

ON ALGEBRAIC MODELS OF DYNAMICAL SYSTEMS

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ABSTRACT. We describe a universal algebraic model which, being read appropriately, yields (periodic and infinite) discrete dynamical systems, as well as their 'continuous limits', which cover all differential scalar Lax systems. For this model we give: Two different constructions of an infinity of integrals; modified equations; deformations; infinitesimal automorphisms. The basic tools are supplied by symbolic calculus and the abstract Hamiltonian formalism.

1. INTRODUCTION

At the present time, the structure of large classes of integrable systems is well understood, both in the differential case [1–3] and for discrete systems of classical mechanics [4–6]. Here by 'integrable' in the differential situation we mean 'with an infinite number of conservation laws'. However, methods employed in both cases are so different that it is not clear what, if any, are the relationships between the two situations. And if we ever want to understand connexions between continuous and discrete, the rich phenomenon of integrability is the first candidate for analysis.

In this note we describe an algebraic scheme which provides, among other things, the above-mentioned connexion between 'continuous' and 'discrete' systems which are, in turn, specializations of the universal model. The plan is as follows. We introduce abstract Lax equations, algebras with automorphisms, corresponding formal calculus (parallel to the differential algebraic case) and Hamiltonian formalism. Then we specialize our situation to symbolic Lax equations, construct their c.l.'s (conservation laws) and Hamiltonian form(s), continuous limits, modified systems and their canonical maps. Finally, we obtain deformations of all previously considered unmodified systems together with their reductions and infinitesimal automorphisms.

2. ABSTRACT LAX DERIVATIONS

Let R be an associative F -algebra over the field F ; $i, \alpha \in \mathbb{Z}_+, j, N \in \mathbb{Z}_+ \cup \{0\}, R[\bar{x}] = R[x_0, x_1, \dots]$ where x_j 's are associative homogeneous generators with $rk x_j = i - \alpha j, R_N[\bar{x}] = R[\bar{x}]/(x_N, x_{N+1}, \dots)$. In the closure $\hat{R}[\bar{x}]$ of $R[\bar{x}]$ fix $L = x_0 + x_1 + \dots$. If $\hat{R}[\bar{x}] \ni P = \sum_s P_s, rk P_s = s$, let $P_+ = \sum_{s \geq 0} P_s, P_- = P - P_+, Res P = P_0$. Let $\gamma = \alpha/(i, \alpha)$. For $k \in \mathbb{Z}_+, \gamma$, take $P = L^k$ and define the derivation X_P of

$\hat{R}[\bar{x}]$ by: $X_P = 0$ on R ; $rk X_P = 0$ and $X_P(L) = [P_+, L] [L, P_-]$.

THEOREM 1. (i) If $l \in \mathbb{Z}_+ \gamma$, $Q = L^l$ then $[X_P, X_Q] = 0$. (ii) $X_P(\text{Res } Q) = \text{Res } [Q_+, P_-]$. (iii) X_P is correctly defined on $\hat{R}_N[\bar{x}]$.

3. SYMBOLIC CALCULUS

Let $\Delta_1, \dots, \Delta_r: K \rightarrow K$ be r commuting automorphisms of an associative algebra K over F . For any $\sigma = (\sigma_1, \dots, \sigma_r) \in \mathbb{Z}^r$ denote $\Delta^\sigma = \Delta_1^{\sigma_1} \dots \Delta_r^{\sigma_r}$, put $\tilde{K} = K^\Delta[\bar{q}] = K[q_j^{(\nu_j)}]$ where $\nu_j \in \mathbb{Z}^r$ and $q_j^{(\nu_j)}$ are independent commuting variables, $q_j^{(0)} = q_j$. The action of Δ 's is uniquely extended on \tilde{K} by $\Delta^\sigma(q_j^{(\nu)}) = q_j^{(\sigma + \nu)}$. Denote $\mathcal{D}_P = 1 - \Delta_P$, $\text{Im } \mathcal{D} = \sum_P \text{Im } \mathcal{D}_P$.

Suppose \tilde{K} is commutative. Consider a Lie algebra of 'evolution fields' $\mathcal{D}^{\text{ev}} = \mathcal{D}^{\text{ev}}(\tilde{K}) = \{\hat{X} \in \mathcal{D}\text{er}(\tilde{K}) \mid \hat{X}\Delta^\sigma = \Delta^\sigma \hat{X}, \forall \sigma \in \mathbb{Z}^r\}$. Any such \hat{X} is uniquely defined by the vector $\tilde{X} = \hat{X}(\bar{q})$: $\tilde{X} = \sum \Delta^\sigma(X_j)(\partial/\partial q_j^{(\sigma)})$. Let $\Omega^1 = \Omega^1(\tilde{K})$ be the usual \tilde{K} -bimodule of differentials with the natural derivation $d: \tilde{K} \rightarrow \Omega^1$, $d(q_j^{(\sigma)}) = dq_j^{(\sigma)}$. Extend Δ 's on Ω^1 such that $d\Delta^\sigma = \Delta^\sigma d$, $\forall \sigma \in \mathbb{Z}^r$. Let $\Omega_0^1 = \sum_j \tilde{K} dq_j$.

THEOREM 2. (i) $\exists! \delta: \tilde{K} \rightarrow \Omega_0^1$ such that $\forall H \in \tilde{K}$, $\forall \hat{X} \in \mathcal{D}^{\text{ev}}$, $\hat{X}(H) \equiv \hat{X} \lrcorner \delta H \pmod{\text{Im } \mathcal{D}}$: $\delta H = \sum_j (\delta H / \delta q_j) dq_j$, $(\delta H / \delta q_j) = \sum_\sigma \Delta^{-\sigma} (\partial H / \partial q_j^{(\sigma)})$. (ii) $\text{Im } \mathcal{D} \subset \text{Ker } \delta$. If the Δ 's act identically on K then $\text{Ker } \delta = \text{Im } \mathcal{D} \oplus K$. (iii) $\exists! \hat{\delta}: \Omega^1 \rightarrow \Omega_0^1$ such that $\tilde{X} \lrcorner \omega \equiv \hat{X} \lrcorner \hat{\delta} \omega \pmod{\text{Im } \mathcal{D}}$, $\forall \hat{X} \in \mathcal{D}^{\text{ev}}$, $\forall \omega \in \Omega^1$. So $\delta = \hat{\delta} d$, cf. (7).

If M, S are \tilde{K} -modules and $A \in \text{Hom}_F(M, S)$, the adjoint operator $A^+ : S^* = \text{Hom}_{\tilde{K}}(S, \tilde{K}) \rightarrow M^*$ is defined by $(A^+ n^*)m \equiv n^*(Am) \pmod{\text{Im } \mathcal{D}}$. If $S^* = M = M^{**}$ then A is called skew (resp. symmetric) if $A^+ = -A$ (resp. $A^+ = A$). If $M = S = \tilde{K}$, $(m\Delta^\sigma)^+ = \Delta^{-\sigma} m$. For skew $B: \Omega_0^1 \rightarrow \mathcal{D}^{\text{ev}}$, $\Gamma = B\delta$ is called Hamiltonian if $\Gamma(\Gamma(H)(\tilde{F})) = [\Gamma(H), \Gamma(\tilde{F})]$, $\forall H, \tilde{F} \in \tilde{K}$, see [7]. In what follows, we identify Ω_0^1 and \mathcal{D}^{ev} with \tilde{K}^∞ , and require $B \in \text{Mat}_\infty(\tilde{K})[\Delta_1, \dots, \Delta_m]$. For $\bar{T} \in \tilde{K}^\infty$, its Frechet Jacobian $D(\bar{T}) \in \text{Mat}_\infty(\tilde{K})[\Delta_1, \dots, \Delta_m]$ is defined by $D(\bar{T})\tilde{X} = \hat{X}(\bar{T})$, $\forall \hat{X} \in \mathcal{D}^{\text{ev}}$.

THEOREM 3. (i) $D(\delta H)$ is symmetric $\forall H \in \tilde{K}$. (ii) For skew $B \in \text{Mat}_\infty(K)$, $B\delta$ is Hamiltonian. (iii) For $C \in \tilde{K}((\Delta_1, \dots, \Delta_m))$, $C = \sum_\sigma c_\sigma \Delta^\sigma$, let $\text{Res } C = c_0$. Then $\text{Res } [C_1, C_2] \in \text{Im } \mathcal{D}$, $\forall C_1, C_2 \in \tilde{K}((\Delta_1, \dots, \Delta_m))$.

4. SYMBOLIC LAX EQUATIONS

Consider an associative graded F -algebra $\tilde{K}((\xi^{-1}))$, $rk \xi = 1$, $rk \tilde{K} = 0$, with relations $\xi^s m = \Delta^s(m)\xi^s$, $\forall s \in \mathbb{Z}$, $\forall m \in \tilde{K}$, where from now on $r=1$, $\Delta_1 = \Delta$ and I write ξ instead of Δ in operators. Pick up some $b_0, b_1, \dots \in \tilde{K}$ and put $x_j = \xi^{j-\alpha_j} b_j$, $R = F$. Then abstract Lax equations associated with the derivation constructed above in the standard fashion (see (1)), turn into symbolic dynamical systems for variables b_0, b_1, \dots . Informal models: (A) Discrete systems on the lattice $\lambda \mathbb{Z} \subset \mathbb{R}^1$: $q_1 \in \text{Map}(\lambda \mathbb{Z}, \cdot)$, $\Delta = S_\lambda^*$ where $S_\lambda: n\lambda \mapsto (n+1)\lambda$ is the shift operator. Since all our constructions commute with S_λ , we can pass

to the periodic lattice $\lambda \mathbb{Z}_n \subset S^1$; (B) Differential systems: $q_j \in \text{Map}(\mathbb{R}^1, \cdot)$, $\Delta = S_\lambda^*$ and λ is considered as a formal ('small') parameter, in other words, $\Delta = \exp(\lambda d/dx) \in \mathbb{Q}[d/dx][(\lambda)]$ and one has to obviously modify the usual differential algebraic constructions to incorporate λ , (8). (Finite λ is not interesting enough).

Now Theorems 1(ii) and 3(iii) yield an infinity of common c.l.'s and Theorem 1(iii) asserts that in fact we are dealing with the universal case: Starting from any N we can put all $b_j \geq N$ to zero. For inessential simplicity we put $i = 1$, $\alpha = 1$ (the case $\alpha > 1$ is similar to the case $\alpha = 2$ which we shall comment upon later). Then $\gamma = 1$ and we have one 'flow' $\bar{q}_t = \hat{X}_L k(\bar{q})$ for every $k \in \mathbb{Z}_+$ (we put $b_0 = 1$, $b_{j+1} = q_j$). We denote c.l.'s $k^{-1} \text{Res } L^k$ by H_k (we assume $\text{char } F = 0$).

Let $B^1 = B_{\bar{q}}^1$ denote the matrix of the (formal) Kirillov form on the dual space of the Lie algebra generated by the associative algebra $\tilde{K}[[\xi]]$ (we write $\beta_{\bar{q}}^1$ to indicate variables involved).

THEOREM 4 (First Hamiltonian structure for symbolic Lax equations). $\hat{X}_{Lk}(\bar{q}) = B^1(\delta H_{k+1})$ and $B^1 \delta$ is Hamiltonian.

REMARK. One can get c.l.'s H_k by the formal analog of the Kostant–Symes theorem via the 'Adler scheme' [9] applied to the Lie algebra generated by $\tilde{K}((\xi^{-1}))$, with nondegenerate ad-invariant form $\langle C_1 | C_2 \rangle = \text{Res}(C_1 C_2)$.

For any $\bar{\eta} \in \tilde{K}^\infty$ denote $Y_{\bar{\eta}} = (\mu \xi^{-1} + \sum_j \eta_j \xi^j) \in \tilde{K}((\xi))$ where $\mu = \sum_j (\Delta^{j+1} - 1)(\Delta - 1)^{-1} \Delta^{-j-1} (q_{j+1} \eta_{j+1})$ and consider the matrix $B^2 = B_{\bar{q}}^2 \in \text{Mat}_\infty(\tilde{K})[\Delta, \Delta^{-1}]$ defined by $\sum_j \xi^{-i} (B^2 \bar{\eta})_j = L(Y_{\bar{\eta}} L)_- - (LY_{\bar{\eta}})_- L$, see [3].

THEOREM 5 (Second Hamiltonian structure for symbolic Lax equations). (i) $\hat{X}_{Lk}(\bar{q}) = B^2(\delta H_k)$ and $B^2 \delta$ is Hamiltonian. (ii) For any $c \in F$ let $\bar{q} + c = (q_0 + c, q_1, \dots)$. Then $B_{\bar{q}+c}^2 = B_{\bar{q}}^2 + cB_{\bar{q}}^1$. (iii) if $\alpha = 2$ then consider $L_{(2)} = \xi + \sum_j \xi^{-2j-1} b_j$ as specialization $S^{(2)}$ of $L = \xi + \sum_j \xi^{-j} q_j$ by $S^{(2)} = \{q_{2j} = 0, q_{2j+1} = b_j\}$. Denote by $\delta_{(2)}$ operator δ on $\tilde{K}_{(2)} = K^\Delta[\bar{b}]$ and let $H'_{2k} = (2k)^{-1} \text{Res } L_{(2)}^{2k}$ (in $\tilde{K}_{(2)}((\xi^{-1}))$). Then $x_{L_{(2)}^{2k}}(\bar{b}) = B^3(\delta_{(2)} H'_{2k})$ where $B^3 = B_{\bar{b}}^3$ is a correctly defined specialization of B^2 , and $B^2 \delta_{(2)}$ is Hamiltonian.

REMARK. Matrix B^1 does not survive specialization $S^{(2)}$.

Define $\beta_j, d_j \in \tilde{K}$ by $\xi = L - \sum_j \beta_j L^{-j}$, $(L^k)_- = \sum_j d_j L^{-j-1}$.

THEOREM 6 (Second construction of c.l.'s for symbolic Lax equations). (i) $\hat{X}_{Lk}[\ln(1 - \sum_j \beta_j Z^{j+1})] = \mathcal{D}(\sum_j d_j Z^{j+1})$ in $\tilde{K}[[Z]]$ where Z commutes with \tilde{K} . (ii) Define $\gamma_j \in \tilde{K}$ by $\sum_j \gamma_j Z^{j+1} = -\ln(1 - \sum_j \beta_j Z^{j+1})$. Then $\gamma_j \equiv H_{j+1} \pmod{\text{Im } \mathcal{D}}$.

REMARK. Theorem 6 is the symbolic counterpart of G. Wilson's treatment of differential case*.

5. CONTINUOUS LIMITS

Let $\phi_p = F[p_0^{(j)}, \dots, p_N^{(j)}]$ and $\phi_u = F[u_0^{(j)}, \dots, u_N^{(j)}]$ be two differential rings over F with

*Wilson, G., 'On Two Constructions of Conservation Laws for Lax Equations', *Quart. J. Math. (Oxford)*, to appear.

derivation ∂ , $\partial(p_j^{(n)}) = p_j^{(n+1)}$, $\partial(u_j^{(n)}) = u_j^{(n+1)}$. We make $K_p = \phi_p((\lambda))$ and $K_u = \phi_u((\lambda))$ into rings with automorphism by defining $\Delta = \exp(-\lambda\partial)$ (λ commutes with everything). In $K_u((\xi^{-1}))$ take $L_N = \xi + \sum_{s=0}^N u_s \xi^{-2s-1}$. Let ϕ_u be the monomorphism of associative rings $K_u((\xi^{-1})) \rightarrow K_u[[\partial]]$ via $\xi \rightarrow \exp(-\lambda\partial)$. Consider an isomorphism ψ of $K_u[[\partial]]$ onto $K_p[[\partial]]$, identical on λ , $(\xi$ and) ∂ and given by $\psi(u_j) = (-1)^j (2j+1)^{-1} \binom{n+1}{j+1} + \sum_{s=0}^j (-1)^{j-s} \binom{N-s}{j-s} p_s \lambda^{s+2}$.

THEOREM 7. $\psi\phi_u(L_N) = \theta_N + \lambda^{N+2} L_{[N]} + 0(\lambda^{N+3})$ where $\theta_N \in \mathbb{Q}$ and $L_{[N]} = (2N+4)^{-1} (-2\partial)^{N+2} + \sum_{s=0}^N p_{N-s} (-2\partial)^s$.

REMARKS. (A) Since $\phi_u(\text{Im } \mathcal{D}) \subset \text{Im } \partial$, we not only cover the basic theory of differential scalar Lax operators $L_{[N]}$ [1, 2] but find the whole λ -series of which it is just the first term (constant θ_N can be ignored). (B) Changing ψ , in the first nontrivial order of λ one can get equations which do not come from any scalar Lax operator.

6. MODIFIED SYSTEMS

These are based on the 2×2 matrix operator \tilde{L} with entries $\tilde{L}_{11} = \tilde{L}_{22} = 0$, $\tilde{L}_{12} = l_1 = \xi + u$, $\tilde{L}_{21} = l_2 = 1 + \sum_j \xi^{-j-1} v_j$; $\forall k \in \mathbb{Z}^+$, $X_{\tilde{L}^{2k}}(\tilde{L}) = [(\tilde{L}^{2k})_+, \tilde{L}] = [\tilde{L}, (\tilde{L}^{2k})_-]$ in $\text{Mat}_2(K'((\xi^{-1})))$, $K' = K^\Delta[u, \bar{v}]$. Since $\tilde{L}^2 = \text{diag}(l_1 l_2, l_2 l_1)$ and $X_{\tilde{L}^{2k}}(\tilde{L}^2) = [(\tilde{L}^2)^k_+, \tilde{L}^2]$, we get two separate symbolic Lax equations for $l_1 l_2$ and $l_2 l_1$ (for each k). Denote by M_1^* and M_2^* the corresponding maps $\tilde{K} \rightarrow K'$ (so that M_1, M_2 send 'solutions' of \tilde{L} into 'solutions' of L), let $h_k = k^{-1} \text{Res}(l_1 l_2)^k$. Let $B^4 = B_{u, \bar{v}}^4$ be a skew matrix with entries: 0 in (u, u) -place; $u(1 - \Delta^{-j-1})v_j$ in (u, v_j) -place; $B_{\bar{v}}^2$ as (\bar{v}, \bar{v}) submatrix.

THEOREM 8. (i) *The modified equations are Hamiltonian: $(u, \bar{v})_t = B^4(\delta h_k)$.* (ii) *Both M_1^* and M_2^* are canonical (i.e., preserve Poisson brackets) between $B^4 \delta$ and $B^2 \delta$.*

REMARKS. (A) $B^4(\delta \ln u) = 0$ so the modified equations have nonpolynomial c.l. in u . (B) The presence of two different canonical maps is the counterpart of the 'permutation of roots' symmetry in the differential case [3].

7. DEFORMATIONS

Theorem 5 (ii) shows that the shift in q_0 by constant ϵ^{-1} can be compensated for by recombination of H 's. Since both M_1^* and M_2^* are quadratic, we can make a simple affine transformation $u = U + \epsilon^{-1}$, $v_j = \epsilon V_j$ such that both M_1^* and M_2^* will have the form: $q_0 \mapsto U + 0(\epsilon)$, $q_{j+1} \mapsto V_j + 0(\epsilon)$. We call M_1 and M_2 'reductions' [10]. Denote by $B^5 = B_{U, \bar{v}}^5$ the (regular in ϵ) matrix that represents $\epsilon B^4 \delta$ in U, \bar{v} variables.

THEOREM 9. *The reductions are canonical maps between $B^5 \delta$ and $(B_{\bar{q}}^1 + \epsilon B_{\bar{q}}^2) \delta$. (ii) $B_{\bar{q}}^1(\delta H_1) = 0$ thus [10] two deformations of the equation $\bar{q}_t = B^2(\delta H_k)$ are*

$(U, \bar{V})_t = B^5 (\delta M_{1,2}^* \Sigma_{j=0}^{k-1} (-1)^j \epsilon^{-1-j} H_{k-j})$. (iii) $M_1^*(M_2^*)^{-1}$ is an infinitesimal automorphism of both the Hamiltonian operator $(B_{\bar{q}}^1 + \epsilon B_{\bar{q}}^2) \delta$ and all scalar Lax equations (extended in $\tilde{K}[[\epsilon]]$).

REMARK. Physicists would consider ϵ as finite, $M_1^*(M_2^*)^{-1}$ as a genuine map and call it a 'Bäcklund transformation'.

EXAMPLE. For $L = \xi + q_0 + \xi^{-1} q_1$ the Toda hierarchy can be written as

$$\begin{pmatrix} q_0 \\ q_1 \end{pmatrix}_t = \begin{pmatrix} q_1 \Delta - \Delta^{-1} q_1 & q_0 (1 - \Delta^{-1}) q_1 \\ q_1 (\Delta - 1) q_0 & q_1 (\Delta - \Delta^{-1}) q_1 \end{pmatrix} (\delta H_k) = \begin{pmatrix} 0 & (1 - \Delta^{-1}) q_1 \\ q_1 (\Delta - 1) & 0 \end{pmatrix} (\delta H_{k+1}).$$

Reductions:

$$M_1^*(q_0) = u + \epsilon v^{(-1)}, \quad M_1^*(q_1) = v + \epsilon u v;$$

$$M_2^*(q_0) = u + \epsilon v, \quad M_2^*(q_1) = v + \epsilon u^{(1)} v.$$

For

$$k = 1: \quad H_1 = q_0, \quad H_2 = (q_0^2 + q_1 + q_1^{(-1)})/2,$$

$$q_{0,t} = q_1 - q_1^{(-1)}, \quad q_{1,t} = q_1 (q_0^{(1)} - q_0), \quad u_t = (1 + \epsilon u)(v - v^{(-1)}),$$

$$v_t = v(u^{(1)} - u).$$

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