_{12}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_2}) \\ \quad + \mu^{RCK}(\mathbf{u}) = C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}), \qquad \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \\ \sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{RCK}_{\beta_1}(\mathbf{u}) C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{RCK}_{\beta_2}(\mathbf{u}) C_{22}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_2}) \\ \quad + \mu^{RCK}(\mathbf{u}) = C_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}), \qquad \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \\ \sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda^{RCK}_{\beta_1}(\mathbf{u}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda^{RCK}_{\beta_2}(\mathbf{u}) = 1 \end{cases} \tag{15}$$

Local Reestimation of the Mean

Unlike ordinary cokriging, rescaled ordinary cokriging requires the knowledge of the stationary means of primary and secondary variables. However, the single unbiasedness constraint (13) leads to the reestimation of the common local mean of primary and rescaled secondary variables within each search neighborhood. Similar to the ordinary cokriging estimator (12), the estimator (14) can be expressed as the sum of the SCK estimator $Z^{(1)*}_{SCK}(\mathbf{u})$ plus a multiple of the difference between local and stationary means of the primary variable at $\mathbf{u}$:

$$Z^{(1)*}_{RCK}(\mathbf{u}) = Z^{(1)*}_{SCK}(\mathbf{u}) + [\lambda^{SCK}_{m_1}(\mathbf{u}) + \lambda^{SCK}_{m_2}(\mathbf{u})][m^*_{RCK}(\mathbf{u}) - m_1]$$

where the local mean $m^*_{RCK}(\mathbf{u})$ can be estimated using an approach similar to that described in the Appendix for ordinary cokriging. Thus, local departures from the overall means also are accounted for by rescaled ordinary cokriging.

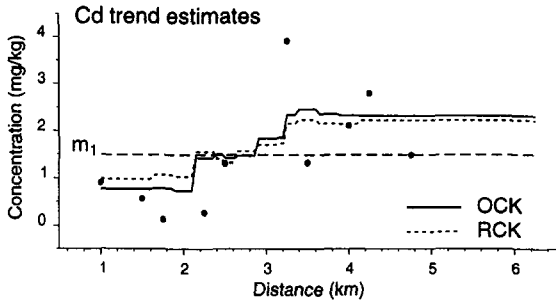

Figure 3. Estimates of Cd local means that is used implicitly in ordinary cokriging (solid line) and rescaled ordinary cokriging (dashed line).

Figure 3 shows the local mean implicitly re-estimated by ordinary cokriging (solid line) and by rescaled ordinary cokriging (dashed line) along the transect of Figure 1.

Unbiasedness Constraint(s) and Cokriging Weights

The impact of using one or two unbiasedness constraints on the cokriging weights is illustrated using the transect of Figure 1 and two criteria: (1) the sum of negative secondary data weights, and (2) the ratio of the sums of absolute values of the secondary and primary data weights:

$$\psi(\mathbf{u}) = \frac{\sum_{\alpha_2=1}^{n_2(\mathbf{u})} |\lambda_{\alpha_2}(\mathbf{u})|}{\sum_{\alpha_1=1}^{n_1(\mathbf{u})} |\lambda_{\alpha_1}(\mathbf{u})|} \cdot \frac{\sigma_2}{\sigma_1} \tag{16}$$

where the rescaling factor σ_2/σ_1 corrects for differences in variances between primary and secondary variables.

Figure 4 shows the values of the two criteria for ordinary cokriging (solid line) and rescaled ordinary cokriging (dashed line). Although the magnitude of negative weights is reduced slightly by rescaled ordinary cokriging (top graph), the use of a single constraint does not increase the relative weight of secondary data (middle graph). Except beyond the extreme right datum (extrapolation situation), traditional and rescaled ordinary cokriging estimates are similar, (see Fig. 4, bottom graph).

As mentioned in previous sections, the magnitude of cokriging weights depends on the ratio of standard deviations of primary and secondary variables. Other things being equal, the smaller the ratio σ_1/σ_2, the smaller the secondary

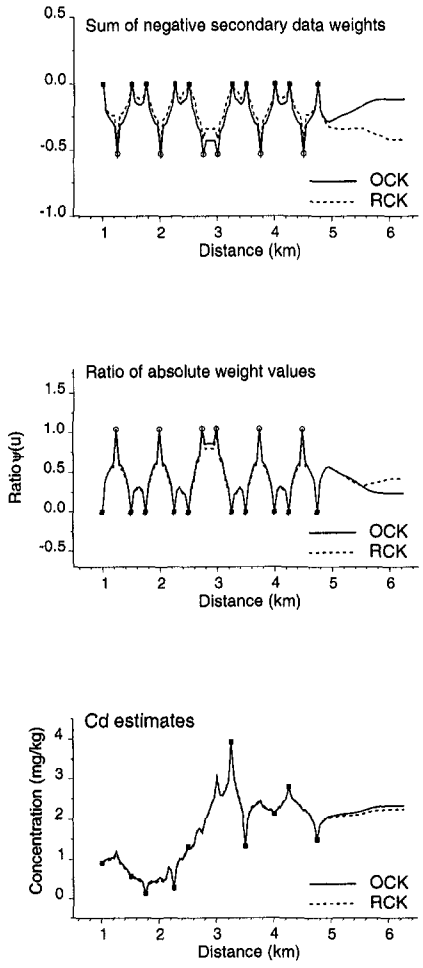

Figure 4. Impact of unbiasedness constraint(s) on ordinary cokriging weights and estimates.

data weights relative to primary data weights, recall relations (10) and (11). In the example of Figure 4, the standard deviation of Ni data is 10 times larger than that of Cd data, hence secondary data weights are much smaller than primary data weights:

$$\sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda_{\alpha_2}^{RCK}(\mathbf{u}) \ll \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda_{\alpha_1}^{RCK}(\mathbf{u})$$

The small secondary data weights entail that the single constraint (13) is actually little different from the traditional constraint that the primary data weights must

sum to one, that is

$$\sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda_{\alpha_1}^{RCK}(\mathbf{u}) \approx 1$$

Thus, using one or two unbiasedness constraints yields similar cokriging weights in the present example.

The impact of the single unbiasedness constraint (13) can be enhanced by rescaling primary and secondary data weights so that they are of the same order of magnitude. A solution consists of standardizing primary and secondary variables to a same zero mean and unit variance. The standardized form of the rescaled ordinary cokriging estimator (14), denoted $Z_{RCK'}^{(1)*}(\mathbf{u})$, is:

$$\frac{Z_{RCK'}^{(1)*}(\mathbf{u}) - m_1}{\sigma_1} = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \nu_{\alpha_1}^{RCK}(\mathbf{u}) \left[\frac{Z_1(\mathbf{u}_{\alpha_1}) - m_1}{\sigma_1}\right] + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \nu_{\alpha_2}^{RCK}(\mathbf{u}) \left[\frac{Z_2(\mathbf{u}_{\alpha_2}) - m_2}{\sigma_2}\right] \tag{17}$$

The data weights $\nu_{\alpha_i}^{RCK}(\mathbf{u})$ are solutions of a rescaled ordinary cokriging system expressed in terms of correlograms:

$$\begin{cases} \sum_{\beta_1=1}^{n_1(\mathbf{u})} \nu_{\beta_1}^{RCK}(\mathbf{u})\rho_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \nu_{\beta_2}^{RCK}(\mathbf{u})\rho_{12}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_2}) \\ \quad + \tau^{RCK}(\mathbf{u}) = \rho_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}), \qquad \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \\ \sum_{\beta_1=1}^{n_1(\mathbf{u})} \nu_{\beta_1}^{RCK}(\mathbf{u})\rho_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \nu_{\beta_2}^{RCK}(\mathbf{u})\rho_{22}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_2}) \\ \quad + \tau^{RCK}(\mathbf{u}) = \rho_{21}(\mathbf{u}_{\alpha_2} - \mathbf{u}), \qquad \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \\ \sum_{\beta_1=1}^{n_1(\mathbf{u})} \nu_{\beta_1}^{RCK}(\mathbf{u}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \nu_{\beta_2}^{RCK}(\mathbf{u}) = 1 \end{cases} \tag{18}$$

Unlike simple or ordinary cokriging, the weights provided by rescaled ordinary cokriging systems written in terms of covariances or correlograms, $\lambda_{\alpha_i}^{RCK}(\mathbf{u})$ and $\nu_{\alpha_i}^{RCK}(\mathbf{u})$, are not linearly related, and so the estimators $Z_{RCK}^{(1)*}(\mathbf{u})$ and $Z_{RCK'}^{(1)*}(\mathbf{u})$ are different!

Figure 5 shows that the single unbiasedness constraint (13) has a greater impact on data weights when the cokriging system is solved in terms of correlograms. A single constraint reduces substantially the sum of negative secondary data weights and increases the influence of the secondary information as measured by the ratio (16). Thus, the difference between traditional and rescaled ordinary cokriging is more pronounced, compare Figures 4 and 5 (bottom graph).

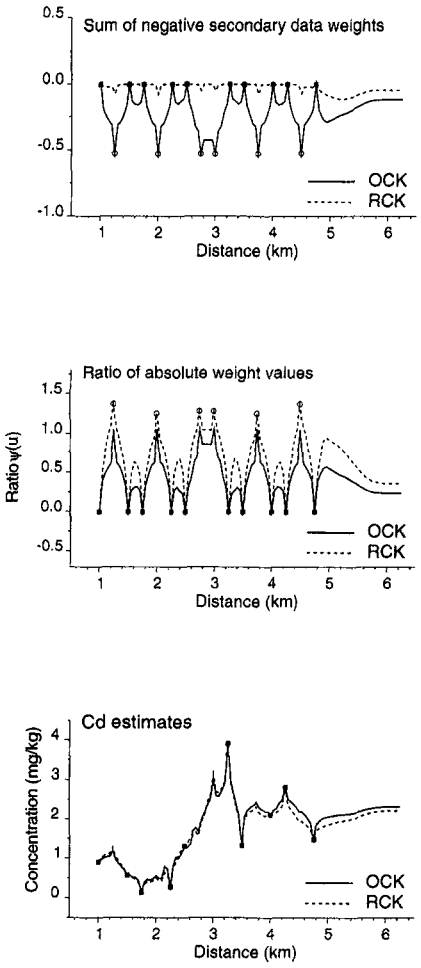

Figure 5. Impact of unbiasedness constraint(s) on ordinary cokriging weights and estimates when both primary and secondary variables are standardized to unit variance. Single unbiasedness constraint (dashed line) yields less negative secondary data weights and gives more importance to secondary information (larger ratio $\psi(\mathbf{u})$) than two nonbias constraints (solid line).

Remarks

- Simple and ordinary cokriging assign zero weights to secondary data when the primary and secondary variables are measured at the same locations (isotopic situation) and the cross covariance $C_{12}(\mathbf{h})$ is proportional to the primary autocovariance $C_{11}(\mathbf{h})$ (Matheron, 1979; Wackernagel, 1994). Such a simplification does not apply to ordinary cokriging with a single unbiasedness constraint in that the secondary variable still contributes to the estimation (Helterbrand and Cressie, 1994).

Figure 6. Location map showing split of 359 data locations into test set (black dots) and prediction set (open circles).

- When the secondary variable is known at the estimated location **u**, the secondary datum $z_2(\mathbf{u})$ collocated with the unknown value $z_1(\mathbf{u})$ tends to screen the influence of more distant secondary data (Asli and Marcotte, 1995). Thus, retaining only that secondary datum in the estimation, that is, $n_2(\mathbf{u}) = 1$, should suffice. Such collocated cokriging (Xu and others, 1992; Almeida and Journel, 1994) cannot be implemented under the two traditional constraints (7) since it would yield a zero weight for the secondary datum. In this situation, rescaled ordinary cokriging must be used.

PERFORMANCE COMPARISON

To investigate the prediction performances of cokriging variants, 100 locations out of the 359 available in the environmental dataset were put aside; these test locations are depicted by black dots in Figure 6. Three sets of primary and secondary variables were considered (see Table 1). The corresponding matrix of linear correlation coefficients is given in Table 2. Figures 7 and 8 show

Table 1. Secondary Variables Used to Reestimate Primary Metals at 100 Test Locations

Primary variable	Secondary variables
Cd	Ni, Zn
Cu	Pb, Ni, Zn
Pb	Cu, Ni, Zn

Table 2. Matrix of Linear Correlation Coefficients

	Cd	Cu	Pb	Ni	Zn
Cd	1.00				
Cu	0.12	1.00			
Pb	0.22	0.78	1.00		
Ni	0.49	0.23	0.31	1.00	
Zn	0.67	0.57	0.59	0.63	1.00

the experimental direct and cross-semivariograms with the linear model of coregionalization fitted using the iterative procedure developed by Goulard and Voltz (1992).

Three different combinations of sampling density and search strategy were used:

(1) primary and secondary variables are available at all 259 data locations, and the 16 closest data of each primary/secondary variable are retained, i.e., $n_i(\mathbf{u}) = 16 \ \forall \ i$ (isotopic situation).

(2) secondary variables are available at the 259 primary data locations plus the 100 test locations, and $n_i(\mathbf{u}) = 16 \ \forall \ i$ (heteropic situation).

(3) secondary variables are available at the 259 primary data locations plus the 100 test locations, and the 16 closest primary data and only the

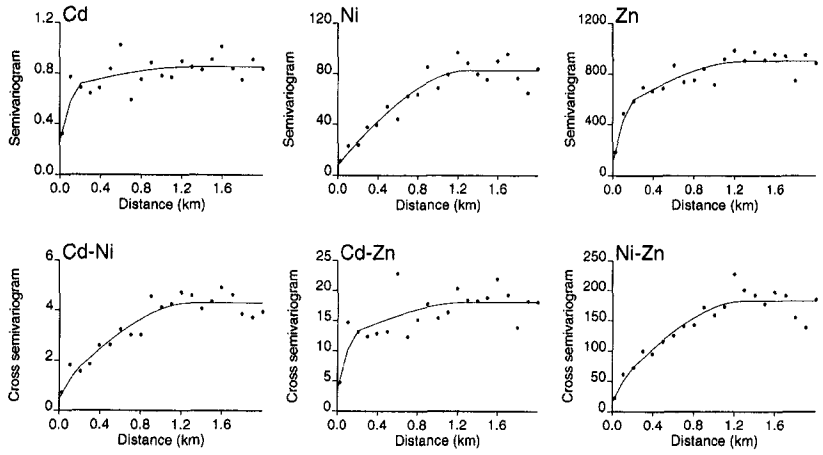

Figure 7. Experimental direct and cross-semivariograms for Cd, Ni, and Zn with linear model of coregionalization fitted.

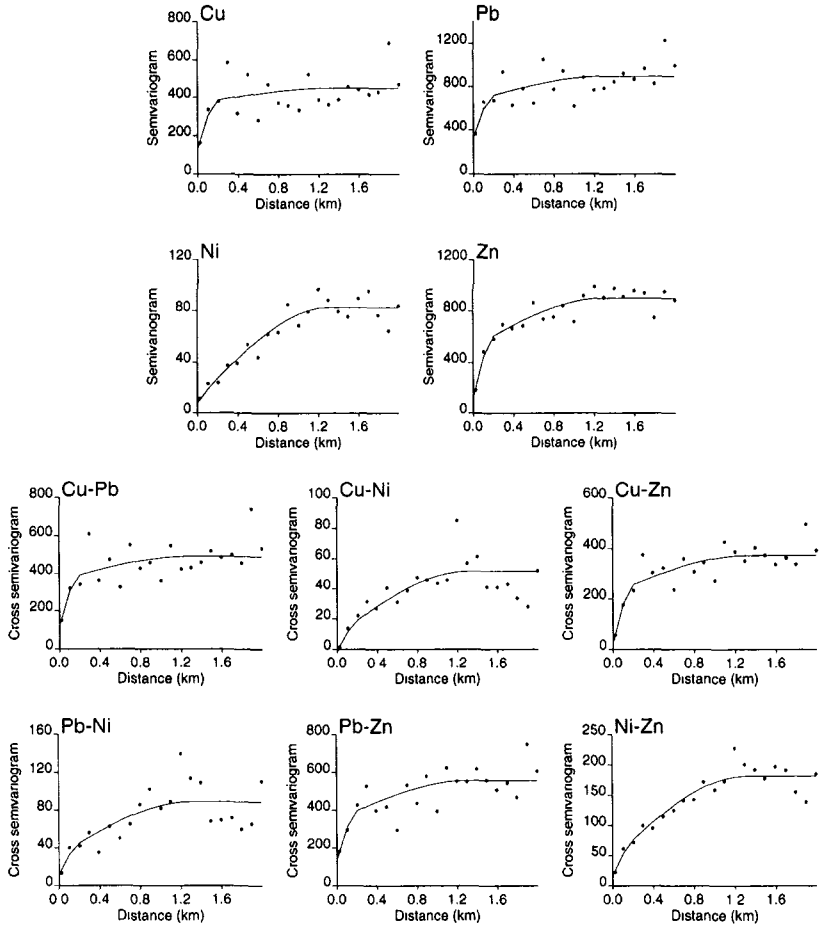

Figure 8. Experimental direct and cross-semivariograms for Cu, Pb, Ni, and Zn with linear model of coregionalization fitted.

collocated secondary data are retained, $n_1(\mathbf{u}) = 16$, $n_i(\mathbf{u}) = 1 \; \forall \; i > 1$ (collocated situation).

In each situation, metal concentrations were estimated at test locations using ordinary kriging (OK, reference), ordinary cokriging (OCK), and rescaled ordinary cokriging (RCK) expressed in terms of covariances (cov.) or correlograms (corr.).

The prediction performances of algorithms are assessed using two criteria:

- the mean absolute error of prediction, that is the average absolute difference between the actual metal concentrations and the estimates,

- the proportion of test locations that are classified wrongly as safe or contaminated on the basis of estimates; the tolerable maxima are 0.8 mg/kg for cadmium and 50 mg/kg for the two other metals.

Results are given in Tables 3–5.

As already observed for indicator variables (Goovaerts, 1994), cokriging does not perform better than kriging in the isotopic situation. Indeed, when all metals are sampled equally, the influence of the secondary information is screened by primary data and so it contributes little to the cokriging estimate. In most situations, there is even a slight decrease in prediction performances when equally sampled secondary information is considered.

Accounting for better sampled secondary metals (heterotopic case) reduces significantly the mean absolute error and the proportion of misclassified locations. The reduction is the most important for copper and lead which are well correlated with each other ($\rho = 0.78$). Better results generally are obtained by using a single unbiasedness constraint (RCK) rather than the traditional constraints requiring the secondary data weights to sum to zero (OCK).

Table 3. Mean Absolute Error of Prediction and Percentage of Test Locations that Are Declared Wrongly Safe or Contaminated with Respect to Cd on Basis of Four (Co)Kriging Estimates. Three Different Data Configurations Are Considered

	Mean absolute error			Perc. of misclassified locations		
Algorithm	Isotopic	Heterotopic	Collocated	Isotopic	Heterotopic	Collocated
OK	0.58	0.58	0.58	35	35	35
OCK	0.60	0.51	—	33	26	—
RCK (cov.)	0.59	0.52	0.59	33	25	29
RCK (corr.)	0.60	0.52	0.50	34	25	27

Table 4. Mean Absolute Error of Prediction and Percentage of Test Locations that Are Declared Wrongly Safe or Contaminated with Respect to Cu on Basis of Four (Co)Kriging Estimates. Three Different Data Configurations Are Considered

	Mean absolute error			Perc. of misclassified locations		
Algorithm	Isotopic	Heterotopic	Collocated	Isotopic	Heterotopic	Collocated
OK	15.4	15.4	15.4	8	8	8
OCK	15.6	7.9	—	9	3	—
RCK (cov.)	15.9	7.8	7.9	9	1	1
RCK (corr.)	16.0	7.4	7.1	9	1	1

Table 5. Mean Absolute Error of Prediction and Percentage of Test Locations that Are Declared Wrongly Safe or Contaminated with Respect to Pb on Basis of Four (Co)Kriging Estimates. Three Different Data Configurations Are Considered

	Mean absolute error			Perc. of misclassified locations		
Algorithm	Isotopic	Heterotopic	Collocated	Isotopic	Heterotopic	Collocated
OK	20.9	20.9	20.9	36	36	36
OCK	21.4	10.8	—	37	20	—
RCK (cov.)	21.5	10.7	10.5	39	23	19
RCK (corr.)	21.5	10.6	10.7	38	23	20

Retaining only the collocated secondary data (collocated situation) reduces slightly the cokriging performances for cadmium. That loss is attenuated by standardizing all variables to zero mean and unit variance. The strong correlation between copper and lead enhances the screening of distant secondary data by colocated secondary data, and so cokriging performs equally in the heterotopic and colocated situations.

CONCLUSIONS

The various cokriging estimators differ in the way the trend component is modeled. Whereas simple cokriging considers the primary and secondary means as known and constant within the study area, rescaled and traditional ordinary cokriging reestimate these means within each search neighborhood through the incorporation of one or several unbiasedness constraints in the cokriging system. Using a single constraint reduces the magnitude of negative secondary data weights and increases the contribution of the secondary information to the estimate.

When primary and secondary variances are not of the same order of magnitude, it is good practice to solve the cokriging system in terms of correlograms to avoid numerical instability. Unlike other cokriging variants, rescaled ordinary cokriging of the original or standardized variables does not yield the same estimate. For the one-dimensional example, the standardization of variables enhances the impact of the single unbiasedness contraint on the cokriging weights: the magnitude of negative secondary data weights is reduced whereas the relative weight of secondary variables vs. the primary variable increases.

The case study has confirmed the current practice that cokriging is not worthwhile if primary and secondary variables are sampled at the same locations (isotopic situation). When the secondary information is available at the estimated

location, the approximation of retaining only the colocated secondary data affects little prediction performance.

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REFERENCES

Almeida, A., and Journel, A. G., 1994, Joint simulation of multiple variables with a Markov-type coregionalization model: Math. Geology, v. 26, no. 5, p. 565–588.

Asli, M., and Marcotte, D., 1995, Comparison of approaches to spatial estimation in a bivariate context: Math. Geology, v. 27, no. 5, p. 641–658.

Atteia, O., Dubois, J.-P., and Webster, R., 1994, Geostatistical analysis of soil contamination in the Swiss Jura: Environmental Pollution, v. 86, no. 1, p. 315–327.

Deutsch, C. V., and Journel, A. G., 1992, GSLIB: Geostatistical Software Library and user's guide: Oxford Univ. Press, New York, 340 p.

Goovaerts, P., 1994, Comparative performance of indicator algorithms for modeling conditional probability distribution functions: Math. Geology, v. 26, no. 3, p. 389–411.

Goulard, M., and Voltz, M., 1992, Linear coregionalization model: tools for estimation and choice of cross-variogram matrix: Math. Geology, v. 24, no. 3, p. 269–286.

Helterbrand, J. D., and Cressie, N., 1994, Universal cokriging under intrinsic coregionalization: Math. Geology, v. 26, no. 2, p. 205–226.

Hevesi, J. A., Istok, J. D., and Flint, A. L., 1992, Precipitation estimation in mountainous terrain using multivariate geostatistics. Part I: Structural analysis: Jour. Applied Meteorology, v. 31, no. 3, p. 661–676.

Isaaks, E., and Srivastava, R., 1989, An introduction to applied geostatistics: Oxford Univ. Press, New York, 561 p.

Journel, A. G., and Rossi, M. E., 1989, When do we need a trend model in kriging?: Math. Geology, v. 21, no. 7, p. 715–739.

Matheron, G., 1970, La théorie des variables régionalisées et ses applications: Fascicule No. 5, Cahier du Centre de Morphologie Mathématique de Fontainebleau, 212 p.

Matheron, G., 1979, Recherche de simplification dans un problème de cokrigeage: Research Rept. N-628, Centre de Géostatistique, Fontainebleau, 18 p.

Stein, A., van Dooremolen, W., Bouma, J., and Bregt, A. K., 1988, Cokriging point data on moisture deficit: Soil Sci. Soc. Am. Jour., v. 52, no. 5, p. 1418–1423.

Wackernagel, H., 1994, Cokriging versus kriging in regionalized multivariate data analysis: Geoderma, v. 62, nos. 1-3, p. 83–92.

Webster, R., Atteia, O., and Dubois, J.-P., 1994, Coregionalization of trace metals in the soil in the Swiss Jura: European Jour. Soil Science, v. 45, no. 1, p. 205–218.

Xu, W., Tran, T., Srivastava, R., and Journel, A. G., 1992, Integrating seismic data in reservoir modeling: the collocated cokriging alternative: SPE Paper No. 24742, unpaginated.

APPENDIX. ORDINARY COKRIGING OF THE LOCAL MEANS

Consider the estimation of the local mean of Z_1 at location **u**. In the situation of a single secondary attribute z_2 the ordinary cokriging estimator is written:

$$m_{OCK}^{(1)*}(\mathbf{u}) = \sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda_{\alpha_1 m_1}^{OCK}(\mathbf{u}) Z_1(\mathbf{u}_{\alpha_1}) + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda_{\alpha_2 m_1}^{OCK}(\mathbf{u}) Z_2(\mathbf{u}_{\alpha_2}) \tag{A1}$$

where $\lambda_{\alpha_i m_1}^{OCK}(\mathbf{u})$ is the weight assigned to the datum $z_i(\mathbf{u}_{\alpha_i})$ for the estimation of the primary mean.

The unbiasedness of estimator (A1) is ensured by constraining the primary data weights to sum to one and the secondary data weights to sum to zero, that is using the same constraints as for the estimation of Z_1:

$$\sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda_{\alpha_1 m_1}^{OCK}(\mathbf{u}) = 1, \qquad \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda_{\alpha_2 m_1}^{OCK}(\mathbf{u}) = 0$$

entails:

$$\begin{aligned} E\{m_{OCK}^{(1)*}(\mathbf{u}) - m_1(\mathbf{u})\} &= \left[\sum_{\alpha_1=1}^{n_1(\mathbf{u})} \lambda_{\alpha_1 m_1}^{OCK}(\mathbf{u}) - 1\right] m_1(\mathbf{u}) + \sum_{\alpha_2=1}^{n_2(\mathbf{u})} \lambda_{\alpha_2 m_1}^{OCK}(\mathbf{u}) m_2(\mathbf{u}) \\ &= 0 \end{aligned}$$

The error variance $\sigma_E^2(\mathbf{u}) = \mathrm{Var}\{m_{OCK}^{(1)*}(\mathbf{u}) - m_1(\mathbf{u})\}$ is expressed as a linear combination of cross-covariance values:

$$\begin{aligned} \sigma_E^2(\mathbf{u}) &= \mathrm{Var}\{m_{OCK}^{(1)*}(\mathbf{u})\} + \mathrm{Var}\{m_1(\mathbf{u})\} - 2\,\mathrm{Cov}\{m_{OCK}^{(1)*}(\mathbf{u}), m_1(\mathbf{u})\} \\ &= \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{\alpha_i=1}^{n_i(\mathbf{u})} \sum_{\alpha_j=1}^{n_j(\mathbf{u})} \lambda_{\alpha_i}^{OCK}(\mathbf{u}) \lambda_{\alpha_j}^{OCK}(\mathbf{u}) C_{ij}(\mathbf{u}_{\alpha_i} - \mathbf{u}_{\alpha_j}) \end{aligned} \tag{A2}$$

Note that the last two terms of the error variance are zero since $m_1(\mathbf{u})$ is viewed as a deterministic component.

The minimization of the error variance (A2) under the two unbiasedness constraints yields the following system of $(n_1(\mathbf{u}) + n_2(\mathbf{u}) + 2)$ linear equations:

$$\begin{cases} \displaystyle\sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda_{\beta_1 m_1}^{OCK}(\mathbf{u}) C_{11}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda_{\beta_2 m_1}^{OCK}(\mathbf{u}) C_{12}(\mathbf{u}_{\alpha_1} - \mathbf{u}_{\beta_2}) \\ \qquad + \mu_{1m_1}^{OCK}(\mathbf{u}) = 0 \qquad \alpha_1 = 1, \ldots, n_1(\mathbf{u}) \\ \displaystyle\sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda_{\beta_1 m_1}^{OCK}(\mathbf{u}) C_{21}(\mathbf{u}_{\alpha_2} - u_{\beta_1}) + \sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda_{\beta_2 m_1}^{OCK}(\mathbf{u}) C_{22}(\mathbf{u}_{\alpha_2} - \mathbf{u}_{\beta_2}) \\ \qquad + \mu_{2m_1}^{OCK}(\mathbf{u}) = 0 \qquad \alpha_2 = 1, \ldots, n_2(\mathbf{u}) \\ \displaystyle\sum_{\beta_1=1}^{n_1(\mathbf{u})} \lambda_{\beta_1 m_1}^{OCK}(\mathbf{u}) = 1 \\ \displaystyle\sum_{\beta_2=1}^{n_2(\mathbf{u})} \lambda_{\beta_2 m_1}^{OCK}(\mathbf{u}) = 0 \end{cases} \tag{A3}$$

where $\mu_{1m_1}^{OCK}(\mathbf{u})$ and $\mu_{2m_1}^{OCK}(\mathbf{u})$ are Lagrange parameters. Interchanging primary and secondary variables in expressions (A1) and (A3) yields the estimator and cokriging system for the secondary local mean at location $\mathbf{u}$, $m_{OCK}^{(2)*}(\mathbf{u})$.

System (A3) is identical to the cokriging system (8) except for the right-hand-side covariance terms $C_{i1}(\mathbf{u}_{\alpha_i} - \mathbf{u})$ being set to zero. The fact that all data-to-unknown covariance terms are zero entails that the specific location $\mathbf{u}$ being estimated does not occur in the cokriging system (A3). Provided the same set of data is used to estimate the local mean at two different locations $\mathbf{u}$ and $\mathbf{u}'$, the system (A3) remains unchanged. Thus, the two sets of cokriging weights and the two trend estimates are identical: $m_{OCK}^{(1)*}(\mathbf{u}) = m_{OCK}^{(1)*}(\mathbf{u}')$.