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THE DISTRIBUTION OF VALUES OF $L(1, \chi_d)$

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1 Introduction

Throughout this paper d will denote a fundamental discriminant, and χ_d the associated primitive real character to the modulus $|d|$. We investigate here the distribution of values of $L(1, \chi_d)$ as d varies over all fundamental discriminants with $|d| \le x$. Our main concern is to compare the distribution of values of $L(1, \chi_d)$ with the distribution of "random Euler products" $L(1, X) = \prod_p (1 - X(p)/p)^{-1}$ where the $X(p)$'s are independent random variables taking values 0 or ± 1 with suitable probabilities, described below. For example, we shall give asymptotics for the probability that $L(1, \chi_d)$ exceeds $e^\gamma \tau$, and the probability that $L(1, \chi_d) \le \frac{\pi^2}{6} \frac{1}{e^\gamma \tau}$ uniformly in a wide range of τ. These results are sufficiently uniform to prove slightly more than a recent conjecture of H.L. Montgomery and R.C. Vaughan [MV]. One important motivation for our work is to make progress towards resolving the discrepancy between extreme values that may be exhibited (the omega results of S.D. Chowla described below) and the conditional bounds on these extreme values (the O-results of J.E. Littlewood, see below). The uniformity of our results provides evidence that the omega results of Chowla may represent the true nature of extreme values of $L(1, \chi_d)$.

These questions have been studied by many authors, most notably by P.D.T.A. Elliott [E1,2,3] and Montgomery and Vaughan [MV]. We begin by reviewing some of the history of the subject which will help place our results in context. Throughout this paper $\log_j$ will denote the j-fold iterated logarithm; that is, $\log_2 = \log\log$, $\log_3 = \log\log\log$, and so on. In [Li] Littlewood showed that on the assumption of the Generalized Riemann Hypothesis (GRH)

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$$\left(\frac{1}{2}+o(1)\right)\frac{\zeta(2)}{e^\gamma \log_2|d|} \le L(1,\chi_d) \le \big(2+o(1)\big)e^\gamma \log_2|d|\,. \tag{1.1}$$

He also showed (again assuming GRH) that there are infinitely many d such that $L(1,\chi_d) \ge (1+o(1))e^\gamma \log_2|d|$, and that for infinitely many d, $L(1,\chi_d) \le (1+o(1))\zeta(2)/(e^\gamma\log_2|d|)$. The latter result was later established unconditionally by Chowla [C]. Thus only a factor of 2 remains at issue regarding the extreme values of $L(1,\chi_d)$, under the assumption of GRH. This question was addressed in a numerical study of D. Shanks [S], but the data there is inconclusive. Recently Vaughan [V] and Montgomery and Vaughan [MV] have returned to this problem, and initiated a finer study of these extreme values.

Write

$$\log L(1,\chi_d) = \sum_p \frac{\chi_d(p)}{p} + \sum_p\sum_{k=2}^\infty \frac{\chi_d(p)^k}{kp^k}\,.$$

The second sum above is rapidly convergent, and hence easy to understand. For a typical d one may expect that the first sum above behaves like $\sum_p X_p/p$, where the X_p are independent random variables taking values ±1 with equal probability. Pursuing this probabilistic model, Montgomery and Vaughan (developing ideas of Montgomery and Odlyzko [14]) suggest that the proportion of fundamental discriminants $|d|\le x$ with $L(1,\chi_d) > e^\gamma\tau$ (say) lies between $\exp(-Ce^\tau/\tau)$, and $\exp(-ce^\tau/\tau)$ for appropriate constants $0<c<C<\infty$. A similar assertion holds for the frequency with which $L(1,\chi_d) < \zeta(2)e^{-\gamma}/\tau$. Extrapolating this model, they formulated three conjectures on the frequencies with which certain extreme values occur.

CONJECTURE 1 (Montgomery–Vaughan). *The proportion of fundamental discriminants $|d|\le x$ with $L(1,\chi_d)\ge e^\gamma\log_2|d|$ is $>\exp(-C\log x/\log_2 x)$ and $<\exp(-c\log x/\log_2 x)$ for appropriate constants $0<c<C<\infty$. Similar estimates apply to the proportion of fundamental discriminants $|d|\le x$ with $L(1,\chi_d)\le\zeta(2)/(e^\gamma\log_2|d|)$.*

CONJECTURE 2 (Montgomery–Vaughan). *The proportion of fundamental discriminants $|d|\le x$ with $L(1,\chi_d)\ge e^\gamma(\log_2|d|+\log_3|d|)$ is $>x^\theta$ and $<x^\Theta$ where $0<\theta<\Theta<1$. Similar estimates apply to the frequency with which $L(1,\chi_d)\le\zeta(2)e^{-\gamma}(\log_2|d|+\log_3|d|)^{-1}$.*

CONJECTURE 3 (Montgomery–Vaughan). *For any $\epsilon>0$ there are only finitely many d with $L(1,\chi_d)>e^\gamma(\log_2|d|+(1+\epsilon)\log_3|d|)$, or with $L(1,\chi_d)$ $\le\zeta(2)e^{-\gamma}(\log_2|d|+(1+\epsilon)\log_3|d|)^{-1}$.*

Notice that Conjecture 3 implies that the true nature of extreme values of $L(1,\chi_d)$ is given by Chowla's omega-results rather than the GRH bounds of Littlewood.

The idea of comparing the distribution of values of $L(1,\chi_d)$ to a random model is quite old. Chowla and P. Erdős [CE] showed first that for a fixed τ the proportion of $|d| \le x$ with $L(1,\chi_d) \ge e^{\gamma}\tau$ tends to a limit as $x \to \infty$. Further this distribution function was shown to be continuous. Elliott [E2] pursued this further and showed that $L(1,\chi_d)$ possesses a smooth distribution function, and he also obtained expressions for its characteristic function (Fourier transform). Indeed Elliott established such results for the distribution of $L(s,\chi_d)$ at any point s with $\mathrm{Re}(s) > 1/2$, and he also allows for d to be restricted to prime discriminants. Elliott's results also allow for τ to grow slowly in terms of x; essentially his methods show that the distribution of values of $L(1,\chi_d)$ approximate the distribution of random Euler products for τ throughout the range $(1/\log_3 x, \log_3 x)$. But these results are not sufficiently uniform to approach the above conjectures of Montgomery and Vaughan.

We now describe our results, starting with the probablistic model we shall use to study $L(1,\chi_d)$. For primes p let $X(p)$ denote independent random variables taking the values 1 with probability $p/(2(p+1))$, 0 with probability $1/(p+1)$, and -1 with probability $p/(2(p+1))$. The reason for this choice of probabilities over the simpler ± 1 with probability $1/2$ is as follows: For odd primes p, fundamental discriminants d are constrained to lie in one of p^2-1 residue classes $(\bmod\, p^2)$, and for $p-1$ of these the character value is 0, and the remaining $p(p-1)$ residue classes split equally into ± 1 values. For the prime 2 note that fundamental discriminants lie in the residue classes $1, 5, 8, 9, 12, 13 \pmod{16}$ and the values $0, \pm 1$ occur equally often. We extend X multiplicatively to all integers n: that is set $X(n) = \prod_{p^\alpha \| n} X(p)^\alpha$. We wish to compare the distribution of values of $L(1,\chi_d)$ with the disrtibution of values of the random Euler products $L(1,X) := \prod_p (1 - X(p)/p)^{-1}$ (these products converge with probability 1).

Before describing how well this model approximates the distribution of $L(1,\chi_d)$, it is helpful to gain an understanding of the distribution of $L(1,X)$. To this end, we define

$$\Phi(\tau) = \mathrm{Prob}\big(L(1,X) \ge e^{\gamma}\tau\big),$$

and

$$\Psi(\tau) = \mathrm{Prob}\left(L(1,X) \le \tfrac{\pi^2}{6}\tfrac{1}{e^{\gamma}\tau}\right).$$

By studying the characteristic function of $\log L(1,X)$ (which may be shown to decay rapidly) one can see that $\Phi(\tau)$ and $\Psi(\tau)$ are smooth functions. This is implicit in Elliott [E2], and follows also from our subsequent work. From the work of Montgomery and Vaughan [MV] we obtain that $\Phi(\tau)$ and $\Psi(\tau)$ decay double exponentially as $\tau\to\infty$: precisely, there exist constants C and c such that

$$\exp\left(-C\tfrac{e^\tau}{\tau}\right)\le\Phi(\tau)\le\exp\left(-c\tfrac{e^\tau}{\tau}\right),$$

and similarly for $\Psi(\tau)$. In section 3 we shall analyse these functions closely, expressing them in terms of the moments of $L(1,X)$ (see Theorem 3.1 below). From our work we obtain the following useful estimates for $\Phi(\tau)$ and $\Psi(\tau)$, which improve upon Montgomery and Vaughan's estimates.

PROPOSITION 1. *For large τ we have*

$$\Phi(\tau)=\exp\Big(-\tfrac{e^{\tau-C_1}}{\tau}+O\left(\tfrac{e^\tau}{\tau^2}\right)\Big),$$

where

$$C_1:=\int_0^1\tanh y\frac{dy}{y}+\int_1^\infty(\tanh y-1)\frac{dy}{y}=0.8187\ldots.$$

The same asymptotic holds also for $\Psi(\tau)$. Further if $0\le\lambda\le e^{-\tau}$ then

$$\Phi(\tau e^{-\lambda})=\Phi(\tau)\big(1+O(\lambda e^\tau)\big)\quad\text{and}\quad\Psi(\tau e^{-\lambda})=\Psi(\tau)\big(1+O(\lambda e^\tau)\big).$$

Below we shall let $\sum^\flat$ indicate that the sum is over fundamental discriminants. A standard argument shows that there are $\frac{6}{\pi^2}x+O(x^{\frac12+\epsilon})$ fundamental discriminants d with $|d|\le x$. Define $\Phi_x(\tau)$ to be the proportion of the fundamental discriminants d with $|d|\le x$, for which $L(1,\chi_d)>e^\gamma\tau$; that is

$$\Phi_x(\tau):=\Big(\sideset{}{^\flat}\sum_{\substack{|d|\le x\\ L(1,\chi_d)>e^\gamma\tau}}1\Big)\Big/\Big(\sideset{}{^\flat}\sum_{|d|\le x}1\Big)$$

(and similarly define $\Psi_x(\tau)$). We would like to compare this with $\Phi(\tau)$ (and analogously to compare the frequency of small values with Ψ). Notice that the viable range for such a correspondence is $\tau\le\log_2x+\log_3x+C_1+o(1)$. Proposition 1 shows that at this juncture the probabilities Φ and Ψ become smaller than $1/x$.

Theorem 1. *Let x be large. Uniformly in the region $\tau\le R(x)$ we have*

$$\Phi_x(\tau)=\Phi(\tau)\left(1+O\left(\frac{1}{(\log x)^5}+e^{\tau-R(x)}\right)\right),$$

and uniformly in $\tau\le R(x)+\log_3x$ we have

$$\Phi_x(\tau)=\Phi(\tau)(\log x)^{O(1)}.$$

Here we may choose $R(x) = \log_2 x - 2\log_3 x + \log_4 x - 20$ *unconditionally, and* $R(x) = \log_2 x - \log_3 x - 20$ *if the GRH is assumed. Analogous results hold replacing* Φ *with* Ψ*.*

Our proof of Theorem 1 relies upon computing the mean moment of $L(1,\chi_d)^z$, as we average over fundamental discriminants d with $|d| \le x$, for complex numbers z in a wide range. We will establish that these moments are very nearly equal to the corresponding moments of the random $L(1,X)$; that is to say the expectation $\mathbb{E}(L(1,X)^z)$. Throughout the paper $\mathbb{E}(\cdot)$ stands for the expectation of the random variable in brackets. Suppose there were a character χ_d with $|d| \asymp x$ for which $L(s,\chi_d)$ has a bad Landau–Siegel zero. In this case we could have $L(1,\chi_d)$ as small as $x^{-\epsilon}$ so that when z is a negative real number the mean moment of $L(1,\chi_d)^z$ would be heavily affected by this particular character. Thus, short of proving the non-existence of Landau–Siegel zeros, we cannot hope for asymptotics for the moments as stated, except in a narrow range of values for z. To circumvent this difficulty, we calculate instead moments of $L(1,\chi_d)$ after first omitting a sparse set of discriminants having Landau–Siegel zeros. Precisely, we define for an appropriate constant $c > 0$

$$\mathcal{L} = \left\{ d : L(\beta,\chi_d) = 0 \quad \text{for some} \quad 1 - c/\log(e|d|) \le \beta < 1 \right\}.$$

We will refer to elements of $\mathcal{L}$ as Landau–Siegel characters, discriminants, or moduli. If c is chosen appropriately, then it is known that there is at most one element d of $\mathcal{L}$ between x and $2x$ (see [D]) so that there are $\ll \log x$ elements of $\mathcal{L}$ with $|d| \le x$.

Theorem 2. *Uniformly in the region* $|z| \le \log x/(500(\log_2 x)^2)$ *we have*

$$\sideset{}{^\flat}\sum_{\substack{|d|\le x\\ d\notin\mathcal{L}}} L(1,\chi_d)^z = \frac{6}{\pi^2} x\mathbb{E}\big(L(1,X)^z\big) + O\left(x\exp\left(-\frac{\log x}{5\log_2 x} \right)\right).$$

We remark that when $\operatorname{Re} z$ is positive then it is not necessary to omit elements of $\mathcal{L}$ while calculating the moments of Theorem 2. This follows because $L(1,\chi_d)$ is easily seen to be $\ll \log(e|d|)$ and so for positive $\operatorname{Re} z$ the error in adding back elements of $\mathcal{L}$ is $\ll \log x (\log x)^{\operatorname{Re} z}$ which may be subsumed in the error term of the theorem. Previously A.F. Lavrik [L] had computed $\sum^\flat_{|d|\le x} L(1,\chi_d)^{2k}$ for integers $k \ll \sqrt{\log x}$, and we see that Theorem 2 goes substantially further than his result. Further if $\operatorname{Re} z$ is negative, but $|\operatorname{Re} z|$ is bounded, then again we need not exclude elements of $\mathcal{L}$. This follows readily from Siegel's famous bound $L(1,\chi_d) \gg_\epsilon |d|^{-\epsilon}$.

We may extend the range of applicability (in $|z|$) of Theorem 2 by excluding a larger (but still very thin) set of characters. We describe this result next, which will be the main ingredient used to prove Theorem 1.

Theorem 3. *Let $\mathcal{E}$ denote a set of $\le \sqrt{x}$ exceptional discriminants $|d| \le x$. Uniformly in the region $|z| \le \log x \log_3 x/(e^{12} \log_2 x)$ we have*

$$\sum_{\substack{|d|\le x\\ d\notin\mathcal{E}}}^{\flat} L(1,\chi_d)^z = \frac{6}{\pi^2} x\mathbb{E}\big(L(1,X)^z\big) + O\left(x\frac{\mathbb{E}(L(1,X)^{Re\, z})}{(\log x)^9}\right).$$

If the GRH is true then the above asymptotic holds uniformly in the larger region $|z| \le 10^{-3} \log x$.

One can show (by modifying Lemma 3.2 below) that the main term in Theorem 3 dominates the error term if and only if $|\mathrm{Im}\, z| \ll \{(1+|\mathrm{Re}\, z|)\log(2+|\mathrm{Re}\, z|)\}^{1/2} \log_2 x$. In contrast the main term in Theorem 2 dominates the error term throughout the region $|z| \le \log x/(500(\log_2 x)^2)$.

Even assuming GRH the range of validity of Theorem 1 is insufficient to penetrate the conjectures of Montgomery and Vaughan. However note that these conjectures ask only that the frequencies of large and small values decay "double exponentially" and not for the more precise information provided by Theorem 1. We now set ourselves the intermediate problem of determining when $\Phi_x(\tau), \Psi_x(\tau) = \exp(-(1+o(1))e^{\tau-C_1}/\tau)$. Exploiting a wonderful result of S.W. Graham and C.J. Ringrose [GR] on character sums to smooth moduli (see Lemma 4.2 below), we settle this problem in a range wider than required for Conjecture 1.

Theorem 4. *Let x be large and let $\log_2 x \ge A \ge e$ be a real number. Uniformly in the range $\tau \le R_1(x) - \log_2 A$ we have*

$$\Phi_x(\tau) = \exp\left(-\frac{e^{\tau-C_1}}{\tau}\left(1 + O\left(\frac{1}{A} + \frac{1}{\tau}\right)\right)\right),$$

and the same asymptotic holds $\Psi_x(\tau)$. Here we may take $R_1(x) = \log_2 x + \log_4 x - 20$ unconditionally, and $R_1(x) = \log_2 x + \log_3 x - 20$ if the GRH is true.

Theorem 4 clearly implies a stronger version of Montgomery and Vaughan's Conjecture 1. Notice also that assuming GRH it implies part of Conjecture 2: namely that the number of $|d| \le x$ with $L(1,\chi_d) \ge e^\gamma(\log_2 |d| + \log_3 |d|)$ is $\le x^\Theta$ for some $\Theta < 1$, and similarly for small values. In [M2] Montgomery established that if the GRH is true then there are infinitely many primes p such that the least quadratic non-residue $(\bmod\ p)$ is $\gg \log p \log\log p$. We adapt his idea to examine extreme values of $L(1,\chi)$ under GRH, which strengthens Theorem 4, but just fails to prove the other half of Conjecture 2.

Theorem 5a. *Assume GRH. For any $\epsilon > 0$, and all large x, there are $\gg x^{1/2}$ primes $q \le x$ such that*

$$L\left(1, \left(\frac{\cdot}{q}\right)\right) \ge e^{\gamma}\big(\log_2 q + \log_3 q - \log(2\log 2) - \epsilon\big),$$

and $\gg x^{1/2}$ primes q such that

$$L\left(1, \left(\frac{\cdot}{q}\right)\right) \le \frac{\zeta(2)}{e^{\gamma}}\big(\log_2 q + \log_3 q - \log(2\log 2) - \epsilon\big)^{-1}.$$

Note that $\log(2\log 2) = 0.3266\ldots$ so that Theorem 5a comes very close to exhibiting the extreme values required in Conjecture 2. We may ask for the extreme values of $L(1,\chi_d)$ that may be obtained unconditionally: that is, for refinements of Chowla's results which would approach the extreme values predicted by the conjectures above. As far as small values are concerned we are unable to go further than the extreme values guaranteed by Theorem 1. However by a judicious use of the pigeonhole principle we are able to exhibit large values of $L(1,\chi_d)$ which are nearly as good as the conjectured truth.

Theorem 5b. *For large x there are at least $x^{1/10}$ square-free integers $d \le x$ such that*

$$L\left(1, \left(\tfrac{\cdot}{d}\right)\right) \ge e^{\gamma}(\log_2 x + \log_3 x - \log_4 x - 10).$$

In summary our results find excellent agreement between the distribution of values of $L(1,\chi_d)$ and the predictions of the probabilistic model. We find (especially on GRH) that the predictions hold true in almost the entire viable range, and this leads us to believe that the extreme values of $L(1,\chi_d)$ behave like Chowla's omega results.

If the asymptotic formula of Theorem 1 holds to the edge of the viable range then perhaps

$$\max_{\substack{d \text{ fundamental} \\ |d|\le x}} L(1,\chi_d) = e^{\gamma}\big(\log_2 x + \log_3 x + C_1 + o(1)\big).$$

It is also plausible that the distribution function changes nature just beyond the range given in Theorem 1 (which is why our methods do not give good results there), and that the maximum value is represented by a slightly different function. Nonetheless given the extraordinary decay of $\Phi(\tau)$ that we have detected in this range we conjecture that, at worst, the above estimate is true with a slightly different constant.

2 Preparatory Lemmas

In this section we collect together some preliminary results which will be useful in our subsequent work. The ideas in this section are standard and classical.

We will show in Proposition 2.2 below that with few exceptions $L(1,\chi)$ may be approximated by the short Euler product $L(1,\chi;y)$ where, throughout the paper, we let $L(1,\chi;y) := \prod_{p\le y}(1-\chi(p)/p)^{-1}$. The primary ingredient is the following classical lemma, which is proved in Lemmas 8.1 and 8.2 of [GrS].

LEMMA 2.1. *Let $s=\sigma+it$ with $|t|\le 3q$, and let $y\ge 2$ be a real number. Let $1/2\le\sigma_0<\sigma$, and suppose that the rectangle $\{z:\sigma_0<\mathrm{Re}(z)\le 1, |\mathrm{Im}(z)-t|\le y+3\}$ contains no zeros of $L(z,\chi)$. Then*

$$\big|\log L(s,\chi)\big| \ll \frac{\log q}{\sigma-\sigma_0}\,.$$

Further, putting $\sigma_1=\min\big(\sigma_0+\frac{1}{\log y},\frac{\sigma+\sigma_0}{2}\big)$,

$$\log L(s,\chi)=\sum_{n=2}^{y}\frac{\Lambda(n)\chi(n)}{n^s\log n}+O\bigg(\frac{\log q}{(\sigma_1-\sigma_0)^2}y^{\sigma_1-\sigma}\bigg).$$

If the GRH holds then we may apply Lemma 2.1 with $\sigma_0=1/2$. We now take $y=(\log q)^2(\log_2 q)^6$ to obtain

$$\log L(1,\chi)=\sum_{n=2}^{y}\frac{\Lambda(n)\chi(n)}{n\log n}+O\left(\frac{1}{\log\log q}\right).$$

Using the prime number theorem to estimate the contribution of the primes between $\log^2 q$ and y, we deduce that

$$L(1,\chi)=\prod_{p\le\log^2 q}\left(1-\frac{\chi(p)}{p}\right)^{-1}\left\{1+O\left(\frac{\log_3 q}{\log_2 q}\right)\right\},$$

which gives Littlewood's result (1.1).

Using this lemma together with the large sieve and zero density results we obtain the following result, essentially due to Elliott [E1].

PROPOSITION 2.2. *Let Q be large, and let $\log_2 Q\ge A\ge 1$ be a real number. Then for all but at most $Q^{2/A}$ primitive characters $\chi \pmod q$ with $q\le Q$ we have for $y\le Q/2$*

$$L(1,\chi)=\prod_{p\le y}\left(1-\frac{\chi(p)}{p}\right)^{-1}\left(1+O\left((A^2+\log^2 y)\frac{\log Q}{y^{1/4A}}\right)\right). \tag{2.1}$$

Further

$$L(1,\chi)=\prod_{p\le(\log Q)^A}\left(1-\frac{\chi(p)}{p}\right)^{-1}\left(1+O\left(\frac{1}{\log\log Q}\right)\right)$$

holds for all but at most $Q^{2/A+5\log_3 Q/\log_2 Q}$ primitive characters $\chi \pmod q$ with $q\le Q$.

Proof. From a standard zero density result (see Theorem 20 of E. Bombieri [B]) we know that there are fewer than $Q^{6(1-\alpha)}(\log Q)^B$ primitive characters with conductor below Q having a zero in the rectangle $1 \geq \mathrm{Re}(s) \geq \alpha$, $|\mathrm{Im}(s)| \leq Q$. Here B is some absolute constant. Thus appealing to Lemma 2.1 with $s = 1$, and $\sigma_0 = 1 - \frac{1}{4A}$ we obtain (2.1) for all but at most $Q^{2/A}$ primitive characters $\chi \pmod q$ with $q \leq Q$.

We now show that

$$\sum_{(\log Q)^A \leq p \leq (\log Q)^{8A}} \frac{\chi(p)}{p} = O\left(\frac{1}{\log_2 Q}\right)$$

for all but $Q^{2/A + 5\log_3 Q/\log_2 Q}$ characters with conductor below Q, which when combined with (2.1) with $y = (\log Q)^{8A}$ gives the proposition. To prove this we shall use the large sieve in the following form:

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \Big| \sum_{m \leq M} a(m)\chi(m) \Big|^2 \ll (Q^2 + M) \sum_{m \leq M} |a(m)|^2, \tag{2.2}$$

where the $\sum^*$ is over primitive characters χ, and the $a(m)$ are arbitrary complex numbers.

For $0 \leq j \leq J := [7A \log_2 Q/\log 2]$ put $z_j = 2^j (\log Q)^A$ and put $z_{J+1} = (\log Q)^{8A}$. Choose $k = [2\log Q/(A\log_2 Q)] + 1$ so that $M_j := z_{j+1}^k \geq Q^2$. Define

$$\sum_{M_j/2^k \leq m \leq M_j} a_j(m) \frac{\chi(m)}{m} = \left(\sum_{z_j < p \leq z_{j+1}} \frac{\chi(p)}{p} \right)^k .$$

Note that $0 \leq a_j(m) \leq k!$ and $\sum_m a_j(m) = (\pi(z_{j+1}) - \pi(z_j))^k \leq (2z_{j+1}/(A\log_2 Q))^k$. Appealing to (2.2) we see that

$$\begin{aligned} \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \Big| \sum_{z_j < p \leq z_{j+1}} \frac{\chi(p)}{p} \Big|^{2k} &\ll z_{j+1}^k \sum_{M_j/2^k \leq m \leq M_j} \frac{|a_j(m)|^2}{m^2} \\ &\ll \frac{k! 2^{2k}}{z_{j+1}^k} \left(\frac{2z_{j+1}}{A\log_2 Q} \right)^k \ll \left(\frac{4k}{A \log_2 Q} \right)^k , \end{aligned}$$

using Stirling's formula. We deduce that

$$\Big| \sum_{z_j \leq p \leq z_{j+1}} \frac{\chi(p)}{p} \Big| \leq \frac{1}{A(\log_2 Q)^2}$$

for all but at most $(10 \log Q (\log_2 Q)^2)^k$ primitive characters with conductor below Q, and the proposition follows at once.

Using Lemma 2.1, we may also derive the following approximation to $L(1,\chi)^z$, provided χ is not a Landau–Siegel character.

Lemma 2.3. *Let χ be a non-principal character* $(\bmod\, q)$ *which is either complex, or real and primitive with its discriminant not in $\mathcal{L}$. Let z be any complex number with $|z| \le (\log q)^2$ and let $Z \ge \exp((\log q)^{10})$ be a real number. Then*

$$L(1,\chi)^z = \sum_{n=1}^{\infty} \chi(n)\frac{d_z(n)}{n}e^{-n/Z} + O\left(\frac{1}{q}\right).$$

Proof. Since $\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} y^s\Gamma(s)ds = e^{-1/y}$, we have

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} L(1+s,\chi)^z Z^s\Gamma(s)ds = \sum_{n=1}^{\infty}\frac{d_z(n)}{n}\chi(n)e^{-n/Z}. \tag{2.3}$$

We shift the line of integration to the contour $s = -C(t)+it$ where $C(t) := -c/(2\log(q(|t|+2)))$, for an appropriately small constant $c > 0$. From our assumption on χ we may choose c such that we encounter no zeros of $L(s,\chi)$ while shifting contours. We encounter a pole at $s = 0$ which leaves the residue $L(1,\chi)^z$. Applying Lemma 2.1 with $\sigma_0 = 1-4C(t)/3$ and $y = 2$ gives $|\log L(s,\chi)| \ll 1/C(t)^2$ and so the left side of (2.3) equals $L(1,\chi)^z + R$ where

$$R \ll \int_{-\infty}^{\infty} Z^{-C(t)}e^{O(|z|/C(t)^2)}\big|\Gamma(-C(t)+it)\big|dt \ll \frac{1}{q},$$

by Stirling's formula.

We end this section by collecting several inequalities for values of $d_z(n)$: the z-th divisor function. Recall that $d_z(n)$ is a multiplicative function given on prime powers by $d_z(p^a) = \Gamma(z+a)/(\Gamma(z)a!)$, and that $\sum_{n=1}^{\infty} d_z(n)/n^s = \zeta(s)^z$ for $\operatorname{Re} s > 1$.

Note that $|d_z(n)| \le d_{|z|}(n)$. For real numbers $k \ge 1$ we observe that $d_k(mn) \le d_k(m)d_k(n)$. For any positive integers a, b, n we have $d_a(n)d_b(n) \le d_{a+b}(n)$. Further for any complex number z and a real number β we have $|d_z(n)|^\beta \le d_{|z|^\beta}(n)$. We also record that $|d(n)d_z(n^2)| \le d_{2|z|+2}(n)^2$ and that $|d_z(n^2)| \le d_{(|z|+1)^2}(n)$. All of these inequalities may be shown by first proving them for prime powers (by induction on the exponent), and then using multiplicitivity to deduce them for all $n \ge 1$.

Lastly we note that for positive integers k and real numbers $x \ge 2$ we have

$$\sum_{n\le x}\frac{d_k(n)}{n} \le \left(\sum_{n\le x}\frac{1}{n}\right)^k \le (\log 3x)^k,$$

and since $d_k(n)e^{-n/x} \le e^{k/x}\sum_{a_1\dots a_k=n} e^{-(a_1+\dots+a_k)/x}$ that

$$\sum_{n=1}^{\infty}\frac{d_k(n)}{n}e^{-n/x} \le \left(e^{1/x}\sum_{a=1}^{\infty}\frac{e^{-a/x}}{a}\right)^k \le (\log 3x)^k. \tag{2.4}$$

3 Random Euler Products and Their Distribution

We put $L(1,X;y) = \prod_{p\le y}(1-X(p)/p)^{-1}$. Since $\mathbb{E}((\sum_{p>y} X(p)/p)^2) \asymp \sum_{p>y} 1/p^2 \asymp 1/(y\log y)$, we see that with probability 1, $L(1,X;y)$ converges to $L(1,X)$ as $y\to\infty$. In this section we investigate the distribution of the random Euler products $L(1,X;y)$. Letting $y=\infty$ in our results gives information on the distribution of $L(1,X)$. Analogously to the definitions of $\Phi(\tau)$ and $\Psi(\tau)$ we define

$$\Phi(\tau;y) = \text{Prob}\big(L(1,X;y)\ge e^\gamma\tau\big) \ \text{ and } \ \Psi(\tau;y) = \text{Prob}\left(L(1,X;y)\le \tfrac{\pi^2}{6}\tfrac{1}{e^\gamma\tau}\right).$$

To facilitate our discussion we define for positive real numbers k

$$I(k;y) := -\sum_{p\le y}\log\left(1-\tfrac{1}{p}\right)\tanh\left(\tfrac{k}{p}\right) \text{ and } I(k) := -\sum_{p}\log\left(1-\tfrac{1}{p}\right)\tanh\left(\tfrac{k}{p}\right).$$

For fixed y plainly $I(k;y)$ is an increasing function of k, and for fixed k it is an increasing function of y. Further writing $g(t)=\tanh(t)$ if $t\le 1$, and $g(t)=\tanh(t)-1$ if $t>1$ we have that

$$I(k;y) = -\sum_{p\le\min(k,y)}\log\left(1-\tfrac{1}{p}\right) - \sum_{p\le y}\log\left(1-\tfrac{1}{p}\right) g\left(\tfrac{k}{p}\right),$$

and now using the prime number theorem and partial summation we see that for large k and y,

$$I(k;y) = \log_2\min(k,y) + \gamma + \frac{C_1(k/y)}{\log k} + O\left(\frac{1}{\log^2 k}\right), \tag{3.1}$$

where

$$C_1(x) := \int_x^\infty \frac{g(t)}{t}dt\,.$$

Note that $C_1(0) = C_1 = 0.8187\ldots$. Define next

$$R(k;y) = \sum_{p\le y}\frac{1}{p^2\cosh^2(k/p)} \quad\text{and}\quad R(k) = \sum_{p}\frac{1}{p^2\cosh^2(k/p)}\,.$$

Again appealing to the prime number theorem and partial summation we see that for large k and y

$$\begin{aligned} R(k;y) &= \frac{1}{k\log k}\int_{k/y}^\infty \frac{dx}{\cosh^2 x} + O\left(\frac{1}{k\log^2 k}\right)\\ &= \frac{1-\tanh(k/y)}{k\log k} + O\left(\frac{1}{k\log^2 k}\right). \end{aligned} \tag{3.2}$$

We are now in a position to state our main result, which obtains an asymptotic formula for $\Phi(\tau;y)$ (and $\Psi(\tau;y)$) in terms of appropriate moments $\mathbb{E}(L(1,X;y)^k)$. Naturally, letting $y\to\infty$ below furnishes asymptotics for $\Phi(\tau)$ or $\Psi(\tau)$.

Theorem 3.1. *Let y be a large real number, and let τ be large and below $\log y - 1$. Let $k = k_{\tau,y}$ denote the unique real number such that $I(k;y) = \gamma + \log\tau$. Then*

$$\Phi(\tau;y) = \frac{\mathbb{E}(L(1,X;y)^k)}{k(e^\gamma\tau)^k}\frac{1}{\sqrt{2\pi R(k;y)}}\left(1+O\left(\frac{\log^2 k}{k^{1/4}}\right)\right),$$

and

$$\Psi(\tau;y) = \frac{\mathbb{E}(L(1,X;y)^{-k})}{k\left(\frac{6}{\pi^2}e^\gamma\tau\right)^k}\frac{1}{\sqrt{2\pi R(k;y)}}\left(1+O\left(\frac{\log^2 k}{k^{1/4}}\right)\right).$$

Further if $0 \le \lambda \le e^{-\tau}$ then

$$\Phi(\tau e^{-\lambda};y) - \Phi(\tau;y) \ll \Phi(\tau;y)\left(\lambda e^\tau + \frac{e^{\frac{3}{4}\tau}\log y}{y}\right),$$

and

$$\Psi(\tau e^{-\lambda};y) - \Psi(\tau;y) \ll \Psi(\tau;y)\left(\lambda e^\tau + \frac{e^{\frac{3}{4}\tau}\log y}{y}\right).$$

We shall focus only on proving the results for $\Phi(\tau;y)$, the argument for Ψ requires only some minor adjustments. For the rest of this section we write $s = k+it$, and we denote $\mathbb{E}((1-X(p)/p)^{-s})$ by $E_p(s)$. Clearly $|E_p(s)| \le E_p(k)$ always.

Lemma 3.2. *Let $s = k+it$, and let k be large. If $p > k/4$ then we have for some positive constant c_0*

$$\left|E_p(s)\right| \le E_p(k)\exp\left(-c_0\left(1-\cos\left(t\log\left(\frac{p+1}{p-1}\right)\right)\right)\right).$$

Further if $k \le 2y$ then there exists a positive constant c such that

$$\left|\frac{\mathbb{E}(L(1,X;y)^s)}{\mathbb{E}(L(1,X;y)^k)}\right| \le \begin{cases} \exp\left(-c\frac{t^2}{k\log k}\right) & \text{if } |t| \le k/4 \\ \exp\left(-c\frac{|t|}{\log|t|}\right) & \text{if } k/4 \le |t| \le y/2 \\ \exp\left(-c\frac{y}{\log^3 y}\right) & \text{if } y/2 \le |t| \le y^2/\log^2 y\,. \end{cases}$$

Note that letting $y \to \infty$ we get that $\mathbb{E}(L(1,X)^s)/\mathbb{E}(L(1,X)^k) \ll \exp\left(-c\frac{t^2}{k\log k}\right)$ if $|t| \le k$ and $\ll \exp\left(-c\frac{|t|}{\log|t|}\right)$ if $|t| > k$.

Proof. Note that for r_1, r_2 and r_3 positive we have $|r_1 + r_2e^{i\theta_2} + r_3e^{i\theta_3}|^2 \le (r_1+r_2+r_3)^2 - 2r_1r_2(1-\cos\theta_3)$, so that $|r_1 + r_2e^{i\theta_2} + r_3e^{i\theta_3}| \le (r_1+r_2+r_3)\exp\left(-\frac{r_1r_3(1-\cos\theta_3)}{(r_1+r_2+r_3)^2}\right)$. We apply this with $r_1 = \frac{p}{2(p+1)}(1-1/p)^{-k}$, $r_2 = 1/(p+1)$, $r_3 = \frac{p}{2(p+1)}(1+1/p)^{-k}$, and $\theta_2 = t\log(1-1/p)$ and $\theta_3 = t\log\left(\frac{p-1}{p+1}\right)$. Since $p > k/4$ the first statement of the lemma follows.

To prove the second part, note that our desired ratio is

$$\leq \prod_{y \geq p \geq k/4} \exp\left(-c_0\left(1-\cos\left(t\log\left(\frac{p+1}{p-1}\right)\right)\right)\right).$$

If $|t| \leq k/4$ then for the primes p in $(k/4, k/2)$ we have that $\left|t\log\left(\frac{p+1}{p-1}\right)\right| \sim 2|t|/p$ lies between $\sim 4|t|/k$ and $\sim 8|t|/k$ so that $1-\cos\left(t\log\left(\frac{p+1}{p-1}\right)\right) \gg |t|^2/k^2$. This gives the bound of the Lemma in this case. Next if $k/4 \leq |t| \leq y/2$ then we apply the above argument with the primes in $(|t|, 2|t|)$ getting the desired bound.

Lastly suppose that $y/2 \leq |t| \leq y^2/\log y$. Here we let $\delta := 10^{-6}/\log y$, and divide the interval $(y/2, y)$ into intervals of length $\delta y^2/|t|$ (with the last interval possibly being shorter). There are $\sim |t|/(2y\delta)$ such intervals. Call an interval good if $\cos(t\log((p+1)/(p-1))) \leq \cos(\delta/10)$ for all primes p in that interval, and bad if otherwise. There are at most $3|t|/y$ bad intervals, and by the Brun–Titchmarsh theorem each bad interval contains at most $3\delta y^2/(|t|\log(\delta y^2/|t|)) \leq y^2/(200|t|\log y)$ primes. Thus there are at least $y/3\log y$ primes in good intervals, and the Lemma follows in this final case.

In addition to Lemma 3.2 we need a result comparing $\mathbb{E}(L(1,X;y)^s)$ with $\mathbb{E}(L(1,X;y)^k)$ for relatively small values of t. First observe that

$$\begin{aligned}\mathbb{E}\Big(\left(1-\tfrac{X(p)}{p}\right)^{-k}\Big) &= \mathbb{E}\left(\exp\left(k\tfrac{X(p)}{p}\right)\right)\exp\left(O\left(\tfrac{k}{p^2}\right)\right)\\ &= \cosh\left(\tfrac{k}{p}\right)\exp\left(O\left(\tfrac{k}{p^2}\right)\right),\end{aligned} \tag{3.3}$$

where the last estimate follows since $\mathbb{E}(\exp(kX(p)/p)) = \frac{p\cosh(k/p)+1}{p+1}$ which equals $\cosh(k/p)(1+O(1/p))$ if $p \leq k$ and equals $\cosh(k/p)+O(k^2/p^3) = \cosh(k/p)(1+O(k^2/p^3))$ if $p > k$. Further note that if $p \leq k/\log k$ then

$$\mathbb{E}\Big(\left(1-\frac{X(p)}{p}\right)^{-k}\Big) = \left(\frac{p}{2(p+1)}\right)\left(1-\frac{1}{p}\right)^{-k}\left(1+O\Big(\frac{e^{-k/p}}{p}\Big)\right). \tag{3.4}$$

Note that, for all primes p we have

$$E_p(s) = \left(1-\frac{1}{p}\right)^{-it}\mathbb{E}\Big(\left(1-\frac{X(p)}{p}\right)^{-k}\left(\frac{p-1}{p-X(p)}\right)^{it}\Big).$$

Now

$$\left(\frac{p-1}{p-X(p)}\right)^{it} = 1-it\frac{(1-X(p))}{p}-t^2\frac{(1-X(p))^2}{2p^2}+O\left((1-X(p))\left(\frac{|t|}{p^2}+\frac{|t|^3}{p^3}\right)\right);$$

this is trivial if $X(p)=1$, otherwise it follows from the Taylor expansion if $p > |t|$, and as $|(p-1)/(p-X(p))^{it}| = 1$ if $p \leq |t|$. By (3.4) we have that $\mathbb{E}((1-X(p))(1-X(p)/p)^{-k}) = (1+1/p)^{-k} + O(1/p)$ and

$\mathbb{E}((1-X(p))^2(1-X(p)/p)^{-k}) = 2(1+1/p)^{-k}+O(1/p)$, and so we deduce that

$$E_p(s) = \left(1-\tfrac{1}{p}\right)^{-it}\left(E_p(k) - \tfrac{it}{p}\left(1+\tfrac{1}{p}\right)^{-k} - \tfrac{t^2}{p^2}\left(1+\tfrac{1}{p}\right)^{-k} + O\left(\tfrac{|t|}{p^2}+\tfrac{|t|^2}{p^3}+\tfrac{|t|^3}{p^3}\right)\right).$$

If $p \le k/3\log k$ then $E_p(k) \ge \frac{1}{3}(1-1/p)^{-k} \gg k^3$ and so the above becomes, using (3.3),

$$E_p(s) = \left(1-\frac{1}{p}\right)^{-it} E_p(k)\left(1+O\left(\frac{|t|+|t|^3}{k^3}\right)\right).$$

If $p \ge k/3\log k$ then $(1+1/p)^{-k} = e^{-k/p}+O(1/p)$. If $k/3\log k \le p \le |t|$ then

$$E_p(s) = \left(1-\tfrac{1}{p}\right)^{-it} E_p(k)\left(1+O\left(\tfrac{|t|^3}{p^3}e^{-k/p}\right)\right).$$

If $p \ge k/3\log k$ and $p \ge |t|$ then the above becomes, using (3.3),

$$\log\frac{E_p(s)}{E_p(k)} = -it\log\left(1-\frac{1}{p}\right)\tanh\left(\frac{k}{p}\right) - \frac{t^2}{2}\frac{1}{p^2\cosh^2(k/p)} + O\left(\left(\frac{|t|}{p^2}+\frac{|t|^3}{p^3}\right)e^{-k/p}\right).$$

It follows that, for $|t| \le k^{2/3}$,

$$\mathbb{E}\big(L(1,X;y)^s\big) = \mathbb{E}\big(L(1,X;y)^k\big)\exp\left(itI(k;y) - \frac{t^2}{2}R(k;y) + O\left(\frac{|t|}{k\log k}+\frac{|t|^3}{k^2\log k}\right)\right). \quad (3.5)$$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\lambda > 0$ be real. For $y>0$ and $c>0$ we have by Perron's formula

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^z\left(\frac{e^{\lambda z}-1}{\lambda z}\right)\frac{dz}{z} = \frac{1}{\lambda}\int_0^\lambda\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(ye^u)^z\frac{dz}{z}\right)du = \begin{cases} 1 & \text{if } y>1, \\ 1+\frac{\log y}{\lambda}\in[0,1] & \text{if } e^{-\lambda}\le y\le 1, \\ 0 & \text{if } y<e^{-\lambda}. \end{cases}$$

Also by Perron's formula we may see that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^z\left(\frac{e^{\lambda z}-1}{\lambda z}\right)\left(\frac{e^{\lambda z}-e^{-\lambda z}}{z}\right)dz$$

is always non-negative, and that it equals 1 if $e^{-\lambda}\le y\le 1$. Applying these identities we obtain that

$$\Phi(\tau;y) \le \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathbb{E}\big(L(1,X;y)^s\big)(e^\gamma\tau)^{-s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\frac{ds}{s} \le \Phi(\tau e^{-\lambda};y)\,, \tag{3.6}$$

and

$$\Phi(\tau e^{-\lambda};y)-\Phi(\tau;y) \le \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathbb{E}\big(L(1,X;y)^s\big)(e^\gamma\tau)^{-s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\left(\frac{e^{\lambda s}-e^{-\lambda s}}{s}\right)ds\,. \tag{3.7}$$

Now suppose that $\tau \le \log y - 1$ is large. We choose $k = k_{\tau,y}$ such that $I(k;y) = \gamma + \log\tau$. Observe that $k \le y$ since $I(k;y)$ is increasing in k and by (3.1) $I(y;y) = \log_2 y + \gamma + C_1(1)/\log y + O(1/\log^2 y) > \log\tau + \gamma$ as $C_1(1) = -0.09\ldots$. Further (3.1) gives us that $k \asymp e^\tau$. We now suppose below that $\lambda \le e^{-\tau} \ll 1/k$, so that $|e^{\lambda s}| = e^{\lambda k} \ll 1$.

Consider first the integral in (3.7). We split the integral into the part when $|t| \le y^2/\log^2 y$, and when $|t| > y^2/\log^2 y$. The contribution of the second segment of the integral is

$$\ll \frac{\mathbb{E}(L(1,X;y)^k)}{(e^\gamma\tau)^k}\int_{|t|\ge y^2/\log^2 y}\frac{1}{\lambda t^2}dt \ll \frac{\mathbb{E}(L(1,X;y)^k)}{(e^\gamma\tau)^k}\frac{\log^2 y}{\lambda y^2}\,.$$

Using Lemma 3.2 and as $(e^{\lambda s}-1)/(\lambda s) \ll 1$ and $(e^{\lambda s}-e^{-\lambda s})/s \ll \lambda$ we may see easily that the contribution of the initial segment of the integral is $\ll \mathbb{E}(L(1,X;y)^k)(e^\gamma\tau)^{-k}\lambda\sqrt{k\log k}$. Thus we conclude that

$$\Phi(\tau e^{-\lambda};y)-\Phi(\tau;y) \ll \frac{\mathbb{E}(L(1,X;y)^k)}{(e^\gamma\tau)^k}\left(\frac{\log^2 y}{\lambda y^2}+\lambda\sqrt{k\log k}\right). \tag{3.8}$$

Consider next the integral in (3.6). We split the integral into three parts: when $|t| \le \sqrt{k}(\log k)^2$, when $\sqrt{k}(\log k)^2 \le |t| \le y^2/\log^2 y$ and when $|t| > y^2/\log^2 y$. In the third case the integrand is $\ll \mathbb{E}(L(1,X;y)^k)(e^\gamma\tau)^{-k}/(\lambda|t|^2)$, and so the contribution of this segment of the integral is $\ll \mathbb{E}(L(1,X;y)^k)(e^\gamma\tau)^{-k}\log^2 y/(\lambda y^2)$. Using Lemma 3.2 we see that the contribution of the second segment of the integral is bounded by $\mathbb{E}(L(1,X;y)^k)(e^\gamma\tau)^{-k}\exp(-c\log^3 k)/\lambda$. Lastly, using (3.5) and our definition of k, we get that the initial segment of the integral contributes

$$\frac{\mathbb{E}(L(1,X;y)^k)}{(e^\gamma\tau)^k}\frac{e^{\lambda k}-1}{\lambda k^2}\left(1+O\left(\frac{\log^5 k}{\sqrt{k}}\right)\right) \cdot\frac{1}{2\pi}\int_{|t|\le\sqrt{k}\log^2 k}\exp\left(-\frac{t^2}{2}R(k;y)\right)dt$$

$$= \frac{\mathbb{E}(L(1,X;y)^k)}{(e^\gamma\tau)^k}\frac{e^{\lambda k}-1}{\lambda k^2}\frac{1}{\sqrt{2\pi R(k;y)}}\left(1+O\left(\frac{\log^5 k}{\sqrt{k}}\right)\right).$$

Observe that $R(k;y) \asymp 1/(k\log k)$ by (3.2), and so choosing $\lambda = k^{-5/4}$ we obtain from the above that

$$\Phi(\tau e^{-\lambda};y) \ge \frac{\mathbb{E}(L(1,X;y)^k)}{(e^\gamma\tau)^k}\frac{1}{k\sqrt{2\pi R(k;y)}}\left(1+O\left(\frac{\log^2 k}{k^{1/4}}\right)\right) \ge \Phi(\tau;y).$$

Further with this same choice of λ we obtain by (3.8) that

$$\Phi(\tau e^{-\lambda};y)-\Phi(\tau;y) \ll \frac{\mathbb{E}(L(1,X;y)^k)}{(e^\gamma\tau)^k}\frac{1}{k\sqrt{2\pi R(k;y)}}\frac{\log^2 k}{k^{1/4}},$$

and so the first part of the theorem follows.

Now noting again that $R(k;y) \asymp 1/(k\log k)$, and using the first part of the theorem, we may write (3.8) as

$$\Phi(\tau e^{-\lambda};y)-\Phi(\tau;y) \ll \Phi(\tau;y)\left(\frac{\sqrt{k}\log^2 y}{y^2\lambda}+\lambda k\right),$$

which gives the second part of the theorem when $\log y/(ye^{\frac{1}{4}\tau}) \asymp \log y/(yk^{\frac{1}{4}}) \le \lambda \le e^{-\tau}$. Since $\Phi(\tau e^{-\lambda};y)-\Phi(\tau;y)$ is a non-decreasing function of λ, thus for $\lambda \le \log y/(ye^{\frac{1}{4}\tau})$ we have $\Phi(\tau e^{-\lambda};y)-\Phi(\tau;y) \ll \Phi(\tau;y)e^{3\tau/4}\log y/y$ as desired.

In order to extract simpler asymptotics for $\Phi(\tau;y)$ we now need some understanding of the asymptotic nature of $\mathbb{E}(L(1,X;y)^k)$. If $k \le y$ then using (3.4) for $p \le k/\log k$, and (3.3) for $k/\log k \le p \le y$ we deduce that

$$\mathbb{E}\big(L(1,X;y)^k\big) \asymp \prod_{p\le k/\log k}\frac{p}{2(p+1)}\left(1-\frac{1}{p}\right)^{-k}\prod_{k/\log k\le p\le y}\cosh(k/p). \tag{3.9}$$

Now note that

$$\prod_{k/\log k\le p\le k} e^{-k/p}\left(1-\tfrac{1}{p}\right)^{-k} \asymp \prod_{k/\log k\le p\le k} e^{O(k/p^2)} \asymp 1,$$

and also that by partial summation using the prime number theorem

$$\prod_{k/\log k\le p\le k} e^{-k/p}\cosh(k/p)\prod_{y\ge p>k}\cosh(k/p)$$
$$= \exp\left(\frac{k}{\log k}\left(\int_{k/y}^{\infty} g_1(t)\frac{dt}{t^2}+O\left(\frac{1}{\log k}\right)\right)\right),$$

where $g_1(t) = \log\cosh(t) - t$ if $t > 1$ and $\log\cosh(t)$ if $t \le 1$. Integration by parts shows easily that

$$\int_{k/y}^{\infty} g_1(t)\frac{dt}{t^2} = C_1(k/y) - 1 + \frac{g_1(k/y)}{k/y}.$$

Using these estimates in (3.9) we conclude that

$$\mathbb{E}\big(L(1,X;y)^k\big) = \prod_{p\le k}\left(1-\frac{1}{p}\right)^{-k} \exp\left(\frac{k}{\log k}\left(C_1(k/y) - 1 + \frac{\log\cosh(k/y)}{k/y} + O\left(\frac{1}{\log k}\right)\right)\right). \quad (3.10)$$

Similarly we find that

$$\mathbb{E}\big(L(1,X;y)^{-k}\big) = \prod_{p\le k}\left(1+\frac{1}{p}\right)^{k} \exp\left(\frac{k}{\log k}\left(C_1(k/y) - 1 + \frac{\log\cosh(k/y)}{k/y} + O\left(\frac{1}{\log k}\right)\right)\right).$$

From these estimates and Theorem 3.1 we may deduce the following corollary which contains Proposition 1.

Corollary 3.3. *Let y be large, and let τ be large with $\tau \le \log y - 1$. Then*

$$\Phi(\tau;y) = \exp\left(-\tfrac{e^{\tau-C_1}}{\tau}\left(1+O\left(\tfrac{e^\tau}{y}+\tfrac{1}{\tau}\right)\right)\right).$$

The same asymptotic holds for $\Psi(\tau;y)$.

Proof. We take $k = k_{\tau,y}$ as in Theorem 3.1. By Mertens' theorem, (3.10), and Theorem 3.1 we see that

$$\Phi(\tau) = \left(\frac{\log k}{\tau}\right)^k \exp\left(\frac{k}{\log k}\left(C_1(k/y) - 1 + \frac{\log\cosh(k/y)}{k/y} + O\left(\frac{1}{\log k}\right)\right)\right),$$

and using (3.1) the definition of k this is

$$= \exp\left(\frac{k}{\log k}\left(-1 + \frac{\log\cosh(k/y)}{k/y} + O\left(\frac{1}{\log k}\right)\right)\right).$$

Now $y \ge k \asymp e^\tau$ and so (3.1) gives that $I(k;y) = \log_2 k + \gamma + C_1(k/y)/\tau + O(1/\tau^2) = \log_2 k + \gamma + C_1/\tau + O(e^\tau/(\tau y) + 1/\tau^2)$ from which it follows that $k = e^{\tau - C_1}(1 + O(e^\tau/y + 1/\tau))$. Using this above we obtain the asymptotic for $\Phi(\tau;y)$, and the proof for $\Psi(\tau;y)$ is similar.

4 Estimates for Real Character Sums

In this section we collect together some estimates for $\sum^{\flat}_{|d|\le x}(d/n)$ for individual n, and also on average over n in a dyadic interval. We begin with a simple application of the Pólya–Vinogradov inequality.

Lemma 4.1. *If n is a positive integer, not a perfect square, then*

$$\Big|\sum_{|d|\le x}^{\flat}\left(\tfrac{d}{n}\right)\Big| \ll x^{\frac12}n^{\frac14}(\log n)^{\frac12}.$$

Proof. We shall confine our attention to d positive and $\equiv 1 \pmod 4$: the cases d negative, or $d\equiv 8$, or $12 \pmod{16}$ are handled similarly. Thus we seek to bound

$$\sum_{\substack{d\le x\\ d\equiv 1 \ (\bmod\ 4)}} \mu^2(d)\left(\tfrac{d}{n}\right) = \tfrac12 \sum_{\psi\ (\bmod\ 4)}\sum_{d\le x}\psi(d)\mu^2(d)\left(\tfrac{d}{n}\right).$$

Writing $\mu^2(d) = \sum_{l^2|d}\mu(l)$, and using the Pólya–Vinogradov inequality (which is applicable since $(\cdot/n)\psi(\cdot)$ is a non-principal character of a conductor at most $4n$) we get that the above is

$$\ll \sum_{\psi\ (\bmod\ 4)}\sum_{l\le\sqrt{x}}\Big|\sum_{\substack{d\le x\\ l^2|d}}\left(\tfrac{d}{n}\right)\psi(d)\Big| \ll \sum_{\psi\ (\bmod\ 4)}\sum_{l\le\sqrt{x}}\Big|\sum_{m\le x/l^2}\left(\tfrac{m}{n}\right)\psi(m)\Big|$$

$$\ll \sum_{l\le\sqrt{x}} \min\left(\tfrac{x}{l^2},\sqrt{n}\log n\right) \ll x^{\frac12}n^{\frac14}(\log n)^{\frac12},$$

which proves the lemma.

Since $\sum^{\flat}_{|d|\le x}(d/n)$ is trivially $\ll x$ we see that Lemma 4.1 furnishes a non-trivial bound only when $x\ge\sqrt{n}\log n$. While this range can be improved using Burgess' character sum estimates, in general non-trivial bounds are known only when x is larger than some fixed power of n. In the special case that n is smooth (that is composed only of small prime factors) then we may use a remarkable result of Graham and Ringrose to obtain non-trivial bounds in the range $x\ge n^{\epsilon}$. This will be a crucial ingredient in section 6 below.

Suppose χ is a non-principal character $(\bmod\ q)$ where $q/(4,q)$ is square-free, and suppose p is the largest prime factor of q. Then for any integer $l\ge 2$ and with $L=2^l$ we have

$$\sum_{n\le N}\chi(n) \ll N^{1-\frac{l}{8L}}p^{\frac13}q^{\frac{1}{7L}}d(q)^{\frac{l^2}{L}}. \tag{4.1}$$

This is a consequence of Theorem 5 of Graham and Ringrose [GR], using there that $\log q \ll p$ and $\prod_{p|q}(1+1/p) \ll \log\log q$, and simplifying their estimates.

Lemma 4.2. *Let n be a positive integer not a perfect square. Write $n = n_0\Box$ where $n_0 > 1$ is square-free, and suppose all prime factors of n_0 are below P. Let $l \geq 2$ be an integer and put $L = 2^l$. Then*

$$\sum_{|d|\leq x}^{\flat} \left(\frac{d}{n}\right) \ll x^{1-\frac{l}{8L}} \prod_{p|n}\left(1+\frac{1}{p^{1-l/8L}}\right) P^{1/3} n_0^{\frac{1}{7L}} d(n_0)^{\frac{l^2}{L}}.$$

Proof. We shall confine our attention to d positive and $\equiv 1 \pmod 4$: the cases d negative, or $d \equiv 8$, or $12 \pmod{16}$ are handled similarly. Thus we seek to bound

$$\sum_{\substack{d\leq x\\ d\equiv 1 \pmod 4}} \mu^2(d)\left(\tfrac{d}{n}\right) = \sum_{\substack{d\leq x\\ d\equiv 1 \pmod 4\\ (d,n)=1}} \mu^2(d)\left(\tfrac{d}{n_0}\right) = \tfrac{1}{2}\sum_{\psi \pmod 4}\sum_{\substack{d\leq x\\ (d,n)=1}} \psi(d)\left(\tfrac{d}{n_0}\right)\mu^2(d).$$

Note that $\sum_{a^2|d,(a,n)=1}\mu(a)\sum_{b|(d,n)}\mu(b) = 1$ if d is square-free and coprime to n, and 0 otherwise. Hence the above is

$$\ll \sum_{\psi \pmod 4}\sum_{b|n}\mu^2(b)\sum_{\substack{a^2b\leq x\\ (a,n)=1}}\mu^2(a)\Big|\sum_{\substack{d\leq x\\ a^2b|d}}\left(\tfrac{d}{n_0}\right)\psi(d)\Big|$$

$$\ll \sum_{\psi \pmod 4}\sum_{b|n}\mu^2(b)\sum_{\substack{a^2b\leq x\\ (a,n)=1}}\mu^2(a)\Big|\sum_{m\leq x/a^2b}\psi(m)\left(\tfrac{m}{n_0}\right)\Big|$$

$$\ll \sum_{b|n}\mu^2(b)\sum_{\substack{a^2b\leq x\\ (a,n)=1}}\mu^2(a)\Big(\frac{x}{a^2b}\Big)^{1-\frac{l}{8L}} P^{\frac{1}{3}} n_0^{\frac{1}{7L}} d(n_0)^{\frac{l^2}{L}},$$

by (4.1), since $\psi(\cdot)(\cdot/n_0)$ is a non-principal character $\pmod{n_0}$ or $\pmod{4n_0}$, which yields the lemma.

We now give results bounding $\sum_{|d|\leq x}^{\flat}(d/n)$ on average over n. To do this we shall use the following consequence of a simple large sieve estimate for real characters.

Lemma 4.3. *For non-zero integers $m \equiv 0,1 \pmod 4$, and natural numbers n, let a_m and b_n denote arbitrary complex numbers, and set $a_0 = 0$ and $a_m = 0$ if $m \equiv 2,3 \pmod 4$. Then*

$$\sum_{n\leq N}\Big|\sum_{|m|\leq M} a_m\left(\tfrac{m}{n}\right)\Big|^2 \ll N \sum_{\substack{|m_1|,|m_2|\leq M\\ m_1m_2=\Box}} |a_{m_1}a_{m_2}| + M\log M\Big(\sum_{|m|\leq M}|a_m|\Big)^2, \tag{4.2}$$

and

$$\sum_{|m|\le M}\Big|\sum_{n\le N} b_n\left(\tfrac{m}{n}\right)\Big|^2 \ll M \sum_{\substack{n_1,n_2\le N\\ n_1n_2=\square}} |b_{n_1}b_{n_2}| + N\log N\Big(\sum_{n\le N}|b_n|\Big)^2. \quad (4.3)$$

Alternative bounds are

$$\sum_{n\le N}\Big|\sum_{|m|\le M} a_m\left(\frac{m}{n}\right)\Big|^2 \ll \big(M^2\sqrt{N}+N^{\frac{3}{2}}M\big)\log(MN)\Big(\sum_{|m|\le M}\frac{d(|m|)|a_m|^4}{|m|}\Big)^{\frac{1}{2}}, \quad (4.4)$$

and

$$\sum_{|m|\le M}\Big|\sum_{n\le N} b_n\left(\frac{m}{n}\right)\Big|^2 \ll \big(N^2\sqrt{M}+M^{\frac{3}{2}}N\big)\log(MN)\Big(\sum_{n\le N}\frac{d(n)|b_n|^4}{n}\Big)^{1/2}. \quad (4.5)$$

Proof. The first two estimates are simple consequences of the Pólya–Vinogradov inequality. Taking (4.2) for instance, the desired quantity is

$$\sum_{|m_1|,|m_2|\le M} a_{m_1}\overline{a_{m_2}}\sum_{n\le N}\left(\frac{m_1m_2}{n}\right).$$

Now $m_1m_2\equiv 0,1 \pmod 4$ so that $(m_1m_2/\cdot$ is a character $(\bmod |m_1m_2|)$, and this character is principal when $m_1m_2=\square$ giving rise to the first term on the RHS of (4.2), and the character is non-principal when $m_1m_2\ne\square$ giving rise (via Pólya–Vinogradov) by the second term there. The proof of (4.3) is similar.

We now prove the second estimates: again we give only the case (4.4), the proof of (4.5) being entirely similar. Note that our desired expression is

$$\sum_{|m|\le M^2}\Big(\sum_{\substack{m_1m_2=m\\ |m_1|,|m_2|\le M}} a_{m_1}\overline{a_{m_2}}\Big)\sum_{n\le N}\left(\tfrac{m}{n}\right)$$

which is by Cauchy–Schwarz

$$\le \Big(\sum_{|m|\le M^2}\Big|\sum_{\substack{m_1m_2=n\\ |m_1|,|m_2|\le M}} a_{m_1}\overline{a_{m_2}}\Big|^2\Big)^{1/2}\Big(\sum_{|m|\le M^2}\Big|\sum_{n\le N}\left(\tfrac{m}{n}\right)\Big|^2\Big)^{1/2}.$$

Applying (4.3) we see that the second factor above is

$$\ll (M^2N\log N + N^3\log N)^{1/2}.$$

Further as

$$\Big|\sum_{\substack{m_1m_2=m\\ |m_1|,|m_2|\le M}} a_{m_1}\overline{a_{m_2}}\Big|^2 \le \Big|\sum_{\substack{d|m\\ |d|\le M}} |a_d|^2\Big|^2 \le 2d(|m|)\sum_{\substack{d|m\\ |d|\le M}}|a_d|^4,$$

by Cauchy–Schwarz, we get that the first factor above is

$$\ll \Big(\sum_{|d|\le M} |a_d|^4 \sum_{\substack{|m|\le M^2 \\ d|m}} d(|m|) \Big)^{1/2} \ll \bigg(M^2 \log M \sum_{|d|\le M} \frac{|a_d|^4 d(|d|)}{|d|} \bigg)^{1/2},$$

completing the proof of the Lemma.

Applying the above lemma we obtain the following estimate for the $2k$-th moment of $\sum^{\flat}_{|d|\le x} (d/n)$ averaged over n. This will form the key input in our first approach to the moments of $L(1,\chi_d)$; see section 5 below.

LEMMA 4.4. *Uniformly for all integers $k \ge 1$ we have*

$$\sum_{n\le N} \Big| \sum^{\flat}_{|d|\le x} \left(\tfrac{d}{n}\right) \Big|^{2k} \ll (x^{2k} N^{\frac12} + x^k N^{\frac32})(2k\log x)^{2k^5},$$

and also

$$\ll (x^k N + x^{3k})(2k\log x)^{3k^2}.$$

Proof. Write

$$\Big(\sum^{\flat}_{|d|\le x} \left(\tfrac{d}{n}\right) \Big)^k = \sum_{|m|\le x^k} a_m \left(\tfrac{m}{n}\right),$$

where $a_m = 0$ unless $1 \le |m| \le x^k$ is $\equiv 0,\ 1 \pmod 4$, and $0 \le a_m \le d_k(|m|)$ for such m. Note also that $\sum_{|m|\le x^k} a_m = \big(\sum^{\flat}_{|d|\le x} 1 \big)^k \le x^k$.

Then from (4.2) we get that

$$\sum_{n\le N} \Big| \sum^{\flat}_{|d|\le x} \left(\tfrac{d}{n}\right) \Big|^{2k} \ll N \sum_{\substack{m_1,m_2\le x^k \\ m_1 m_2 = \square}} d_k(m_1) d_k(m_2) + x^k \log(x^k) \Big(\sum_{|m|\le x^k} a_m \Big)^2. \tag{4.6}$$

The second term above is $\le x^{3k} \log(x^k)$. As for the first term above, note that $m_1 m_2 = \square$ means that we may write $m_1 = n\alpha^2$, and $m_2 = n\beta^2$ where $n \le x^k$ and α and β are $\le \sqrt{x^k/n}$. Since $d_k(n\alpha^2) d_k(n\beta^2) \le d_k(n)^2 d_k(\alpha^2) d_k(\beta^2) \le d_{k^2}(n) d_{k^2}(\alpha) d_{k^2}(\beta)$, and $n\alpha\beta \le x^k$, we see that

$$\sum_{\substack{m_1,m_2\le x^k \\ m_1 m_2 = \square}} d_k(m_1) d_k(m_2) \le x^k \sum_{n,\alpha,\beta\le x^k} \frac{d_{k^2}(n) d_{k^2}(\alpha) d_{k^2}(\beta)}{n\alpha\beta} \le x^k \bigg(\sum_{n\le x^k} \frac{d_{k^2}(n)}{n} \bigg)^3$$

$$\le x^k \Big(\sum_{n\le x^k} \tfrac{1}{n} \Big)^{3k^2} \le x^k (2k\log x)^{3k^2}.$$

Inputting these estimates in (4.6) we get the second bound claimed in the lemma.

The first bound of the lemma is trivial when $N \geq x^{2k}$ since the sum is $\leq N x^{2k} \leq x^k N^{3/2}$. Suppose now that $N \leq x^{2k}$. Using (4.4) we get that

$$\sum_{n \leq N} \Big| \sideset{}{^\flat}\sum_{|d| \leq x} \left(\frac{d}{n}\right) \Big|^{2k} \ll \big(x^{2k}\sqrt{N} + N^{\frac{3}{2}} x^k\big)(3k \log x)\bigg(\sum_{n \leq x^k} \frac{d(n) d_k(n)^4}{n} \bigg)^{1/2}.$$

Since $d(n)d_k(n)^4 \leq d(n) d_{k^4}(n) \leq d_{k^4+1}(n) \leq d_{2k^5}(n)$ we have

$$\sum_{n \leq x^k} \frac{d(n) d_k(n)^4}{n} \leq \sum_{n \leq x^k} \frac{d_{2k^5}(n)}{n} \leq \bigg(\sum_{n \leq x^k} \frac{1}{n} \bigg)^{2k^5} \leq (2k \log x)^{2k^5},$$

and the lemma follows.

Lastly we give a bound for $\sum^\flat_{|d| \leq x} (d/n)$ for non-square integers n, conditional on the generalized Riemann hypothesis.

Lemma 4.5. *Assume GRH. Fix $\varepsilon > 0$. Let χ be a non-principal character* (mod q). *Then for $x \geq 2$*

$$\sum_{m \leq x} \chi(m) \ll x^{\frac{3}{4}+\varepsilon} \exp\left((\log q)^{1/2-\varepsilon} \right).$$

Consequently, for any positive integer n which is not a perfect square we have

$$\sideset{}{^\flat}\sum_{|d| \leq x} \left(\tfrac{d}{n}\right) \ll x^{\frac{3}{4}+\varepsilon} \exp\left((\log n)^{1/2-\varepsilon} \right).$$

Proof. We may suppose that $x \leq q$, and also that x is half an odd integer. Then by Perron's formula

$$\begin{aligned} \sum_{n \leq x} \chi(n) &= \frac{1}{2\pi i} \int_{1+\frac{1}{\log x} - i\infty}^{1+\frac{1}{\log x}+i\infty} L(s,\chi) x^s \frac{ds}{s} \\ &= \frac{1}{2\pi i} \int_{1+\frac{1}{\log x} - iq}^{1+\frac{1}{\log x}+iq} L(s,\chi) x^s \frac{ds}{s} + O\left(\frac{x \log^2 x}{q} \right). \end{aligned}$$

Appealing to Lemma 2.1 with $\sigma_0 = 1/2$ there (by GRH) and $y = (\log q)^2 (\log_2 q)^6$, we deduce that if $\mathrm{Re}\,(s) \geq \frac{3}{4} + \varepsilon$, and $\mathrm{Im}\,(s) \leq q$ then $\log L(s,\chi) \leq (1/2)(\log q)^{1/2-\varepsilon}$. Using this bound, and moving the line of integration above to the $\mathrm{Re}\,(s) = \frac{3}{4} + \varepsilon$ line we obtain the first part of the lemma. To obtain the second part of the lemma, we proceed along the lines of the proof of Lemma 4.1, replacing the use of Pólya–Vinogradov by the GRH bound for character sums above.

Different bounds may be found by moving to the line $\mathrm{Re}(s) = \sigma$ for any $1 > \sigma > 1/2$, but the present bound is adequate for our applications.

5 Moments of $L(1,\chi_d)$: Proof of Theorem 2

By Lemma 2.3 we see that with $Z=\exp((\log x)^{10})$

$$\sum_{\substack{|d|\le x\\ d\notin\mathcal{L}}}^{\flat} L(1,\chi_d)^z=\sum_{n=1}^{\infty}\frac{d_z(n)}{n}e^{-n/Z}\sum_{\substack{|d|\le x\\ d\notin\mathcal{L}}}^{\flat}\left(\frac{d}{n}\right)+O(\log x)\,.$$

Since there are $\ll \log x$ discriminants $d\in\mathcal{L}$, $|d|\le x$, we see that the above is

$$\sum_{n=1}^{\infty}\frac{d_z(n)}{n}e^{-n/Z}\sum_{|d|\le x}^{\flat}\left(\frac{d}{n}\right)+O\bigg(\log x\sum_{n=1}^{\infty}\frac{|d_z(n)|}{n}e^{-n/Z}\bigg)$$
$$=\sum_{n=1}^{\infty}\frac{d_z(n)}{n}e^{-n/Z}\sum_{|d|\le x}^{\flat}\left(\frac{d}{n}\right)+O(x^{\epsilon})\,,\quad (5.1)$$

using $|d_z(n)|\le d_{|z|}(n)$ and (2.4), and since $|z|\le \log x/(\log_2 x)^2$.

We now handle the contribution of the terms $n=\square$ to (5.1), which gives the main term. When $n=m^2$ we have

$$\sum_{|d|\le x}^{\flat}\left(\frac{d}{m^2}\right)=\sum_{\substack{|d|\le x\\ (d,m)=1}}^{\flat}1=\frac{6}{\pi^2}x\prod_{p|m}\left(\frac{p}{p+1}\right)+O\big(x^{\frac12+\epsilon}d(m)\big)\,.$$

Thus the contribution of such terms to (5.1) is

$$\frac{6}{\pi^2}x\sum_{m=1}^{\infty}\frac{d_z(m^2)}{m^2}\prod_{p|m}\left(\frac{p}{p+1}\right)e^{-m^2/Z}+O\bigg(x^{\frac12+\epsilon}\sum_{m=1}^{\infty}\frac{|d_z(m^2)d(m)|}{m^2}e^{-m^2/Z}\bigg).$$

Since $|d_z(m^2)d(m)|\le d_{2|z|+2}(m)^2$ (see section 2) and $e^{-m^2/Z}\le 1$ the error term above is

$$\ll x^{\frac12+\epsilon}\sum_{m=1}^{\infty}\frac{d_{2|z|+2}(m)^2}{m^2}\ll x^{\frac12+\epsilon}\prod_{p}\bigg(\int_0^1\left|1-\frac{e(\theta)}{p}\right|^{-4(|z|+1)}d\theta\bigg)$$
$$\ll x^{\frac12+\epsilon}\prod_{p\le|z|+2}\left(1-\tfrac1p\right)^{-4(|z|+1)}\prod_{p>|z|+2}\left(1+O\left(\tfrac{|z|^2}{p^2}\right)\right)$$
$$\ll x^{\frac12+\epsilon}\big(10\log(|z|+2)\big)^{4|z|+4}\ll x^{\frac23},$$

say, since $|z|\le\log x/(\log_2 x)^2$. Further since $1-e^{-t}\le t^{1/4}$ for any $t>0$, and $|d_z(m^2)|\le d_{(|z|+1)^2}(m)$, we have

$$\sum_{m=1}^{\infty}\frac{|d_z(m^2)|}{m^2}(1-e^{-m^2/Z})\le\sum_{m=1}^{\infty}\frac{d_{(|z|+1)^2}(m)}{m^2}\left(\frac{m^2}{Z}\right)^{1/4}=\frac{\zeta(3/2)^{(1+|z|)^2}}{Z^{1/4}}\le\frac{1}{x}\,.$$

We conclude that the contribution of the terms $n=\square$ to (5.1) is

$$\frac{6}{\pi^2}\sum_{m=1}^{\infty}\frac{d_z(m^2)}{m^2}\prod_{p|m}\left(\frac{p}{p+1}\right)+O(x^{\frac{2}{3}})=\frac{6}{\pi^2}\mathbb{E}\big(L(1,X)^z\big)+O(x^{\frac{2}{3}}). \tag{5.2}$$

We now handle the $n\neq\square$ terms in (5.1). Let k be a positive integer ≥ 2 to be fixed shortly. Using Hölder's inequality with exponents $\alpha:=2k$ and $\beta:=2k/(2k-1)$ (so that $1/\alpha+1/\beta=1$) we see that

$$\sum_{\substack{n=1\\ n\neq\square}}^{\infty}\frac{|d_z(n)|}{n}e^{-\frac{n}{Z}}\Big|\sideset{}{^\flat}\sum_{|d|\leq x}\left(\frac{d}{n}\right)\Big|$$
$$\leq\left(\sum_{n=1}^{\infty}\frac{|d_z(n)|^\beta}{n}e^{-\frac{n}{Z}}\right)^{1/\beta}\left(\sum_{\substack{n=1\\ n\neq\square}}^{\infty}\frac{e^{-\frac{n}{Z}}}{n}\Big|\sideset{}{^\flat}\sum_{|d|\leq x}\left(\frac{d}{n}\right)\Big|^{2k}\right)^{1/2k}. \tag{5.3}$$

Since $|d_z(n)|^\beta\leq d_{|z|^\beta}(n)$, (2.4) gives that the first factor here is

$$\leq(\log 3Z)^{(|z|+1)^\beta}=\exp\big(20(|z|+1)^\beta\log\log x\big). \tag{5.4}$$

We split the second factor into dyadic blocks:

$$\sum_{\substack{n=1\\ n\neq\square}}^{\infty}\frac{e^{-n/Z}}{n}\Big|\sideset{}{^\flat}\sum_{|d|\leq x}\left(\frac{d}{n}\right)\Big|^{2k}\leq\sum_{j=0}^{\infty}\frac{e^{-2^j/Z}}{2^j}\sum_{\substack{n=2^j\\ n\neq\square}}^{2^{j+1}-1}\Big|\sideset{}{^\flat}\sum_{|d|\leq x}\left(\frac{d}{n}\right)\Big|^{2k}.$$

To bound the contribution of the dyadic block $[2^j,2^{j+1})$, we use the bound of Lemma 4.1 when $2^j\leq x^{2k/(k+1)}$, the first bound of Lemma 4.4 when $x^{2k/(k+1)}<2^j\leq x^{4k/3}$ and the second bound of Lemma 4.4 when $2^j\geq x^{4k/3}$. Summing these bounds we deduce that the above is $\ll x^{k(2k+1)/(k+1)}(2k\log x)^{2k^5}$. Thus the second factor in (5.3) is $\ll x^{(2k+1)/(2k+2)}(2k\log x)^{k^4}$. Using this with (5.4) we conclude that the contribution of the $n\neq\square$ terms is

$$\ll x^{1-\frac{1}{2k+2}}(2k\log x)^{k^4}\exp\big(20(|z|+1)^\beta\log\log x\big).$$

Choose $k=[\log_2 x]$ and recall that $|z|\leq\log x/(500(\log_2 x)^2)$. Then the above error is $\ll x\exp(-\log x/(5\log_2 x))$, and this when combined with (5.2) proves our theorem.

6 Moments of Short Euler Products: Proof of Theorem 3

Theorem 6.1. *Let $2\leq y\leq\exp(\sqrt{\log x})$, and let z be a complex number. Write*

$$\sideset{}{^\flat}\sum_{|d|\leq x}L(1,\chi_d;y)^z=\tfrac{6}{\pi^2}x\mathbb{E}\big(L(1,X;y)^z\big)+\theta x\mathbb{E}\big(L(1,X;y)^{\operatorname{Re} z}\big)E(z,y),$$

where $|\theta| = 1$, *and* $E(z,y)$ *is a positive real number. If* $y \ge 4|z|+4$ *then*

$$E(z,y) \ll y^{1/3}\exp\biggl(-\frac{\log x \log_2 y}{12\log y}+\frac{30|z|}{\log(4|z|+4)}+10|z|\log\left(\frac{\log y}{\log(4|z|+4)}\right)\biggr),$$

and if $y \le 4|z|+4$ *then*

$$E(z,y) \ll x^{-\frac{1}{40}}\,, \quad \text{if } y \le \frac{\log x}{2}\,,$$

and

$$E(z,y) \ll \exp\left(-\frac{\log x}{(\log_2 x)^{3/4}}\right), \quad \text{if } y \le \tfrac{1}{3}\log x \log_3 x\,.$$

Proof. Let $\delta = \pm 1$ indicate the sign of $\operatorname{Re} z$. Denote $\sum_{|d|\le x}^{\flat}\chi_d(n)$ by $S(x;n)$, and define $\mathcal{S}(y)$ to be the set of integers whose prime factors are all $\le y$. Then

$$\sum_{|d|\le x}^{\flat} L(1,\chi_d;y)^z = \sum_{|d|\le x}^{\flat}\sum_{\substack{n=1\\ n\in\mathcal{S}(y)}}^{\infty}\frac{d_z(n)}{n}\chi_d(n) = \sum_{\substack{n=1\\ n\in\mathcal{S}(y)}}^{\infty}\frac{d_z(n)}{n}S(x;n)\,.$$

Decompose $n = n_1n_2^2n_3^2$ where n_1 and n_2 are square-free, with $(n_1,n_2)=1$, and $p|n_3 \Rightarrow p|n_1n_2$: that is n_1 is the product of all primes dividing n to an odd power, and n_2 is the product of all primes dividing n to an even power ≥ 2. Observe that $S(x;n_1n_2^2n_3^2) = S(x;n_1n_2^2)$. Thus our sum above is

$$\begin{aligned}&\sum_{n_1\in\mathcal{S}(y)}\mu^2(n_1)\sum_{\substack{n_2\in\mathcal{S}(y)\\(n_1,n_2)=1}}\mu^2(n_2)S(x;n_1n_2^2)\sum_{p|n_3\Rightarrow p|n_1n_2}\frac{d_z(n_1n_2^2n_3^2)}{n_1n_2^2n_3^2}\\ &=\sum_{n_1\in\mathcal{S}(y)}\mu^2(n_1)\sum_{\substack{n_2\in\mathcal{S}(y)\\(n_1,n_2)=1}}\mu^2(n_2)S(x;n_1n_2^2)\prod_{p|n_1}s_p(z)\prod_{p|n_2}\bigl(c_p(z)-1\bigr)\,, \qquad (6.1)\end{aligned}$$

where

$$s_p(z) := \frac{(1-1/p)^{-z}-(1+1/p)^{-z}}{2} \quad\text{and } c_p(z) := \frac{(1-1/p)^{-z}+(1+1/p)^{-z}}{2}\,.$$

Note that $|s_p(z)|$ and $|c_p(z)|$ are always $\le (1-\delta/p)^{-\operatorname{Re} z}$, and that when $p > 4|z|+4$ we have $|s_p(z)| \le 2|z|/p$, and $|c_p(z)-1| \le 2|z|^2/p^2$. To evaluate (6.1), we now distinguish the cases $n_1 = 1$ which give rise to the main term, and $n_1 > 1$ which contribute the error term.

We first handle the terms $n_1 > 1$. By Lemma 4.2 we see that for any integer $l \ge 2$ and with $L = 2^l$ we have

$$\bigl|S(x;n_1n_2^2)\bigr| \ll x^{1-\frac{l}{8L}}\prod_{p|n_1n_2}\left(1+\frac{1}{p^{1-l/8L}}\right)y^{\frac{1}{3}}n_1^{\frac{1}{7L}}d(n_1)^{\frac{l^2}{L}}.$$

Using this estimate in (6.1) we see that the contribution of the terms $n_1 > 1$ is

$$\ll x^{1-\frac{l}{8L}} y^{\frac{1}{3}} \prod_{p\le y}\left(1+2^{\frac{l^2}{L}} p^{\frac{1}{7L}}\left(1+\frac{1}{p^{1-1/8L}}\right)\left|s_p(z)\right|+\left(1+\frac{1}{p^{1-1/8L}}\right)\left|c_p(z)-1\right|\right)$$

$$\ll x^{1-\frac{l}{8L}} y^{\frac{1}{3}} \prod_{p\le \min(4|z|+4,y)}\left(8p^{\frac{1}{7L}}\left(1-\frac{\delta}{p}\right)^{-\mathrm{Re}\, z}\right)\prod_{\min(4|z|+4,y)\le p\le y}\left(1+\frac{8|z|p^{\frac{1}{7L}}}{p}\right).$$

We now handle the terms arising from $n_1 = 1$. Note that $S(x;n_2^2)$ counts the number of fundamental discriminants $|d| \le x$ that are coprime to n_2. Thus we see easily that $S(x;n_2^2) = \frac{6}{\pi^2}x\prod_{p|n_2}\left(\frac{p}{p+1}\right) + O\big(\sqrt{x}d(n_2)\big)$. Hence the contribution of the terms $n_1 = 1$ to (6.1) equals

$$\sum_{n_2\in\mathcal{S}(y)} \mu^2(n_2)\prod_{p|n_2}\left(c_p(z)-1\right)\left(\frac{6}{\pi^2}x\prod_{p|n_2}\left(\frac{p}{p+1}\right)+O\big(\sqrt{x}d(n_2)\big)\right)$$

$$= \tfrac{6}{\pi^2}x\mathbb{E}\big(L(1,X;y)^z\big) + O\Big(\sqrt{x}\prod_{p\le y}\big(1+2|c_p(z)-1|\big)\Big).$$

The error term here may be subsumed into our estimate for the contribution of the $n_1 > 1$ terms.

Thus we have an asymptotic formula for $\sum^{\flat}_{|d|\le x} L(1,\chi_d;y)^z$ with an error

$$\ll x^{1-\frac{l}{8L}} y^{\frac{1}{3}} \prod_{p\le \min(4|z|+4,y)}\left(8p^{\frac{1}{7L}}\left(1-\frac{\delta}{p}\right)^{-\mathrm{Re}\, z}\right)\prod_{\min(4|z|+4,y)\le p\le y}\left(1+\frac{8|z|p^{\frac{1}{7L}}}{p}\right)$$

$$\ll x^{1-\frac{l}{8L}} y^{\frac{1}{3}}\mathbb{E}\big(L(1,X;y)^{\mathrm{Re}\, z}\big) \prod_{p\le \min(4|z|+4,y)}(24p^{\frac{1}{7L}})\prod_{\min(4|z|+4,y)\le p\le y}\left(1+\frac{8|z|p^{\frac{1}{7L}}}{p}\right),$$

since $\mathbb{E}(L(1,X;y)^{\mathrm{Re}\, z}) \ge \prod_{p\le\min(4|z|+4,y)}\frac{1}{2}\left(\frac{p}{p+1}\right)(1-\delta/p)^{-\mathrm{Re}\, z}$.

Suppose that $y \ge 4|z|+4$. Then choosing the integer l such that $2\log y > L = 2^l \ge \log y$ and using the prime number theorem we see easily that the error above meets the bound prescribed in the theorem. Suppose now that $y \le 4|z|+4$. If $y \le \frac{1}{2}\log x$ then we choose $l = 2$ and then a simple calculation gives that the error above is $\ll x^{1-\frac{1}{40}}$, proving the theorem in this case. Lastly, when $y \le \frac{1}{3}\log x\log_3 x$, we take $l = [\log_3 x]$, and again the error above meets the bound in the theorem.

In applications, the case $y > 4|z|+4$ is the most useful case, but we have included the other cases mainly for the sake of completeness. If the GRH is true then the bounds for $E(z,y)$ may be improved considerably.

Theorem 6.2. *Let y, z and $E(z,y)$ be as in Theorem 6.1 above. Suppose that the GRH is true. If $y \geq 4|z|+4$ then*

$$E(z,y) \ll x^{-\frac{1}{10}} \exp\left(\frac{30|z|}{\log(4|z|+4)} + 10|z| \log\left(\frac{\log y}{\log(4|z|+4)} \right) \right),$$

while if $y \leq 4|z|+4$ then

$$E(z,y) \ll x^{-\frac{1}{10}} \exp\left(\frac{4y}{\log y} \right).$$

Proof. We follow the proof of Theorem 6.1 closely. The main change is that for $n_1 > 1$ we have the following improved bound for $S(x; n_1 n_2^2)$ arising from Lemma 4.5:

$$S(x; n_1 n_2^2) \ll x^{\frac{3}{4}+\varepsilon} \exp\left(\sqrt{\log(n_1 n_2^2)} \right) \ll x^{\frac{9}{10}} (n_1 n_2^2)^{\frac{10}{\log x}},$$

where the last estimate follows upon distinguishing the cases $n_1 n_2^2 \geq \exp((\log x)^2/100)$ and $n_1 n_2^2 \leq \exp((\log x)^2/100)$.

Using Theorems 6.1 and 6.2, we may now prove Theorem 3.

Proof of Theorem 3. Taking $A = 4$ and $y = (\log x)^{180}$ in (2.1) of Proposition 2.2 we see that for all but at most $x^{1/2}$ fundamental discriminants $|d| \leq x$ we have

$$L(1, \chi_d) = L(1, \chi_d; y) \left(1 + O\left(\frac{1}{(\log x)^{10}} \right) \right).$$

We take $\mathcal{E}$ to be the set of exceptional discriminants d for which the above asymptotic does not hold. Then for a complex number z with $|z| \leq \log x$ we have

$$\sideset{}{^\flat}\sum_{\substack{|d| \leq x \\ d \notin \mathcal{E}}} L(1, \chi_d)^z = \sideset{}{^\flat}\sum_{\substack{|d| \leq x \\ d \notin \mathcal{E}}} L(1, \chi_d; y)^z + O\biggl(\frac{1}{(\log x)^9} \sideset{}{^\flat}\sum_{|d| \leq x} L(1, \chi_d; y)^{\operatorname{Re} z} \biggr).$$

In the notation of Theorem 6.1 we have that

$$\begin{aligned} \sideset{}{^\flat}\sum_{\substack{|d| \leq x \\ d \notin \mathcal{E}}} L(1, \chi_d)^z &= \frac{6}{\pi^2} x \mathbb{E}\bigl(L(1, X; y)^z \bigr) \\ &\quad + O\left(x \mathbb{E}\bigl(L(1, X, y)^{\operatorname{Re} z} \bigr) \left(\frac{1 + E(\operatorname{Re} z, y)}{(\log x)^9} + E(z, y) \right) \right) \\ &\qquad + O\biggl(|\mathcal{E}| \prod_{p \leq y} \left(1 - \tfrac{\delta}{p} \right)^{-\operatorname{Re} z} \biggr). \end{aligned}$$

Since $|\mathcal{E}| \leq x^{1/2}$ and $\mathbb{E}(L(1, X; y)^{\operatorname{Re} z}) \geq \prod_{p \leq 4|z|+4} (p/2(p+1))(1 - \delta/p)^{-\operatorname{Re} z}$ we see that in the range $|z| \leq 10^{-3} \log x$ the last error term above is

$$O\left(x^{\frac{1}{2}}\mathbb{E}\big(L(1,X;y)^{\operatorname{Re} z}\big)\exp\left(\frac{8|z|}{\log(4|z|+4)}+|z|\log\left(\frac{\log y}{\log(4|z|+4)}\right)\right)\right)$$
$$= O\big(x^{2/3}\mathbb{E}(L(1,X;y)^{\operatorname{Re} z})\big)\,.$$

Further, applying Theorem 6.1 we get that if $|z| \le e^{-12}\log x \log_3 x/\log_2 x$ then $E(z,y)$ and $E(\operatorname{Re} z, y)$ are $\ll (\log x)^{-9}$. If the GRH is true then Theorem 6.2 gives that $E(z,y)$ and $E(\operatorname{Re} z,y)$ are $\ll (\log x)^{-9}$ in the extended range $|z| \le 10^{-3}\log x$. Thus we have shown that

$$\sum_{\substack{|d|\le x\\ d\notin\mathcal{E}}}^{\flat} L(1,\chi_d)^z = \frac{6}{\pi^2}x\mathbb{E}\big(L(1,X;y)^z\big) + O\left(x\frac{\mathbb{E}(L(1,X;y)^{\operatorname{Re} z})}{(\log x)^9}\right),$$

in the appropriate conditional and unconditional ranges for z. Observe that

$$\mathbb{E}\big(L(1,X)^z\big) = \mathbb{E}\big(L(1,X;y)^z\big)\prod_{p>y}\big(1+O(|z|^2/p^2)\big)$$
$$= \mathbb{E}\big(L(1,X;y)^z\big)\big(1+O((\log x)^{-178})\big)\,,$$

and further that $\mathbb{E}(L(1,X;y)^{\operatorname{Re} z}) \le \mathbb{E}(L(1,X)^{\operatorname{Re} z})$, and so Theorem 3 follows.

7 The Distribution Function: Proof of Theorem 1

As with Proposition 1 we prove only the statements involving $\Phi_{\mathbb{R}}$, the corresponding results for $\Psi_{\mathbb{R}}$ are established similarly. Let $\lambda > 0$ be a real number, and let $N \ge 1$ be an integer. Observe that for any $y > 0$ and $c > 0$ we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s\left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N\frac{ds}{s}$$
$$= \frac{1}{\lambda^N}\int_0^\lambda\cdots\int_0^\lambda \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(ye^{t_1+\dots+t_N})^s\frac{ds}{s}dt_1\dots dt_N$$

so that by Perron's formula we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s\left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N\frac{ds}{s}\begin{cases} =1 & \text{if } y\ge 1\,,\\ \in[0,1] & \text{if } 1\ge y\ge e^{-\lambda N}\,,\\ =0 & \text{if } e^{-\lambda N}\ge y\,.\end{cases}$$

Write $s = k + it$ from now on, where k is positive. We put

$$I_1 = \frac{1}{2\pi i}\int_{(k)}\frac{\pi^2}{6x}\sum_{\substack{|d|\le x\\ d\notin\mathcal{E}}}^{\flat} L(1,\chi_d)^s(e^\gamma\tau)^{-s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N\frac{ds}{s}\,,$$

where $\mathcal{E}$ is the set in Theorem 3, so that $|\mathcal{E}| \le x^{1/2}$. From the above we know that

$$\Phi_x(\tau) + O(x^{-1/2}) \le I_1 \le \Phi_x(\tau e^{-\lambda N}) + O(x^{-1/2})\,. \tag{7.1}$$

Further if we set

$$I_2 = \frac{1}{2\pi i}\int_{(k)} \mathbb{E}\big(L(1,X)^s\big)(e^\gamma\tau)^{-s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s}\,,$$

then we have that

$$\Phi(\tau) \le I_2 \le \Phi(\tau e^{-\lambda N})\,. \tag{7.2}$$

We shall now estimate $|I_1 - I_2|$. We put $T := 10^{-6}\log x \log_3 x/\log_2 x$ if we are to prove the unconditional parts of Theorem 1, and $T := 10^{-4}\log x$ if we are to prove the GRH part of Theorem 1. We take $k = k_\tau$ as in Theorem 3.1 (thinking of y there as $\to\infty$; thus $I(k) = \gamma + \log\tau$), so that when $\tau \le \log T + C_1 - 20$ we have $k_\tau = e^{\tau - C_1}\{1 + O(1/\tau)\} \le e^{-15}T$. Observe that

$$|I_1 - I_2| \ll (e^\gamma\tau)^{-k}\left(\int_{|t|\le T} + \int_{|t|>T}\right) \cdot \left|\frac{\pi^2}{6x}\sideset{}{^\flat}\sum_{\substack{|d|\le x\\ d\notin\mathcal{E}}} L(1,\chi_d)^s - \mathbb{E}\big(L(1,X)^s\big)\right| \left|\frac{e^{\lambda s}-1}{\lambda s}\right|^N \left|\frac{ds}{s}\right|. \tag{7.3}$$

From Theorem 3.1 we know that $\Phi(\tau) \asymp \big(\sqrt{\frac{\log k}{k}}\big)\frac{\mathbb{E}(L(1,X)^k)}{(e^\gamma\tau)^k}$, and so applying Theorem 3 we get that when $|t| \le T$

$$(e^\gamma\tau)^{-k}\left|\frac{\pi^2}{6x}\sideset{}{^\flat}\sum_{\substack{|d|\le x\\ d\notin\mathcal{E}}} L(1,\chi_d)^s - \mathbb{E}\big(L(1,X)^s\big)\right| \ll \frac{1}{(\log x)^9}\frac{\mathbb{E}(L(1,X)^k)}{(e^\gamma\tau)^k} \ll \frac{\Phi(\tau)}{(\log x)^7}\,.$$

Further note that $|(e^{\lambda s}-1)/(\lambda s)| \le 1 + e^{\lambda k}$, which is easily seen by looking at the cases $|\lambda s| \le 1$ and $|\lambda s| > 1$. Hence it follows that the $|t| < T$ integral contributes to (7.3) an amount

$$\ll \frac{\Phi(\tau)}{(\log x)^8}(1+e^{\lambda k})^N \log T \ll (1+e^{\lambda k})^N \frac{\Phi(\tau)}{(\log x)^7}\,.$$

Next note that, by Theorem 3,

$$(e^\gamma\tau)^{-k}\left|\frac{\pi^2}{6x}\sideset{}{^\flat}\sum_{\substack{|d|\le x\\ d\notin\mathcal{E}}} L(1,\chi_d)^s - \mathbb{E}\big(L(1,X)^s\big)\right| \ll (e^\gamma\tau)^{-k}\left(\frac{\pi^2}{6x}\sideset{}{^\flat}\sum_{\substack{|d|\le x\\ d\notin\mathcal{E}}} L(1,\chi_d)^k + \mathbb{E}\big(L(1,X)^k\big)\right)$$

$$\ll \frac{\mathbb{E}(L(1,X)^k)}{(e^\gamma \tau)^k} \ll \Phi(\tau)\sqrt{\log x}\,.$$

Hence the contribution to (7.3) from the $|t| > T$ integral is

$$\ll \sqrt{\log x}\Phi(\tau)\int_{|t|>T} \frac{(1+e^{\lambda k})^N}{(\lambda|t|)^N}\frac{dt}{|t|} \ll \sqrt{\log x}\Phi(\tau)\left(\frac{1+e^{\lambda k}}{\lambda T}\right)^N.$$

We now choose $\lambda = 4e^{10}/T$, and $N = [\log_2 x]$. From the above estimates we then conclude that

$$|I_1 - I_2| \ll \Phi(\tau)(1+e^{\lambda k})^N\left(\frac{1}{(\log x)^7} + \frac{\sqrt{\log x}}{(\lambda T)^N}\right) \ll \frac{\Phi(\tau)}{(\log x)^5}\,.$$

Using this in (7.1), and (7.2) we deduce that

$$\Phi_x(\tau) \le \Phi(\tau e^{-\lambda N}) + O\left(x^{-\frac{1}{2}} + \frac{\Phi(\tau)}{(\log x)^5}\right),$$

and that

$$\Phi_x(\tau e^{-\lambda N}) \ge \Phi(\tau) + O\left(x^{-\frac{1}{2}} + \frac{\Phi(\tau)}{(\log x)^5}\right).$$

In the lower bound above, replace τ by $\tau e^{\lambda N}$. Then we get that

$$\Phi(\tau e^{\lambda N})\big(1+O((\log x)^{-5})\big) \le \Phi_x(\tau) + O(x^{-1/2}) \le \Phi(\tau e^{-\lambda N})\big(1+O((\log x)^{-5})\big)\,,$$

since $\Phi(\tau)\le\Phi(\tau e^{-\lambda N})$. We now observe that $\Phi(\tau e^{\pm\lambda N}) = \Phi(\tau)\exp(O(\lambda N e^\tau))$ which follows directly from Proposition 1 if $\lambda N \le e^{-\tau}$, and also in the range $\lambda N \le 1/\tau$ upon iterating the estimate in Proposition 1. Theorem 1 now follows upon recalling our choices of λ and N.

8 The Distribution Function: Proof of Theorem 4

To prove Theorem 4 we shall first show that for most discriminants d we may approximate $L(1,\chi_d)$ by a short Euler product. Then we shall use Theorems 6.1 and 6.2 to compare the distribution of these short Euler products with the random Euler products of section 3.

Lemma 8.1. *Let $y^{100} \ge z \ge y$ be real numbers, with y large. Then for integers k, $l \ge 2$*

$$\sum_{|d|\le x}^{\flat}\left|\sum_{y\le p\le z}\frac{d/p}{p}\right|^{2k} \ll x\left(\frac{2k}{y\log y}\right)^k + x^{1-\frac{l}{8L}}y^{35+30\frac{k}{L}}e^{6k}$$

uniformly, where $L = 2^l$. Further if the GRH is assumed then

$$\sum_{|d|\le x}^{\flat}\left|\sum_{y\le p\le z}\frac{d/p}{p}\right|^{2k} \ll x\left(\frac{2k}{y\log y}\right)^k + x^{\frac{4}{5}}\exp(\sqrt{2k\log z})\left(\frac{\log(ez/y)}{\log y}\right)^{2k}.$$

Proof. The quantity we seek to estimate is

$$\sum_{\substack{p_1,\dots,p_{2k}\\ y\le p_i\le z}} \frac{1}{p_1\cdots p_{2k}} \sum_{|d|\le x}^{\flat} \left(\frac{d}{p_1\cdots p_{2k}}\right).$$

The terms where $p_1\cdots p_{2k}=\square$ (so that the p_i occur in pairs) contribute

$$\ll x\frac{(2k)!}{2^k k!}\bigg(\sum_{y\le p\le z}\frac{1}{p^2}\bigg)^k \ll x\left(\frac{2k}{y\log y}\right)^k,$$

using Stirling's formula and $\sum_{p>y}1/p^2\le 2/(y\log y)$. By Lemma 4.2, the terms where $p_1\dots p_{2k}\ne\square$ contribute

$$\ll \sum_{\substack{p_1,\dots,p_{2k}\\ y\le p_i\le z}} \frac{1}{p_1\cdots p_{2k}} x^{1-\frac{l}{8L}} y^{35} (y^{200k})^{\frac{1}{7L}} (2^{2k})^{\frac{l^2}{L}} \prod_{i=1}^{2k}\left(1+\frac{1}{p_i^{1-l/8L}}\right)$$

$$\ll x^{1-\frac{l}{8L}} y^{35+30\frac{k}{L}}\bigg(\sum_{y<p<y^{100}}\tfrac{e}{p}\bigg)^{2k} \ll x^{1-\frac{l}{8L}} y^{35+30\frac{k}{L}} e^{6k},$$

which gives the first part of the lemma. If the GRH is assumed then we may estimate the contribution of the terms $p_1\dots p_{2k}\ne\square$ using Lemma 4.5. This gives for these terms an amount

$$\ll x^{\frac45}\exp\big(\sqrt{2k\log z}\big)\bigg(\sum_{y\le p\le z}\tfrac1p\bigg)^{2k} \ll x^{\frac45}\exp\big(\sqrt{2k\log z}\big)\left(\frac{\log(ez/y)}{\log y}\right)^{2k},$$

proving the lemma.

PROPOSITION 8.2. *Let $(\log_2 x)^{10}\ge A\ge e$ be a real number, put $y_0=A^2\log x\log_3 x$, and $y_1=eA^3\log x\log_2 x$. Then there is a positive constant c such that*

$$L(1,\chi_d)=L(1,\chi_d;y_0)\left(1+O\left(\frac{1}{A\log_2 x}+\frac{1}{\log x}\right)\right),$$

for all but $x\exp(-c\log x\log_3 x/\log_2 x)$ fundamental discriminants $|d|\le x$. If the GRH is true then

$$L(1,\chi_d)=L(1,\chi_d;y_1)\left(1+O\left(\frac{1}{A\log_2 x}+\frac{1}{(\log_2 x)^2}\right)\right),$$

for all but $x^{24/25}$ fundamental discriminants $|d|\le x$.

Proof. Setting $z=(\log x)^{80}$, we know that $L(1,\chi_d)=L(1,\chi_d;z)(1+O(1/\log x))$ for all but at most $x^{1/5}$ discriminants $|d|\le x$ (by (2.1) with $A=10$).

Since $L(1,\chi_d;z)=L(1,\chi_d;y_0)\exp\big(\sum_{y_0\le p\le z}(\chi_d(p)/p+O(1/p^2))\big)$, we see that the first part of the proposition follows from showing that

$$\bigg|\sum_{y\le p\le z}\frac{\chi_d(p)}{p}\bigg|\ge\frac{1}{A\log y}\quad\text{for }\ll x\exp\left(-c\frac{\log x\log_3 x}{\log_2 x}\right)$$

fundamental discriminants $|d| \le x$. Choosing $k = [\log x \log_3 x/(333 \log_2 x)]$ and $l = [\log_3 x/(2\log 2)]$ in Lemma 8.1, and keeping in mind $y \ge A^2 \log x \log_3 x$, we deduce that

$$\sum_{|d|\le x}^{\flat} \Big| \sum_{y\le p\le z} \frac{\chi_d(p)}{p} \Big|^{2k} \ll x \left(\frac{1}{10A\log y}\right)^{2k} + x^{1-\frac{1}{\sqrt{\log_2 x}}}.$$

From this the first part of the proposition follows easily.

To prove the second part, as above it suffices to show that

$$\Big| \sum_{y_1\le p\le z} \frac{d/p}{p} \Big| \le \frac{4}{A\log_2 x}, \quad \text{for all but } \le x^{\frac{24}{25}} \text{ exceptions } |d| \le x\,.$$

Put $y_r{=}e^{r-1}y_1$, and $z_r{=}e^r y_1$ where $r{=}1,\dots,R$ where $R := [80\log_2 x - \log y_1]$, and set $y_{R+1} = e^R y$, and $z_{R+1} = z$. For each $1 \le r \le R+1$, we use the conditional part of Lemma 8.1 appropriately, choosing the exponent $k_r = [\log x/(50\log(Ae^{r/3+1}))]$. Keeping in mind that $y_r = A^3 e^r \log x \log_2 x$, and that $z_r \le \log^{80} x$, we deduce easily that

$$\Big| \sum_{y_r\le p\le z_r} \frac{d/p}{p} \Big| \le \frac{1}{e^{r/3}A\log_2 x}, \quad \text{for all but } \ll x^{\frac{19}{20}+o(1)} \text{ exceptions } |D| \le x\,.$$

Summing this over $1 \le r \le R+1$, we obtain the second part of the proposition.

Keep y_0 and y_1 as in Proposition 8.2. Then applying Theorem 6.1 we see that uniformly for $|z| \le \log x \log_3 x/(10^6 \log A)$

$$\frac{\pi^2}{6x} \sum_{|d|\le x}^{\flat} L(1,\chi_d;y_0)^z = \mathbb{E}\big(L(1,X;y_0)^z\big) + O\left(\frac{\mathbb{E}(L(1,X;y_0)^{\operatorname{Re} z})}{(\log x)^{10}} \right). \tag{8.1}$$

Thus upon using the argument of section 7 together with Corollary 3.3 we easily obtain that uniformly in $\tau \le \log_2 x + \log_4 x - \log_2 A - 20$ we have

$$\frac{\pi^2}{6x} \sum_{\substack{|d|\le x \\ L(1,\chi_d;y_0)\ge e^\gamma \tau}}^{\flat} 1 = \exp\left(-\frac{e^{\tau-C_1}}{\tau}\left(1 + O\left(\frac{1}{\tau} + \frac{e^\tau}{y_0}\right)\right)\right).$$

Combining this with Proposition 8.2 above, we get the unconditional part of Theorem 4 for the frequency of large values. The frequency of small values is obtained similarly. As for the part of Theorem 4 conditional on GRH we may use the above argument, applying Theorem 6.2 instead of Theorem 6.1. Then (8.1) holds, with y_0 replaced by y_1, in the wider range $|z| \le \log x \log_2 x/(10^6 \log A)$, and hence we get the appropriately stronger conclusion.

9 Extreme Values of $L(1,\chi_d)$ under GRH: Proof of Theorem 5a

PROPOSITION 9.1. *Suppose the GRH is true. Let $z \ge 2$ be a real number, and let $P(z) = \prod_{p\le z} p = e^{z+o(z)}$. Let $\epsilon_p = \pm 1$ for each prime $p \le z$, and let $\mathcal{P}(x, \{\epsilon_p\})$ denote the set of primes $q \le x$ such that $p/q = \epsilon_p$ for each $p \le z$. Then, for $z \ll x^{1/2}$,*

$$\sum_{q\in\mathcal{P}(x,\{\epsilon_p\})} \log q = \frac{x}{2^{\pi(z)}} + O\big(x^{\frac12}\log^2(xP(z))\big)\,, \tag{9.1}$$

and

$$\sum_{q\in\mathcal{P}(x,\{\epsilon_p\})} L\left(1,\left(\frac{\cdot}{q}\right)\right)\log q = \frac{x\zeta(2)}{2^{\pi(z)}}\prod_{p\le z}\left(1+\frac{\epsilon_p}{p}\right) + O\big(x^{\frac12}\log^3(xP(z))\big)\,. \tag{9.2}$$

Proof. We shall only demonstrate (9.2): (9.1) is similar and simpler. For $l|P(z)$ put $\epsilon_l = \prod_{p|l}\epsilon_p$. Note that for $q \le x$

$$\frac{1}{2^{\pi(z)}}\sum_{l|P(z)}\epsilon_l\left(\frac{l}{q}\right) = \begin{cases}1 & \text{if} \quad q\in\mathcal{P}(x,\{\epsilon_p\})\,,\\ 0 & \text{if otherwise}\,.\end{cases}$$

By partial summation we easily see that

$$L\left(1,\left(\tfrac{\cdot}{q}\right)\right) = \sum_{n\le N}\frac{(n/q)}{n} + O\left(\frac{q}{N}\right). \tag{9.3}$$

Using (9.3) with $N = x^2P(z)^2$, it follows that the LHS of (9.2) equals

$$\frac{1}{2^{\pi(z)}}\sum_{l|P(z)}\epsilon_l\sum_{n\le x^2P(z)^2}\frac{1}{n}\sum_{q\le x}\left(\frac{ln}{q}\right)\log q + O\big(1/P(z)^2\big)\,. \tag{9.4}$$

If $ln = \square$ then the inner sum over q above is $\psi(x)+O\big(\sqrt{x}+\sum_{p|ln}\log p\big) = x + O(x^{1/2}\log^2 x + \log(xP(z))) = x + O(x^{1/2}\log^2 x)$, since RH is assumed. Moreover we may write $n = lm^2$ since l is squarefree. Thus the contribution of these terms to (9.4) is

$$\frac{1}{2^{\pi(z)}}\sum_{l|P(z)}\epsilon_l\sum_{m\le xP(z)/\sqrt{l}}\frac{1}{lm^2}\big(x+O(x^{1/2}\log^2 x)\big)$$
$$= \frac{\zeta(2)}{2^{\pi(z)}}\prod_{p\le z}\left(1+\frac{\epsilon_p}{p}\right)\big(x+O(x^{\frac12}\log^2 x)\big)$$

This accounts for the main term in (9.2), with an acceptable error term.

If $ln \ne \square$ then $(ln/\cdot)$ is a non-principal character χ say, of conductor ln or $4ln$. Thus the inner sum over q in (9.4) is now

$\psi(x,\chi) + O(x^{1/2}) \ll x^{1/2}\log^2(xP(z))$, by GRH. It follows that the contribution of these terms to (9.4) is

$$\ll x^{\frac{1}{2}}\log^2(xP(z))\frac{1}{2^{\pi(z)}}\sum_{l|P(z)}\sum_{n\le x^2P(z)^2}\frac{1}{n} \ll x^{\frac{1}{2}}\log^3(xP(z))\,.$$

This proves the proposition.

We are now ready to prove Theorem 5a. Let z be such that $2^{\pi(z)} \le x^{\frac{1}{2}-3\epsilon}$. Then for any choice of $\epsilon_p = \pm 1$ for $p \le z$, we get by Proposition 9.1 that

$$\sum_{q\in\mathcal{P}(x,\{\epsilon_p\})}\log q = \frac{x}{2^{\pi(z)}}\big(1+O(x^{-2\epsilon})\big)\,,$$

and

$$\sum_{q\in\mathcal{P}(x,\{\epsilon_p\})} L\left(1,\left(\tfrac{\cdot}{q}\right)\right)\log q = \frac{x\zeta(2)}{2^{\pi(z)}}\prod_{p\le z}\left(1+\frac{\epsilon_p}{p}\right)\big(1+O(x^{-2\epsilon})\big)\,.$$

Since $L(1,(\cdot/q)) \le 2\log q$ for large q (take $N=q$ in (9.3)), we deduce easily from the above estimates that for $\delta > 0$ there are

$$\ge \frac{\delta}{2\log^2 x}\frac{x}{2^{\pi(z)}} + O\left(\frac{x^{1-\epsilon}}{2^{\pi(z)}}\right) \tag{9.5}$$

primes $q \in \mathcal{P}(x,\{\epsilon_p\})$ with

$$L\left(1,\left(\tfrac{\cdot}{q}\right)\right) \ge \zeta(2)\prod_{p\le z}\left(1+\tfrac{\epsilon_p}{p}\right) - \delta\,.$$

Similarly we deduce that the sum of $\log q$, over those primes $q \in \mathcal{P}(x,\{\epsilon_p\})$ with

$$L\left(1,\left(\tfrac{\cdot}{q}\right)\right) \le \zeta(2)\prod_{p\le z}\left(1+\tfrac{\epsilon_p}{p}\right)(1+\delta)\,,$$

is

$$\ge \frac{\delta}{(1+\delta)}\frac{x}{2^{\pi(z)}} + O\left(\frac{x^{1-\epsilon}}{2^{\pi(z)}}\right). \tag{9.6}$$

Take $z = \log x\log\log x/(2\log 2 + 10\epsilon)$, $\delta = \epsilon$, and $\epsilon_p = 1$ for all $p \le z$. By (9.5) that there are $\gg x^{1/2}$ primes $q \le x$ such that

$$\begin{aligned} L\left(1,\left(\tfrac{\cdot}{q}\right)\right) &\ge \zeta(2)\prod_{p\le z}\left(1+\tfrac{1}{p}\right) - \epsilon \\ &\ge e^{\gamma}\big(\log\log x + \log\log\log x - \log(2\log 2) - 100\epsilon\big)\,, \end{aligned}$$

by using the prime number theorem, which implies the first part of Theorem 5a. The second part follows from the analogous argument using (9.6) with the same z, and with $\delta = \epsilon/\log x$, and $\epsilon_p = -1$ for all $p \le z$.

10 Large Values of $L(1,\chi_d)$: Proof of Theorem 5b

Let x be large, and put $L = [100 \log_3 x]$, and $z = \log x \log_2 x/(10L)$.

Lemma 10.1. *There are at least $x^{1/3L}$ pairwise coprime integers of the form pq with p and q primes below $x^{1/2L}$ such that $\ell/pq = 1$ for all primes $\ell \le z$.*

Proof. For each prime $\log^2 x \le p \le x^{1/2L}$ let $\delta(p)$ denote the vector $((2/p),(3/p),\dots,(\ell_{\max}/p))$ where $\ell_{\max}$ denotes the largest prime below z. Note that the total number of possible vectors is $2^{\pi(z)}$. Also observe that if p and q have the same vector $\delta(p)=\delta(q)$ then $\ell/pq = 1$ for all primes $\ell \le z$.

Given a vector δ of ±1's let $N(\delta)$ denote the number of primes p with $\delta(p)=\delta$. By considering the products of these primes taken two at a time we get $\ge [N(\delta)/2]$ coprime integers of the form pq, satisfying $\ell/pq = 1$ for all $\ell \le z$. Thus the total number of such pq exceeds

$$\sum_{\delta}\big(N(\delta)/2-1\big) = \tfrac{1}{2}\big(\pi(x^{1/2L})-\pi(\log^2 x)\big) - 2^{\pi(z)} \ge x^{13L},$$

proving the lemma.

Lemma 10.2. *Let $d_1,\dots,d_L$ be any L numbers constructed in Lemma 10.1. Then there exists a square-free integer $d \le x$ which is the product of at least $L/3$ of these d_i's such that*

$$\left(\frac{\ell}{d}\right) = 1\,, \quad \text{for all primes } \ell \le z\,; \quad \text{and} \quad \sum_{z\le p\le \exp((\log x)^3)} \frac{p/d}{p} \ge -\frac{1}{\log_2 x}\,.$$

Proof. If d is a product of distinct d_i let $\nu(d)$ denote the number of d_i's involved in this product. Note that

$$\sum_{z\le p\le \exp((\log x)^3)} \frac{1}{p}\prod_{i=1}^{L}\left(1+\left(\frac{p}{d_i}\right)\right) \ge 0\,,$$

and so

$$\sum_{\substack{d\\ \nu(d)\ge L/3}}\ \sum_{z\le p\le \exp((\log x)^3)} \tfrac{1}{p}\big(\tfrac{p}{d}\big) \ge -\sum_{\substack{d\\ \nu(d)\le L/3}}\ \sum_{z\le p\le \exp((\log x)^3)} \tfrac{1}{p} \ge -10\log_2 x \sum_{\substack{d\\ \nu(d)\le L/3}} 1\,.$$

Now $\#\{d : \nu(d)\ge L/3\} \le \#\{d\} = 2^L$, and $\#\{d : \nu(d)\le L/3\} = \sum_{j\le L/3}\binom{L}{j} \le (1.9)^L$ using Stirling's formula. It follows that there exists d with $\nu(d)\ge L/3$ and

$$\sum_{z\le p\le \exp((\log x)^{100})} \frac{1}{p}\left(\frac{p}{d}\right) \ge -10(\log_2 x)(0.95)^L \ge -\frac{1}{\log_2 x}\,,$$

as desired.

LEMMA 10.3. *There are $\geq x^{2/19}$ integers $d \leq x$ as in Lemma 10.2.*

Proof. Counted with multiplicity the number of d's constructed in Lemma 10.2 is $\binom{[x^{1/3L}]}{L}$. The number d is counted $\binom{[x^{1/3L}]-\nu(d)}{L-\nu(d)} \geq \binom{[x^{1/3L}]-[L/3]}{L-[L/3]}$ times, since each such d has $\nu(d) \leq L/3$. So we are left with at least $\binom{[x^{1/3L}]}{L}/\binom{[x^{1/3L}]-[L/3]}{L-[L/3]}$ distinct d, which suffices.

Proof of Theorem 5b. If d is such that there is no Landau–Siegel character (mod d) then by a simple application of the prime number theorem for arithmetic progressions we see that

$$\log L\left(1, \left(\frac{\cdot}{d}\right)\right) = \sum_{2 \leq n \leq \exp(\log^3 x)} \frac{\Lambda(n)}{n \log n} \left(\frac{n}{d}\right) + O\left(\frac{1}{\log x}\right),$$

so that

$$L\left(1, \left(\frac{\cdot}{d}\right)\right) = \prod_{p \leq \exp((\log x)^3)} \left(1 - \frac{p/d}{p}\right)^{-1} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Use this for the d in Lemma 10.3 which are not Landau–Siegel moduli, and the result follows.

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