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THE FORMULATION OF SOME ELECTRONIC WARFARE PROBLEMS AS PROBLEMS IN MODERN CONTROL THEORY

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ABSTRACT

This memorandum considers several electronic warfare problems which can be treated using the techniques of modern control theory. It is shown that these electronic warfare problems, which deal with jamming techniques, systems organization, communications systems and direction-finding systems, can be formulated as discrete-time optimal control problems. In some cases the formulations allow solutions to be obtained immediately. The formulation itself yields new insight in other instances where solutions have not yet been attempted.

1. INTRODUCTION

Many electronic warfare problems can be treated using the techniques of modern control theory. This memorandum illustrates several such problems which can be formulated as discrete-time optimal control problems.[1] The use of the discrete-time formulation is not a serious restriction on the class of physical problems which can be considered, as will be shown in Section 2; the reasons for using the discrete-time formulation are also presented there.

The systems considered are assumed to have the form illustrated by the block diagram in Fig. 1. Because of the presence of noise in the plant, the state[2] of the plant is random at each of the sampling instants. The measurement device is used to feed back to the controller some known function (called an observation) of the state of the plant. This observation is also subject to random noise.

The controller operates in the following way: It is given in advance a sequence of desired plant states for a finite time interval. The plant input is then determined by the controller at each instant of time in such a way that the state of the plant is in some sense "close" to the desired state. The mathematical definition of "close" will be given in Section 2 in terms of a quadratic form in the plant error. The effect of the noise on the plant state measurement is to reduce the amount of information concerning the plant state which is fed back to the controller. However, the information that is fed back is used to minimize the cost of controlling the plant. Using this information the controller chooses that control which makes the plant state at the next instant of time "close" to the desired state. Of course,

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[1]A discrete-time optimal control problem, as intended here, is an optimal control problem in which the plant is described at discrete instants of time by a difference equation in the plant state, the plant input, and the plant noise.

[2]See Tou, Ref. 2, for a discussion of state variables and the formulation of problems in terms of state variables.

since random processes are involved, "close" will be defined with the aid of some averaging operation.

The feedback of the observations to the controller gives rise to closed loop operation of the system. The value of this closed loop operation lies in the possibility of better control of the plant in the sense that the actual outputs can usually be made to approach more closely the desired outputs than if the system were to operate open loop.

The mathematical formulation of the optimal control problem is given in the next section. Section 3 considers several jamming problems of interest in electronic warfare, such as the jamming of a tracking radar system which has a conical-scan receiving antenna and a transmitting antenna which does not scan. Section 4 considers a problem in systems organization. This problem deals with the efficient operation of a large computing system which services programs having different priority levels. The three problems considered in Section 5 deal with communications systems in which some of the received information is fed back to the transmitter. In the third of these, an interesting approach to noisy communications system operation is presented. The problem treated in Section 6 is concerned with the design of direction-finding systems.

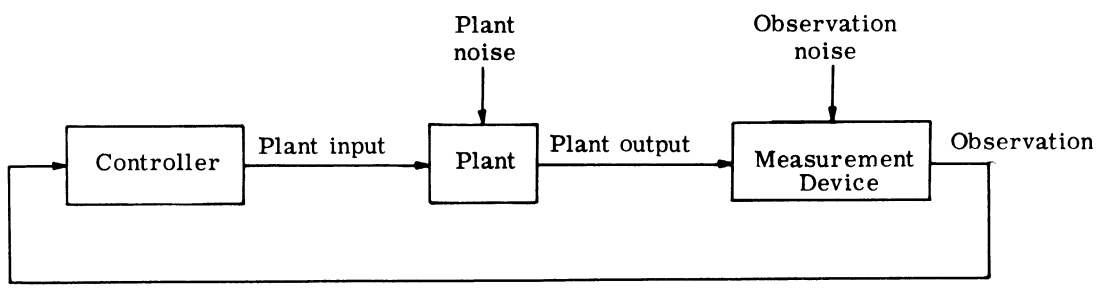

Fig. 1 Block diagram of the general system.

2. THE MATHEMATICAL FORMULATION OF THE OPTIMAL CONTROL PROBLEM

It will be helpful to refer to the block diagram given in Fig. 2 in connection with the following description of the optimal control problem: the dynamic operation of the plant is assumed to be governed by the difference equation

$$x(k+1) = f(k, x(k), u(k), w(k)) \quad , \quad k=0,\ldots,N-1 \tag{1}$$

where

k is an integer index for N equally spaced instants of time (dependence on time is denoted using k rather than t_k),

the plant is to be controlled in the time interval $[0, N-1]$,

x is an (n x 1) vector representing the state of the plant,[3]

u is an (r x 1) input or control vector,

w is an (n x 1) random disturbance or noise vector,[4]

f is an (n x 1) vector function called the state transition function.

The initial state x(0) is in general considered to be a random vector with a known probability distribution. The probability distribution of the plant noise w(k) is also known for each k in the interval $[0, N-1]$. We assume here that the plant noise vectors at different sampling instants are independent.[5] Thus the joint distribution of the plant noise sequence

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[3]All vectors and matrices considered in this memorandum are assumed to have components which are real numbers unless otherwise stated.

[4]A random vector is here defined as one whose components are random variables.

[5]A method for extending this problem to include dependent noise will be mentioned in Section 5.

$w(0), \ldots, w(N-1)$ equals the product of the individual distributions.[6]

The plant is to be controlled by selecting inputs during the interval $[0, N-1]$ in such a way that the state $x(k)$ is (in a sense to be given explicitly below) "close" to some known desired state $d(k)$, for $k=1, \ldots, N$. The sequence of desired states $d(1), \ldots, d(N)$ is assumed to be known at time $k = 0$. From Eq. 1 it is clear that for $u(k)$ to be chosen so that $x(k + 1)$ is "close" to $d(k + 1)$, it is desirable that $x(k)$ be known. In the case of open-loop control, $x(k)$ is random due to the random noise $w(k - 1)$ and/or the random initial state $x(0)$. It is possible to estimate $x(k)$ using the given distribution of the plant noise and the given distribution of the initial state.

Closed-loop control is often preferable for physical systems. Ideally, the plant state itself should be fed back to the controller. Unfortunately, it often happens that the plant state cannot be measured directly. In such cases some function of the plant state (called an observation), which may also include measurement noise, must be used instead. Nonetheless, closed-loop operation using these observations is usually superior to open-loop control.

We assume for the optimal control problem being considered here that observations are made at each discrete-time instant during the control interval, $[0, N-1]$. The observations are given by

$$y(k) = g(k, x(k), v(k)) \quad , \quad k=0, \ldots, N-1 \tag{2}$$

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[6] It should be remarked at this point that the discrete-time formulation of this optimal control problem does not restrict the class of physical systems which can be treated. The analogous continuous-time formulation considers plants described by differential equations of the form

$$\frac{dx}{dt} = F[t, x(t), u(t), w(t)] \ , \tag{1a}$$

where F satisfies certain conditions which guarantee the existence and uniqueness of solutions (Refs. 1, 2). The discrete-time representation of Eq. 1a can be shown to have the form of Eq. 1 (Ref. 1). The discrete-time formulation in Eq. 1 can be shown to approach Eq. 1a for many physical systems by making the discrete time intervals (i. e., the sampling intervals) sufficiently small.

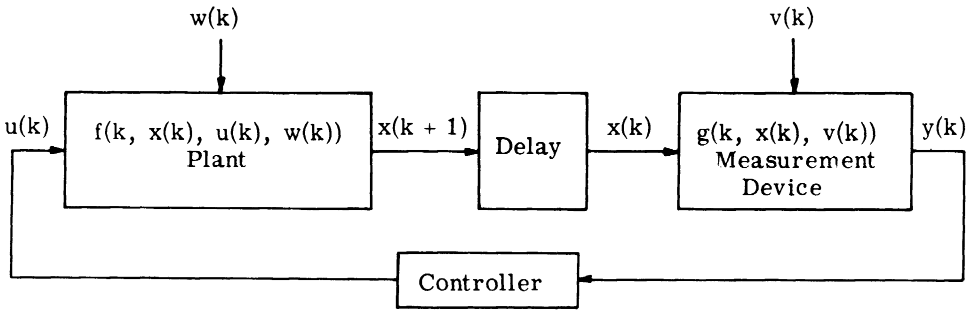

Fig. 2 Block diagram of discrete-time control system.

where

y is a (p x 1) observation vector,

v is a (p x 1) measurement random noise vector, and

g is a (p x 1) vector function describing the observation.

The probability distribution of v(k), k = 0, . . . , N-1, is given. We assume that the observations can be stored by the controller for use at a later time.[7] We assume that the measurement noise vectors at different sampling instants are independent; thus the joint distribution of v(0), . . . , v(N-1) is simply the product of the individual distributions.

In order to complete the formulation of the optimal control problem it is necessary to define what is meant by the requirement that x(1), . . . , x(N) be "close" to d(1), . . . , d(N). Note first that the observations y(0), . . . , y(k) are available to the controller at time k. Thus the input at the k^{th} instant can be chosen conditional to the observations y(0), . . . , y(k) for the interval [0, N-1]. The optimal input sequence is then defined to be that sequence of controls u(k) (for k = 0, . . . , N - 1) chosen conditional to the observations y(0), . . . , y(k) which minimizes the expected value of the cost function

$$J_N[x(0)] = \sum_{k=0}^{N-1} \left[\|x(k+1) - d(k+1)\|^2_{Q_1(k+1)} + \|u(k)\|^2_{Q_2(k)} \right] . \tag{3}$$

[7] It should be noted that taking observations at discrete instants of time rather than continuously is not a serious limitation. If a digital computer is to be used to process the observations only a finite number of observations can be used anyway. This will be the case in several of the problems to be treated in this memorandum. The case of discrete observations can in many cases be made to approach the continuous case by making the discrete time intervals sufficiently small.

The quantity $\|z\|_Q^2$ is defined as the quadratic form $\|z\|_Q^2 = z'Qz$, where z is any (m x 1) vector and Q is any (m x m) matrix.[8] The matrices Q_1 and Q_2 are assumed to be positive semi-definite at each sampling instant. $J_N[x(0)]$ is then non-negative and has zero as a lower bound for its expected value.

The matrix Q_1 is a weighting function for the plant error. For example, the frequently used quadratic error function is obtained by letting $Q_1(k) = I$, $k=1,\ldots,N$, where I is the identity matrix. The more general plant-error term is used in the present cost function to allow for cases in which more emphasis is placed on some components of the state vector than on others. For example, rocket position is of primary importance in an intercept problem; here Q_1 should weight the position coordinates more heavily than the other components of the state vector.

The term $\|u(k)\|_{Q_2(k)}^2$ is included in the cost function for two reasons:

1) For many physical systems the actual energy input to the plant during the k^{th} discrete-time interval can be expressed as a quadratic form in u(k). For example, if u(k) is a scalar representing the voltage applied to a resistive network, then $[u(k)]^2$ is proportional to the energy input to the network during the k^{th} discrete-time interval.

2) It is true that in some cases the energy input to the plant cannot be described by a quadratic form. However, it is also true that the inputs to any physical system must be bounded in magnitude. If the weighting matrices $Q_2(k)$ are positive definite for $k=0,\ldots,N-1$ they will have the effect of bounding the inputs. To see this, one need only note that the optimal input sequence is to be chosen to minimize the expected value of a sum of nonnegative terms. Minimizing this cost function (which includes the inputs) will tend to make the inputs small and therefore bounded.

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[8]The notation z' represents the transpose of z.

The expectation of $J_N[x(0)]$ is to be taken with respect to the given distributions of the plant noise, the observation noise, and the initial state. The cost function is written $J_N[x(0)]$ to emphasize its dependence on the initial state of the plant and on the length of the control interval $[0, N-1]$.

The above formulation describes a fairly broad class of optimal control problems. The general solution to this class of non-linear problems has not been obtained. However, special cases can be solved. Moreover, when the plant is linear and the disturbances are gaussian, a very convenient solution is known. It is therefore worthwhile to give the mathematical formulation of the optimal control problem for this special case. The notation will be made to correspond as closely as possible to that used above for the general problem.

The operation of the plant is described by the linear difference equation

$$x(k+1) = \Phi(k)\, x(k) + \Delta(k)\, u(k) + w(k) , \qquad (4)$$

where

- x is an $(n \times 1)$ plant state vector,
- u is an $(r \times 1)$ input or control vector,
- w is an $(n \times 1)$ gaussian random vector,
- Φ is an $(n \times n)$ plant state transition matrix, and
- Δ is an $(n \times r)$ plant input distribution matrix of maximum rank.

It is assumed that the observations $y(k)$ take the form

$$y(k) = M(k)\, x(k) + v(k) , \qquad (5)$$

where

- y is a $(p \times 1)$ observation vector,
- M is a $(p \times n)$ plant state observation matrix, and
- v is a $(p \times 1)$ gaussian random measurement noise vector.

The gaussian random vectors $w(k)$ and $v(k)$ are assumed to have known mean values $\bar{w}(k)$ and $\bar{v}(k)$ and known covariance matrices $W(k)$ and $V(k)$, respectively. We assume that the plant noise (and the measurement noise) at different sampling instants are independent. The initial state x(0) is assumed to be a gaussian random vector with known mean $\bar{x}(0)$ and covariance matrix X. The cost function is taken to be the quadratic type given in Eq. 3.

As stated, this particular case of the optimal control problem has been solved in detail. If d(k) = 0 for k = 1,...,N, Tou (Ref. 3) and Gunckel and Franklin (Ref. 4) have shown that the optimal input sequence u(k), k = 0,...,N - 1 is given by

$$u(k) = -C(k)\, x(k \mid k) \quad , \tag{6}$$

where $x(k \mid k)$ is the expected value of x(k) conditional to the observations y(0),..., y(k). The feedback matrix C(k) is uniquely determined by the matrices, Φ, Δ, Q_1, Q_2, M, W, V, and X;[9] it does not depend on the observations of the plant state. Thus the C(k) can be computed in advance. Then, using Kalman's filtering technique to compute $x(k \mid k)$, the optimal control can be computed (Refs. 3, 5).

Admittedly, this linear problem is a rather special case. However, it does include a fairly broad class of electronic warfare problems, as we shall show below. The first electronic warfare problem to be considered in Section 3 is a nonlinear problem. The other problems can be treated in terms of the linear problem introduced in the previous paragraph.

[9] For uniqueness it is necessary that either Q_1 or Q_2 be positive definite at each sampling instant.

3. JAMMING PROBLEMS

3.1 The Problem of Jamming a Radar System Having a Non-scanning Transmitting Antenna

An important problem in modern electronic warfare is the determination of techniques for jamming various types of radar detection systems. There are many different types of radar systems presently being used for military purposes. The particular radar system to be considered here is a tracking radar with a conical-scan receiving antenna and a transmitting antenna which does not scan.

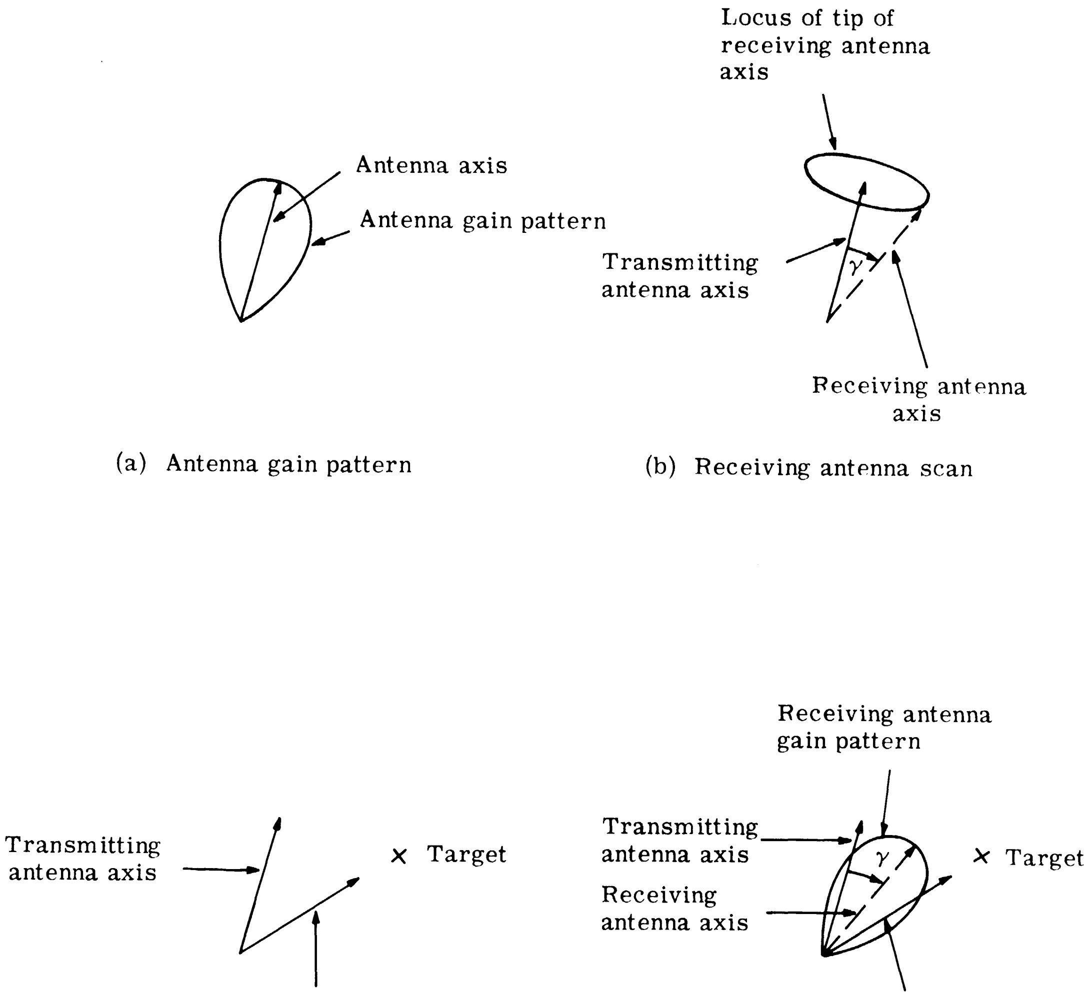
Conventional jamming techniques rely on the jammer's knowing the scanning frequency and phase of the receiving antenna. If the transmitting and receiving antennas of a radar system are scanning at the same frequency, then the illumination intensity of the fixed target will be periodic at this frequency. In this case the necessary frequency and phase information is available to the jammer. In the radar system considered here the transmitting antenna does not scan. Thus, the receiving antenna scanning frequency and phase are not readily available to the jammer.[10]

It is interesting to consider this radar system in detail to determine possible jamming procedures. This problem is most conveniently formulated as a continuous-time problem; the transition to the discrete-time formulation can be made if desired. We do not claim to have a complete solution to the problem at this time. However, the mere formulation as an optimal control problem yields important insight into possible methods of solution, as we shall show below.

We now describe the operation of the radar system. The antennas for the receiver and the transmitter are assumed to have the voltage gain pattern indicated in Fig. 3(a); the gain pattern is assumed to be symmetric about the antenna axis. The receiver

[10]It should be noted that this is also true if the transmitting antenna scans but at a different frequency than the receiving antenna. Our problem is really a special case of this in which the transmitter does not scan at all. We assume that the jammer is located at the target.

antenna scans conically about the transmitter axis as indicated in Fig. 3(b). The squint angle, γ, is assumed to be constant. The target axis is defined by the line segment joining the target to the radar antennas as indicated in Fig. 3(c). The receiving and transmitting antennas are assumed to be at the same location.



(a) Antenna gain pattern

(b) Receiving antenna scan

(c) The target axis

(d) Receiving antenna scan causes modulation appearing on the received signal

Fig. 3. Radar operation notation

The transmitting antenna is positioned by an error-signal-driven servo system. The error signal for the servo is derived from the modulation on the radar signal reflected from the target. This modulation is the result of the conical scan of the receiving antenna, as can be seen from Fig. 3(d). It is also clear from this figure, and from the symmetry of the antenna gain pattern, that the modulation will go to zero as the transmitter axis approaches the target axis. Thus, when these two axes coincide, there will be no error signal and the transmitter antenna will be locked onto the target.

Before proceeding to the mathematical description of the operation of the radar system, it is desirable to state several assumptions which will be made:

1) The transmitter operates at a constant power level,
2) The radar cross-section of any given target is constant as a function of viewing angle and time,
3) The relative velocity between the radar transmitting antenna and the target is small enough that the distance between the two can be taken to be constant during the time interval considered. We denote this fixed distance by r.

These three assumptions allow us to conclude that the level of the signal refelcted from the target is constant when the transmitting antenna is not moving.[12] The actual transmitter power level, radar cross-section of the target, and radar-to-target distance will be parameters entering into the problem, as discussed below. These assumptions are quite plausible for those cases in which the radar encounters a given target for a relatively short period of time. We now proceed to the mathematical formulation of this problem. The description of the operation of the unjammed radar system will be somewhat lengthy. This description is necessary in order to make clear exactly how the radar system operates in order that the effects of the jamming signal are immediately apparent. The reader should keep in mind that our primary interest here is in developing techniques for jamming such a radar system.

Referring to Fig. 4(a) we let the (u, v, w) coordinates denote the usual

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[12]That is, when the time constant of the transmitting antenna positioning servo is much larger than the receiving antenna scanning period, the level of the reflected signal can be considered constant during several scanning periods of the receiving antenna.

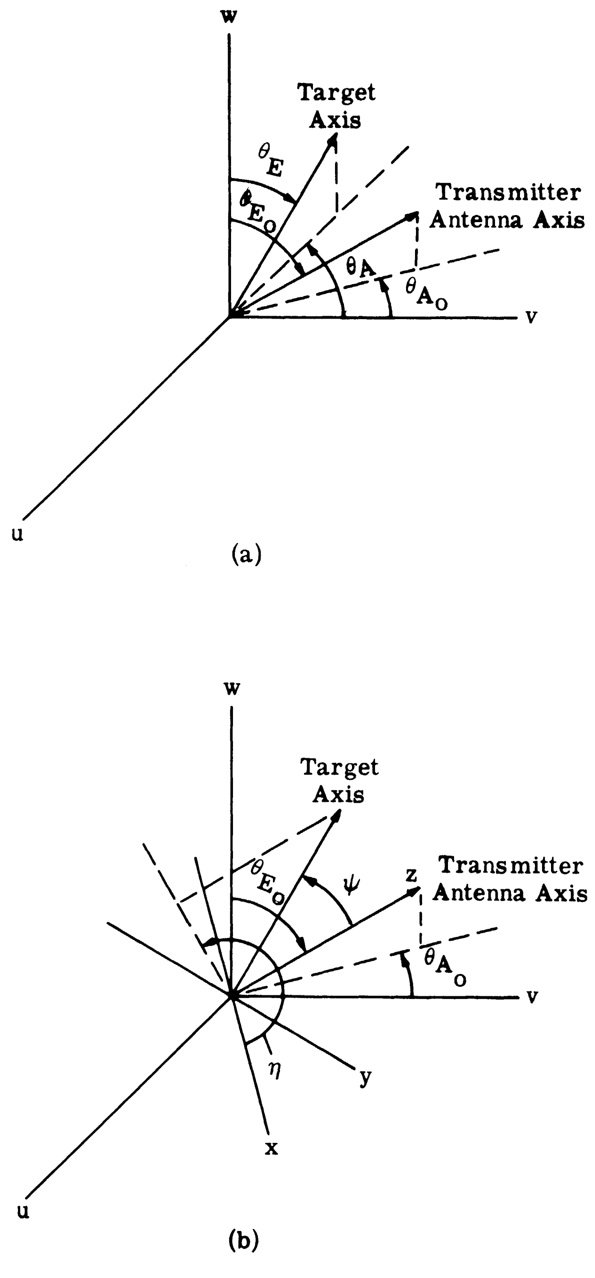

Fig. 4. Coordinate systems used to describe the operation of the radar system.

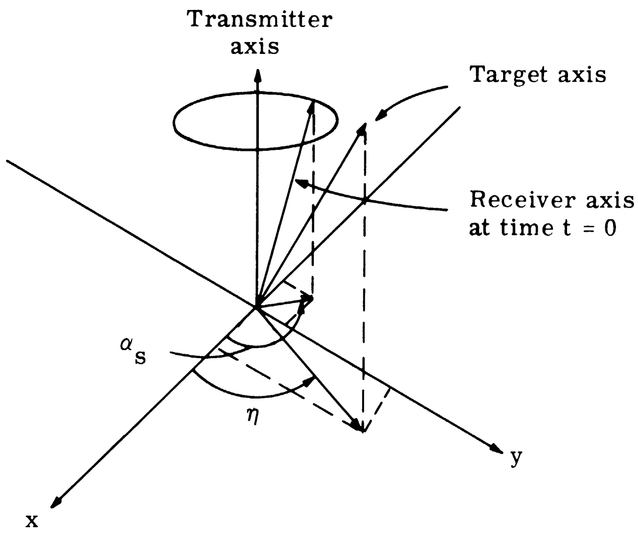

Fig. 4(c) Phase reference for receiving antenna scan.

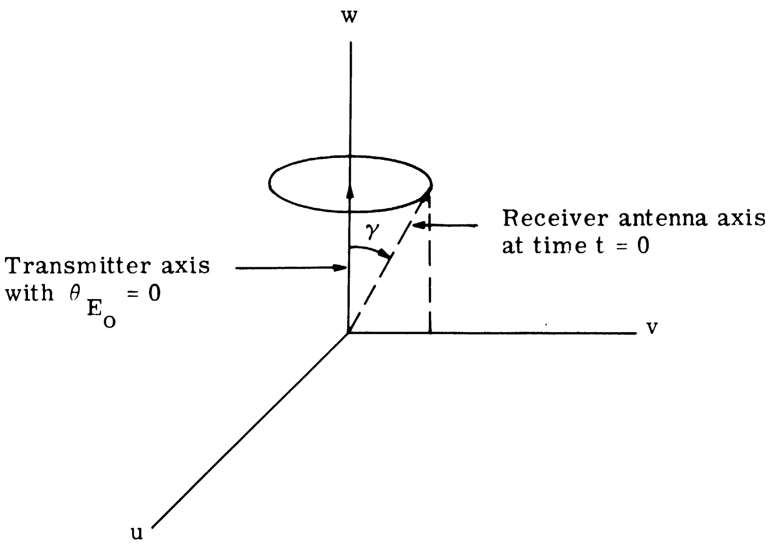

Fig. 4(d). Orientation of uv plane.

3-dimensional Cartesian coordinate system with its origin located at the transmitting antenna.[13] The directions of the transmitter antenna axis and the target axis can be described by unit vectors. We let the azimuth angle θ_{A_o} represent the angle which the projection of the transmitter antenna axis into the uv-plane makes with v-axis. The positive direction for azimuth angle is that indicated in Fig. 4(a). We let the elevation angle θ_{E_o} denote the angle between the w-axis and the transmitter antenna axis. The azimuth and elevation angles θ_A and θ_E for the target are similarly defined as indicated Fig. 4(a).

We let S_o and S denote the unit vectors in the direction of the transmitter axis and the target axis, respectively. Then, using the notation of Fig. 4(a), we have

$$[S_o]_{uvw} = \begin{bmatrix} -\sin\theta_{E_o} \sin\theta_{A_o} \\ \sin\theta_{E_o} \cos\theta_{A_o} \\ \cos\theta_{E_o} \end{bmatrix} \qquad [S]_{uvw} = \begin{bmatrix} -\sin\theta_E \sin\theta_A \\ \sin\theta_E \cos\theta_A \\ \cos\theta_E \end{bmatrix}, \tag{7}$$

where the uvw subscripts are used to emphasize that these vectors represent the transmitter axis and the target axis directions in the uvw-coordinate system.

Since the target is fixed, the angles θ_A and θ_E will be constant. The transmitter antenna positioning servo is designed to vary the elevation angle θ_{E_o} and the azimuth angle θ_{A_o} in such a way that the transmitter axis aligns with the target axis. A detailed description of the operation of the transmitter antenna positioning servo will be given below. However, before giving this description it is desirable to introduce a second coordinate system whose orientation will change as the transmitter axis changes. This second coordinate system will provide more insight into the effects of jamming on the operation of the radar system.

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[13] The (u, v, w) coordinate system is assumed to remain fixed throughout this discussion. The orientation of the (u, v, w) coordinate system is assumed to be such that the uv-plane coincides with the plane of the antenna platform. Let the phase reference axis for the receiver scan be the direction of the projection of the receiving antenna onto the uv-plane at time zero if the transmitting antenna is aligned with the positive w-axis. The uv-plane is oriented by taking the positive v-axis along the phase reference axis of the receiver scan. (See Fig. 4(d)).

Let us consider the (x, y, z) coordinate system with its origin coinciding with the origin of the uvw coordinate system as indicated in Fig. 4(b). The z-axis coincides with the transmitter axis and the xy-plane is normal to the transmitter axis with the x-axis lying in the uv-plane.[14] The projection of the target axis onto the xy-plane will make an azimuth angle η with the positive x-axis. (The direction of positive azimuth is that indicated in Fig. 4(b).) The elevation angle in this (x, y, z) coordinate system is given by ψ. Note that although the target is assumed to be stationary in the fixed (u, v, w) coordinate system its position in the (x, y, z) coordinate system will change as the transmitter axis moves. In particular, the target axis will coincide with the transmitter axis when ψ equals zero. In terms of the elevation and azimuth angles ψ and η the direction of the target is given by the unit vector

$$[S]_{xyz} = \begin{bmatrix} \sin\psi \quad \cos\eta \\ \sin\psi \quad \sin\eta \\ \cos\psi \end{bmatrix} , \tag{8}$$

where the xyz subscript again is used to emphasize that the xyz coordinates are used as a basis.

It is now desirable to obtain a relation between variations in θ_{A_o} and θ_{E_o} and the variations in η and ψ. The angles θ_{A_o} and θ_{E_o} represent the azimuth and elevation angles of the transmitter axis in the fixed (u, v, w) coordinate system. These angles are known by and controlled by the transmitting positioning servo. The angles η and ψ are not known by the transmitter servo. However, we show that the angles η and ψ vary with changes in θ_{A_o} and θ_{E_o} in such a way that ψ goes to zero. Moreover, we can show that the motion of the unit vector in the transmitter axis direction takes place along a great circle of the unit sphere centered at the origin. The relations between changes in θ_{A_o} and θ_{E_o} and changes in ψ and η can be derived in the following way:

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[14]The x and y axis are assumed to be normal to each other so that the (x, y, z) coordinate system is uniquely determined by the orientation of the transmitter axis. The (x, y, z) coordinate system is assumed to be a right-handed coordinate system.

Let us refer again to Fig. 4(a). If we rotate the uv-plane through an angle θ_{A_o} about the w-axis and then rotate the wv'-plane though an angle θ_{E_o} about the u'-axis (where the prime indicates the axis after it has been rotated through θ_{A_o} radians) then the uvw coordinates of any point will be transformed into the xyz coordinates of the same point. We let $R_{\theta_{A_o}}$ denote the rotation of the uv-plane through θ_{A_o} radians and let $R_{\theta_{E_o}}$ denote the rotation of the wv'-plane through θ_{E_o} radians. Then we have the following equation involving the direction of the target:

$$[S]_{xyz} = R_{\theta_{E_o}} R_{\theta_{A_o}} [S]_{uvw} \tag{9}$$

The two rotations can be represented in matrix form in the following way:

$$R_{\theta_{A_o}} = \begin{bmatrix} \cos\theta_{A_o} & \sin\theta_{A_o} & 0 \\ -\sin\theta_{A_o} & \cos\theta_{A_o} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{\theta_{E_o}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{E_o} & -\sin\theta_{E_o} \\ 0 & \sin\theta_{E_o} & \cos\theta_{E_o} \end{bmatrix} \tag{10}$$

If we substitute Eqs. (7), (8), and (10) into Eq. (9) and carry out the indicated multiplications, we get

$$\sin\psi\cos\eta = -\cos\theta_{A_o}\sin\theta_E\sin\theta_A + \sin\theta_{A_o}\sin\theta_E\cos\theta_A , \tag{11a}$$

$$\sin\psi\sin\eta = \sin\theta_{A_o}\cos\theta_{E_o}\sin\theta_E\sin\theta_A + \cos\theta_{A_o}\cos\theta_{E_o}\sin\theta_E\cos\theta_A - \sin\theta_{E_o}\cos\theta_E , \tag{11b}$$

$$\cos\psi = \sin\theta_{A_o}\sin\theta_{E_o}\sin\theta_E\sin\theta_A + \cos\theta_{A_o}\sin\theta_{E_o}\sin\theta_E\cos\theta_A + \cos\theta_{E_o}\cos\theta_E . \tag{11c}$$

We now differentiate Eq. 11c with respect to time to get

$$-\sin\psi \frac{d\psi}{dt} = [\cos\theta_{A_o} \sin\theta_{E_o} \sin\theta_E \sin\theta_A - \sin\theta_{A_o} \sin\theta_{E_o} \sin\theta_E \cos\theta_A] \frac{d\theta_{A_o}}{dt}$$

$$+ [\sin\theta_{A_o} \cos\theta_{E_o} \sin\theta_E \sin\theta_A$$

$$+ \cos\theta_{A_o} \cos\theta_{E_o} \sin\theta_E \cos\theta_A - \sin\theta_{E_o} \cos\theta_E] \frac{d\theta_{E_o}}{dt} .$$

If we substitute Eqs. 11a and 11b and simplify, we have

$$\frac{d\psi}{dt} = \cos\eta \sin\theta_{E_o} \frac{d\theta_{A_o}}{dt} - \sin\eta \frac{d\theta_{E_o}}{dt} . \qquad (12)$$

Similarly, we can obtain the equation

$$\frac{d\eta}{dt} = -[\cot\psi \sin\eta \sin\theta_{E_o} + \cos\theta_{E_o}] \frac{d\theta_{A_o}}{dt} - \cot\psi \cos\eta \frac{d\theta_{E_o}}{dt} \qquad (13)$$

by differentiating Eq. 11b with respect to time and substituting Eqs. 11a, 11c, and 12.

Note that Eqs. 12 and 13 give relations for $\frac{d\psi}{dt}$ and $\frac{d\eta}{dt}$ in terms of $\frac{d\theta_{A_o}}{dt}$ and $\frac{d\theta_{E_o}}{dt}$. The transmitting antenna positioning servo controls θ_{A_o} and θ_{E_o} in the following way. The radar signal reflected from the target will be received by the receiving antenna as it scans. We denote the received waveform after the RF has been removed as the video signal. The video signal will be periodic with frequency ω_s, where ω_s is the scanning frequency of the receiving antenna. The phase of the receiving antenna scan is given by α_s where α_s is taken to be the angle determined by the projection of the receiving antenna onto the xy-plane at time zero. (See Fig. 4(c)). The video signal is given by[15]

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[15]Recall that at this point of the discussion we are assuming that there is no jamming signal present.

$$e_o(t) = K_1(\psi, r) + K_2(\psi, r) \cos(\omega_s t + \alpha_s - \eta) , \tag{14}$$

where $K_1(\psi, r)$ is the average value of e_o and $K_2(\psi, r)$ is the coefficient of the modulation due to the receiving antenna scan. K_1 and K_2 depend on r (distance between radar antennas and target) because the transmitter power level is constant. Note that K_1 and K_2 also depend on ψ. K_2 approaches zero as ψ goes to zero since the modulation on the video signal goes to zero with ψ; K_1 will increase as ψ decreases since the average value of e_o increases as ψ decreases. The functions K_1 and K_2 also depend on such parameters as 1) the squint angle γ, 2) the transmitter power level, 3) the radar cross-section of the target, and 4) the voltage gain pattern of the antennas.

If we now pass $e_o(t)$ through a quadrature-axis phase detector we get two output signals of the form

$$\frac{1}{2} K_2(\psi, r) \cos \eta = \langle e_o(t) , \cos (\omega_s t + \alpha_s)\rangle ,$$

$$\frac{1}{2} K_2(\psi, r) \sin \eta = \langle e_o(t) , \sin (\omega_s t + \alpha_s)\rangle ,$$

where the triangular brackets represent the mixing and time averaging operation of the phase detector. The reference signals for the phase detector, $\cos (\omega_s t + \alpha_s)$ and $\sin (\omega_s t + \alpha_s)$, can be derived from the receiver antenna scan. The phase detector outputs are then fed into the transmitter antenna positioning servo. It is assumed that the positioning servo varies θ_{A_o} and θ_{E_o} in the following way

$$\frac{d\theta_{A_o}}{dt} = - \frac{1}{2} K_2(\psi, r) \cos \eta / \sin \theta_{E_o} = -\langle e_o(t), \cos(\omega_s t + \alpha_s)\rangle / \sin \theta_{E_o} \tag{15}$$

$$\frac{d\theta_{E_o}}{dt} = \frac{1}{2} K_2(\psi, r) \sin \eta = \langle e_o(t), \sin(\omega_s t + \alpha_s)\rangle \tag{16}$$

The factor $\sin\theta_{E_o}$ in the denominator of Eq. 15 is due to the fact that, as θ_{E_o} becomes small, a very small change in the direction of the transmitter axis results in a very large change in θ_{A_o}. Note that the positioning servo holds θ_{A_o} and θ_{E_o} constant when $\psi = 0$, since $K_2(0, r) = 0$. If we substitute these relations for $\frac{d\theta_{A_o}}{dt}$ and $\frac{d\theta_{E_o}}{dt}$ into Eqs. 12 and 13, we have

$$\frac{d\psi}{dt} = -\frac{1}{2} K_2(\psi, r) \quad , \tag{17}$$

$$\frac{d\eta}{dt} = \frac{1}{2} K_2(\psi, r) \cos\eta \cot\theta_{E_o} \quad . \tag{18}$$

Equations 17 and 18 describe the motion of the target in the (x, y, z) coordinate system.[16] Note that the transmitter axis is moved into coincidence with the target axis (ψ approaches zero).

We have said nothing thus far concerning the effects of jamming signals on the operation of the radar system. The above discussion gives a detailed mathematical description of the operation of the transmitting-antenna positioning servo when no jamming signal is present. The effects of a jamming signal on the operation of the transmitter positioning servo can now be clearly shown. The video signal will now have the form

$$e(t) = e_o(t) + e_1(t) \quad ,$$

where $e_o(t)$ is given by Eq. 14 and $e_1(t)$ is the component of the video signal due to the jamming signal.[17] Note that $e_1(t)$ will differ from the modulation $e_j(t)$ on the signal transmitted by the jammer due to the modulation caused by the receiver antenna scan. The relation

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[16]Recall that this coordinate system rotates as the transmitter axis is rotated as described above. This results in the somewhat complicated expression for Eq. 18.

[17]The RF of the jamming signal is assumed to be of the same frequency as that of the radar transmitter. Moreover, we assume that the jammer is located at the target.

between the two is given by

$$e_1(t) = K_3(r)[K_1(\psi, r) + K_2(\psi, r) \cos(\omega_s t + \alpha_s - \eta)] \, e_j(t) ,$$

where $K_3(r)$ depends on the power level of the jammer.[18]

Instead of having $e_o(t)$ as its input (as when no jamming signal was present) the phase detector now has $e_o(t) + e_1(t)$ fed into it. The transmitter servo will still vary θ_{A_o} and θ_{E_o} according to Eqs. 15 and 16, but with $e_o(t)$ replaced by $e_o(t) + e_1(t)$. Thus the jamming signal $e_1(t)$ causes Eqs. 17 and 18 to take the form

$$\frac{d\psi}{dt} = -\frac{1}{2} K_2(\psi, r) - \cos \eta \langle e_1(t), \cos(\omega_s t + \alpha_s)\rangle - \sin \eta \langle e_1(t), \sin(\omega_s t + \alpha_s)\rangle , \tag{17a}$$

$$\frac{d\eta}{dt} = \frac{1}{2} K_2(\psi, r) \cos \eta \cot \theta_{E_o} + [\cot \psi \sin \eta + \cot \theta_{E_o}] \langle e_1(t), \cos(\omega_s t + \alpha_s)\rangle$$
$$- \cot \psi \cos \eta \langle e_1(t), \sin(\omega_s t + \alpha_s)\rangle . \tag{18a}$$

In order to illustrate more clearly the effect of a jamming signal on the motion of the transmitter axis, we consider a specific example.[19] Let the jamming signal be an RF signal modulated by a sinusoid of the form $e_j(t) = \sin(\omega t + \alpha)$. The component of the video signal due to jamming will then be given by

$$e_1(t) = K_3(r)[K_1(\psi, r) + K_2(\psi, r) \cos(\omega_s t + \alpha_s - \eta)] \sin(\omega t + \alpha) .$$

By substituting this expression for $e_1(t)$ in Eqs. 17a and 18a it can be shown that the jamming signal will cause the transmitting antenna of the radar system to "wobble" at a frequency

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[18] It should be kept in mind that the jamming signal travels only a distance r while the signal from the radar system itself must travel twice as far.

[19] It should be kept in mind that the jammer does not know the receiving antenna scanning frequency ω_s or phase α_s in advance although he will usually know the frequency range in which ω_s lies.

$\omega - \omega_s$ if this difference frequency is sufficiently small to pass through the radar's video filter and through the phase detector. This wobble will give rise to varying illumination of the target. Thus, by observing the transmitter antenna wobble at the difference frequency $\omega - \omega_s$, the jammer can learn something about ω_s.

The operation of the radar jamming system has thus become closed-loop with the path from radar to jammer being the wobbling target illumination and the path from jammer to radar being the jamming signal. The effects of noise on the observations (as well as on the operation of the radar system) have been neglected until now. If the presence of noise is taken into account the above radar-jammer system becomes a closed-loop system subject to random disturbances. The optimal control of such a system is a classic problem in modern control theory. That jamming signal must be selected which on a statistical basis "controls" the tracking radar system in an optimal manner. It is important that this control be optimal, for only limited feedback (e.g. small wobbles) and limited time are available to the jammer.

It should be emphasized that it has not been shown that the overall optimal jamming signal will necessarily have the sinusoidal modulation considered here. The sinusoidal form does, however, allow a clear description of the effect of a jamming signal on the operation of the radar system. The difficulties arising in this non-linear problem prevent us from giving a complete solution at this time. We now consider an analogous jamming problem for a linear system.

3.2 A Parametric Amplifier Jamming Problem

It is possible to use parametric amplifiers as triggering devices in various electronic systems. One possible application is as a trigger for an electronic fuze of a warhead. We consider here the problem of jamming the parametric amplifier when it is used as a triggering device.

The parametric amplifier is indicated schematically in Fig. 5. The operation of the parametric amplifier is described by the differential equation

$$a_n(t)\frac{d^{(n)}z}{dt^n} + a_{n-1}(t)\frac{d^{(n-1)}z}{dt^{n-1}} + \dots + a_1(t)\frac{dz}{dt} + a_o z = e_1(t) \ , \tag{19}$$

where

$e_1(t)$ is a scalar input signal, and

$z(t)$ is the scalar output of the amplifier.

The time-varying coefficients in Eq. 19 account for the effect of the pumping signal, $e_p(t)$, indicated in Fig. 5. Thus, the pumping signal is not considered as an input in the usual sense.

It is desirable to write Eq. 19 in vector-matrix form. We denote the output z by x_1 and the input e_1 by u. Then, if $a_n(t) \neq 0$, we have

$$\frac{dx_1}{dt} = x_2 \ ,$$

$$\frac{dx_2}{dt} = x_3 \ ,$$

$$\frac{dx_{n-1}}{dt} = x_n \ ,$$

$$\frac{dx_n}{dt} = -\frac{1}{a_n(t)}\left[a_o(t)x_1 + a_1(t)x_2 + \dots + a_{n-1}(t)x_n\right] + \frac{1}{a_n(t)}u(t) \ . \tag{20}$$

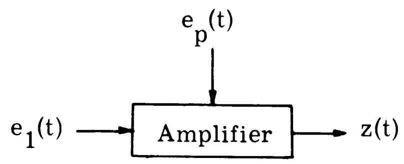

Fig. 5. The parametric amplifier.

This system of equations can be rewritten in the form

$$\frac{dx}{dt} = A(t)\,x(t) + B(t)\,u(t) \quad , \tag{21}$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & & 1 \\ \frac{-a_o(t)}{a_n(t)} & \frac{-a_1(t)}{a_n(t)} & \frac{-a_2(t)}{a_n(t)} & \cdots & \frac{-a_{n-1}(t)}{a_n(t)} \end{bmatrix} B(t) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \frac{1}{a_n(t)} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad .$$

The transition to the discrete-time formulation is straightforward for this linear problem. This discrete-time formulation is given in order that the optimal control problem correspond to the general formulation given in Section 2. If we let

$$\Phi(k) = \phi(t_{k+1}, t_k) \quad ,$$

and

$$\Delta(k) = \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \tau)\, B(\tau)\, d\tau \quad ,$$

where $\phi(t, t_o)$ is the n x n matrix satisfying $\frac{d\phi(t, t_o)}{dt} = A(t)\,\phi(t,t_o)$ and $\phi(t,t) = I$ for all t, then Eq. 21 can be written as

$$x(k+1) = \Phi(k)\,x(k) + \Delta(k)\,u(k) \ . \tag{22}$$

The initial state x(0) is assumed to be a random vector with known probability distribution.

We assume that the parametric amplifier is to be used as a triggering device. The jamming problem is then to "control" the amplifier in such a way that it does not trigger at the proper time. That is, the jamming signal is to be chosen so that the amplifier triggers either too soon or too late to be effective. Thus there are two cases to consider. It will be shown that the two jamming problems differ only in the definition of the cost function.

The parametric amplifier is assumed to trigger at time k if the output $x_1(k)$ is greater than or equal to some desired value d. To make the amplifier trigger early, one might use the cost function defined by

$$J_{N_o}[x(0)] = \sum_{k=0}^{N_o-1} [q_1(k+1)[x_1(k+1) - d]^2 + q_2(k)[u(k)]^2] \ , \tag{23}$$

where q_1 and q_2 are positive numbers at each sampling instant. The time N_o is that discrete-time instant before which the amplifier is to be triggered. Jamming is to cease at time N_o whether or not triggering has occurred. The relative magnitudes of q_1 and q_2 will be determined by how important it is to pre-trigger the amplifier as compared to minimizing jamming signal energy.

The jamming signal energy will not be directly proportional to $[(u(k)]^2$ since u(k) is the sum of the jamming signal plus the input $e_1(t)$ to the unjammed system. In many cases the magnitude of the jamming signal is much greater than the input to the unjammed system. In such cases, $[u(k)]^2$ will approximate the energy of the jamming signal, so that our interpretation of the quadratic cost function as minimizing the sum of input energy and system error will remain true. Any jamming signal which minimizes the expected value of the quadratic form given by Eq. 23 will tend to force the amplifier output x_1 towards the triggering value d.

Although the cost function defined by Eq. 23 may lead to a convenient mathematical solution of the jamming problem, the value of such a solution is somewhat questionable. In particular, a control sequence which minimizes the expected value of $J_{N_o}[x(0)]$ (which has $q_1(k+1)[x_1(k+1) - d]^2$ as one of its summands) may not force x_1 close enough to d at any one sampling instant to trigger the amplifier. Such a jamming sequence would be of little practical value. Therefore, it is desirable to define a more suitable cost function despite the increased mathematical difficulty likely to be encountered in solving for the optimal jamming sequence.

Let the energy of the input (jamming) sequence be limited by some constant U_o; that is,

$$\sum_{k=0}^{N_o-1} [u(k)]^2 \leq U_o \ . \tag{24}$$

Any input sequence which satisfies this energy constraint is said to be an admissible input sequence. We define a new cost function by

$$J_{N_o}^o [x(0)] = \min_{k=1,\ldots,N_o} C_o(x_1(k), d) \ , \tag{25}$$

where

$$C_o(x_1(k), d) = \begin{cases} d - x_1(k) & \text{for } x_1(k) < d \\ 0 & \text{otherwise .} \end{cases}$$

The optimal jamming sequence is then defined to be any admissible input sequence which minimizes the expected value of $J_{N_o}^o[x(0)]$. Clearly, an optimal jamming sequence will in this case force the output $x_1(k)$ as close to the triggering value d (for at least one $k=1,\ldots,N_o$) as the energy constraint U_o on the jamming sequence will allow.

Suppose now that delayed triggering of the amplifier is desirable, rather than pre-triggering. Let N_1 be the discrete time instant to which trigger delay is sought. We can then define a cost function analogous to the quadratic type (Eq. 23) which was defined for the pre-triggering case.

$$J_{N_1}[x(0)] = \sum_{k=0}^{N_1-1} \{q_1(k+1)\,[x_1(k+1) - d]^2 + q_2(k)\,[u(k)]^2\} \quad , \tag{26}$$

where $q_1(k+1)$ is positive and $q_2(k)$ is negative for $k = 0, \ldots, N_1-1$. To prevent triggering before time N_1 it is desirable to keep the output away from the triggering level. The optimal jamming signal is therefore defined as that which maximizes the expected value of $J_{N_1}[x(0)]$. The negative coefficient $q_2(k)$ causes the optimal jamming signal to minimize the input energy consistent with keeping x_1 away from the triggering level.

Again, the use of the quadratic type cost function (Eq. 26) may lead to a convenient mathematical solution of the delayed-triggering jamming problem. However, it may also happen that the jamming sequence that maximizes $J_{N_1}[x(0)]$ forces the output to exceed the triggering level at some sampling instant $k < N_1$ and thus trigger the amplifier. It is therefore desirable to define a more practical cost function. By analogy with the second cost function defined for the pre-triggering case (Eq. 25), we introduce the cost function

$$J^{o}_{N_1}[x(0)] = \min_{k=1,2,\ldots,N_1} \{C_0[x_1(k), d]\} \quad , \tag{27}$$

where $C_0[x_1(k), d]$ is defined as before. The admissible jamming sequences are again assumed to be those satisfying the inequality (24). In contrast with the optimal jamming sequence for Eq. 25, the optimal jamming sequence is here defined to be an admissible sequence which maximizes the expected value of $J^{o}_{N_1}[x(0)]$. In this case it is clear that the optimal jamming signal will force the output as far below the triggering level at each instant as is allowed by the energy limit U_0.

So far we have said nothing concerning what type of observations of the state of the amplifier will be available to the jammer. It can easily happen that no meaningful observation will be available; consequently, open-loop control must be used. On the other hand, there are cases in which the jammer can observe a signal derived from the amplifier state (e.g., an active as opposed to passive detection system). In these cases closed-loop control can be employed. In any event, it is important that the jammer use all available information in an optimal manner.

3.3 A General Jamming Problem for Linear Systems

At this point we formulate a rather general type of jamming problem for linear systems. The problem deals with the interaction of two given linear systems. The formulation can be extended to include more than two interacting systems.

System Number One has been designed to determine some of the state variables of System Number Two. System Number One will be called the detection system and System Number Two will be called the target. We assume that the target is "located" when the detection system has determined some "desired" state variables[20] of the target within some acceptable degree of accuracy.

The target has as one of its objectives the deceiving of the detection system as to the true value of some of the state variables (e.g. position, velocity, etc.) of the target. To this end, the target is assumed to include a jammer which sends to the detection system a jamming signal e(k) based on the state of the target and on observations made of the state of the detection system. Both systems are assumed to be subject to random noise.

We now give the mathematical formulation of this problem; a solution of this linear problem will yield insight into various methods for solving nonlinear problems such as the radar problem considered above.

The "motion" of the state of each system will be described by a linear difference equation. The state of each system includes the electrical and mechanical coordinates of the system. Letting x and z represent the state of the detection system and the target, respectively, we can write the state transition equations

$$x(k+1) = \Phi_x(k)\, x(k) + \Delta_x(k)\, u_x(k) + w_x(k) ,$$

$$z(k+1) = \Phi_z(k)\, z(k) + \Delta_z(k)\, u_z(k) + w_z(k) , \qquad (26a)$$

[20] It is assumed that the detection system knows what the "desired" state variables of the target are. These may be, for example, the position and velocity coordinates of the target.

where

x is an (n x 1) state vector,

z is an (m x 1) state vector,

Φ_x, Φ_z are (n x n) and (m x m) state transition matrices,

u_x, u_z are (p x 1) and (q x 1) control or input vectors,

w_x, w_z are (n x 1) and (m x 1) random noise vectors, and

Δ_x, Δ_z are (n x p) and (n x q) control distribution matrices.

The probability distributions of w_x and w_z are assumed to be known: the distributions of w_x (and of w_z) are assumed to be independent at different sampling instants.

It is assumed that the state of each system is observed by the other system. The observations are assumed to have the form

$$y_x(k) = M_x(k)\, x(k) + v_x(k) \,,$$

$$y_z(k) = M_z(k)\, z(k) + v_z(k) \,, \qquad (27a)$$

where

y_x, y_z are (p_1 x 1) and (q_1 x 1) state observation vectors for the detection system and target, respectively,

M_x, M_z are (p_1 x n) and (q_1 x m) observation matrices, and

v_x, v_z are (p_1 x 1) and (q_1 x 1) measurement noise vectors, respectively.

The probability distributions of v_x (and of v_z) are assumed to be known and to be independent at different sampling instants.

The interaction between the target and the detection system is assumed to be as shown in Fig. 6. Since it is desirable from the standpoint of the target that the behavior of the target depend to some extent on what the detection system knows about the target, we take $y_x(k)$, the observation of the state of the detection system at time k, as the input $u_z(k)$ to the target's control system. This same information is also fed into the target's jammer to help it to jam the detection system more effectively.

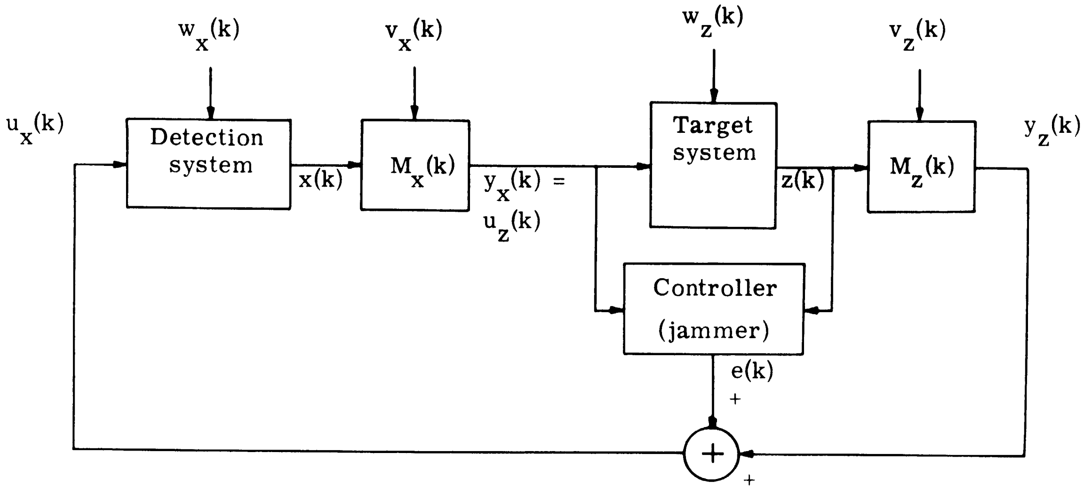

Fig. 6. Interacting systems.

The jamming signal e(k) is also taken to be a function of the state of the target. Note that e(k) will therefore depend indirectly on the state of the detection system, as can be seen from the discussion in the previous paragraph. The jamming signal is to be used to prevent the detection system from locating the target. The input $u_x(k)$ to the detection system is taken to be the sum of the observation $y_z(k)$ and the jamming signal e(k). Thus the detection system is "partially controlled" by the jamming signal and "partially controlled" by the observations y_z.

The two interacting systems can also be described by a single composite system. The dynamic operation of this composite system is most easily obtained in the following way: We substitute $y_x(k)$ for $u_z(k)$ and $y_z(k) + e(k)$ for $u_x(k)$ in Eq. 26a to obtain

$$x(k+1) = \Phi_k(k)\, x(k) + \Delta_x(k)\, [M_z(k)\, z(k) + v_z(k)] + w_x(k) + \Delta_x(k)\, e(k)$$

$$z(k+1) = \Phi_z(k)\, z(k) + \Delta_z(k)\, [M_x(k)\, x(k) + v_x(k)] + w_z(k)$$

The operation of the composite system is then given by (see Fig. 7)

$$s(k+1) = \Phi(k)\, s(k) + \Delta(k)\, u(k) + w(k) \tag{26b}$$

$$y(k) = M(k)\, s(k) + v(k) \tag{27b}$$

where

$$s(k) = \begin{bmatrix} x(k) \\ \\ z(k) \end{bmatrix}, \quad \Phi(k) = \begin{bmatrix} \Phi_x(k) & \Delta_x(k)\, M_z(k) \\ \\ \Delta_z(k)\, M_x(k) & \Phi_z(k) \end{bmatrix}, \quad \Delta(k) = \begin{bmatrix} \Delta_k(k) \\ \\ 0 \end{bmatrix}, \quad u(k) = e(k)$$

$$M(k) = \begin{bmatrix} M_x(k) & 0 \\ \\ 0 & I \end{bmatrix}, \quad v(k) = \begin{bmatrix} v_x(k) \\ \\ 0 \end{bmatrix}, \quad w(k) = \begin{bmatrix} w_x(k) + \Delta_x(k)\, v_z(k) \\ \\ w_z(k) + \Delta_z(k)\, v_x(k) \end{bmatrix}$$

The optimal control problem will be formulated after we have explained the operation of this composite system. Note that the jamming signal is the only input to the composite system. The effect of using the observations as inputs has been incorporated into the transition matrix Φ.

Let us now explain what is happening in the detection part of the composite system. Suppose for the moment that no jamming signal is present. Then, although there is no input to the composite system, the state transition matrix Φ forces some of the state variables (which we denote by s_x) of the detection system towards the values of some of the coordinates of the target's state (which we denote by s_z). We assume here that s_x (and therefore s_z) is an r dimensional vector. The detection system is said to have "located" the target when s_x comes within some small "distance" of s_z.

It is clear from this description of the detection system that the jammer should try to force s_x away from s_z. A reasonable and mathematically convenient cost function is therefore defined by

$$J_N[s(0)] = \sum_{k=0}^{N-1} [\|s_x(k+1) - s_z(k+1)\|^2\, Q_3(k+1) + \lambda\, \|u(k)\|^2\, Q_2(k)] \tag{28a}$$

where

$Q_3(k + 1)$ is a symmetric positive semidefinite (r x r) matrix,

$Q_2(k)$ is a symmetric positive definite (p x p) matrix,

λ is a negative constant, and

$[0, N]$ is the discrete-time interval during which jamming takes place.

The optimal jamming signal is then defined to be any sequence u(0), . . . , u(N-1) which maximizes the expected value of $J_N[s(0)]$ conditional to the observations available at each sampling instant. Since λ is negative, maximizing J_N will result in an optimal jamming signal which uses a minimum of energy[21] consistent with the requirement that s_x be kept away from s_z.

Suppose that the s_x and s_z represent the first r components of the state vectors x and z, respectively. Then the state vector of the composite system is given by $s = \left[s_{1_x} \cdots s_{r_x} \; x_{r+1} \cdots x_m \; s_{z_1} \cdots s_{z_r} \; z_{r+1} \cdots z_n\right]$. The cost function can then be re-written as

$$J_N[s(0)] = \sum_{k=0}^{N-1} \left[\|s(k+1)\|^2_{Q_1(k-1)} + \lambda \|u(k)\|^2_{Q_2(k)}\right] \qquad (28b)$$

where

$$Q_1(k) = \begin{bmatrix} Q_3(k) & 0 & -Q_3(k) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -Q_3(k) & 0 & Q_3(k) & 0 \end{bmatrix}$$

For the case in which v(k) and w(k) are independent Gaussian random vectors this problem is closely related to the linear optimal control problem considered in the introduction to this memorandum. The two problems differ primarily because of the negative constant λ appearing in Eq. 28b. The problem becomes more difficult, yet tractable, if we allow arbitrary probability distributions for the noise vectors w(k) and v(k).

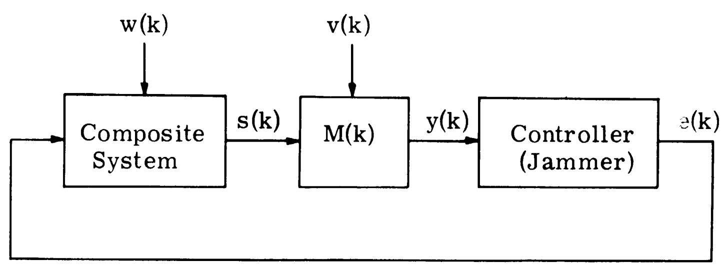

Fig. 7. Composite system.

[21] We assume here that Q (k) is so chosen that $\|u(k)\|^2_{Q_2(k)}$ is proportional to the jamming signal energy at each sampling instant.

4. SYSTEM ORGANIZATION

Many of the high-speed digital computing systems presently in operation perform computation for users having different priority levels. In order that such a computing system operate efficiently, it is necessary to specify a procedure for determining which job should be serviced next when several jobs await servicing. We give here a formulation of this problem within the framework of modern control theory.

Let each job be placed in one of n classes. The classification is to be made according to 1) the computation time required for the job[22] and 2) the cost rate of the job. The cost rate of a job represents the charge per unit time to the computer system for the time that the job must wait for servicing. The classification is to be made so that the jth class contains all jobs having the same computation time and the same cost rate. Thus it is possible to assign a rate to each class according to the cost rate of the jobs in the class, with a higher rate being assigned to those classes containing jobs with a higher cost rate.

It is assumed that the arrival of jobs for servicing is random. The number of jobs of class j arriving at the computer for servicing during the kth discrete-time interval $[k - 1, k]$ is assumed to be a random variable having Poisson distribution with mean value $m_j(k)$. The probability density function for a random variable having Poisson distribution with mean value m is given by

$$p(s) = \frac{m^s e^{-m}}{s!} .$$

A very convenient description of the job servicing procedure can be given if we define a state vector for the queue. (The queue is the collection of jobs waiting to be serviced.) Let an (n + 1)-dimensional state vector x(k) have components defined by

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[22] We assume here that all computation times are known in advance. Although this assumption does not apply to all computing systems, it is essential to the formulation of the optimal control problem to be introduced below.

$x_j(k)$ = number of jobs of class j waiting to be serviced at time $k(j = 1, \ldots, n)$, and

$x_{n+1}(k)$ = time remaining to finish the job presently being serviced.

Let u(k) be an (n + 1)-dimensional input vector with components

$$u_j(k) = \begin{cases} 1 & \text{if a job of class j is taken for servicing at time k} \\ 0 & \text{otherwise} \qquad j = 1, \ldots, n \end{cases}$$

$$u_{n+1}(k) = \begin{cases} -t_j & \text{if a job of class j is taken for servicing at time k} \\ 1 & \text{otherwise} \end{cases}$$

The computation time vector $T = (t_1, \ldots, t_n)$ represents the computation time required for jobs according to their respective class. Using these definitions the dynamic operation of the queue is described by

$$x(k + 1) = x(k) - u(k) + a(k) , \tag{29}$$

where a(k) is an (n + 1)-dimensional vector-valued random variable representing the random arrival of jobs at the queue. The component $a_j(k)$ is a Poisson random variable with mean $m_j(k)$, $j = 1, \ldots, n$. The $(n + 1)^{st}$ component of a(k) is assumed to be identically zero.

There are several constraints on the computer which limit the choice of inputs, u(k). It is assumed that the computer can take only one job for servicing at any time and that no new job can be taken for servicing until the previous job is completed. Thus at most one of the $u_j(k)$ can be equal to 1, $j = 1, \ldots, n$. If $x_{n+1}(k) \neq 0$, then $u_j(k) = 0$, $j = 1, \ldots, n$, since the previous job has not been completed. If $u_j(k) = 1$ then the time until this job is completed is given by t_j; thus we let $u_{n+1}(k) = -t_j$. One more obvious constraint is that $u_j(k)$ cannot equal one if $x_j(k) = 0$ since $x_j(k) = 0$ implies that there are no jobs of class j waiting to be serviced at time k. Thus it is clear that the constraints on the computer operation considerably restrict the allowable inputs.

In order to complete this formulation as an optimal control problem it is necessary to define a cost function. First let us define a diagonal "rate matrix" R(k) with diagonal elements representing the cost rate of the various classes of jobs. Thus, we let

$$R(k) = \begin{bmatrix} r_1(k) & & & 0 \\ & r_2(k) & & \\ & & \ddots \; r_n(k) & \\ 0 & & & 0 \end{bmatrix} ,$$

where $r_i(k) > r_j(k)$ if at time k the ith class has higher priority than the jth class, $i, j = 1, \ldots, n$. The rate $r_i(k)$ represents the charge to the computer for not servicing jobs of the ith class which are in the queue at time k. The cost function can then be taken as

$$J_N[x(0)] = \sum_{k=1}^{N} \|x(k)\|^2 R(k) \; . \tag{30}$$

The quadratic cost function used here discourages the computer from allowing a large number of jobs of any one class to accumulate in the queue. Such a cost function is desirable if each class contains jobs from a single project. To see this observe that if many jobs from one project are waiting to be serviced, people working on this project are likely to be idled. It is clear from Eq. 29 that $J_N[x(0)]$ is a random variable. An optimal input sequence is defined to be any sequence of allowable inputs which minimizes the expected value of $J_N[x(0)]$.

We assume that the state of the queue can be determined exactly at each discrete-time instant. Thus the observation of the queue can be described by

$$y(k) = x(k) \; . \tag{31}$$

This problem appears to be very similar to the linear problem introduced in Section 2. The cost function given by Eq. 30 is a special case of that given in Eq. 4 with $Q_2(k) \equiv 0$ and $d(k) \equiv 0$. Moreover, the observation matrix, M(k), is just the identity matrix, and $v(k) \equiv 0$.

There are, however, very important differences between this problem and that introduction in Section 2. In particular, for this problem we have that:

1) The random arrivals have a Poisson distribution. Thus the well-developed theory of linear systems subject to Gaussian noise cannot be used.

2) In this problem the components of the state variables and the input variables are restricted to integer values.[23] Therefore, any analytical techniques which require that x and u be allowed to vary over a continuum of values (e.g. differentiation with respect to x or u) are not applicable.

Thus, techniques other than those used to solve the general linear problem must be used.

A complete solution to this problem is not known at present. However, the solution of a closely related problem (one where the cost function is linear in x instead of quadratic in x) is known (Refs. 6, 11) and easily implemented. Obviously, the soluble problem is very meaningful in its own right. Moreover, it may be that its solution can be used as a step in solving the quadratic cost function problem.

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[23] In fact, with exception of u_{n+1}, all these variables are restricted to being nonnegative integers.

5. COMMUNICATIONS SYSTEMS WITH FEEDBACK

5.1 Systems With Channels Having Constant Delay

There is a class of communications systems in which part of the received information is fed back to the transmitter. Many satellite communications systems belong to this class. In such systems the satellite will, upon receiving a signal from the ground, return a signal to the ground station indicating that a certain message has been received. This helps the ground station to determine whether or not the proper message was received. We consider here feedback communications systems with constant-delay channels. This restriction is not essential; however, it makes the initial formulation of the problem more tractable. We allow the system to have different forward and feedback channels. Channels having time-varying delay will be considered in the problem treated next.

The block diagram for systems of this class is given in Fig. 8. We assume that the input signal u(k) is to be determined by the controller. In order to use the discrete-time formulation for this problem, it is necessary for the input signal to be represented as a sequence of discrete pulses. This representation can be made if the input signal is essentially bandlimited so that it is determined by its sampled waveform (Ref. 12). We assume that the input signal can be described by a finite sequence of samples. It is desirable to use the discrete-time formulation of this problem because it permits a very clear description of the effects of channel delay on the output of the receiver filter and on the information fed back to the controller. This discrete-time formulation is also more suitable if solutions are to be obtained using a digital computer. In order to describe channel delay, it is convenient to assume that the lengths of the discrete-time intervals are equal. The length of the sampling interval is also to be inversely proportional to the bandwidth of the input signal (Ref. 12).

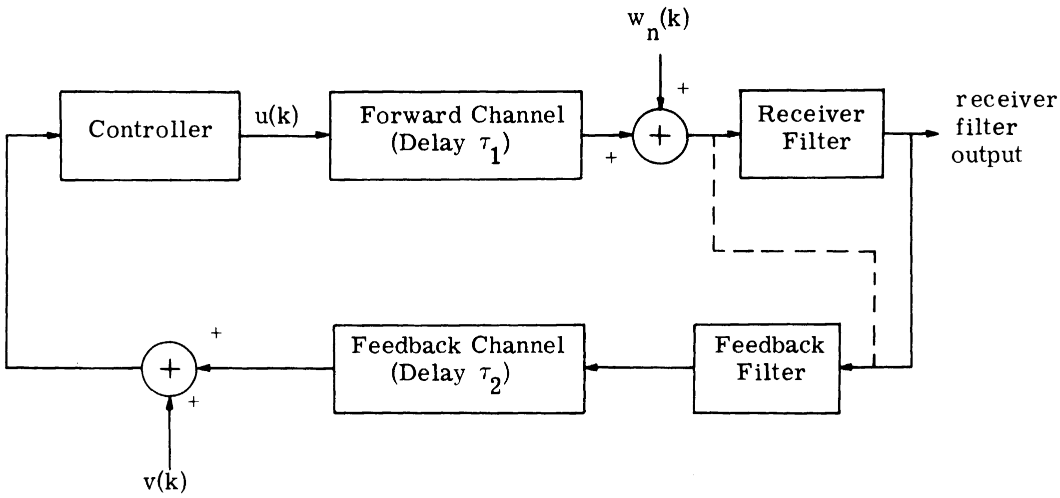

Fig. 8. Block diagram of feedback communications system.

The operation of the communications systems can be described by a set of first-order difference equations. It is convenient to give these equations, which describe the dynamic operation of the entire system, and then explain the operation of the several individual subsystems. The system equations are

$$\left.\begin{aligned} x_1(k+1) &= u(k) \\ x_2(k+1) &= x_1(k) \\ &\vdots \\ x_n(k+1) &= x_{n-1}(k) + w_n(k) \end{aligned}\right\}$$ equations characterizing the relations among the signals at n points along the forward channel separated by equal units of delay

$$x_{n+1}(k+1) = f[k, x_{n+1}(k), x_n(k)]$$ receiver filter equation

$$x_{n+2}(k+1) = g[k, x_{n+2}(k), x_{n+1}(k)]$$ feedback filter equation

$$\left.\begin{aligned} x_{n+3}(k+1) &= x_{n+2}(k) \\ &\vdots \\ x_{n+m+2}(k+1) &= x_{n+m+1}(k) \end{aligned}\right\}$$ equations characterizing the relations among the signals at m points along the feedback channel separated by equal units of delay

The initial state $x(0) = [x_1(0) \ldots x_{n+m+2}(0)]'$ is assumed to be a Gaussian random vector with known mean and known covariance.

The Input Signal Channel

Due to u, voltages $x_1, \ldots, x_n$ are observed at n points along the input channel $x_1, \ldots, x_n$ separated by equal units of delay. The effect of the channel delay is, then, to make $x_i(k)$ equal to the input signal at time (k - i). The number of state variables required to describe the channel is therefore proportional to the delay time of the channel and is inversely proportional to the length of the discrete-time intervals.[24]

The scalar $w_n(k)$ is the sampling of zero-mean, Gaussian random noise at the receiver input. The covariance of this noise is assumed to be known. It is also assumed that the values of the noise samples at different times are independent.[25]

The Receiver Filter

The output of the receiver filter[26] is defined by x_{n+1}. This output represents the actual received message. The desired receiver output signals $d(n + 1), \ldots, d(n + N)$ are assumed to be specified in advance.

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[24] That is, the required number of state variables will be proportional to the product of delay time and bandwidth of the channel.

[25] This independent noise assumption is not essential to the treatment of this problem. Dependent Gaussian noise can be obtained from the output of a linear filter excited by independent Gaussian noise. Thus dependent noise can be handled in this problem by viewing it as the output of a linear filter excited by independent noise (Ref. 5). This would require that the dimension of the state vector be increased to include the linear noise filter. The extension to include dependent noise adds little to the understanding of this problem. We therefore consider only independent noise.

[26] For purposes of this discussion, this filter and the feedback filter are assumed to be of first order. Thus x_{n+1} also describes the state of the filter. More general filters are discussed later.

The inputs u(0), . . . , u(N - 1) are to be chosen so that $x_{n+1}(k)$ is "close" to d(k) for k = n + 1, . . . , n + N.[27]

The function, $f[k, x_{n+1}(k), x_n(k)]$, represents the dynamic operation of the receiver filter. The explicit form of this function will depend on the particular type of filter being used. For example, a time-varying linear filter can be represented by

$$f[k, x_{n+1}(k), x_n(k)] = r_1(k)\, x_{n+1}(k) + r_2(k)\, x_n(k) \tag{33}$$

where $r_1(k)$ and $r_2(k)$ are the time-varying weighting coefficients of the filter.

The receiver filter output at each time instant will be random due to the random initial state, x(0), and the noise $w_n(k)$ at the input to the receiver. It is therefore desirable to design the communications system to feed back part of the received information. This information can be used to learn something about the effects of previous inputs on the output of the receiver filter. This knowledge can then be used to aid in the selection of succeeding inputs.

The Feedback Filter

The general form for the filter used to feed back information on the receiver filter output is given by $g[k, x_{n+2}(k), x_{n+1}(k)]$. For the linear filter, this takes the form

$$g[k, x_{n+2}(k), x_{n+1}(k)] = b_1(k)\, x_{n+2}(k) + b_2(k)\, x_{n+1}(k) \tag{34}$$

where $b_1(k)$ and $b_2(k)$ are the weighting coefficients of the filter.[28]

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[27] It would not be realistic to try to control $x_{n+1}(k)$ for $k \leq n$ since the channel delay makes it impossible for an input to affect x_{n+1} until (n + 1) discrete-time instants later.

[28] Note that the input to the feedback filter could also be taken as the input (rather than the output) to the receiver filter as indicated by the dashed line in Fig. 8. The formulation of the problem would proceed along the same line in either case except that $x_n(k)$ would replace $x_{n+1}(k)$ in Eq. 34.

It should be noted that the output at each instant of a filter described by Eq. 33 or Eq. 34 depends only on the previous output sample and on the previous input to the filter input. The present problem formulation can be extended to include systems having filters whose outputs depend explicitly on the last p inputs, i. e., to pth-order systems. However, if the state transition equation is to be expressed in terms of first-order difference equations, it is then necessary to increase the dimension of the state vector by (p - 1). The extension is not given here because it leads to notational complexity which tends to obscure the underlying communications problem.

The feedback channel:

The output x_{n+2} of the feedback filter is returned to the controller through the feedback channel. The state of this feedback channel is described by the components $x_{n+3}, \ldots, x_{n+m+2}$. The delay-time bandwidth product of the feedback channel determines the number of components, m, required to describe the channel.

The input u at time k = 0 does not reach the receiver filter until time k = n. However, the observations of the receiver filter output are being fed back to the controller during this time. The observations are assumed to be described by

$$y(k) = x_{n+m+2}(k) + v(k) \tag{35}$$

where v(k) is the noise incurred in reception of the geedback signal. We assumed that v(k) is an independent zero-mean Gaussian random variable.[29]

We assume that the receiver filter and feedback filter are described by Eqs. 33 and 34, respectively. Then Eqs. 32 and 35 can be rewritten in vector-matrix form as

[29]The remarks made (see Footnote 25) following the description of the input signal channel also apply here.

$$x(k+1) = \Phi(k)\,x(k) + \Delta(k)\,u(k) + w(k)\ , \tag{36}$$

$$y(k) = M(k)\,x(k) + v(k)\ , \tag{37}$$

$$\Phi(k) = \begin{bmatrix} 0 & 0 & 0 & & & & & \\ 1 & 0 & 0 & & & 0 & & \\ & 1 & \ddots & 0 & & & & \\ & & \ddots & 1 & 0 & & & \\ & & & & r_2(k) & r_1(k) & & \\ & & & & & b_2(k) & b_1(k) & \\ & 0 & & & & & 1 & 0 \ \ddots \\ & & & & & & \ddots & 1 \ \ddots \ 0 \end{bmatrix}$$

$$\Delta(k) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad M'(k) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad w(k) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ w_n(k) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

The sequence of input samples is to be chosen which will make the sequence of receiver filter output samples $x_{n+1}(n+1), \ldots, x_{n+1}(n+N)$ close, in some sense, to the known desired output sequence $d(n+1), \ldots, d(n+N)$. The input energy is assumed to be proportional to $[u(k)]^2$. A reasonable and mathematically convenient cost function is therefore defined as

$$J_N[x(0)] = \sum_{k=0}^{N-1} [x_{n+1}(n+1+k) - d(n+1+d)]^2 + \lambda(k)[u(k)]^2 \quad , \tag{38}$$

where $\lambda(k) \geq 0$. The number N is equal to the number of samples required to specify the continuous-time signal.

If $\lambda(k) = 0$, it is clear that minimizing the expected value of $J_N[x(0)]$ just minimizes the mean square error of the receiver output filter. If $\lambda(k)$ is positive then the minimization procedure weights the squared error and the input energy accordingly. The input $u(k)$ is to be chosen conditional to the delayed feedback-filter output signals, $y(0), \ldots, y(k)$, in such a way that the expected value of $J_N[x(0)]$ is minimized. The amount of information contained in these feedback signals depends on the filters used, on the relative distributions of the receiver filter input noise, w_n, and on the observation noise v. For example, if $v = 0$ the observations can be used to determine the exact receiver filter output. Of course, the delayed feedback prevents the receiver filter output at time k from being determined until time $k + m + 1$.

The discrete-time formulation allows us to obtain solutions to this linear problem using a digital computer. The problem becomes considerably harder to solve when nonlinear filters are used. The extension to channels having time-varying delay can be made rather simply. We consider this problem next.

5.2 System With Channels Having Time-Varying Delay

Many communications systems have signal channels with time-varying delay. This is usually true of satellite communications. The changing distance between satellite and ground station (or between two satellites) gives rise to time-varying delay in radio communication. We show here that linear systems having variable-delay channels can also be described by Eqs. 36 and 37. However, in this case the matrices Δ and M will be time-varying, and the number of variables required to specify the state of the forward channel on the feedback channel will depend on the maximum delay of the respective channel. The extension of the above formulation is immediate:

Let $\tau_1(t)$ and $\tau_2(t)$ describe the continuous-time variation of the input-channel delay and the feedback-channel delay, respectively. It will be convenient to assume that the initial delays are n units of discrete time in the forward channel and m units of discrete time in the feedback channel. We assume that both τ_1 and τ_2 vary slowly so that the continuous change in delay of τ_1 (or τ_2) during ℓ (or j) discrete time intervals can be replaced by a single discrete change in delay. The number ℓ (or j) will be inversely proportional to the rate of change of τ_1 (or τ_2) and the length of the discrete-time intervals. To facilitate this discussion we assume that all discrete-time intervals of of unit length. It is convenient to assume that all discrete changes in delay also equal one unit of discrete time.

In order to facilitate the description of this problem, we consider a specific case. Figure 9 illustrates the delay variations for a system in which the delay in both channels is decreasing linearly with time. The delay in the feedback channel, τ_2, is decreasing twice as fast as that in the input channel. We assume that at $k = 0$, the system is described by Eq. 32. Then, assuming that the delay can be considered constant during the interval $[0, j]$, the matrices $\Phi(k)$, $\Delta(k)$ and $M(k)$ are the same as those given following Eq. 37.

We assume that, in this approximate model of the actual system, the discrete changes in delay for the feedback channel are made at times $j_1, j_2, \ldots, j_r$ and for the input channel at times $\ell_1, \ell_2, \ldots, \ell_s$. It is reasonable to assume that these discrete changes in delay will be made twice as often for the feedback channel since its delay decreases twice as fast as that of the input channel.

Let us suppose that $j_1 < \ell_1$. Then at time j_1 the observation matrix will be given by

$$M(j_1) = [0\,0 \ldots 0\,1\,0] , \tag{39}$$

since the observations will be fed back one discrete-time interval earlier due to the unit decrease in the feedback channel delay. Thus, the observation will be given by

$$y(j_1) = x_{n+m+1}(j_1) + v(j_1) \tag{40}$$

The unit decrease in delay allows the feedback signal to reach the controller one discrete-time interval sooner. It must be noted that this change in delay also results in the loss of

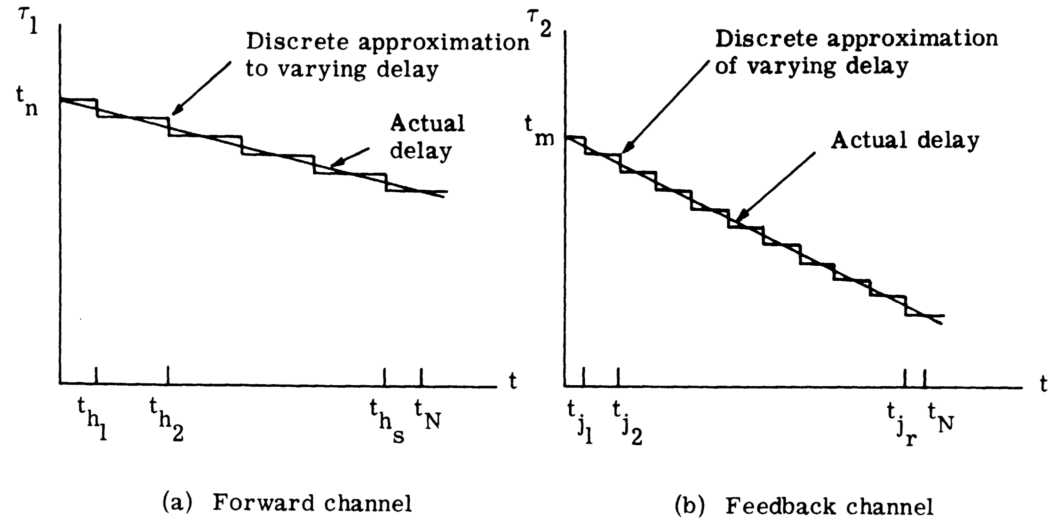

Fig. 9. Time varying delay in communication channels

the observation that would have otherwise been made at time j_1. The effect of this information loss will be discussed later.

Thus for $j_1 \leq k < \ell_1$ the system is described by

$$x(k+1) = \Phi(k)\, x(k) + \Delta(k)\, u(k) + w(k)$$

$$y(k) = M(k)\, x(k) + v(k) \tag{41}$$

where $\Delta(k) = [1\,0\,0 \ldots 0]$ and M(k) is given by Eq. 39.[30] At time ℓ_1, we must account for the change in delay of the input channel. This can be taken care of in the model by letting the input signal enter the channel at a point one unit closer (in time) to the receiver filter. Thus for $\ell_1 \leq k < j_2$, the communications system is described by Eq. 41 with $\Delta(k) = [0\,1\,0 \ldots 0]$ and M(k) given by Eq. 39. In this case we note that $u(\ell_1 - 1)$ and $u(\ell_1)$ are added together at time ℓ_1. Thus the input to the receiver filter at time $\ell_1 + n - 1$ will be a distorted sum of these two inputs.

[30] $\Phi(k)$ is the same as in the case of systems having constant-delay channels.

We continue in this way, accounting for the changes in feedback delay in M(k) and for changes in delay in the input channel in $\Delta(k)$.

Other forms of delay variation in the two channels are treated in the same way. The cost function and the optimum sequence of input signals can be defined just as for systems having constant-delay channels. Given a sequence, $d(n + 1), \ldots, d(n + N)$, of desired receiver filter outputs, the optimum sequence of input signals is that which minimizes the expected value of Eq. 38.

There are several problems peculiar to this formulation. The procedure used to account for time varying feedback delay results in the loss of some of the signals in the feedback channel. Thus the amount of information available for estimating the receiver filter output is reduced. Moreover, the varying delay in the input channel makes it more difficult to control the output of the receiver filter.[31] The effect of these two difficulties will depend on how much the delay in the two channels varies in the given problem. Thus the practical significance of this formulation depends on the variable delay in the two channels. It might be tempting at this point to conclude that a constant-delay system can be controlled more effectively (in the sense of yielding lower expected cost) than a similar system with time-varying delay. This temptation can be dismissed by comparing two systems. In the first system, the delay in both channels is constant at n units of discrete time. The two channels of the second system are assumed to have equal delays, each initially n units long and falling off to zero delay in j discrete-time intervals. Then for N (the length of the sequence of desired receiver filter outputs) much greater than j we would expect better control from the system with decreasing delay. That is, as N gets large the effect of the loss of observations during the early stages of control is offset by the improved control of the system after the delay has decreased to zero.

In conclusion it should be noted that there are numerous extensions of this problem. We have restricted our consideration to linear systems subject to Gaussian noise. For

[31] This can be seen by noting that a discrete increase in delay will result in one instant of time at which there is no input (except noise) to the receiver filter.

some applications, consideration of nonlinear filters may be necessary. The particular procedure used here to account for time varying delay is not unique. It does, however, provide a rather convenient formulation of the problem.

5.3 Communication Through a Channel With Unknown Statistics

It is sometimes desirable to transmit information through channels containing noise whose probability distribution is not known (or is only partially known). The present problem is primarily concerned with a method for 1) learning something about the probability distribution of the channel noise and 2) simultaneously communicating the desired information to the receiver.

We allow the communications system to be multichanneled so that the receiver output is in general vector valued, but assume channel delay is negligible. The operation of the communications system is described by a difference equation; in particular, the state of the receiver is governed by

$$x(k+1) = \Phi(k)\, x(k) + \Delta(k)\, [u(k) + w'(k)] \tag{42}$$

where

- $x(k)$ is an $(n \times 1)$ vector representing the state of the receiver,
- $\Phi(k)$ is an $(n \times n)$ matrix describing the receiver operation,
- $\Delta(k)$ is an $(n \times r)$ input distribution matrix,
- $u(k)$ is an $(r \times 1)$ vector representing the signal transmitted to the receiver, and
- $w'(k)$ is an $(r \times 1)$ input noise vector.

If we let $w(k) = \Delta(k)\, w'(k)$ then Eq. 42 can be rewritten as

$$x(k+1) = \Phi(k)\, x(k) + \Delta(k)\, u(k) + w(k) \ . \tag{43}$$

The initial state of the receiver, $x(0)$, is assumed to be a random vector with known probability distribution. The output of the receiver filter, which we denote by $y(k)$, is assumed to be a linear function of the state of the filter. Thus we have that

$$y(k) = M_1(k)\, x(k) \ , \tag{44}$$

where $M_1(k)$ is an (m_1 x n) output matrix.

The function of the transmitter controller is to make the sequence of receiver filter outputs y(1),..., y(N) close, in some sense, to a known desired output sequence d(1),..., d(N). We assume that in order to help the transmitter-controller to control the receiver filter output more effectively [i.e., to make y(k) close, in some sense, to d(k)], some of the received information is fed back to the transmitter-controller (see Fig. 10).

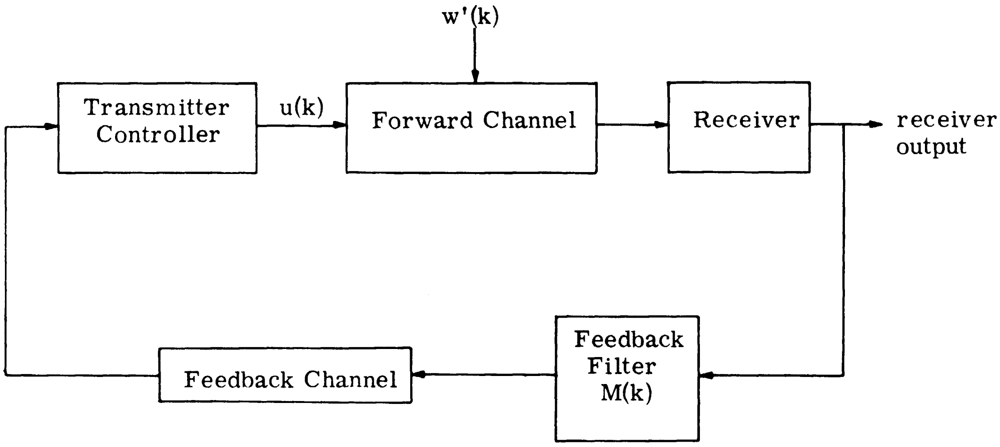

Fig. 10. Block diagram of feedback communications system with noise of unknown distribution in forward channel.

The feedback channel is assumed to have sufficiently narrow bandwidth so that any noise in the channel can be neglected. (This noiseless feedback channel cannot be used for the forward channel because of its narrow bandwidth.)

We assume that the signal fed back (called an observation) is a linear function of the output of the receiver filter. Thus the observation fed back at time k is given by

$$z(k) = M_2(k)\, y(k) , \tag{45}$$

where

z(k) is a (p x 1) observation vector, and

$M_2(k)$ is a (p x m_1) observation matrix.

Thus the description of the communications system given by Eqs. 42 and 46 is the same as that for the general linear system given in Section 2.

The observations z(k) are processed by the controller to provide aposteriori information concerning the parameters of the noise distribution in the forward channel. Observations will generally contribute information; thus, the transmitter-controller can control the output of the receiver more effectively by taking more observations. However, it is also true that making an observation available to the transmitter-controller requires some energy. For example, suppose that we are transmitting information from a ground station to a satellite. If the satellite is to provide an observation to the ground controller, some energy will be required to transmit this observation to the ground station. Thus, it may be desirable to limit the number of observations taken.

If it is known a priori that the parameters of the noise distribution vary slowly with time,[32] we can limit our observations to the initial instants of signal transmission. These observations can then be used to estimate the parameters of the noise distribution for the entire transmission time interval. For example, if the signal to be transmitted requires one-hundred sampling intervals, it may be desirable to estimate the parameters of the noise distribution on the basis of observations made at the first ten sampling instants. No observations would be maed during the last ninety sampling intervals, and the variation in the parameters of the noise probability distribution during this time would be neglected.[33] With these remarks in mind we can now define a reasonable cost function for this communications problem.

Let the desired sequence of receiver outputs be denoted by $d(1), \ldots, d(N)$. Let $\alpha(k)$ be a vector of parameters describing the distribution of the noise in the forward channel. We assume that the noise parameter vector has some known a priori distribution and that the noise parameters vary slowly with time. This last assumption allows us to restrict our consideration to the cases in which all observations are made at the initial sampling instants.

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[32] Slowly is here taken to mean that the parameters of the noise distribution do not change significantly during the time required for signal transmission.

[33] It may happen that the parameters of the noise distribution vary too much during the N sampling intervals required for signal transmission to be treated as constant. In such cases we can use either of two approaches: 1) take observations intermittenly so that up-to-date information is available concerning the noise parameters, or 2) use only initial observations and a priori knowledge concerning the variation of the noise distribution parameters. We shall consider only the latter case in our discussion.

That is, i ten observations are to be made they will be made at the first ten sampling instants.

Under these restrictions, a reasonable cost function is defined by

$$J_N[x(0)] = \sum_{k=0}^{N-1} [\|x(k+1) - d(k+1)\|^2 Q_1(k) + \|u(k)\| Q_2(k)] ,$$

where $Q_1(k)$ and $Q_2(k)$ are positive semidefinite matrices. The cost of making observations at the first n sampling instants is defined by $f(n) = n f_0$ where f_0 is a nonnegative constant. Suppose, momentarily, that we make exactly n observations where $0 \leq n \leq N$. Then the optimum input sequence for n observations is defined to be that which minimizes

$$R_N^n[x(0)] = E_{x(0)\,\alpha(k\ n)} \{J_N[x(0)]\} , \qquad (45)$$

where $E_{x(0)\,\alpha(k\ n)}$ is used to denote the expectation over the distribution of the initial state and the aposteriori distributions of the noise parameters. (This latter distribution will depend on n the number of observations taken; thus we use the subscript notation $\alpha(k\ n)$.)

The risk function $R_N^n[x(0)]$ defined by Eq. 45 will, in general, depend on the number of observations, n. Since it is desirable to make some charge observations, we define the optimum input sequence as that which minimizes, over $n = 0, 1, \ldots, N-1$, the function

$$R_N^n[x(0)] + n f_0 .$$

The cost of making observations for communication can then be varied by changing the constant f_0.

There are two features of this problem which distinguish it from the general linear problem whose solution was given in Section 2. The more important of these is that the probability distribution of the noise in the secure channel is not known in advance. The controller must perform the dual functions of: 1) estimating the parameters of the noise distribution and 2) determining the optimum input sequence. The other distinguishing feature of this problem is the presence of the "observation cost" term f(k). A technique known as "dual control" introduced by Feldbaum (Ref. 10) can be used to treat this problem. The description of this technique is too lengthy to be given in detail here, but a discussion of the basic ideas involved is given in connection with the example considered in the next section.

6. DIRECTION-FINDING SYSTEMS

Direction-finding systems are of considerable interest in electronic warfare. We consider here systems having the form illustrated in Fig. 11. Many of the ideas discussed below can be extended to other types of direction-finding systems.

The system illustrated in Fig. 11 operates in the following way. An incoming signal is received by each of the m antennas of an array. There will be a phase difference between the signals received by the various antennas. The phase difference between the signals on any two antennas depends on: 1) the relative positions of the two antennas within the array and 2) the location of the signal source with respect to the array. For example, consider the linear antenna array shown in Fig. 12. Suppose that the received signal results from a plane wave making an angle α with the normal to the line containing the antenna array. Then the phase difference between the signals received by adjacent antennas will be equal to $(d \cos \alpha)/c$, where d is the distance between the antennas and c is the propagation velocity of the incoming wavefront.

Again referring to the system of Fig. 11, we assume that each of the m antennas in the array is connected to an electronically variable phase shifter. The phase shifter of each antenna is controlled by a signal determined by the control unit. The outputs of the phase shifters are fed into a signal processor which performs some known operation on these signals. The output(s) of the signal processor are forwarded to the controller-estimator unit. The controller-estimator uses the information contained in these signals to control the phase shifters and to specify the "location" of the signal source (in a sense to be discussed below).

The signal processor is used to transform the received signals (with their corresponding phase shifts) into a form suitable for use by the controller-estimator. We assume that the signal processing is subject to noise and that the probability distribution of this noise is known in advance. A very simple form of signal processor is one which has a single output equal to the sum of its inputs with added noise. We do not restrict our attention to any one form of signal processor in this discussion.

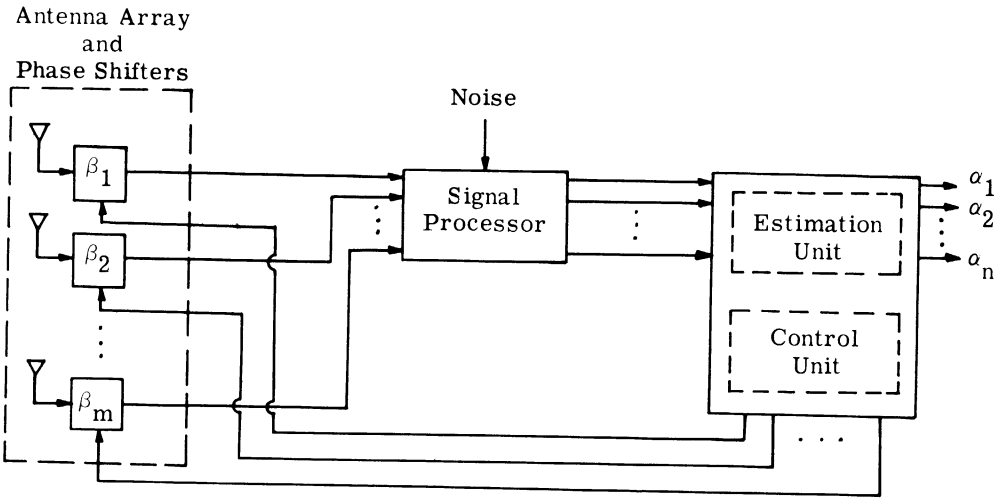

Fig. 11. Block diagram of a direction-finding system.

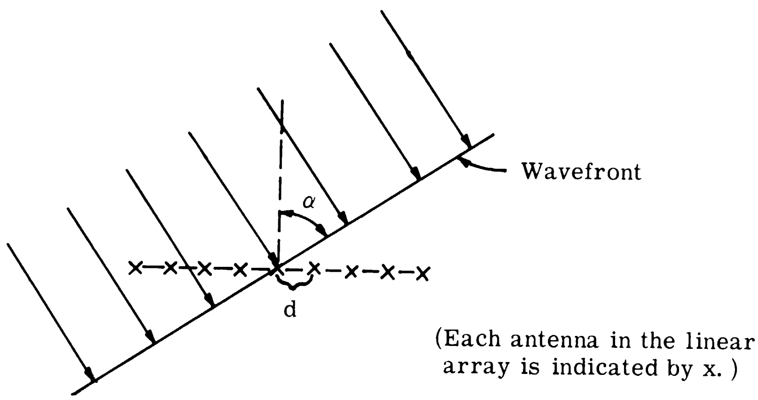

Fig. 12. Linear antenna array with incident plane wave.

The "location" of the source of the received signals is assumed to be specified by a vector α.[34] The function of the estimation unit is to compute an estimate of α which is optimal in some sense. We use the notation $\hat{\alpha}$ to denote an optimal estimate.

The control unit determines what signal is to be used to control the antenna phase shifters. However, varying the phase shifts of the received signals will vary the output(s) of the signal processor. Thus, the information available to the estimation unit will depend on the control signal. Since the estimation unit uses this information to determine $\hat{\alpha}$, it is clear that the optimal estimate will depend on both the control unit and the estimation unit. For this reason, it is necessary that the design of the two units be carried out jointly in order to obtain an optimal estimate of α. We denote the two units as the controller-estimator system. We assume that the antenna array, the antenna phase shifters, and the signal processor are given. The remainder of this discussion is therefore concerned only with the design of the controller-estimator.

Several techniques in modern control theory can be used to design the controller-estimator. The optimal system in each case will be that which minimizes some given cost function. It is not necessary for the present discussion to restrict our attention to any one particular cost function. The cost function will in general depend on the intended application of the direction-finding system being considered.[35]

Before considering methods for designing the controller-estimator, it is desirable to point out a difficulty inherent in this problem: The received signals are usually corrupted by noise. As mentioned above, the signal processor introduces additional noise. Although we assume that the probability distributions of both noise components are known, such noise tends to obscure the phase relations between the signals received by the various antennas, thus reducing the amount of information available for estimating α. This difficulty makes it desirable to design the controller-estimator unit so that it operates in some optimal manner.

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[34] We assume here that the "location" of a source may require specification of only some of its position coordinates. For example, the "location" of a ground source may only require that a single angle be specified even though the actual position of the source is given by an angle and a radial distance.

[35] It should be pointed out, however, that time will usually be a factor in the cost function since the received signals may be of short duration.

As noted above, there are several techniques which can be used to carry out a jointly optimal design of the control unit and the estimation unit. We mention here a technique known as "dual control" which was introduced by Feldbaum (Ref. 10). This method can be used whether the controller-estimator is to be a continuous-time or a discrete-time system.

It is sufficient here to discuss the ideas behind the dual-control-theory approach. (The mathematical details are somewhat lengthy and will not be given in this memorandum.) The main idea behind the dual control method is to view the control signal as simultaneously performing two functions. The usual function of the control signal is to control the system in some way; however, the control signal can be used simultaneously to probe the system. That is, the control signal can also be used to allow the controller-estimator to learn something about some unknown or partially known parameters of the system.

The present discussion will be restricted to a degenerate form of dual control. In this case we require that the controller and estimator be jointly designed to yield an optimal estimate $\hat{\alpha}$ of the location of the source of the received signal.[36] The control signal will then vary the phase shifters so that the output(s) of the signal processor are optimal in the sense that they provide sufficient information to the estimation unit so that it can determine $\hat{\alpha}$.

It may happen at some point in the design of the optimal controller-estimator that the problem separates into the individual designs of the optimal estimation unit and the optimal control unit. There are well-known techniques for carrying out the individual designs in such cases. The estimation of vector valued random variables has played a key role in the development of modern control theory. Some of these techniques can be found in the works of Kalman (Ref. 7), Ho (Ref. 8) and Cox (Ref. 9). The method used to estimate α will usually depend on the cost function which is to be minimized by the optimal estimation unit.

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[36] In a more general case it may, for example, also be desirable that the control unit vary the antenna phase shifters so that the signal-to-noise ratio(s) of the signal processor output(s) are maximized, thus allowing information to be extracted from the received signal. Actually, if we are interested in both estimating the location of the source of the received signal and extracting information from the signal itself, some trade-off between signal-to-noise ratio and information content of the signal processor outputs must be made. This is usually a considerably more complicated problem.

Some of the ideas introduced above can be clarified by considering the direction-finding system illustrated in Fig. 13. We assume that the system has a linear antenna array and that the received signals result from a plane wave. The line segment joining the source of this plane wave to the antenna array makes some angle α with the line containing the linear array. The outputs of the antenna phase shifters are operated on by the signal processor. The output of the signal processor in Fig. 13 is equal to the sum of the outputs of the phase shifters. We assume that any noise entering the signal processor can be neglected.

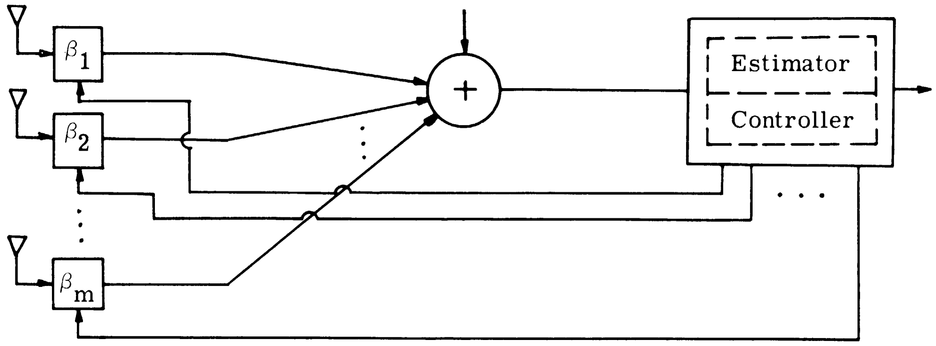

Fig. 13. Direction-finding system using an adder as a signal processor.

The controller-estimator unit is to be designed to have an output $\hat{\alpha}$ which is an optimal estimate (in a sense specified below) of the angle α. The control unit is to be designed to vary the phase shifts $\beta_1, \ldots, \beta_m$ in such a way that the signal-to-noise ratio at the output of the signal processor is maximized. The "effective direction" of maximum gain (see Fig. 14) of the antenna array can be varied by changing the phase shifts $\beta_1, \ldots, \beta_m$.[37]

[37] To see this, note that the phase difference between the signals received by different antennas in the array depends on the spacing between the antennas and the angle of incidence of the plane wave on the linear array. (See the discussion given above concerning Fig. 12.) Thus, if the phase shifts of the antenna filters $\beta_1, \ldots, \beta_m$ are chosen to cancel these phase differences, the signal level at the output of the signal processor will be a maximum. Thus we refer to the effective direction of maximum gain of the antenna array as the direction from which the received signal will yield maximum signal level at the output of the signal processor.

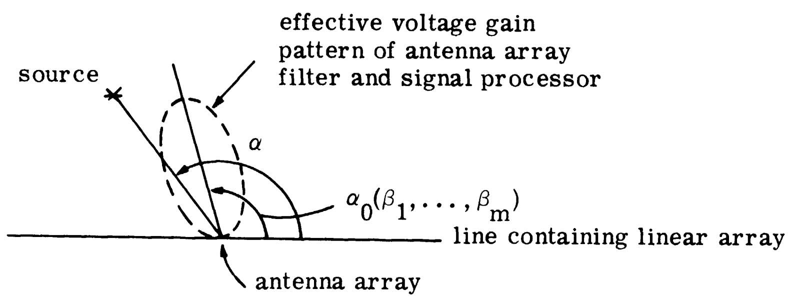

Fig. 14. Effective antenna gain pattern.

The effective direction of maximum gain is taken to be the direction of maximum array gain as measured at the output of the signal processor. We let $\alpha_0 = \alpha_0(\beta_1, \ldots, \beta_m)$ (not to be confused with the source angle α) denote this angle corresponding to the effective direction of maximum gain.

We assume that the spacing of the antennas in the array is such that the noise present on different antennas is independent. Then it is reasonable to define the optimal estimate $\hat{\alpha}$ as that which maximizes the signal-to-noise ratio at the output of the signal processor. Thus the control unit must be designed to choose the phase shifts $\beta_1, \ldots, \beta_m$ which maximize the signal-to-noise ratio at the output of the signal processor. The corresponding angle $\alpha_0(\beta_1, \ldots, \beta_m)$ will then equal $\hat{\alpha}$.

Note that in this case we have defined the optimal estimate in such a way that the optimal controller and the optimal estimator can be designed separately. The details of determining how the maximum signal-to-noise ratio is to be obtained are not considered here.

In conclusion, there is generally no one "best" method for designing a direction-finding system have a given form. It is usually difficult to say what cost function is best for any particular system. There are other factors such as reliability of operation, or (dollar) cost of building the system, which help to determine which "optimal" system should be used for a particular direction-finding application.

7. CONCLUSIONS

This memorandum illustrates a variety of electronic warfare problems which can be treated using the techniques of modern control theory. A quadratic type cost function was used for those problems in which the resulting optimal control problem was practically significant. The discrete-time formulation was used for the problems considered because the resulting difference equations are more easily handled on a digital computer and because of the added insight it provides into possible methods of solution.

The problems formulated in this memorandum are only representative of the types of problems which can be treated within the framework of modern control theory; they do not exhaust the class of electronic warfare problems which can be considered. Moreover, it should be kept in mind that solutions have not yet been obtained for all of the problems formulated in this memorandum.

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13. ABSTRACT

This memorandum considers several electronic warfare problems which can be treated using the techniques of modern control theory. It is shown that these electronic warfare problems, which deal with jamming techniques, systems organization, communications systems and direction-finding systems, can be formulated as discrete-time optimal control problems. In some cases the formulations allow solutions to be obtained immediately. The formulation itself yields new insight in other instances where solutions have not yet been attempted.

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