CLASSIFYING RELATIVE EQUILIBRIA. III

JULIAN I. PALMORE*
Dept. of Mathematics, University of Michigan, Ann Arbon, Mich. 48104, U.S.A.

ABSTRACT. We announce several theorems on the evolution of relative equilibria classes in the planar n-body problem. In an earlier paper [1] we announced a partial classification of relative equilibria of four equal masses. In [2] we described these new relative equilibria classes and showed the way in which a degeneracy arose in the four body problem. These results point the way toward classifying relative equilibria for any $n > 4$.

1. DEGENERATE RELATIVE EQUILIBRIA CLASSES

For any $n \geqslant 4$ and for any choice of positive masses $m = (m_1, \ldots, m_n) \varepsilon R^n_+$ we study the degenerate critical points of a real analytic function $\tilde{V}_m < 0$ which is defined on a real analytic manifold X_m [1, 2, 3]. Each such critical point corresponds to a degenerate relative equilibria class.

Let $\Sigma_n \subset R^n_+$ be the set of all m such that $\tilde{V}_m$ has a degenerate critical point.

We show in Theorem 1 the existence of degenerate relative equilibria classes of $\tilde{V}_m$ for some $m \varepsilon R^n_+$ and for any $n \geqq 4$. In Theorem 2 we state a sharpened result on the nature of Σ_n for $n \geqq 4$. In Theorem 3 places an upper bound on values of k for which Σ_n has positive k-dimensional (Hausdorff) measure.

Finally, in light of Theorem 1 in the case of $n = 4$ masses $m = (1, 1, 1, m_4)$ we count classes of relative equilibria.

2. MAIN THEOREMS

In the plane E^2 we place $n - 1$ unit masses at the vertices of a regular polygon of $n - 1$ sides with center at the origin. We place at

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the origin an arbitrary positive mass m_n. It follows from the definition of relative equilibrium for all values of $m_n > 0$ that this configuration is a relative equilibrium of n masses.

Let E^2 be identified with C so that we write the configuration above as $(x_1, \ldots, x_n) \in C^n$, where $x_n = 0$ and for each i, $1 \leqq i \leqq n-1$, $x_i = \omega^{i-1}$, where $\omega \neq 1$ is the first primitive n-1 root of unity. Let $x \in X_m$ be the relative equilibria class which contains this configuration for any given $m = (1, \ldots, 1, m_n)$. In the theorem below $v \in T_x X_m$ is the tangent vector which corresponds to $(v_1, \ldots, v_n) \in C^n$, where $v_n = 0$ and for each i, $1 \leqq i \leqq n-1$, $v_i = \omega^{2(i-1)}$.

THEOREM 1. Let $x \in X_m$ be a relative equilibria class as defined above for any $n \geqq 4$. Then $x \in X_m$ is a degenerate critical point of $\tilde{V}_m$ if and only if $m = (1, \ldots, 1, m_n{}^*)$ where

$$m_n^* = \frac{A(B-(n-1)^2)}{6(n-1)^3-3(n-1)(A+B)},$$

where $A = D^2\tilde{V}_{m'}(x)(v, v)$ and $B = \tilde{V}_{m'}(x)$ for $m' = (1, \ldots, 1, 0)$.

COROLLARY 1.1. For $m = (1, \ldots, 1, m_n)$ and $m_n < m_n^*$ the index of x (i.e. the index of $D^2\tilde{V}_m(x)$) equals $2n-4$ and x is a nondegenerate local maximum of $\tilde{V}_m$. For $m_n > m_n^*$ the index of x equals $2n-6$ and x is a nondegenerate saddle. When $m_n = m_n^*$ the rank of x (= index of x) equals $2n-6$.

Let $\Sigma_{n,i} \subset \Sigma_n$ denote the set of masses $m \in \Sigma_n$ such that $\tilde{V}_m$ has a degenerate critical point with rank which does not exceed $2n-4-i$, $i \geqq 1$. In particular $\Sigma_n = \Sigma_{n,1} \supset \Sigma_{n,2} \supset \ldots \supset \Sigma_{n,n-2}$ holds and by [1, Theorem 2] $\Sigma_{n,n-1} = \emptyset$ holds for any $n \geqq 4$.

By Theorem 1 we have shown that $\Sigma_{n,2} \neq \emptyset$ and consequently, $\Sigma_n \neq \emptyset$ for any $n \geqq 4$. We sharpen this result to include $\Sigma_{n,i}$ for any i, $1 \leqq i \leqq n-2$.

THEOREM 2. $\Sigma_{n,i} - \Sigma_{n,i+1} \neq \emptyset$ for any i, $1 \leqq i \leqq n-2$, and $\Sigma_{n,i} = \emptyset$ for any $i > n-2$ and for any $n \geqq 4$.

Finally, we state a result on the k-dimensional (Hausdorff) measure of Σ_n [4, Theorem 3].

Let A be a subset of R^n. We say that A has k-dimensional measure 0 provided that for each $\varepsilon > 0$ there is a cover of A by a sequence of sets $\{A_i\}$ such that

$$\sum_{i=1}^{\infty} (\operatorname{diam} A_i)^k < \varepsilon.$$

If A has k-dimensional measure 0, then A has r-dimensional measure 0 for all r, $k \leqq r \leqq n$. If A is a closed subset of R^n, we say that A has positive k-dimensional (Hausdorff) measure provided that A fails to have k-dimensional measure 0.

THEOREM 3. Σ_n *has positive k-dimensional (Hausdorff) measure for any k,* $0 \leqq k \leqq n - 1$ *and for any* $n \geqq 4$.

3. CLASSIFYING RELATIVE EQUILIBRIA

Let $m = (1, 1, 1, m_4)$ be chosen with $m_4 > 0$. We now count the number of relative equilibria classes in the case of $n = 4$ masses.

THEOREM 4. For $m = (1, 1, 1, m_4)$ *and for any* $m_4 < m_4^*$ *there are* 38 *classes of relative equilibria. Their distribution is* 8 *maxima of index* 4, 18 *saddles of index* 3 *and* 12 *saddles (the Moulton classes) of index* 2.

COROLLARY 4.1. *For* $m = (1, 1, 1, m_4)$ *and for any* $m_4 < m_4^*$, $\tilde{V}_m$ *is a Morse function.*

COROLLARY 4.2. *When* $m = (1, 1, 1, m_4^*)$, $\tilde{V}_m$ *has* 32 *critical points. There are* 6 *maxima (index* 4), 12 *saddles (index* 3), 12 *saddles (index* 2) *and* 2 *degenerate saddles of type* (0, 2, 0).

Remark. The classification given by Corollary 4.2 corresponds to the minimal classification of [3, Theorem 4].

For $m_4 > m_4^*$ new classes of relative equilibria exist in addition to those 38 classes for $m_4 < m_4^*$. Compare [1, Theorem 5]. In particular there are other degenerate critical points of $\tilde{V}_m$ for a unique $m = (1, 1, 1, m_4^+)$, where $m_4^* < m_4^+ < 1$.

By analyzing this second degeneracy we are able to account by evolution for the existence of precisely 146 classes of relative equilibria in the case of $n = 4$ equal masses.

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