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HARMONIC CURRENTS OF FINITE ENERGY AND LAMINATIONS

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Abstract. We introduce a notion of energy for harmonic currents of bidegree $(1,1)$ on a complex Kähler manifold (M,ω). This allows us to define $\int T \wedge T \wedge \omega^{k-2}$, for positive harmonic currents. We then show that for a lamination with singularities of a compact set in $\mathbb{P}^2$, without directed positive closed currents, there is a unique positive harmonic current which minimizes energy. If X is a compact laminated set in $\mathbb{P}^2$ of class $\mathcal{C}^1$ it carries a unique positive harmonic current T of mass 1. The current T can be obtained by an Ahlfors type construction starting with an arbitrary leaf of X. When X has a totally disconnected set of singularities, contained in a countable union of analytic sets, the above construction still gives positive harmonic currents.

1 Introduction

The notion of invariant probability measure is a quite central notion in dynamical systems. It permits in particular to study statistical properties of orbits. The corresponding notion for smooth foliations of a compact Riemannian manifold was introduced by L. Garnett [G]: it is the notion of harmonic measure. She proved their existence and studied their ergodic properties. The article by A. Candel [Ca] contains a more recent approach to that theory.

Let X be a compact set in a complex manifold M. When X is laminated by Riemann surfaces, the Garnett result implies the existence of a positive current T of bidimension $(1,1)$ which is harmonic, i.e. such that $i\partial\overline{\partial}T = 0$. Moreover in a flow box B, the current can be expressed as

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha) \,,$$

the functions h_α are positive and harmonic on the local leaves V_α, and μ is a measure on the transversal. On the other hand there is a well-known

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problem. Does there exist a compact laminated set X in $\mathbb{P}^2$ which is not a compact Riemann surface? See [CLS], [BoLM], [Gh] and [Z], where the problem is discussed. If such an X does not exist then the closure of any leaf L of a holomorphic lamination in $\mathbb{P}^2$ will contain a singularity.

For nonexistence of Levi flat hypersurfaces in $\mathbb{P}^2$, see [S2], [CaoSW] and [I]. When X is a $\mathcal{C}^1$ laminated compact set in $\mathbb{P}^2$, so that no leaf is a compact Riemann surface, a result by Hurder–Mistumatsu [HM] implies that it supports no positive *closed* current directed by the lamination. Deroin [Der], using a result by S. Frankel [Fr] has shown that an immersed Levi-flat hypersurface X in $\mathbb{P}^2$ does not carry a positive harmonic current with a strictly positive continuous density with respect to Lebesgue measure.

In a recent paper Loray and Rebelo [LoR] have made important progress in the study of holomorphic foliations of $\mathbb{P}^k$ by Riemann surfaces. They show in particular that for any degree $d \geq 2$, there is a non-empty open set of foliations with singularities of degree d, such that *every* leaf is dense in $\mathbb{P}^k$. It seems that this should be the generic case for this class of holomorphic foliations with singularities. It is then of interest to extend Garnett's theory of harmonic measure to laminations with singularities.

In a recent work with B. Berndtsson, the second author, [BS] proved the existence of directed positive harmonic currents for holomorphic foliations with singularities. It is however useful to develop an intersection theory of such positive harmonic currents.

In this paper we study harmonic currents. For example in $\mathbb{P}^2$ a harmonic current T can be written in the form

$$T = c\omega + \partial S + \overline{\partial S}$$

where ω is the standard Kähler form on $\mathbb{P}^2, c \in \mathbb{R}$, S is a $(0,1)$ current. It turns out that $\overline{\partial} S$ depends only on T and that for a harmonic current one can define the energy $E(T)$ of T by the following integral, when $\overline{\partial} S$ is in L^2,

$$E(T) = \int \overline{\partial} S \wedge \partial \overline{S}$$

and that $0 \leq E(T) < \infty$. It is hence possible to introduce a Hilbert space of classes of currents of finite energy. We prove that positive harmonic currents have finite energy. With this in hand, the integral

$$Q(T) = \int T \wedge T$$

makes sense for positive harmonic currents and has the usual meaning when T is smooth. The theory extends to compact Kähler manifolds. We apply this intersection theory to positive harmonic currents directed by a laminated set with singularities in a complex Kähler surface M. More precisely we consider compact sets X laminated by Riemann surfaces out of an exceptional set E. We will assume that E is locally pluripolar and that $\overline{X \setminus E} = X$. We will call such a set $(X, \mathcal{L}, E)$ a laminated compact set with singularities. On such sets, we consider harmonic currents T of order 0 and bidegree $(1,1)$ in M. We then prove (Theorem 3.9)

Theorem 1.1. *Let $(X, \mathcal{L}, E)$ be a laminated compact set with singularities on a compact Kähler surface. There is a closed positive laminated current on X or there is a* unique *positive harmonic laminated current T on X minimizing energy.*

We then study the geometric intersection of laminated currents. We show, see Theorem 3.9 and Theorem 6.2,

Theorem 1.2. *If X is a $\mathcal{C}^1$ laminated compact set in $\mathbb{P}^2$, not containing a compact curve, then X carries a* unique *laminated positive harmonic current T of mass 1. The class of the current T is extremal in the cone of positive harmonic currents. Moreover $\int T \wedge T = 0$.*

A problem that is left open in our approach is how to estimate

$$c(\mathbb{P}^2) := \inf \left\{ \int T \wedge T \, ; \int T \wedge \omega = 1 \, , \ T \geq 0 \, , \ i\partial\overline{\partial} T = 0 \right\} .$$

If $c(\mathbb{P}^2) > 0$, then there is no $\mathcal{C}^1$ laminated set in $\mathbb{P}^2$.

When X is not $\mathcal{C}^1$ we have to assume "finite transverse energy" to get the result in Theorem 1.2. The main tool is that the quadratic form Q is negative definite on the hyperplane $\{T; \int T \wedge \omega = 0\}$, Corollary 2.10.

It follows from results of Lins Neto and Soares [LS] that a generic holomorphic foliation of $\mathbb{P}^k$ by Riemann surfaces does not admit a directed positive *closed* current. So in particular the above results apply to the foliations studied by Loray–Rebelo. In that case the minimal set X is equal to $\mathbb{P}^2$.

When X is a compact space and $f : X \to X$ is a continuous map one can construct invariant measures by taking cluster points of $\frac{1}{N} \sum_{i=1}^{N-1} \delta_{f^i(x)}$, where δ_a denotes the Dirac mass at the point a. When f is uniquely ergodic all such cluster points are equal to the unique invariant measure μ.

We introduce an averaging process à la Ahlfors along the leaves. We recall the classical Ahlfors process. Given a holomorphic map $\phi : \mathbb{C} \to M$, let Δ_r denote the disc of radius r. We define the current $S_r = \frac{[\phi_*(\Delta_r)]}{\|[\phi_*(\Delta_r)]\|}$ where $\|[\phi_*(\Delta_r)]\|$ denotes the mass of the current of integration on $\phi_*(\Delta_r)$.

It follows from the estimates of Ahlfors that there are sequences $r_n \to \infty$ such that S_{r_n} converges weakly to a positive closed current S. In our case, there is no positive closed current on X directed by $\mathcal{L}$. Hence the leaves are covered by the unit disc. Mimicking the Ahlfors approach we construct instead a positive harmonic current. Let $\phi : \Delta \to L$ be the universal covering map from the unit disc to a leaf L. Let $G_r(z) = \frac{1}{2\pi} \log^+ \frac{r}{|z|}$. Define $T_r := \phi_*(G_r[\Delta])$, $r < 1$. If $A(r)$ is the mass of T_r we define

$$\tau_r := \frac{T_r}{A(r)} .$$

When $(X, \mathcal{L}, E)$ is a laminated set with singularities such that E is totally disconnected, and contained in a countable union of analytic sets disjoint from $X \setminus E$, we show that all cluster points of (τ_r) are positive harmonic currents directed by $\mathcal{L}$. For that purpose we need to estimate the derivative of ϕ, the estimates are valid for any laminated set. This is the analogue of the Krylov–Bogoloubov construction of invariant measures. It will be of interest to prove an ergodic theorem using this averaging procedure. When X is a $\mathcal{C}^1$ laminated set in $\mathbb{P}^2$ we get, since it carries a unique positive harmonic current T of mass one, that

$$\lim_{r\to 1} \tau_r = T .$$

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2 Harmonic Currents

2.1 Harmonic currents.

DEFINITION 2.1. *Let M be a compact complex manifold of dimension k. For $0 \le p, q \le k$, let T be a (p, q) current on M of order 0. We say that T is harmonic if $i\partial\overline{\partial}T = 0$.*

Observe that if T is harmonic then $\overline{T}$, the conjugate, is also harmonic. A current is real if $T = \overline{T}$, in which case $p = q$.

Decomposition of harmonic currents. We want to prove a representation theorem for real harmonic currents on compact Kähler manifolds. So from now on M will be a compact Kähler manifold.

There is also a notion of $\Box$ harmonic forms in the $\overline{\partial}$ literature [FK], [De1]. These are smooth (p, q) forms Ω for which $\Box\Omega = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\Omega = 0$. These forms consist of the common null space of $\overline{\partial}$ and $\overline{\partial}^*$. Note that when $p = q$, $\overline{\Box} = \Box$ so the conjugate of a $\Box$ harmonic form is also $\Box$ harmonic.

Since $\square$ harmonic forms are $\overline{\partial}$ closed they are also harmonic in the sense of currents as defined above. The operator $\square$ is elliptic, selfadjoint. For basics on Hodge theory we refer to [De1]. According to the theory of elliptic operators there is an operator G so that $\mathrm{Id} = H + G\square = H + \square G$. The operator $H = H^2$ is a projection on $\square$ harmonic forms and G is a Green operator, which extends to currents and is continuous for the weak topology of currents.

On a compact Kähler manifold, for a closed current u of bidegree (p,q), the following are equivalent:

i) u is exact;
ii) u is $\overline{\partial}$ exact;
iii) u is $\partial\overline{\partial}$ exact.

See Demailly [De1, p. 41] for smooth forms. The proof is the same for currents since cohomology groups for currents and smooth forms are the same [De2].

PROPOSITION 2.2. *Let T be a harmonic (p,q) current on a compact Kähler manifold M of dimension k. Then*

$$T = \Omega + \partial S + \overline{\partial} R \tag{1}$$

where Ω is a unique closed smooth $\square$ harmonic form of bidegree (p,q), and S is a current of bidegree $(p-1,q)$, R is of bidegree $(p,q-1)$. When T is real we can choose $R = \overline{S}$. If T and dT are of order 0, *one can choose S, R of order* 0. *Moreover the linear map*

$$L : T \to (\Omega, S, R)$$

is continuous in the topology of currents.

Proof. The current $\overline{\partial}T$ is ∂ closed, hence d closed and is $\overline{\partial}$ exact. It follows from the $\partial\overline{\partial}$ lemma that $\overline{\partial}T$ is $\partial\overline{\partial}$ exact so there is a current S_0 of bidegree $(p-1,q)$ such that

$$\overline{\partial}T = \overline{\partial}\partial S_0 \,.$$

One can choose the canonical solution given by Hodge theory, hence it satisfies $\langle S_0, \beta\rangle = 0$ for all $\square$ harmonic forms β. Then S_0 depends linearly on T and is of order zero if $\overline{\partial}T$ is of order zero.

Let Ω be a smooth $\square$ harmonic representative of the Dolbeault cohomology class of $T - \partial S_0$. (See [De1] and [V].) Then $T - \partial S_0 - \Omega$ is $\overline{\partial}$ exact. Hence

$$T = \Omega + \partial S_0 + \overline{\partial} R \,.$$

If T is real we obtain that T can be expressed as claimed.

The current T acts on $H^{n-p,n-q}$, the Dolbeault cohomology group, because of the $\partial\overline{\partial}$ lemma. Hence Ω is uniquely determined by T.

The continuity follows from the ellipticity of $\square$ [De1, p. 17]. $\square$

In [DiS1, Proposition 2.1] an explicit kernel is given to solve $\partial\overline{\partial}$ on a compact Kähler manifold when the right-hand side is a difference of positive closed currents. In this case the solution is a difference of negative forms.

PROPOSITION 2.3. *If T is as in (1), then S, R are not unique, but any other S', R' can be obtained as $S' = S + \Omega' + \partial v + \overline{\partial} u$, similarly for R'.*

Proof. The cohomology class Ω is defined uniquely. If S', R' is another solution we get $\partial(S - S') + \overline{\partial}(R - R') = 0$. Assume $\partial\sigma + \overline{\partial}\sigma' = 0$. Then σ is harmonic. Using the above construction for a harmonic $(p-1, q)$ form, we get

$$S - S' = \sigma = \Omega' + \partial v + \overline{\partial} u\,.$$

Hence

$$S = S' + (\Omega' + \partial v + \overline{\partial} u)\,.$$

$\square$

COROLLARY 2.4. *Let T be a harmonic current of bidegree $(1, p)$ on (M, ω). Let $T = \Omega + \partial S + \overline{\partial} R$ be any decomposition as in (1). Then $\overline{\partial} S$ is uniquely determined by T. In particular it does not depend on the Kähler structure on M. If $p = 1$ and T is real, then T is closed if and only if $\overline{\partial} S = 0$.*

Proof. If S', R' also satisfy $T = \Omega + \partial S' + \overline{\partial} R'$ then, following Proposition 2.3, for bidegree reasons $S' = S + \Omega' + \overline{\partial} u$. Consequently $\overline{\partial} S' = \overline{\partial} S$. Assume that T is a real $(1,1)$ current. If T is closed, then $\overline{\partial} T = 0$; hence, S_0 in the proof of Proposition 2.2 can be chosen to be zero, i.e. $\overline{\partial} S = 0$. Conversely, if $T = \Omega + \partial S + \overline{\partial S}$ and $\overline{\partial} S = 0$, and θ is any test form, we have

$$\langle T, \overline{\partial}\theta\rangle = -\langle \overline{\partial} T, \theta\rangle = \langle -\overline{\partial}\partial S, \theta\rangle = 0\,.$$

Hence T is closed. $\square$

As Proposition 2.2 shows, the space of harmonic currents is infinite dimensional. The cone of positive harmonic currents is also infinite dimensional. It posseses however some properties reminiscent of rigidity properties of analytic objects.

- The complement of the support of a positive harmonic current of bidimension $(k-1, k-1)$ is pseudoconvex. For bidegree (s, s) it is $(k-s, k-s)$ pseudoconvex. See [FoS, Corollary 2.6].
- It is an observation by Skoda [Sk] that for a positive harmonic current T of bidimension (p, p) it is possible to define a Lelong number $\nu(T, a)$ at every point.

$$\nu(T, a) = \lim_{r\to 0} \frac{1}{r^{2p} c_p} \int_{B(a,r)} T \wedge \omega^p.$$

This permits us to show that positive harmonic currents give no mass to sets of $2p$-Hausdorff dimension 0 [BS, p. 389].

- For positive harmonic currents of bidimension (1,1) one can solve the $\overline{\partial}$ equation on the current with L^2 estimates [BS].

For a positive current T of bidimension (p,p) we define the mass norm of T as $\|T\| = \int T \wedge \omega^p$.

In order to compute with positive harmonic currents on compact Kähler manifolds, it is useful to approximate such currents by smooth ones. This is easy when the manifold is homogeneous, but false in general. The following result recently proved by Dinh and the second author is sufficient for most computations.

Theorem 2.5 [DiS2]**.** *Let T be a positive harmonic current of bidegree (p,p) on a compact Kähler manifold (M,ω). Then there exist smooth positive harmonic forms $T_n^{\pm}$ such that $T_n^+ - T_n^- \to T$ weakly. Moreover, $\|T_n^{\pm}\| \leq C_M \|T\|$, where C_M is independent of T.*

2.2 Energy of harmonic currents. In this paragraph we introduce a notion of energy of harmonic currents of bidegree $(1,1)$ on a compact Kähler manifold (M,ω) of dimension k. We normalize ω so that $\int \omega^k = 1$.

We showed above that if T is a real harmonic $(1,1)$ current on M, then it can be represented as

$$T = \Omega + \partial S + \overline{\partial S}$$

with S of bidegree $(0,1)$, Ω is a $\Box$ harmonic form and $\overline{\partial} S$ is uniquely determined. We define the energy $E(T) = E(T,T)$ of T as

$$E(T,T) = \int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2}$$

when $\overline{\partial} S \in L^2$. Observe that $\overline{\partial} S$ is a $(0,2)$ form. Hence $0 \leq E(T,T) < \infty$. We have seen in Corollary 2.4 that the energy depends on T only, not on the choice of S.

Let $\langle \Omega_1, \Omega_2 \rangle$ be some scalar product on the finite dimensional space of $\Box$ harmonic $(1,1)$ forms.

We define $\mathcal{H}_e$ to be the space of real harmonic $(1,1)$ currents on M of finite energy. We consider on $\mathcal{H}_e$ the following (real) inner product and semi norm:

$$\langle T_1, T_2 \rangle_e = \langle \Omega_1, \Omega_2 \rangle + \tfrac{1}{2} \int \overline{\partial} S_1 \wedge \partial \overline{S}_2 \wedge \omega^{k-2} + \tfrac{1}{2} \int \overline{\partial} S_2 \wedge \partial \overline{S}_1 \wedge \omega^{k-2}$$

$$\|T\|_e^2 = \langle \Omega, \Omega \rangle + \int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2}.$$

PROPOSITION 2.6. *Let* $T = \Omega + \partial S + \overline{\partial S}$, $\overline{\partial} S \in L^2$, *a* $(1,1)$ *real harmonic current of order* 0 *in* M. *Then* $\|T\|_e = 0$ *if and only if* $T = i\partial\overline{\partial}u$, *for* $u \in L^1$, u *real.*

Proof. If $\|T\|_e = 0$, then $\overline{\partial} S = 0$, hence T is closed. Since $\Omega = 0$, T is exact and therefore $T = i\partial\overline{\partial}u$ [De1]. Regularity of the Laplace equation shows that $u \in L^1$. Conversely, suppose that $T = i\partial\overline{\partial}u$, $u \in L^1$, u real, so we can set $S = \frac{1}{2}i\overline{\partial}u$. Then $\overline{\partial}S = 0$, hence $\int \overline{\partial}S \wedge \partial\overline{S} \wedge \omega^{k-2} = 0$. Clearly also, the corresponding $\Box$ harmonic form vanishes, so $\|T\|_e = 0$. □

PROPOSITION 2.7. *There is a constant* $C = C(M)$ *so that if* T *is a real harmonic* $(1,1)$ *current of order* 0 *on* M, *with finite energy, then there is an element* $\tilde{T}$ *in the equivalence class of* T, *i.e.* $\|T - \tilde{T}\|_e = 0$, *which can be written as* $\tilde{T} = \Omega + \partial S + \overline{\partial S}$ *with* $S, \partial S, \overline{\partial} S \in L^2$, $\|S\|_{L^2}, \|\partial S\|_{L^2}, \|\overline{\partial} S\|_{L^2} \leq C\|T\|_e$. *Hence* $T = \tilde{T} + i\partial\overline{\partial}u$ *and* $i\partial\overline{\partial}u$ *is of order* 0.

Proof. We can write $T = \Omega + \partial S_1 + \overline{\partial S_1}$ with $\overline{\partial} S_1 \in L^2$. Since $\overline{\partial} S_1$ is in L^2, we can find an $S \in L^2_{01}$ for which $\overline{\partial} S_1 = \overline{\partial} S$. Moreover, $\partial S \in L^2$ as well. Recall that by Hodge theory on compact Kähler manifolds, if a form α is in L^2 and is $\overline{\partial}$ exact, then the equation $\overline{\partial}u = \alpha$ admits a solution in the Sobolev space H^1. So there is a gain of one derivative. The L^2 estimates are classical [FK], [De1].

By Proposition 2.3, there is a $\Box$ harmonic form Ω' and a distribution v for which $S_1 - S = \Omega' + \overline{\partial} v$. Therefore we have the decomposition

$$\begin{aligned} T &= \Omega + \left[\partial(S + \overline{\partial} v)\right] + \overline{\left[\partial(S + \overline{\partial} v)\right]} \\ &= \Omega + \partial S + \overline{\partial S} + \partial\overline{\partial} v + \overline{\partial}\partial\overline{v} \\ &= \tilde{T} + i\partial\overline{\partial}\left(\frac{v - \overline{v}}{i}\right). \end{aligned}$$

The distribution $u := \frac{v - \overline{v}}{i}$ is real. Since $T, \tilde{T}$ have order 0, $i\partial\overline{\partial}u$ is also of order 0.

By Proposition 2.6, $\|\partial\overline{\partial}u\|_e = 0$. It follows that $\tilde{T} := \Omega + \partial S + \overline{\partial S}$ is in the equivalence class of T as desired. □

Let H_e denote the quotient space of equivalence classes $[T]$ in $\mathcal{H}_e$.

PROPOSITION 2.8. *The space* H_e *is a real Hilbert space. Every element* $[T]$ *in* H_e *can be represented as*

$$T = \Omega + \partial S + \overline{\partial S}$$

where S *is a* $(0,1)$ *form in* L^2, *with* ∂S *and* $\overline{\partial} S$ *in* L^2. *Convergence in* H_e *implies weak convergence of currents. More precisely, if* $[T_n] \to [T]$ *in* H_e

then there are representatives $\tilde{T}_n \in [T_n]$ such that $\tilde{T}_n \to T$ in the weak topology of currents. In fact the mass norms $\|\tilde{T}_n - T\| \to 0$.

Proof. We show first that H_e is complete. Let $\{[T_n]\}$ be a Cauchy sequence of equivalence classes, $\lim_{n,m\to\infty} \|T_n - T_m\|_e = 0$. We can suppose $\|T_{n+1} - T_n\|_e < 1/2^n$. Inductively, we can (Proposition 2.7) choose representatives $\tilde{T}_n$ so that

$$\tilde{T}_{n+1} = \tilde{T}_n + \Omega_n + \partial S_n + \overline{\partial S_n}\,, \quad \|\Omega_n\|, \|\partial S_n\|_{L^2}, \|S_n\|_{L^2}, \|\overline{\partial} S_n\|_{L^2} \le C\tfrac{1}{2^n}\,.$$

Hence $\{[T_n]\}$ converges in H_e. This shows that H_e is complete. The last statement is similar. □

Next, we will introduce a notion of wedge product of real harmonic currents of finite energy. Let T, T' be representatives of equivalence classes as above,

$$T = \Omega + \partial S + \overline{\partial S}\,, \quad T' = \Omega' + \partial S' + \overline{\partial S'}\,.$$

Then a formal calculation gives

$$\begin{aligned}\int T \wedge T' \wedge \omega^{k-2} &= \int \Omega\wedge\Omega'\wedge\omega^{k-2} + \int \partial S\wedge\overline{\partial S'}\wedge\omega^{k-2} + \int \overline{\partial S}\wedge\partial S'\wedge\omega^{k-2} \\ &= \int \Omega\wedge\Omega'\wedge\omega^{k-2} - \int \overline{\partial} S\wedge\partial\overline{S'}\wedge\omega^{k-2} - \int \partial\overline{S}\wedge\overline{\partial} S'\wedge\omega^{k-2}.\end{aligned}$$

Notice that if T, T' have finite energy, the last expression is well defined. We define in this case the quadratic form $Q(T, T')$ for currents T, T' of finite energy:

$$Q(T, T') = \int \Omega\wedge\Omega' \wedge \omega^{k-2} - \int \overline{\partial} S \wedge \partial\overline{S'} \wedge \omega^{k-2} - \int \partial\overline{S} \wedge \overline{\partial} S' \wedge \omega^{k-2}$$

and, motivated by the formal calculation, we define

$$\int T \wedge T' \wedge \omega^{k-2} := Q(T, T')$$

when T, T' are harmonic $(1,1)$ currents on M with finite energy. Recalling the definition of energy, we get

$$\int T \wedge T \wedge \omega^{k-2} = Q(T, T) = \int \Omega \wedge \Omega \wedge \omega^{k-2} - 2E(T, T)\,.$$

Note that $Q(T, T')$ is well defined on equivalence classes in H_e.

Theorem 2.9. *Any positive harmonic current T of bidegree $(1,1)$ on M is of finite energy. If a sequence (T_n) of positive harmonic currents converge weakly to T, then $[T_n] \to [T]$ weakly in H_e. The quadratic form $Q(T, T')$ is continuous on H_e. If M is homogeneous then $\int T \wedge T \wedge \omega^{k-2} \ge 0$ when T is a positive harmonic current.*

Proof. Assume first that T is smooth, then $T = \Omega + \partial S + \overline{\partial S}$ with Ω smooth and S smooth. We get, after integration by parts,

$$\begin{aligned}\int T \wedge T \wedge \omega^{k-2} &= \int \Omega \wedge \Omega \wedge \omega^{k-2} + 2 \int \partial S \wedge \overline{\partial S} \wedge \omega^{k-2} \\ &= \int \Omega \wedge \Omega \wedge \omega^{k-2} - 2 \int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2} \\ &\geq 0 .\end{aligned}$$

So

$$2 \int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2} \leq \int \Omega \wedge \Omega \wedge \omega^{k-2}.$$

Assume (T_n^+) is a sequence of smooth harmonic currents, $T_n^+ \to T^+$ weakly.

Then we can write $T_n^+ = \Omega_n + \partial S_n + \overline{\partial S_n}$ and $T^+ = \Omega + \partial S + \overline{\partial S}$, with $\Omega_n \to \Omega$ and $S_n \to S$ weakly. Indeed S_n and S are constructed using canonical solutions to the equations $\overline{\partial} T_n^+ = \partial \overline{\partial} S_n, \overline{\partial} T^+ = \partial \overline{\partial} S$.

We have seen that

$$2 \int \overline{\partial} S_n \wedge \partial \overline{S}_n \wedge \omega^{k-2} \leq \int \Omega_n \wedge \Omega_n \wedge \omega^{k-2}.$$

The T_n^+ have bounded mass, hence their classes Ω_n are bounded. Using a basis for the finite dimensional space $H^{(1,1)}$, it follows that $-C\omega \leq \Omega_n \leq C\omega$.

Hence the left-hand side is bounded by a fixed constant. So any weak limit of $\overline{\partial} S_n$ is equal to $\overline{\partial} S$ and hence

$$2 \int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2} \leq \int \Omega \wedge \Omega \wedge \omega^{k-2}.$$

Consequently T^+ is in H_e. The same argument shows that $\overline{\partial} S_n \rightharpoonup \overline{\partial} S$ in L^2 and hence $[T_n] \rightharpoonup [T]$ weakly in H_e.

Theorem 2.5 implies that any positive harmonic current T on M, can be written $T = T^+ - T^-$, with $T^\pm = \lim T_n^\pm$, and $\|T_n^\pm\| \leq C_M \|T\|$, $T_n^\pm$, T positive, $T_n^\pm$ smooth. So T is in H_e and

$$\int \overline{\partial} S \wedge \partial \overline{S} \leq \int \Omega^+ \wedge \Omega^+ \wedge \omega^{k-2} + \int \Omega^- \wedge \Omega^- \wedge \omega^{k-2} \leq C\|T\| ,$$

where C depends only on M. It remains to show the last part of the theorem. First we show that $\int T \wedge T \wedge \omega^{k-2} \geq 0$ if $T \geq 0$ can be approximated by positive smooth harmonic currents. Secondly we show that smoothing works when M is homogeneous.

If T, T' are in H_e then we have the estimate

$$\left| Q(T, T') \right| \leq C \|T\|_e \|T'\|_e .$$

So Q is continuous on H_e.

If $T = \lim T_n$, with $T_n \geq 0$ smooth and harmonic, then $\int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2} \leq \liminf \int \overline{\partial} S_n \wedge \partial \overline{S}_n \wedge \omega^{k-2}$. Hence $\int T \wedge T \wedge \omega^{k-2} \geq \limsup \int T_n \wedge T_n \wedge \omega^{k-2} \geq 0$. So $\int T \wedge T \wedge \omega^{k-2} \geq 0$.

When M is homogeneous, any positive harmonic current T is the limit of a sequence of positive smooth harmonic currents. Indeed define $T_\epsilon = \Omega_\epsilon + \partial S_\epsilon + \overline{\partial S_\epsilon}$ with

$$S_\epsilon = \int \rho_\epsilon(g) g_* S d\nu(g) \,,$$

ν is the Haar measure on the component of the identity in $\mathrm{Aut}_0(M)$, and ρ_ϵ is an approximation of unity on $\mathrm{Aut}_0(M)$.

In this case we have $\overline{\partial} S_\epsilon \to \overline{\partial} S$ in L^2 and hence $E(T_\epsilon, T_\epsilon) \to E(T,T)$. $\square$

One can define a notion of positivity in H_e. A class $[T]$ is positive if there is a positive harmonic current in the class $[T]$. We denote the convex cone of positive classes by $\mathcal{K}_e$. If $T = T_1 + i\partial\overline{\partial} u$ and $-T = T_2 + i\partial\overline{\partial} v$, with $T_j \geq 0$, then $T_1 + T_2 = i\partial\overline{\partial} w$. It follows that $\int (T_1 + T_2) \wedge \omega^{k-1} = 0$ so $T_1 = T_2 = 0$. Consequently $\mathcal{K}_e \cap (-\mathcal{K}_e) = 0$. The cone $\mathcal{K}_e$ is closed in H_e. If $T_n \geq 0$ and $[T_n] \to [T_0]$ then $\int T_n \wedge \omega^{k-1}$ is bounded. So we can assume that (T_n) converges weakly to the positive current T_0 and by Theorem 2.9 $[T_0'] = [T_0]$.

We have seen that any element T of H_e can be written

$$T = \Omega + \partial S + \overline{\partial S}$$

where Ω is a $\square$ harmonic form. By Proposition 2.2, Ω is unique, so the map $T \to \Omega$ is linear. Moreover $T \to \Omega$ is continuous in the topology of currents.

We consider as in Hodge theory the "primitive" classes of harmonic currents. Let $[\Omega]$ and $[\omega]$ denote the classes in the Dolbeault cohomology group $H^{(1,1)}(M)$. Note that $[\Omega] \wedge [\omega]^{k-1}$ is a (k,k) form proportional to $[\omega]^k$.

Define

$$\mathcal{H} = \big\{T; T \in H_e \,,\ [\Omega] \wedge [\omega]^{k-1} = 0\big\} = \Big\{T; T \in H_e \,,\ \int T \wedge \omega^{k-1} = 0\Big\}.$$

Clearly $\mathcal{H}$ is a hyperplane in H_e. It is the hyperplane of harmonic currents with primitive class Ω, in fact this is how primitive classes are defined, i.e. by $[\Omega] \wedge [\omega]^{k-1} = 0$. In this context, we give a version of the Riemann–Hodge theorem.

Corollary 2.10. *On H_e, the quadratic form*

$$Q(T_1, T_2) = \int T_1 \wedge T_2 \wedge \omega^{k-2}$$

is strictly negative definite on the hyperplane $\mathcal{H}$. *If* M *is homogeneous and if* T, T' *are positive harmonic currents, non-proportional, then* $\int T \wedge T' \wedge \omega^{k-2} > 0$. *If* $[\Omega]$ *is the class of a positive harmonic current* T *and* $\int \Omega \wedge \Omega \wedge \omega^{k-2} = 1$, *then* T *is non-closed if and only if* $0 \leq \int T \wedge T \wedge \omega^{k-2} < 1$.

Proof. Assume that the $\Box$ harmonic form corresponding to T satisfies $[\Omega] \wedge [\omega]^{k-1} = 0$. The classical Riemann–Hodge theorem [GrH, p. 123] asserts that when $[\Omega] \neq 0$, we have

$$\int \Omega \wedge \Omega \wedge \omega^{k-2} < 0\,.$$

Hence

$$Q(T,T) = \int \Omega \wedge \Omega \wedge \omega^{k-2} - 2\int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2} < 0\,.$$

It is zero only if $T = 0$ in H_e. Hence Q is strictly negative definite on $\mathcal{H}$. Suppose the space generated by T', T is of dimension 2. There is an $a > 0$, so that $T' - aT \in \mathcal{H}$. Hence

$$\begin{aligned} 0 &> Q(T' - aT, T' - aT) \\ &= Q(T', T') + a^2 Q(T,T) - 2aQ(T', T)\,. \end{aligned}$$

When M is homogeneous we have seen in Theorem 2.9 that $Q(T', T'), Q(T,T) \geq 0$. It follows that $Q(T', T) > 0$.

The last part is an immediate consequence of Corollary 2.4. $\Box$

Proposition 2.11. *Let* (M, ω) *be a compact Kähler manifold. The function* $T \to Q(T,T)$ *is upper semi-continuous for the weak topology on* H_e *and for the weak topology on positive harmonic currents. It is strictly concave on* $\{T; \int T \wedge \omega = 1\}$.

Proof. First, we prove that $T \to Q(T,T)$ is upper semi-continuous for the weak topology of H_e. This means: Fix $T_1 \in H_e$ and $\epsilon > 0$. Let $T' \in H_e$. Then there is a $\delta > 0$ so that if

$$T_2 \in H_e\,, \quad \big|\langle T_2, T'\rangle_e - \langle T_1, T'\rangle_e\big| \leq \delta\,, \quad \text{then} \quad Q(T_2, T_2) < Q(T_1, T_1) + \epsilon\,.$$

The function Q is the sum of a continuous function $\int \Omega \wedge \Omega \wedge \omega^{k-2}$, and of $-\int \overline{\partial} S \wedge \partial \overline{S} \wedge \omega^{k-2} = -\|T\|_e^2 + \langle \Omega, \Omega \rangle$. Hence it is upper semi-continuous for the weak topology on H_e. We have shown in Theorem 2.9 that if $T_n \geq 0$ and $T_n \rightharpoonup T$ weakly as positive currents, then $T_n \to T$ weakly in H_e.

To prove concavity, observe that if $\int (T - T') \wedge \omega^{k-1} = 0$ then $T - T' \in \mathcal{H}$. Hence $Q(T - T', T - T') < 0$, so $2Q(T, T') > Q(T,T) + Q(T', T')$. Hence $Q\big(\frac{T+T'}{2}, \frac{T+T'}{2}\big) > \frac{1}{2}Q(T,T) + \frac{1}{2}Q(T', T')$. $\Box$

Notice that $C = \mathcal{K}_e \cap \{T; \int T \wedge \omega^{k-1} = 1\}$ is weakly compact in H_e.

Proposition 2.12. *Let M be a compact Kähler manifold. Let $\mathcal{C} = \mathcal{K}_e \cap \{T; \int T \wedge \omega^{k-1} = 1\}$. If $T_0 \in \mathcal{C}$ and $Q(T_0, T_0) = \inf\{Q(T,T); T \in \mathcal{C}\}$ $=: m$, then $[T_0]$ is extremal in $\mathcal{C}$. In particular, if M is homogeneous and $Q(T_0,T_0) = 0$, then $[T_0]$ is extremal in $\mathcal{K}_e$.*

Proof. If $[T_0] = \frac{[T_1]+[T_2]}{2}$ and $Q(T_j,T_j) \geq m, j = 1,2$ then $Q(T_0,T_0) > m$ by strict concavity of Q. When M is homogeneous we have seen in Theorem 2.9 that $Q(T,T) \geq 0$ for every $T \in \mathcal{K}_e$. □

The next proposition shows that the energy norm on currents gives a different topology than the weak topology on positive harmonic currents.

Proposition 2.13. *The map $T \to \overline{\partial} S$ is not continuous for the weak topology on positive harmonic currents T of bidegree $(1,1)$ on $\mathbb{P}^2$ and L^2 topology on $\overline{\partial} S$. However, the map is continuous for the H_e topology on T and the L^2 topology on $\overline{\partial} S$.*

Proof. The standard Kähler form ω on $\mathbb{P}^2$ is a generator for $H^{(1,1)}(\mathbb{P}^2)$. In the case of $(1,1)$ harmonic currents T on $\mathbb{P}^2$, one can decompose $T = c\omega + \partial S + \overline{\partial S}$ if T is real, $c \in \mathbb{R}$. Let $T = \omega + \epsilon(\partial S + \overline{\partial S})$ for a smooth $(0,1)$ form S supported in the unit bidisc. For $\epsilon > 0$ small enough, T is positive and $\int T \wedge T = \int \omega \wedge \omega - 2\epsilon^2 \int \overline{\partial} S \wedge \partial \overline{S} < 1$. If the map $T \to \overline{\partial} S$ with weak topology on T and L^2 topology on $\overline{\partial} S$ were continuous then, when $T_n \to T_0$, $\int T_n \wedge T_n \to \int T_0 \wedge T_0$. Let f be an endomorphism of $\mathbb{P}^2$ of algebraic degree d. The map $f^* : \mathcal{H}_e \to \mathcal{H}_e$ is a linear map of norm d if the algebraic degree of f is d. Indeed

$$\begin{aligned}\left|\int f^*T \wedge \omega\right|^2 &= \left|\int T \wedge f_*\omega\right|^2 \\ &= d^2 \left|\int T \wedge \omega\right|^2,\end{aligned}$$

because $f_*\omega \sim d\omega$. We also have

$$\left|\int f^*(\overline{\partial} S \wedge \partial \overline{S})\right| = d^2 \left|\int \overline{\partial} S \wedge \partial \overline{S}\right|.$$

This can be obtained by smoothing.

Therefore $E(f^*T/d) = E(T)$ so $\int f^*T/d \wedge f^*T/d = \int T \wedge T$.

Let $f[z : w : t] = [z^2 : w^2 : t^2]$ and $T = \omega + \epsilon(\partial S + \overline{\partial S})$ as above. Then

$$\begin{aligned}1 &> \int T \wedge T \\ &= \int (f^n)^*T/2^n \wedge (f^n)^*T/2^n\end{aligned}$$

$$= \int (f^n)^*\omega/2^n \wedge (f^n)^*\omega/2^n - 2\int \overline{\partial}(f^n)^*(S)/2^n \wedge \partial(f^n)^*(\overline{S})/2^n$$
$$= 1 - 2\int \overline{\partial}(f^n)^*(S)/2^n \wedge \partial(f^n)^*(\overline{S})/2^n.$$

If we choose $S = a(|z|,|w|)d\overline{z}$ it is easy to check that $(f^n)^*S/2^n \to 0$ weakly. Hence $(f^n)^*T/2^n$ converges weakly to a closed current $A = \lim (f^n)^*\omega/2^n$ whose class in $H^{(1,1)}$ is $[\omega]$. If $T \to \overline{\partial}S$ were continuous, the second integral would converge to zero. Since $\int \omega\wedge\omega = 1$, we get that the map $T \to \overline{\partial}S$ is not continuous, since the first integral is equal to 1 and the second converges to zero.

The last statement follows from the definition of the weak topology in H_e. □

3 Laminated Compact Sets with Singularities

Let M be a compact complex manifold of dimension k. We will say that the triple $(X, \mathcal{L}, E)$ is a laminated compact with singularities if it satisfies the following conditions:

(i) X, E are compact subsets of M with $E \subset X$ and $\overline{X \setminus E} = X$;
(ii) E is locally a pluripolar set, called the singular set;
(iii) $X \setminus E$ has the structure of a lamination $\mathcal{L}$ by Riemann surfaces.

Recall that a closed set E is locally pluripolar if any point a in E has a neighborhood $U(a)$ such that $E \cap U(a) \subset \{z; z \in U(a), u(z) = -\infty\}$ where u is plurisubharmonic in $U(a)$. It is locally complete pluripolar if $E \cap U(a) = \{z; z \in U(a), u(z) = -\infty\}$. An analytic set is locally complete pluripolar. For basics on pluripolar sets and extension results of positive closed currents, see the book by Demailly [De2].

A closed subset Y of a complex manifold M is laminated by Riemann surfaces if it admits an open covering $\{U_i\}$ and on each U_i there is a homeomorphism $\phi_i = (h_i, \lambda_i) : U_i \cap Y \to \Delta(z_i) \times T_i(t_i)$ where Δ is the unit disc and T_i is a topological space. The ϕ_i^{-1} are holomorphic in z. Moreover,

$$\phi_{ij}(z_i, t_i) = \phi_j \circ \phi_i^{-1}(z_i, t_i) = \big(h_{ij}(z_i, t_i), \lambda_{ij}(t_i)\big)$$

where the $h_{ij}(z_i, t_i)$ are holomorphic with respect to z_i.

When T_i is in a Euclidean space and ϕ_i extend to $\mathcal{C}^k$ diffeomorphisms, we say that the lamination is $\mathcal{C}^k$. We call the U_i flow boxes, $\{\lambda_i = t_0\}$ is a plaque. A leaf is a minimal connected set such that if L intersects a plaque, then L contains the plaque. We will say that $(X, \mathcal{L}, E)$ is oriented if there are continuous non-vanishing $(1,0)$ forms γ_j, $j = 1, \dots, k-1$, defined on $X \setminus E$ such that $\gamma_j \wedge [\Delta_\alpha] = 0$ for every plaque Δ_α; $[\Delta_\alpha]$ denotes the current of integration on the disc Δ_α and the γ_j are pairwise linearly independent over $\mathbb{C}$. Otherwise, we can still find a larger finite set of γ_j of rank $k-1$ everywhere.

A positive current T of bidimension $(1,1)$ with support in X is said to be directed by $\mathcal{L}$ if on any open set U where $\mathcal{L}$ is defined by non-vanishing continuous $(1,0)$ forms γ_j, $j = 1 \dots, k-1$, i.e. $\gamma_j \wedge [\Delta_\alpha] = 0$, we have

$$T \wedge \gamma_j = 0\,.$$

The wedge product makes sense because T has measure coefficients and the forms γ_j are continuous.

It is interesting to introduce the notion of a minimal set for X.

Definition 3.1. *A minimal set for $(X, \mathcal{L}, E)$ is a compact subset $Y \subset X$ such that Y is not contained in E, moreover $Y \setminus E$ is a union of leaves L and for every leaf $L \subset Y \setminus E$, $\overline{L} = Y$.*

Proposition 3.2. *Let $(X, \mathcal{L}, E)$ be such that there is a neighborhood $\mathcal{V}$ of E, so that no leaf is contained in $\mathcal{V}$. Then there are minimal sets $Y \subset X$. Two different minimal sets intersect only on E. If M is a surface where the Levi problem is solvable and E is locally contained in a complex hypersurface, then any two minimal sets intersect.*

Proof. Let $\mathcal{V}$ be an open neighborhood of E such that no leaf is contained in $\mathcal{V}$. Let (X_α) be an ordered decreasing chain saturated for $\mathcal{L}$. Let $X'_\alpha := X_\alpha \cap (X \setminus \mathcal{V})$. Then $X'_\alpha \neq \emptyset$. Hence $\cap X_\alpha \neq \emptyset$. So Zorn's lemma applies. This shows that minimal sets exist.

If M is a surface, it follows that each $M \setminus Y$ is locally pseudoconvex away from E, and since E is locally contained in a complex hypersurface, $M \setminus Y$ is pseudoconvex [GrR]. If the Levi problem is solvable on M, i.e. if pseudoconvex domains are Stein, each $M \setminus Y$ is Stein. Since Stein manifolds cannot have two ends, it follows that any two minimal sets must intersect. □

Remark 3.3. *Fujita [Fu] solved the Levi problem in $\mathbb{P}^k$. See [E] for a proof of this and some generalizations. The condition on E in the above proposition, can be relaxed to assuming that E is meager in the sense of [GrR].*

Example 3.4. Let $\mathcal{L}_\alpha$ be the foliation in $\mathbb{P}^2$ defined in an affine chart by $wdz - \alpha z dw = 0$, with α real and irrational. Then $Y_c = \overline{\{|z| = c|w|^\alpha\}}$ is minimal for every c, the closure is in $\mathbb{P}^2$. Let π denote the projection from $\mathbb{C}^3 \setminus 0$ to $\mathbb{P}^2$. The associated positive closed current T is defined by $\pi^* T = i\partial\overline{\partial} u, u(z,w,t) = \log(\max\{|z||t|^{\alpha-1}, c|w|^\alpha\})$ if $\alpha > 1$ and is directed by $\mathcal{L}_\alpha$.

Garnett has shown in [G] the existence of positive currents T, satisfying $i\partial\overline{\partial} T = 0$ and directed by foliations. In [BS] a version of this result is given allowing singular sets. Here we are interested in constructing laminar currents for a foliation with singularities that are only holomorphic motions in flow boxes, the holomorphic case is treated in [BS].

Theorem 3.5. *Let $(X, \mathcal{L}, E)$ be a directed set with singularities in a surface M. Then there is a laminated harmonic positive current T, i.e. of the form $T = \int_\alpha h_\alpha [\Delta_\alpha] d\mu(\alpha)$ in flow boxes. Here $\mu(\alpha)$ is a measure on transversals, h_α are strictly positive harmonic functions, uniformly bounded above and below by strictly positive constants, h_α are Borel measurable with respect to α.*

Proof. Let $\{\gamma_j\}$ be a rank 1 finite collection of continuous $(0,1)$ forms such that $\gamma_j \wedge [\Delta] = 0$ for every plaque Δ. In [BS, Theorem 1.4] a current $T \geq 0$ supported on X satisfying $i\partial\overline{\partial} T = 0$ and $T \wedge \gamma_j = 0$ is constructed. The fact that E is locally pluripolar is used. It is shown that the current is laminar in flow boxes when the foliation is holomorphic. We consider here the general case, we need to modify the argument.

Let C be the convex compact set of positive currents of mass one, directed by the lamination and supported on X. Any element $T \in C$, can be written in a flow box B as $T = \|T\| i\gamma \wedge \overline{\gamma}$ where $\|T\|$ is a positive measure, possibly zero.

Assume $z = 0$ is a transversal and π is the projection along leaves in B, on $(z = 0)$. Let (ν_w) be the disintegration of $\|T\|$ along π. If ϕ is a smooth test form of bidegree $(1,1)$ supported in B, then

$$\begin{aligned}\langle T, \phi\rangle &= \int \langle \nu_w i\gamma \wedge \overline{\gamma}, \phi\rangle d\mu(w) \\ &= \int \big\langle \tilde{\nu}_w [D_w], \phi\big\rangle d\mu(w)\,,\end{aligned}$$

here $[D_w]$ denotes the current of integration on the plaque through w and $\tilde{\nu}_w$ is a measure. Let V_N denote the space of continuous functions on X, compactly supported in a union N of finitely many (open) flow boxes and $\mathcal{C}^2$ along leaves, with the topology of uniform convergence on X and $\mathcal{C}^2$ along leaves. We can use the previous formula to extend the action of T on $\partial_b\overline{\partial}_b\psi$ for $\psi \in V_N$. On a plaque where $w = f(z)$ we can identify $\partial_b\overline{\partial}_b\psi$ with $\frac{\partial^2\psi(z,f(z))}{\partial z\partial\overline{z}} dz \wedge d\overline{z} = \frac{\partial^2\psi(z,f(z))}{\partial z\partial\overline{z}}$ with abuse of notation. If (χ_j) is a partition of unity associated to $\cup B_j = N$ and $\psi \in V_N$, then we define

$$\langle T, \partial_b \overline{\partial}_b \psi \rangle := \sum_j \int \big\langle [D_w]\tilde{\nu}_w, \Delta_w(\chi_j \psi)\big\rangle d\mu(w)$$

where Δ_w denotes the Laplacian on D_w. Then T is continuous on V_N. Moreover if a sequence T_n in C converges weakly to T then it converges also weakly in the dual of V_N. Define $W_N = \{\beta\} \oplus V_N$. Here $\{\beta\}$ refers to the space of $(1,1)$ forms where β is a $\mathcal{C}^\infty$ $(1,1)$ form on M and we use the supnorm topology on X for the forms $\{\beta\}$. Let $\tilde{W}_N$ denote the Banach space completion of W_N. Then each $T \in C$ extends as a continuous linear functional on $\tilde{W}_N$. Hence we have a natural map $\Lambda : C \to \tilde{W}'_N$, the dual space of $\tilde{W}_N$. If $\{T_n\} \subset C$ is any sequence, replacing with a suitable subsequence we can assume that $T_n \to T \in C$ in the weak topology of currents. In addition to the norm topology on $\tilde{W}'_N$ there is also the weak topology from W_N, i.e. the weak topology for which all elements of W_N give rise to continuous linear functionals on $\tilde{W}'_N$. Since this topology is weaker, we obtain that $\Lambda(C)$ is a convex compact set in this weak topology. Let $B_N := \{i\partial\overline{\partial}\phi\} \oplus V_N$ as a subspace of W_N.

Suppose that $\Lambda(C) \cap B_N^{\perp} = \emptyset$. Then, by the Hahn–Banach theorem, it follows that there is a continuous linear functional $L : \tilde{W}'_N[W_N] \to \mathbb{R}$ so that $L \circ \Lambda(T) > \delta > 0$ for all $T \in C$. Moreover this linear functional is given by an element of B_N.

Hence there are functions $\phi_N \in \mathcal{C}^\infty(M)$ and $\psi_N \in V_N$ [we extend the latter trivially to X], such that $\langle T, i\partial\overline{\partial}\phi_N \oplus i\partial_b\overline{\partial}_b\psi_N\rangle \geq \delta > 0$ for every T in C. Hence the function $u_N := \phi_N + \psi_N$ is continuous and strictly subharmonic on leaves of $\mathcal{L}$. The max of u_N can be reached only at a point $z_0 \in E$. Choose r such that $E \cap B(z_0, r) \subset \{v = -\infty\}$ for v plurisubharmonic near $\overline{B}(z_0, r)$. Then $u - \frac{\delta}{2}|z - z_0|^2 + \epsilon v$ will still have a maximum at z_1 near z_0, $z_1 \notin E$, a contradiction.

Hence there is a current T_N in C vanishing on B_N. Taking a weak limit we get a harmonic current T such that for any ψ continuous in a flow box B and $\mathcal{C}^2$ along leaves

$$\int \big\langle [D_w]\tilde{\nu}_w, \Delta_w \psi \big\rangle d\mu(w) = 0\,.$$

Hence $\langle \tilde{\nu}_w[D_w], \Delta_w \psi\rangle = 0$ μ a.e. So $\tilde{\nu}_w$ is a positive harmonic function on the leaf $[D_w]$ μ a.e. □

The following corollary was suggested by T.C. Dinh.

Corollary 3.6. *Let $(X, \mathcal{L}, E)$ be a laminated set with singularities in M. Assume E does not support a non-zero positive harmonic current of* *bidimension* $(1,1)$. *Then there are minimal sets* $Y \subset X$. *In particular for* $X \subset \mathbb{P}^2$ *it is enough to assume that* $\mathbb{P}^2 \setminus E$ *is not contained in a Stein domain.*

Proof. Let X_α be an ordered decreasing chain saturated for $\mathcal{L}$. Let (T_α) be positive harmonic currents of bidimension $(1,1)$ of mass 1 supported on X_α. If T is a cluster point of (T_α), then T is supported by $\cap X_\alpha$, and not only on E. Hence $\cap(X_\alpha \setminus E)$ is non-empty and saturated for $\mathcal{L}$. In section 2, we have seen recalled that the complement of the support of a positive, harmonic current is pseudoconvex, hence Stein in $\mathbb{P}^2$ [FoS]. □

PROPOSITION 3.7. *Let* $(X, \mathcal{L}, E)$ *be a compact laminated set in a compact Kähler manifold* M. *Assume* E *is locally complete pluripolar. Let* T *be a positive laminated harmonic current on* X, *which is extremal in the cone of such currents. If* A *is a measurable set which is a union of leaves then* A *has zero or full mass for* T.

Proof. Define $T_1 = \mathbf{1}_A T$. Then T_1 is clearly harmonic on $M \setminus E$ and $T_1 \leq T$. Hence T_1 has bounded mass and the extension by zero defines a current. Since E is complete pluripolar, according to [DEE] the trivial extension $\tilde{T}_1$ satisfies $i\partial\overline{\partial}\tilde{T}_1 \leq 0$. The manifold being compact and Kähler, we get by Stokes that $i\partial\overline{\partial}\tilde{T}_1 = 0$. Hence $\tilde{T}_1 = cT$, $c \geq 0$. So A is of zero or full measure with respect to T. □

Theorem 3.8. *Let* $(X, \mathcal{L}, E)$ *be a laminated set with singularities in a complex Kähler surface* M. *There is a* unique equivalence class of harmonic currents directed by $\mathcal{L}$ of mass one and minimal energy.

Observe that we don't assume X is minimal.

Proof. Let $C_1 = \{T; T \geq 0, \int T \wedge \omega = 1, T \text{ is } \mathcal{L} - \text{directed, harmonic}\}$. Then C_1 is compact in the weak topology of currents since the forms γ_j directing the lamination are continuous. From Theorem 3.5 we know that C_1 is non-empty.

The energy is a lower semi-continuous function on C_1 by Proposition 2.11. Since C_1 is compact it now follows that $E(T,T)$ takes on a minimum value c on C_1.

If $E(T,T) = c$ and $\int T \wedge \omega = 1$, then $Q(T,T) = 1 - 2c$. If T, T' are two elements of non-collinear equivalence classes of currents where the minimum c is reached, then $Q\left(\frac{T+T'}{2}, \frac{T+T'}{2}\right) > 1 - 2c$ by strict concavity of Q (Proposition 2.10), a contradiction. So the minimum is unique. □

We next show that under mild extra hypotheses, equivalence classes of laminated positive currents contain only one current. Recall that a current on a laminated compact X is said to be a laminated positive harmonic current if in local flow boxes it has the form $\int h_\alpha [V_\alpha] d\mu(\alpha)$ for a positive measure $\mu(\alpha)$ on the space of plaques, and $h_\alpha > 0$, harmonic functions on plaques V_α. The current is closed and laminated if the h_α are constant.

Theorem 3.9. *Let $(X, \mathcal{L}, E)$ be a laminated compact set with singularities in a Kähler surface M. Suppose E is locally complete pluripolar with 2-dimensional Hausdorff measure $\Lambda_2(E) = 0$. Assume there is no non-zero positive closed laminated current on X. Consider the convex compact set C of laminated positive harmonic $(1,1)$ currents of mass 1. Then there is a unique element T in C minimizing energy.*

LEMMA 3.10. *Let $(X, \mathcal{L}, E)$ be as above. If T_1, T_2 are two positive laminated harmonic currents such that $[T_1] = [T_2]$, then $T_1 = T_2$.*

Proof of the Lemma. Since $[T_1] = [T_2]$, then it follows from Proposition 2.6 that $T_1 - T_2$ is closed. In a flow box we have $T_j = \int h_j^\alpha [V_\alpha] d\mu_j(\alpha)$, $j = 1, 2$. Let $\nu(\alpha) = \mu_1(\alpha) + \mu_2(\alpha)$, so $\mu_j = r_j(\alpha)\nu$. Then

$$T_1 - T_2 = \int \big(h_1^\alpha r_1(\alpha) - h_2^\alpha r_2(\alpha)\big)[V_\alpha] d\nu(\alpha)\,.$$

Since $d(T_1 - T_2) = 0$ it follows that

$$h_1^\alpha r_1(\alpha) - h_2^\alpha r_2(\alpha) \equiv c(\alpha)\,.$$

We decompose the measure $c(\alpha)\nu(\alpha)$ on the space of plaques, $c(\alpha)\nu(\alpha) = \lambda_1 - \lambda_2$ for positive mutually singular measures λ_j. Then

$$T_1 - T_2 = \int [V_\alpha]\lambda_1(\alpha) - \int [V_\alpha]\lambda_2(\alpha) = T^+ - T^-$$

for positive closed currents $T^\pm$. These locally defined currents fit together to global positive closed currents on $X \setminus E$. Observe that the mass of $T^\pm$ is bounded by the mass of $T_1 + T_2$.

Since E is locally complete pluripolar the trivial extensions of $T^\pm$ are also closed [De2].

Consequently $T^\pm \equiv 0$ and $T_1 = T_2$ on $X \setminus E$. It follows from a theorem by Skoda [Sk], mentioned in section 2, that no positive harmonic current of bidimension $(1,1)$ can have mass on a set of 2-dimensional Hausdorff measure $\Lambda_2 = 0$. So $T_1 = T_2$. □

Proof of the theorem. Observe that we do not assume that X is minimal. We know C is non-empty. We showed in Theorem 3.8 that the class of currents minimizing energy is unique. The lemma implies that the current T is unique. □

Corollary 3.11. *Let $(X, \mathcal{L}, E)$ be a laminated set with singularities in a homogeneous compact Kähler surface M. Assume $\Lambda_2(E) = 0$. At least one of the following statements holds:*

i) *There is a closed positive laminated current of mass 1 on X.*
ii) *All positive laminated harmonic currents of mass one on X satisfy $\int T \wedge T = 0$. In this case the space they generate is of dimension one.*
iii) *There is a positive laminated harmonic current T of mass one on X such that $\int T \wedge T > 0$. In particular the current T_0 minimizing energy satisfies $\int T_0 \wedge T_0 > 0$.*

Proof. If i), ii) don't hold, then we can assume $\int T_j \wedge T_j = 0$, the $\{T_j\}$ being harmonic and linearly independent. By Corollary 2.10, $\int T_1 \wedge T_2 > 0$. Hence $Q\big(\frac{T_1+T_2}{2}, \frac{T_1+T_2}{2}\big) > 0$. □

Remark 3.12. *The above corollary holds for surfaces such that every positive harmonic current is a limit of smooth positive harmonic ones. Otherwise we can only say that either there is a* unique *current such that $\int T \wedge T = 0$ or there are currents with $\int T \wedge T \neq 0$.*

In view of the previous corollary it is interesting to show that generically for a holomorphic foliation by curves on $\mathbb{P}^k$ the first case does not occur.

Let $\mathcal{F}$ be a holomorphic foliation of $\mathbb{P}^k$ of degree d. It was proved by Lins Neto and Soares, [LS] see also [LoR], that for a Zariski open set of foliations of degree d, all the singular points are hyperbolic and no leaf is contained in an algebraic curve. Let $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ be the eigenvalues of the linear part of the vector field Z_p defining the foliation near a singular point p. Then the foliation is hyperbolic at p if for every (i, j), $i \neq j$, $\lambda_i/\lambda_j \notin \mathbb{R}$. In such case Chaperon [Ch] has shown that the foliation is topologically linearizable in a neighborhood of p. The following result is well known according to the referee. We give some indications for the non-experts.

Theorem 3.13. *Let X be a minimal set for a holomorphic foliation $\mathcal{F}$ of dimension* 1*, in the complex projective space. Assume X is not an analytic set and that X contains a singular point p of hyperbolic type. Then there is no non-zero positive closed current directed by $\mathcal{F}$.*

Proof. Suppose τ is a positive closed current on X directed by $\mathcal{F}$. In [Br] it is shown that if a closed current is directed by the foliation, then near the hyperbolic point p, it should be supported by the separatrices. One has to consider the case where p is in the Siegel domain. One can conclude as follows. Let $\nu(T, q)$ denote the Lelong number of T at q. A theorem of Siu [S1], [De2] asserts that $Y := \{q, \nu(T, q) \geq c\}$ is an analytic set. In our case we know that this analytic set contains $\cup_{j=1}^{\ell} A_j$, a union of local separatrices. Any irreducible branch of Y containing one of the A_j' gives a global analytic set of M, contained in X and directed by $\mathcal{F}$, contradicting the assumption on X. Hence $\tau = 0$. □

The following is an analogue (for singular laminations) of Theorem 1 in [G].

Theorem 3.14. *Let $(X, \mathcal{L}, E)$ be a lamination with singularities in a compact Kähler manifold M. Let $C_{\mathcal{L}}$ denote the convex compact set of $\mathcal{L}$-directed positive harmonic currents of mass 1. Let T be an extreme point in $C_{\mathcal{L}}$. If $u \in L^1(T)$ and $i\partial\overline{\partial}(uT) = 0$, then u is constant $|T|$ a.e.*

Proof. If u is bounded, u must be constant a.e. by extremality. Assume next that u is unbounded. Let (u_n) be a sequence of smooth functions $u_n \to u$ in $L^1(T)$. Let χ be a smooth convex function such that $\chi(t) = |t|$ for $|t|$ large. Then $\chi(u_n) \to \chi(u)$ in $L^1(T)$. We want to show that $i\partial\overline{\partial}(\chi(u)T) = 0$. We compute

$$\begin{aligned}\partial\overline{\partial}\big(\chi(u_n)T\big) &= \chi''(u_n)\partial u_n \wedge \overline{\partial} u_n \wedge T + \chi'(u_n)\partial\overline{\partial} u_n \wedge T - \chi'(u_n)\overline{\partial} u_n \wedge \partial T \\ &\quad + \chi'(u_n)\partial u_n \wedge \overline{\partial} T + \chi(u_n)\partial\overline{\partial} T \\ &= \chi''(u_n)\partial u_n \wedge \overline{\partial} u_n \wedge T + \chi'(u_n)\big[\partial\overline{\partial}(u_n T)\big] .\end{aligned}$$

Since $\partial\overline{\partial}(u_n T) \to 0$, we get since $\chi'' \geq 0$ and $i\partial u_n \wedge \overline{\partial u_n} \wedge T \geq 0$, that $i\partial\overline{\partial}(\chi(u)T) \geq 0$. Since M is compact $\partial\overline{\partial}(\chi(u)T) = 0$. Observe that $\chi(u)T$ is $\mathcal{L}$-directed. If χ_1, χ_2 are convex functions as above, then $i\partial\overline{\partial}((\chi_1(u) - \chi_2(u))T) = 0$. Since $\chi_1(u) - \chi_2(u)$ is bounded, it follows that $\chi_1(u) - \chi_2(u) \equiv$ some constant for all such χ_1, χ_2. This is only possible if u is constant. □

Remark 3.15. *Assume $T \geq 0$ is harmonic and extremal in the convex cone of positive harmonic currents on M of bidegree (p, p), $p \leq k-1$. The above proof shows that if $u \in L^1(T)$ and $\partial\overline{\partial}(uT) = 0$ then u is constant $|T|$ a.e.*

4 Intersections of Laminar Currents

4.1 $\mathcal{C}^1$ laminations. Here we assume that $X \subset \mathbb{P}^2$ is a laminated compact covered by finitely many flow boxes B_i. We suppose that X locally extends to a $\mathcal{C}^1$ lamination of an open neighborhood. Since X is laminated it contains a unique minimal laminated set X'. Choose $p \in X'$. We can assume that $p := [0:0:1] \in B_1 \subset X$ and that the leaf L through p has the form $w = \mathcal{O}(z^2)$. So we will assume that the local extension of the lamination of X is of the form $w = w_0 + f_{w_0}(z)$, $f_{w_0}(0) = 0$, where the map $\Psi(z, w_0) = (z, w_0 + f_{w_0}(z))$ is a $\mathcal{C}^1$ diffeomorphism in a neighborhood of p.

Let $\Phi_\epsilon([z : w : t]) = [z : w + \epsilon z : t]$ denote a family of automorphisms of $\mathbb{P}^2$. Notice that each of these automorphisms fixes the w axis.

For two graphs L_1, L_2 given by $w = g_1(z)$, $w = g_2(z)$, $z \in S$, we define the vertical distances over S between the two as $d_S^{\max}(L_1, L_2) = \sup_{z\in S} |f_1(z) - f_2(z)|$ and $d_S^{\min}(L_1, L_2) = \inf_{z\in S} |f_1(z) - f_2(z)|$.

Theorem 4.1. *There exists an integer N independent of $\epsilon \neq 0$, $|\epsilon|$ small, so that in any of the flow boxes B_i local leaves L_1 and $\Phi_\epsilon(L_2)$ can at most intersect in N points, counted with multiplicity. Moreover there exist neighborhoods U_ϵ of Id in $U(3)$ so that the same conclusion holds for $\Psi_1(L_1)$ and $\Psi_2(\Phi_\epsilon(L_2))$, $\Psi_1, \Psi_2 \in U_\epsilon$.*

The main idea of the proof is that if plaques L_1 and $\Phi_\epsilon(L_2)$ have many intersection points, then their distance is at most a very small multiple of ϵ. Following the leaves back to B_1, their distance is still at most a small multiple of ϵ. We use the following lemma repeatedly:

LEMMA 4.2. a) *There is a constant $0 < c_0 < 1$ so that the following holds: Let g be a holomorphic function on the unit disc with $|g| < 1$ and suppose that g has N zeroes in $\Delta(0, 1/2)$. Then $|g| \leq c_0^N$ on $\Delta(0, 1/2)$.*

b) *Let g denote a holomorphic function on the unit disc and suppose that $|g| < 1$ and that $|g| < \eta < 1$ on $\Delta(0, 1/4)$. Then $|g| < \sqrt{\eta}$ on $\Delta(0, 1/2)$.*

Proof. To prove a) set

$$c_0 = \sup_{|\alpha|\leq 1/2, |z|\leq 1/2} \frac{|z - \alpha|}{|1 - z\overline{\alpha}|} < 1\,.$$

To prove b) observe that $\log |g| < \log \eta$ when $|z| < 1/4$. Hence by subharmonicity

$$\log |g| \leq \max \left\{ \log \eta \frac{\log |z|}{\log 1/4}, \log \eta \right\}.$$

This implies that if $|z| = 1/2$, then $\log |g| \leq \log \eta/2$. □

Using this lemma, we obtain that the continuation of the plaques to B_1 have distance at most a small multiple of ϵ. This contradicts the explicit form of Φ_ϵ.

Proof of the theorem. Fix a $\delta > 0$. Let L_w denote the plaque through $[0 : w : 1]$ and let L_w^ϵ denote its image under Φ_ϵ. Say L_{w_0} is given by $w = w_0 + f_{w_0}(z)$ and $L_{w_0}^\epsilon$ is given by $w = w_0 + f_{w_0}(z) + \epsilon z$. Note that the vertical distance $d_S^{\max}$ between L_{w_0} and $L_{w_0}^\epsilon$ is $|\epsilon|\delta$ at the boundary S of the disc $|z| \leq \delta$. In order to use vertical distance we need ϵ to be small enough. Because the lamination is of class $\mathcal{C}^1$, there exists a constant $K > 1$ so that if $(0, w_0), (0, w_1) \in B_1$, then, using that the graphs satisfy

$$\tfrac{1}{K}|w_1 - w_0| \leq \big|w_0 + f_{w_0}(z) - w_1 - f_{w_1}(z)\big| \leq K|w_1 - w_0|\,,$$

we get that there is a constant $C > 1$ such that

$$\sup_{|z|\leq\delta} \big|w_0 + f_{w_0}(z) - w_1 - f_{w_1}(z)\big| \leq C \inf_{|z|\leq\delta} \big|w_0 + f_{w_0}(z) - w_1 - f_{w_1}(z)\big| \quad (2)$$

If plaques L_1 and $\Phi_\epsilon(L_2)$ intersect in a flow box B_i, then L_1 and L_2 must be at most $a|\epsilon|$ apart in B_i for some fixed large a. We are going to show by contradiction that the number of points of intersection cannot be arbitrarily large. Let $c > 0$ be any small constant. If L_1, $\Phi_\epsilon(L_2)$ intersect in N points in B_i and N is sufficiently large then L_1 and $\Phi_\epsilon(L_2)$ can be at most at distance $c|\epsilon|$ apart from each other in B_i. This follows from the lemma with $c = c_0^N$. Note that the same conclusion holds if the number of intersections is counted for $\Psi_1(L_1)$ and $\Psi_2(\Phi_\epsilon(L_2))$ for a small enough U_ϵ. But this again implies that L_1 and $\Phi_\epsilon(L_2)$ have distance at most $c|\epsilon|$ also in this case. Hence we only have to consider L_1 and $\Phi_\epsilon(L_2)$. Since X' is minimal, there is a path of at most a fixed length along these leaves ending in the flow box B_1 containing $[0:0:1]$. It follows that continuing these leaves to this neighborhood, they will have to stay at most $bc|\epsilon|$ apart, for a fixed constant b determined by the length of the path. Choosing c small enough, we get $bc < \delta/2C$. Let $L_1 = \{w = w_1 + f_{w_1}(z)\}$, $L_2 = \{w = w_2 + f_{w_2}(z)\}$ denote the continuations to B_1. Then

$$\big|w_2 + f_{w_2}(z) + \epsilon z - w_1 - f_{w_1}(z)\big| \leq bc|\epsilon| < \frac{|\epsilon|\delta}{2C} \quad \text{when } |z| \leq \delta\,. \quad (3)$$

Hence, using (2)

$$\begin{aligned} d^{\max}_{|z|\leq\delta}(L_1, L_2) &\leq C d^{\min}_{|z|\leq\delta}(L_1, L_2) \\ &\leq C d^{\min}_{z=0}(L_1, L_2) \\ &= C d^{\max}_{z=0}\big(L_1, \Phi_\epsilon(L_2)\big) \\ &\leq C d^{\max}_{|z|\leq\delta}\big(L_1, \Phi_\epsilon(L_2)\big) \\ &\leq \tfrac{|\epsilon|\delta}{2}\,. \end{aligned} \quad (4)$$

For the third inequality we use that 0 is fixed for Φ_ϵ, and the last inequality follows from (3). Applying this estimate when $|z| = \delta$, we get using (4)

$$\begin{aligned} \tfrac{|\epsilon|\delta}{2} &> \tfrac{|\epsilon|\delta}{2C} \\ &> \big|w_2 + f_{w_2}(z) + \epsilon z - w_1 - f_{w_1}(z)\big| \\ &\geq |\epsilon|\delta - \big|w_2 + f_{w_2}(z) - w_1 - f_{w_1}(z)\big| \end{aligned}$$

$$\geq |\epsilon|\delta - \tfrac{|\epsilon|\delta}{2}$$
$$= \tfrac{|\epsilon|\delta}{2},$$

a contradiction. Hence N cannot be arbitrarily large. □

REMARK 4.3. *We only used that the lamination is transversally bilipschitz in a neighborhood of a point in the associated minimal set.*

4.2 Laminations by holomorphic motions. Now we consider the case of laminations which are not $\mathcal{C}^1$. We recall the following result by Bers–Royden [BeR].

PROPOSITION 4.4. *We are given a lamination of a neighborhood of the unit polydisc in $\mathbb{C}^2$. Assume that the leaves are of the following form:*

$$L_t\,,\ t \in \mathbb{C}\,,\ |t| < C\,,\ w = F_t(z)\,,\ F_t(0) = t\,,\ F_0(z) \equiv 0\,.$$

The map $\Phi(z)(t) = F_t(z)$ is a holomorphic motion and we have the estimate

$$\tfrac{1}{K}|t-s|^{\frac{1+|z|}{1-|z|}} \leq \big|F_t(z) - F_s(z)\big| \leq K|t-s|^{\frac{1-|z|}{1+|z|}}\,.$$

Theorem 4.5. *Let $X \subset \mathbb{P}^2$ be a compact subset laminated by Riemann surfaces. Then there exists a holomorphic family $\Phi_\epsilon : \mathbb{P}^2 \to \mathbb{P}^2$ for $\epsilon \in \mathbb{C}$, $\Phi_0 \equiv Id$ with the following properties. There are finitely many flow boxes $\{B_i\}_{i=1,\dots,\ell}$ covering X and an $\epsilon_0 > 0$ and a constant A such that if L, L' are any plaques in any flow box B_i then:*

If $0 < |\epsilon| < \epsilon_0$, the number of intersection points counted with multiplicity of $L, \Phi_\epsilon(L')$ is at most $A \log 1/|\epsilon|$. Moreover there exist neighborhoods U_ϵ of Id in $U(3)$ so that the same conclusion holds for $\Psi_1(L_1)$ and $\Psi_2(\Phi_\epsilon(L_2))$, with $\Psi_1, \Psi_2 \in U_\epsilon$.

Proof. We first choose a finite cover by flow boxes, B_i. We can do this so that for each flow box there is a linear change of coordinates in $\mathbb{P}^2$ so that $[z : w : t] = [0 : 0 : 1] \in B_i \cap X$. Moreover, we can arrange that if L is any local leaf intersecting $\Delta(0,2)$ then $L \cap \Delta(0,2)$ is contained in a local leaf $\tilde{L}$ of the form $\{w = f_\alpha(z), |z| < 3\}$, $(0,\alpha) \in \tilde{L}$, and $\|f_\alpha\|_\infty < 3$. Moreover we can assume that each $\|f'_\alpha\| < 0.1$ and that $f'_0(0) = 0$. Redefining the flow boxes, we can let B_i denote the union of those graphs over $|z| < 3$ intersecting $\Delta(0,2)$. We can assume that the smaller flow boxes B'_i consisting of those graphs over $|z| < 1$ for which the graph is in $\Delta^2(0,1)$, already cover X.

Next we fix the coordinates z, w, t on $\mathbb{P}^2$ used for the first flow box B_1 such that $p_0 = [0 : 0 : 1]$ is on the minimal set $X' \subset X$. Define the family Φ_ϵ by

$$\Phi_\epsilon[z : w : t] = [z : w + \epsilon z : t]\,.$$

LEMMA 4.6. *There exists a $\delta > 0$ and $C > 0$ so that if $w = f_\alpha(z)$, $w = f_\beta(z)$ are two local leaves in B_1, then*

$$\frac{|\alpha-\beta|^2}{C} \le \big|f_\alpha(z) - f_\beta(z)\big| \le C|\alpha-\beta|^{1/2}\,, \quad \forall\, z, |z| \le \delta\,.$$

Proof. This is a special case of the Bers–Royden result. □

LEMMA 4.7. *Fix $\delta > 0$. Let $\epsilon_0 > 0$ be small enough. Then if L_1, L_2 are plaques in the first flow box then $d^{\max}_{\{|z|\le\delta\}}(\Phi_\epsilon(L_1), L_2) > |\epsilon|^3$ for all $|\epsilon| \le \epsilon_0$.*

Proof. Let L_i be given by $w = f_i(z)$, $f_i(0) = w_i$. Then $\Phi_\epsilon(L_1)$ is the graph $w = f_1(z) + \epsilon z$. Suppose that $|f_1(z) + \epsilon z - f_2(z)| \le |\epsilon|^3$ for all $|z| \le \delta$. Then, we get that $|w_1 - w_2| \le |\epsilon|^3$. Hence, by the previous lemma, we have that $|f_1(z) - f_2(z)| \le C|f_1(0) - f_2(0)|^{1/2} \le C|\epsilon|^{3/2}$ for all $|z| \le \delta$. Hence if $|z| = \delta$,

$$\begin{aligned}
|\epsilon|^3 &\ge \big|f_1(z) + \epsilon z - f_2(z)\big| \\
&\ge |\epsilon|\delta - \big|f_2(z) - f_1(z)\big| \\
&\ge |\epsilon|\delta - C|\epsilon|^{3/2} \\
&\ge |\epsilon|\big(\delta - C\sqrt{|\epsilon|}\big)\,, \\
&\Rightarrow \\
\epsilon_0^2 &\ge |\epsilon|^2 \ge \delta - C\sqrt{|\epsilon|} \ge \delta - C\sqrt{\epsilon_0} \\
&\Rightarrow \\
\epsilon_0^2 + C\sqrt{\epsilon_0} &\ge \delta\,,
\end{aligned}$$

a contradiction if ϵ_0 is small enough. □

Continuation of the proof of Theorem 4.5. Pick $\rho > 0$. Let $p \in X$. Since every leaf clusters everywhere on X', there is a (nonunique) continuous curve $\gamma_p(t)$, $0 \le t \le 1$, from $\gamma_p(0) = p$ to a point $\gamma_p(1) = (0, w_p) \in B_1'$ which is contained in the leaf through p. By continuity, for every $q \in X$ close enough to p, a curve γ_q can be chosen so that $\mathrm{dist}(\gamma_q(t), \gamma_p(t)) \le \rho$ for all $0 \le t \le 1$.

A chain of flow boxes is a finite collection $C = \{B_{i(j)}\}_{j=1}^k$. Let $p \in X$. We say that the leaf through p follows the chain $\{B_{i(j)}\}_{j=1}^k$ if there are local leaves $L_j \subset B_{i(j)}$, $\hat L_j := L_j \cap B'_{i(j)}$, $p \in \hat L_1$, $\hat L_j \cap \hat L_{j+1} \ne \emptyset\ \forall\, j < k$, $i(k) = 1$.

By compactness there are finitely many chains of flow boxes $C_1, \dots, C_\ell$ such that for each $p \in X$, there is an open neighborhood $U(p)$ and a chain C_r so that the leaf through q follows C_r for any $q \in U(p) \cap X$.

We will apply Lemma 4.6 repeatedly along a chain. We need to apply Lemma 4.6 at most a fixed number of times, m depending on the length of each chain. Note that every time we switch flow box there is a change of coordinates which distorts distances by at most a factor $C > 1$.

LEMMA 4.8. *Let ϵ be sufficiently small and suppose that $N = N(\epsilon)$ is an integer such that $C^2 c_0^{N/2^m} \leq |\epsilon|^3$. Then no local leaves of the laminations $L_1, \Phi_\epsilon(L_0)$ can intersect more than N times in any flow box.*

Proof. Suppose that local components of L_1 and $\Phi_\epsilon(L_0)$ intersect in more than N points in some local flow box B'_s. Then these local graphs differ by at most c_0^N. Using Lemma 4.6 they differ by at most $c_0^{N/2}$ in a suitable larger flow box. Changing to the coordinates of another flow box might increase the difference to $Cc_0^{N/2}$. Applying Lemma 4.6 the difference increases to at most $C^{1/2}c_0^{N/4}$ and after another change of flow box to $C^{3/2}c_0^{N/4}$. Following the leaves along a chain of flow boxes we see inductively that the distance between continuations of the leaves grows at most like $C^2 c_0^{N/2^k}$ after k steps. Hence once we are in the first flow box, the leaves differ by at most $|\epsilon|^3$. By Lemma 4.7, this is impossible for any pair of leaves. □

End of proof of theorem. From the above Lemma there is a constant A so that for all small enough ϵ local leaves $L_1, \Phi_\epsilon(L_0)$ have at most $N_\epsilon := A \log 1/|\epsilon|$ intersection points. The construction is stable under small perturbations by Ψ_1, Ψ_2 close to the identity. □

5 Construction from Discs. Ahlfors Type Construction

In this section we consider a laminated set $(X, \mathcal{L}, E)$ in a compact complex Hermitian manifold M of dimension k. We assume that E is a compact set with $\Lambda_2(E) = 0$.

We want to construct harmonic currents using the Ahlfors exhaustion technique.

5.1 When leaves are not uniformly Kobayashi hyperbolic. We consider only the case when X does not contain a compact Riemann surface, possibly singular. Consider the universal covering for each leaf. We can assume that the covering is $\mathbb{C}$ or the unit disc Δ. Let $\phi : \Delta \to L$ be a holomorphic map.

PROPOSITION 5.1. *Let $(X, \mathcal{L}, E)$ be a laminated set with singularities. If there is no positive closed current on X, laminated on $X \setminus E$, directed by the lamination, then there is a constant C such that $|\phi'(z)| \leq \frac{C}{1-|z|}$ for every holomorphic map $\phi : \Delta \to L$.*

Proof. If $\{|\phi'(0)|\}$ is not uniformly bounded, then using the Brody technique, one can construct an image of $\mathbb{C}$. The part of the image not in E is locally contained in a leaf of X.

The Ahlfors exhaustion technique furnishes a positive closed current of mass 1 directed by the lamination. Since $\Lambda_2(E) = 0$, the current gives no mass to E. So there is a constant C such that $|\phi'(0)| \leq C$ for every holomorphic map $\phi : \Delta \to L$, where L is a leaf of $X \setminus E$. Using automorphisms of the unit disc, one gets the above estimate. □

When the $|\phi'(0)|$ are uniformly bounded, we say that the leaves are uniformly hyperbolic.

5.2 The case with no positive closed current directed by $(X, \mathcal{L}, E)$. Let $(X, \mathcal{L}, E)$ be a laminated set with singularities. We assume that there is no non-zero positive closed current on X directed by $\mathcal{L}$. We showed in Proposition 5.1 that there is a constant C so that for all holomorphic maps $\phi : \Delta \to X \setminus E$, directed by $\mathcal{L}$, we have $|\phi'(0)| \leq C$.

Let $\phi : \Delta \to X \setminus E$ be a covering map for a leaf. As above, we define the (1,1) positive current T_r by

$$\langle T_r, \theta \rangle = \tfrac{1}{2\pi} \int_\Delta \log^+ \tfrac{r}{|z|} \phi^*(\theta)\,.$$

We want to interpret the concept of ergodicity within this context. The currents T_r can be thought of as averages on an orbit $\phi(\Delta)$. They correspond to partial sums of Dirac masses on an orbit, $\sum_1^N \delta_{f^n(x_p)}$ for a continuous map f from a compact space to itself. We want to show that after normalization the limits when $r \to 1$ are "invariant", i.e. define harmonic currents. The ergodic theorem would be to show that the limits exist for almost every leaf of a positive harmonic current. We start with a general result.

Theorem 5.2. *Let $\phi : \Delta \to M$ be a holomorphic map where M is a compact complex Hermitian manifold. If*

$$\int_\Delta \big(1 - |x|\big)\big|\phi'(x)\big|^2 d\lambda(x) = \infty\,,$$

where λ is the Lebesgue measure, then there is a positive harmonic current T, supported on $\overline{\phi(\Delta)}$. If $\phi(\Delta)$ is contained in a leaf of a lamination $(X, \mathcal{L}, E)$, then the current T is directed by the lamination on $X \setminus E$.

Proof. Assume that $\phi(0) = p$. Define

$$G_r(x) := \tfrac{1}{2\pi} \log^+ \tfrac{r}{|x|}\,, \quad T_r := \phi_*\big(G_r[\Delta]\big)\,, \quad r < 1\,.$$

where $\log^+ = \sup(\log x, 0)$. If θ is a $(1,1)$ test form on M

$$\langle T_r, \theta\rangle = \frac{1}{2\pi}\int_\Delta \log^+ \frac{r}{|x|}\phi^*(\theta)\,.$$

So T_r is positive of bidimension $(1,1)$. The mass of T_r is comparable to $\int_\Delta \log \frac{r}{|x|}|\phi'(x)|^2 \sim \int_{\{|x|<r\}}(r-|x|)|\phi'(x)|^2 =: A(r)$. A direct computation gives

$$i\partial\overline{\partial}T_r = \phi_*(\nu_r) - \delta_p\,,$$

where δ_p is the Dirac mass at p and ν_r is the angular measure on the circle of radius r. Let T be a cluster point of $T'_r := T_r/A(r)$. Since $A(r) \to \infty$ we have $i\partial\overline{\partial}T = 0$.

If $\phi(\Delta)$ is contained in a leaf of a lamination and γ is a continuous $(1,0)$ form so that $[\Delta_\alpha]\wedge\gamma = 0$ for all plaques, then $T_r \wedge \gamma = 0$. Hence T is directed by $\mathcal{L}$ on $X \setminus E$. □

Theorem 5.3. *Let $(X, \mathcal{L}, E)$ be a laminated set with singularities. Assume E is totally disconnected, and that E is contained in a countable union of irreducible analytic sets $\cup A_j$, with $\cup A_j \cap X \setminus E = \emptyset$. Then*

$$\int_\Delta \big(1-|x|\big)\big|\phi'(x)\big|^2 d\lambda(x) = \infty\,,$$

and all cluster points of the normalized currents $\tau_r := T_r/\|T_r\|$ are harmonic.

Let X be a minimal laminated compact set in M. Suppose that there is no non-zero positive closed current on X, directed by $\mathcal{L}$. In particular, X does not contain any non-constant holomorphic image of $\mathbb{C}$. Let B_i be a locally finite covering of $X \setminus E$ by flow boxes which in local coordinates are of the form $w_i = f_\alpha(z_i)$, $|z_i| < 1$ [but the graphs extend uniformly to $|z_i| < 2$]. Let $\phi : \Delta \to L$ denote the universal covering of an arbitrary leaf in $X \setminus E$. We say that $x \in \Delta$ is a center point if $\phi(x) = (0, w_i)$ in some B_i. We can normalize ϕ for any center point, say $\phi_x : \Delta \to L$, $\phi_x(0) = \phi(x)$ [i.e. we move x to 0 with an automorphism of the unit disc]. Let $w_i = f_x(z_i)$ denote the associated graph in the flow box. Denote by $U_x := \phi_x^{-1}(\{(z_i, f_x(z_i)); |z_i| < 1\})$. If ϕ is a multisheeted covering, we let U_x denote the connected component containing 0. Then $U_x \subset \Delta$ is a relatively compact open subset of Δ containing 0. Let $0 < r_x \le R_x < 1$ denote the largest, respectively smallest, radii such that $\Delta(0, r_x) \subset U_x \subset \Delta(0, R_x)$.

Proposition 5.4. *Let $(X, \mathcal{L}, E)$ be as above. Consider the family of universal covering maps ϕ from Δ to an arbitrary leaf. Given $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that $|\phi'(z)| \ge \frac{c_\epsilon}{1-|z|}$, for all ϕ, if $\operatorname{dist}(\phi(z), E) \ge \epsilon$.*

LEMMA 5.5. $R_x \leq 1/2$.

Proof. Since ϕ_x^{-1} maps $\Delta(0,2)$ into $\Delta(0,1)$ and sends 0 to 0 this follows from the Schwarz's lemma. □

LEMMA 5.6. *Fix a finite number of flow boxes $\overline{B}_i \cap E = \emptyset$, $i = 1, \dots, \ell$. Then $\inf\{r_x; x$ is a center point, $\phi(x) \in \cup_{i=1}^{\ell} B_i\} > 0$. In fact the same holds if we take the infimum over all leaves and all covers of the leaves by discs. We also have $|\phi_x'(0)| \geq 1/4r_x$.*

Proof. Fix an x and a covering $\phi : \Delta \to L$ of the leaf through x. Note that $\phi_x^{-1}(\Delta(0,1)) \supset \Delta(0, r_x)$ hence by the Koebe 1/4 theorem, $[\phi_x^{-1}]'(0) = \alpha$, $|\alpha| \leq 4r_x$. Hence $\phi_x'(0) = 1/\alpha$, $|1/\alpha| \geq 1/4r_x$. If $r_x \to 0$ then $\alpha \to 0$. Using the Brody technique we construct an image of $\mathbb{C}$ contained in X. □

Proof of Proposition 5.4. Using automorphisms of the unit disc, we only have to show that if $\operatorname{dist}(\phi(0), E) \geq \epsilon$, then $|\phi'(0)| \geq c_\epsilon$. But $\phi(0)$ belongs to a finite number of flow boxes. Hence the estimate follows from Lemma 5.6, since $r_z \leq R_z < 1/2$. The constant c_ϵ appears when we compare the metric in normalized cordinates with the Hermitian metric on M. □

Next we prove a density theorem for the above minimal laminations with only Kobayashi hyperbolic leaves:

Theorem 5.7. *Assume $(X, \mathcal{L}, E)$ is a minimal lamination. Fix a locally finite cover of $X \setminus E$ by flow boxes B_i. Fix a compact set $K \subset X \setminus E$. There are constants R, N so that if $\phi : \Delta \to X$ is a covering of any leaf and $\phi(x) \in K$, $B_i \cap K \neq \emptyset$, then $\phi(\Delta_{kob}(x, R))$ intersects at most N of the graphs over $\{|z_i| < 3/2\}$ in B_i and contains at least one complete graph over $|z_i| < 1$.*

Proof. The estimates in Proposition 5.4 show that as long as $\phi(z)$ is away from E, then ϕ is a bilipschitz map, from the unit disc with the Kobayashi metric to $X \setminus E$ with Hermitian metric induced from M. □

Proof of Theorem 5.3. The main difficulty is to deal with the behaviour of leaves near E. If $\phi(0) = p$, we get as in the proof of Theorem 5.2,

$$i\partial\overline{\partial} T_r = \phi_*(\nu_r) - \delta_p \tag{5}$$

where δ_p is the Dirac measure at p and ν_r is the Lebesgue measure on the circle of radius r. The mass $\|T_r\|$ is comparable to

$$\int_\Delta \log^+ \tfrac{r}{|x|} \big|\phi'(x)\big|^2 \sim \int_{|x|<r} \big(r - |x|\big)\big|\phi'(x)\big|^2 =: A(r)\,.$$

It's enough to show that $A(r) \to \infty$ when $r \to 1$, i.e.

$$\int_\Delta \big(1 - |x|\big)\big|\phi'(x)\big|^2 d\lambda(x) = \infty\,.$$

For simplicity we consider first the case where E is just one point $q = [0:0:1]$. Let $\phi = [\phi_1 : \phi_2 : \phi_3]$ in homogeneous coordinates.

Let

$$A := \Big\{e^{it}\,;\ e^{it} \in \partial\Delta\,,\ \text{such that}\ \lim_{r\to 1} \phi(re^{it}) = [0:0:1]\Big\}\,.$$

We show first that $\nu_1(A) = 0$. Let $e^{it_0} \in A$. Since $|\phi'(x)| \le \frac{C}{1-|x|}$, then ϕ is bounded in an angle with vertex at e^{it_0}. Since ϕ has radial limit at e^{it_0}, by the Lindelöf theorem [CoL, p. 19], ϕ has non-tangential limit zero in some smaller angle. By a theorem of Privalov [CoL] for meromorphic functions, the functions $\phi_1/\phi_3, \phi_2/\phi_3$ are identically zero if A has positive measure.

Privalov's theorem [CoL, p. 166] asserts that if a meromorphic function f on the unit disc has non-tangential limits equal to zero for $e^{i\theta}$ of positive measure, then f is identically zero. Since the cluster set of $\phi(re^{i\theta})$ is connected and E is totally disconnected, $\phi(re^{i\theta})$ has radial limit at a point p is equivalent to $\lim_{r\to 1} \operatorname{dist}(\phi(re^{i\theta}), E) = 0$. If ϕ has radial values in E for a set of positive measure we can choose a neighborhood of a point $p \in E$ such that ϕ has radial limits in A_j on a set of positive measure. Then we apply Privalov's theorem to the function $h_j \circ \phi$ where h_j is a holomorphic function defining A_j in U_j. Since $\cup A_j$ does not intersect $X \setminus E$ we get a contradiction.

As a consequence there exists $\delta > 0$ and a set $F \subset \partial\Delta$ with $\nu_1(F) > 1/2$, such that for every $e^{it} \in F$ there is a sequence $r_j \to 1$ and $\operatorname{dist}(\phi(r_j e^{it}), E) \ge 2\delta$.

Let $E_\delta = \{q; \operatorname{dist}(q, E) \le \delta\}$. Since $|\phi'| \le \frac{C}{1-|x|}$ there is a constant $c > 0$ such that $\phi(\Delta(r_j e^{it}, c(1 - r_j)))$ does not intersect E_δ. It follows from Proposition 5.4 that on $\phi^{-1}(L \setminus E_{2\delta})$ there exists a constant $c_\delta > 0$ such that $|\phi'(x)| \ge \frac{c_\delta}{1-|x|}$. Hence for $e^{it} \in F$, we have

$$\int_0^1 \big|\phi'(re^{it})\big|^2 (1-r)dr \ge \sum_j \int_{r_j - c(1-r_j)}^{r_j + c(1-r_j)} \frac{c_\delta}{1-r} = c_\delta \sum_j \log\frac{1+c}{1-c} = +\infty\,.$$

Hence $A(r) \to \infty$ as $r \to 1$.

Equation (5) implies that

$$\lim_{r\to 1} i\partial\overline{\partial}\tau_r = \lim_{r\to 1} \frac{i\partial\overline{\partial} T_r}{A(r)} = 0\,. \qquad \square$$

Remark 5.8. *In Theorem 5.3 we don't need the assumption that* $\Lambda_2(E)=0$.

Remark 5.9. *If we assume that* $\lim_{r\to 1}(1-r)\int_{D_r}|\phi'(z)|^2=\infty$, *it follows from a result of Ahlfors that* $\underline{\lim}_{r\to 1}\frac{\ell(\phi(\partial\Delta_r))}{\mathrm{Area}(\phi(\Delta_r))}\to 0$, ℓ *represents length. Hence one can choose the current* T *to be closed.*

In the usual Ahlfors procedure to construct a closed current starting from an image of $\mathbb{C}$ *one has to first extract good subsequences from* $\frac{\Phi_*[\Delta_R]}{\mathrm{Area}\ \Phi_*[\Delta_R]}$ *when* $R\to\infty$. *Then cluster points of these give closed currents. In our case, there is no need to first take a subsequence.*

Remark 5.10. *Let* $(X,\mathcal{L},E)$ *be a compact laminated set out of a totally disconnected set* E, *satisfying the assumption of Theorem 5.3. Theorem 5.3 gives a new construction of positive harmonic currents directed by* $\mathcal{L}$. *Indeed the currents constructed by the averaging process in Theorem 5.3 are harmonic. We only have to show that they cannot be supported only on* E. *But Stoke's formula shows that a positive current cannot be completely supported on a Euclidean ball,*

$$0<\int T\wedge dd^c\big(|z|^2-R^2\big)=0\,.$$

Proposition 5.11. *If a positive harmonic current on a laminated compact* X *gives mass to a leaf, then this leaf is a compact Riemann surface.*

Recall that in a flow box $T=\int h_\alpha[L_\alpha]d\mu(\alpha)$, $\{L_\alpha\}$ are the plaques and $\{h_\alpha\}$ are the harmonic positive functions.

Lemma 5.12. *Let* T *be a laminated harmonic current. Let* ϕ *denote the covering map* $\Delta\to L$. *If* H_α *denotes the analytic continuation of* $h_\alpha\circ\phi$ *in a flow box, then we have the estimate*

$$c\big(1-|x|\big)\le H_\alpha\le C\frac{1}{(1-|x|)}\,.$$

Proof. The harmonic function h defined by the current on the leaf L is determined up to a multiplicative constant. So H is positive. The estimate follows from Harnack's inequality and the Hopf lemma. □

Proof of the proposition. We assume first that the leaf is hyperbolic. Let ϕ, H_α be as in the lemma. Suppose at first that H_α is unbounded. Then we can choose a sequence $p_n\to\partial\Delta$ and $H_\alpha(p_n)\to\infty$ such that H_α is uniformly large on $\Delta(p_n,R)$ by Harnack, where R is as in Theorem 5.7. Hence T will have infinite mass on a flow box. If H is bounded and nonconstant, we can choose θ_n so that $\lim_{r\to 1}H(re^{i\theta_n})$ are different. We can again choose $p_n\to\partial\Delta$ so that $\phi(\Delta(p_n,R))$ are disjoint and again T will have infinite mass on a flow box. If H is constant, we get a positive closed current and the leaf has an analytic closure. The same argument applies to the case when the leaf is not hyperbolic. □

6 Vanishing of $\int T \wedge T$

Suppose that T is a positive laminated harmonic current on $(X, \mathcal{L}, E)$ which in local flow boxes can be written as $\int h_\alpha[\Delta_\alpha]d\mu(\alpha), h_\alpha(0,\alpha) = 1$.

Definition 6.1. *The harmonic laminated current has finite transverse energy if in some local flow box* $\int \log|\alpha - \beta| d\mu(\alpha)d\mu(\beta) > -\infty$.

Having finite transverse energy is well defined and independent of the choice of flow box.

Recall that $\Phi_\epsilon([z : w : t]) = [z : w + \epsilon z : t]$. If T is a current, let $T_\epsilon := (\Phi_\epsilon)_*(T)$. Observe that $T_\epsilon \to T$ in H_e.

Theorem 6.2. *If a harmonic current for a minimal laminated compact in* $\mathbb{P}^2$ *has finite transverse energy, then the geometric intersection* $T \wedge T_\epsilon \to 0$. *The same conclusion holds for* $\mathcal{C}^1$*-laminations without the hypothesis of finite transverse energy. In both cases we have* $\int T \wedge T = 0$.

Proof. We calculate the geometric wedge product $T \wedge T_\epsilon$ in a flow box. Set $T = \int h_\alpha[\Delta_\alpha]d\mu(\alpha), T_\epsilon = \int h^\epsilon_\beta[\Delta^\epsilon_\beta]d\mu(\beta)$. Note that functions defined on plaques $w = k(z)$, can be locally extended to be independent of w. Let ϕ be a test function supported in a flow box. To avoid confusion, we index with g during the proof when wedge products are geometric. We define

$$\langle T \wedge T_\epsilon, \phi\rangle_g := \int \sum_{p \in J^\epsilon_{\alpha,\beta}} \phi h_\alpha(p) h^\epsilon_\beta(p) d\mu(\alpha)d\mu(\beta)$$

where $J^\epsilon_{\alpha,\beta}$ consists of the intersection points of Δ_α and Δ^ϵ_β. Assume at first that μ has finite transverse energy. The estimate on the size of $J^\epsilon_{\alpha,\beta}$ in Theorem 4.5, implies that the number of points in $J^\epsilon_{\alpha,\beta}$ is of order at most $\log 1/\epsilon$. Since $h_\alpha, h^\epsilon_\beta$ are uniformly bounded we get

$$\begin{aligned}\big|(T \wedge T_\epsilon)_g(\phi)\big| &\le C_1\|\phi\|_\infty \int_{\mathrm{dist}^{\min}(\Delta_\alpha,\Delta_\beta)\le C\epsilon} A \log \tfrac{1}{|\epsilon|} d\mu(\alpha)d\mu(\beta) \\ &\le C_2\|\phi\|_\infty \int_{\mathrm{dist}^{\min}(\Delta_\alpha,\Delta_\beta)\le C\epsilon} \log \frac{1}{\mathrm{dist}(\Delta_\alpha,\Delta_\beta)} d\mu(\alpha)d\mu(\beta) \\ &\to 0 \text{ as } \epsilon \to 0\,.\end{aligned}$$

In the $\mathcal{C}^1$ case the number of intersection points is bounded by N independent of ϵ (Theorem 4.1). Hence

$$\big|(T \wedge T_\epsilon)_g(\phi)\big| \le C\|\phi\|_\infty \int_{\mathrm{dist}(\Delta_\alpha,\Delta_\beta)\le C\epsilon} N d\mu(\alpha)d\mu(\beta) \to 0\,,$$

since μ has no pointmasses by Proposition 5.11.

Next we show that $Q(T,T) = \int T \wedge T = 0$. It suffices to show by Theorem 2.9 that $Q(T,T_\epsilon) \to 0$ or even that for smoothings $T^\delta, T_\epsilon^{\delta'}$, $Q(T^\delta, T_\epsilon^{\delta'}) \to 0$ when δ,δ' are sufficiently small compared to ϵ and $\delta,\delta',\epsilon \to 0$. Indeed, $T_\epsilon \to T$ in H_e, and for δ,δ' small enough, $T_\epsilon^{\delta'}, T^\delta$ converge also to T.

Note that the estimate on the geometric wedge product is stable under small translations of T, T_ϵ. This is what allows us to smooth.

Let ϕ be a test function supported in some local flow box. As above, the value of the geometric wedge product on ϕ is

$$\langle T \wedge T_\epsilon, \phi\rangle_g = \int \sum_{p\in J_{\alpha,\beta}} \phi h_\alpha(p) h_\beta^\epsilon(p) d\mu(\alpha)d\mu(\beta)\,.$$

We can write this as

$$\langle T \wedge T_\epsilon, \phi\rangle_g = \int \bigg(\int_{\Delta_\beta^\epsilon} [\phi h_\alpha h_\beta^\epsilon](p) i\partial\overline{\partial}\log\big|w - f_\alpha(z)\big|\bigg) d\mu(\alpha)d\mu(\beta)\,.$$

Note that these expressions are small when ϵ is small. The same applies when we do this for slight translations within small neighborhoods $U(\epsilon)$ of the identity in $U(3)$ and their smooth averages T^δ. Considering ϕT^δ as a smooth test form, we get

$$\langle T_\epsilon, \phi T^\delta\rangle = \int \bigg(\int_{\Delta_\beta^\epsilon} [\phi h_\beta^\epsilon](p) T^\delta\bigg) d\mu(\beta)\,.$$

Averaging also over small translations of T_ϵ we get that $\langle T_\epsilon^{\delta'}, \phi T^\delta\rangle \to 0$ when $\delta,\delta' \ll \epsilon, \epsilon \to 0$. Adding up for a partition of unity of ϕs, we get $\langle T_\epsilon^{\delta'}, T^\delta\rangle = Q(T_\epsilon^{\delta'}, T^\delta) \to 0$. Hence $Q(T,T) = 0$. □

Corollary 6.3. *If a laminated compact set in $\mathbb{P}^2$ carries a positive closed laminar current T, then T has infinite transverse energy.*

Proof. If $T \ne 0$ has finite transverse energy, then $0 = \int T \wedge T = |\int T \wedge \omega|^2 - E(T,T)$ but $E(T,T) = 0$ since T is closed. Hence $T = 0$, a contradiction. □

J. Duval has independently obtained this corollary. Hurder and Mitsumatsu proved that there is no nontrivial $\mathcal{C}^1$ lamination in $\mathbb{P}^2$ which carries a positive closed current [HM].

Corollary 6.4. *If $(X, \mathcal{L})$ is a $\mathcal{C}^1$ lamination on $\mathbb{P}^2$ with only hyperbolic leaves, then*

$$T = \lim_{r \nearrow 1} \phi_* \frac{\big(\log^+ \big(\frac{r}{|z|}\big)[D_r]\big)}{A_r}$$

uniformly with respect to ϕ.

Proof. We know from [HM] that there is no positive closed current directed by $\mathcal{L}$. It follows from Corollary 3.11 and Theorem 6.2 that there is a unique harmonique current of mass 1 on $(X, \mathcal{L})$. Hence the result follows. □

7 Examples of Harmonic Current

If X is a non-singular $\mathcal{C}^1$ lamination in $\mathbb{P}^2$, then there are positive harmonic currents T such that $\int T \wedge T = 0$. The problem of existence of such currents is interesting in itself. A possible candidate is a current of the form $T = i\partial u \wedge \overline{\partial} u$. In this section we investigate harmonic currents on $\mathbb{P}^2$ of the form $T = i\partial u \wedge \overline{\partial u}$. Our main result is that if $u \in \mathcal{C}^2_{\mathbb{R}}(\mathbb{P}^2)$ and $i\partial\overline{\partial}T = 0$, then u is constant, hence $T \equiv 0$. We also compute the energy of some positive harmonic currents.

Let M be a complex manifold of dimension m. For $1 \leq k \leq m$, we define $\mathcal{P}^{(k)}_-(M)$ as the cone of upper semi-continuous real functions v on M such that for every $p \in M$, there is an open neighborhood U of p and $\{v_n\} \subset \mathcal{C}^2(U)$ such that $v_n \searrow v$ in U and $(-1)^k(i\partial\overline{\partial}v_n)^k \leq \epsilon_n \omega^k$, $\epsilon_n \searrow 0$. We say that U is associated to v. Here ω denotes a strictly positive hermitian form. Notice that this condition implies that when $\epsilon_n = 0$, not all eigenvalues of $i\partial\overline{\partial}v_n$ can be strictly negative. We define $\mathcal{P}^{(k)}_+ := -\mathcal{P}^{(k)}_-(M)$ and $\mathcal{P}^{(k)} = \mathcal{P}^{(k)}_+ \cap \mathcal{P}^{(k)}_-$. In particular, $\mathcal{P}^{(1)}_-$ consists of the plurisubharmonic functions and $\mathcal{P}^{(1)}$ are the pluriharmonic functions. In dimension 2, a smooth function v belongs to $\mathcal{P}^{(2)}_-$ if and only if its Levi form has at most one eigenvalue of each sign. These functions then, also belong to $\mathcal{P}^{(2)}_+$ and hence $\mathcal{P}^{(2)}$. Pseudoconvex domains are usually characterized by plurisubharmonic functions, i.e. $\mathcal{P}^{(1)}_-$. We show here that $\mathcal{P}^{(2)}_-$ works as well, and that there are similar results for $\mathcal{P}^{(k)}_-$, $k > 2$.

Let $\Phi : M \to N$ be a holomorphic map between complex manifolds. If $v \in \mathcal{P}^{(k)}_-(N)$, then $v \circ \Phi \in \mathcal{P}^{(k)}_-(M)$. In particular, if $k > \dim N$, any upper semi-continuous v is in $\mathcal{P}^{(k)}_-(N)$, hence $v \circ \Phi \in \mathcal{P}^{(k)}_-(M)$.

We give some examples of compact complex manifolds for which $\mathcal{P}^{(2)}(M) \neq \mathbb{R}$:

1. Tori: Let T be a torus. Then $T = \mathbb{C}^k$ mod a lattice generated by $\{v_i\}_{i=1}^{2k}$. Let $\pi : \mathbb{C}^k \to \mathbb{R}$, $\pi\big(\sum x_i v_i\big) = x_i$. Let $v = \phi(x_1)$ where ϕ is a smooth function supported in $]0,1[$. Then $(i\partial\overline{\partial} v)^2 = 0$.
2. Hopf manifolds: Let $M = \mathbb{C}^2 \setminus \{0\}/(\phi)$ where (ϕ) denotes the group generated by $\phi(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2)$ with α_1, α_2 fixed, $0 < |\alpha_1| \leq |\alpha_2| < 1$. Fix r such that $|\alpha_1| = |\alpha_2|^r$. Define
$$v(z_1, z_2) = \frac{|z_1|^2}{|z_1|^2 + |z_2|^r}\,.$$
The function v is well defined on M and $(i\partial\overline{\partial} v)^2 = 0$.
3. For any surface admitting a projection on a Riemann surface, $\mathcal{P}^{(2)}(M) \neq \mathbb{R}$, for example ruled surfaces. Actually the Hopf surfaces above admit such a projection $\mathbb{C}^2 \setminus \{0\}/(\phi) \to \mathbb{P}^1$, $(z_1, z_2) \to [z_1^q : z_2^p]$ if $\alpha_1^q = \alpha_2^p$.

For $k > 1$, set $z = (z_1, \dots, z_{k-1})$, $|z| = \max\{|z_1|, \dots, |z_{k-1}|\}$ and let H_{k-1}^r denote the Hartogs figure

$$H_{k-1}^r := \big\{(z,w) \in \mathbb{C}^{k-1} \times \mathbb{C} = \mathbb{C}^k\,;\ |z| \leq 1+r\,,\ |w| \leq 1\big\} \setminus \big\{r < |w| \leq 1\,,\ |z| < 1\big\}\,.$$

Let $\hat{H}_{k-1}^r := \{(z,w) \in \mathbb{C}^k; |z| \leq 1 + r, |w| \leq 1\}$.

Definition 7.1. *Let $2 \leq k \leq m$. We say that an open set $N \subset M$ is $(k-1)$-pseudoconvex if whenever $\Phi : U \to M$ is a rank $k, 1-1$, holomorphic map of a neighborhood $U \supset \hat{H}_{k-1}^r$ onto its image and $\Phi(H_{k-1}^r) \subset N$, then $\Phi(\hat{H}_{k-1}^r) \subset N$.*

Remark 7.2. *In the case $k = 2$, the definition is equivalent to N being pseudoconvex.*

Proposition 7.3. *Let M be a complex manifold of dimension $m \geq 2$. Let N be an open set in M. Assume $v \in \mathcal{P}_-^k(M)$, $2 \leq k \leq m$ and $v < 0$ on N, $v_{|\partial N} \equiv 0$, $v \leq 0$ on M. Let $\mathcal{U} = \{U_\alpha\}$ be a cover of M associated to v. If $\Phi(\hat{H}_{k-1}^r) \subset U_\alpha$ and $\Phi(H_{k-1}^r) \subset N$, then $\Phi(\hat{H}_{k-1}^r) \subset N$. In particular, if $k = 2$, then N is pseudoconvex in M.*

Proof. Assume that $\Phi(\hat{H}_{k-1}^r) \setminus N \neq \emptyset$. Then also, $\Phi(\hat{H}_{k-1}^r) \cap \partial N \neq \emptyset$. We can assume that $M = U$, $\Phi = \mathrm{Id}$, $H_{k-1}^r \subset N$ and that there is an interior point of $\hat{H}_{k-1}^r$ in ∂N. Assume that $v_n \searrow v$, $v_n \in \mathcal{C}^2(U)$, $(-1)^k (i\partial\overline{\partial} v_n)^k \leq \epsilon_n \omega^k$. Let $(z_0, w_0) \in \partial N$, $|z_0|, |w_0| < 1$. Then $v < 0$ on H_{k-1}^r and $v(z_0, w_0) = 0$. This is where we use that $v_{|\partial N} \equiv 0$. We would like to get a contradiction.

Let $X := \{|z| \le 1, r < |w| \le 1\}$. Define the function u by

$$u(z,w) := \eta\left(1 - \sum_{j=1}^{k-1} |z_j|^2\right) - \frac{\epsilon}{|w|^2} + \delta$$

where $\eta \ll \epsilon < \delta \ll 1$. We will choose the constants so that $v < u$ on ∂X and $u(z_0, w_0) < 0$. First observe that if δ is small enough, then automatically $v < u$ on all of ∂X except possibly where $|w| = 1$ and $|z| < 1$. Fix any such δ. Let $\epsilon < \delta$ be chosen big enough that $-\frac{\epsilon}{|w_0|^2} + \delta < 0$. Since $\epsilon < \delta$ and $v \le 0$ on $|w| = 1$ and $|z| < 1$, $v < u$ on all of ∂X if we choose $\eta = 0$. To finish the choice of constants, η, ϵ, δ, choose $\eta > 0$ small enough that $v < u$ still on ∂X and in, addition $u(z_0, w_0) < 0$. Then $i\partial\overline{\partial}(-u) \ge a\omega$ on X for some constant $a > 0$.

Next choose n large enough so that $v_n < u$ on ∂X. We have $v_n(z_0, w_0) \ge v(z_0, w_0) = 0 > u(z_0, w_0)$. Then if we add a strictly positive constant c_n to u we can assume that

$$\begin{aligned} v_n &< u + c_n \text{ on } \partial X \\ v_n &\le u + c_n \text{ on } X \\ v_n(z_1^n, w_1^n) &= u(z_1^n, w_1^n) + c_n, (z_1^n, w_1^n) \in X\,. \end{aligned}$$

This implies that $i\partial\overline{\partial}v_n(z_1^n, w_1^n) \le i\partial\overline{\partial}u(z_1^n, w_1^n)$. Hence $i\partial\overline{\partial}(-v_n)(z_1^n, w_1^n) \ge a\omega$.

Hence

$$\begin{aligned} \epsilon_n \omega^k &\ge (-1)^k (i\partial\overline{\partial}v_n)^k (z_1^n, w_1^n) \\ &= \big(i\partial\overline{\partial}(-v_n)\big)^k (z_1^n, w_1^n) \\ &\ge a^k \omega^k, \end{aligned}$$

a contradiction. □

Corollary 7.4. *If M is a compact manifold, $v \in \mathcal{P}_-^{(k)}$ and $K(v) := \{p; v(p) = \max_M v\}$, then the open set $M \setminus K(v)$ is $k-1$ pseudoconvex.*

Remark 7.5. *Let v be a $\mathcal{C}^2$ function on a compact complex manifold of dimension k. Stokes' theorem implies that if $(-1)^k (i\partial\overline{\partial}v)^k \le 0$, then $(i\partial\overline{\partial}v)^k = 0$. In the case of compact Kähler manifolds of dimension m, Stokes' theorem applied to $(-1)^k (i\partial\overline{\partial}v)^k \wedge \omega^{m-k}$ shows that the same conclusion holds.*

Remark 7.6. *The proof above shows that if an upper semi-continuous function v is locally a decreasing limit of $\mathcal{C}^2$ functions v_n such that at each point $i\partial\overline{\partial}v_n$ has $m-1$ non-negative eigenvalues, then $M \setminus K(v)$ is* *pseudoconvex. Namely, we get by the above construction with a Hartogs figure of dimension two:*

$$v_n < u + c \text{ on } \partial X\,,$$
$$v_n \leq u + c \text{ on } X$$
$$v_n(z_1, w_1) = u(z_1, w_1) + c \text{ at some point of } X\,. \text{ Hence}$$
$$i\partial\overline{\partial}(v_n - u)(z_1, w_1) \leq 0\,.$$

Since $-u$ is strictly plurisubharmonic there is at least one eigenvalue of the Levi form of $(v_n - u)$ at (z_1, w_1) which is positive.

Corollary 7.7. *If v is a continuous function on $\mathbb{P}^m$ such that $v \in \mathcal{P}^{(2)}_-$, then v is constant. In particular there are no nonconstant functions in $\mathcal{C}^2_{\mathbb{R}}$ such that $T = i\partial v \wedge \overline{\partial v}$ satisfies $i\partial\overline{\partial}T = 0$.*

Proof. We know that $\mathbb{P}^m \setminus K(v)$ is pseudoconvex. We show next that also $\mathbb{P}^m \setminus K(-v)$ is pseudoconvex. It suffices to show that $-v$ is also locally a decreasing limit of $\mathcal{C}^2$ functions w_n, $(i\partial\overline{\partial}w_n)^2 \leq \epsilon_n\omega^2$. For this, let v_n be such a sequence for v. Taking a subsequence if necessary we can assume that $v \leq v_n \leq v+1/2^n$. Then $w_n := -v_n+1/n$ works. Since on $\mathbb{P}^m$ the Levi problem has a positive solution, this implies that the complements of $K(v)$ and $K(-v)$ are both Stein. But then the intersection of the two domains is a Stein manifold of dimension > 1 with two ends, unless $K(v)$ and $K(-v)$ have a non-empty intersection. But then v must be constant. □

The only property of $\mathbb{P}^m$ we have used is that the Levi problem is solvable in $\mathbb{P}^m$. So we get

Corollary 7.8. *If the Levi problem is solvable on a compact complex manifold M, then $\mathcal{P}^{(2)}$ only contains constant functions.*

Recall [BS] that given a positive closed current S on M, an upper semi-continuous function ϕ defined on $\mathrm{Supp}(S)$ is S-plurisubharmonic if for every $p \in \mathrm{Supp}(S)$ there is an open set $U \subset M$, $p \in U$ and a sequence $\phi_n \in \mathcal{C}^2(U)$, such that $\phi = \lim_{\searrow} \phi_n$ on $U \cap \mathrm{Supp}(S)$ and $i\partial\overline{\partial}\phi_n \wedge S \geq 0$. A function ϕ is said to be S-pluriharmonic if both ϕ and $-\phi$ are S-plurisubharmonic.

Theorem 7.9. *Let S be a positive closed current of bidegree $(1,1)$ in $\mathbb{P}^2$. Assume ϕ is S-plurisubharmonic and let $K(\phi) = \{p \in \mathrm{Supp}(S); \phi(p) = \max\phi\}$. Then $\mathbb{P}^2 \setminus K(\phi)$ is pseudoconvex. If ϕ is S-pluriharmonic, then ϕ is constant.*

Proof. Recall that for S-plurisubharmonic functions, the local maximum principle is valid [BS, Prop. 3.1]. [The local maximum principle says that for every ball $\max_B v \le \max_{\partial B} v$.] We claim that $\mathbb{P}^2 \setminus K(\phi)$ is pseudoconvex. We modify the proof of Proposition 7.3. Assume that $\mathbb{P}^2 \setminus K(\phi)$ is not pseudoconvex. We can assume in local coordinates that $K(\phi)$ contains a point (z_0, w_0), $|z_0|, |w_0| < 1$ and that $K(\phi)$ does not intersect the Hartogs figure $H = \{(z,w); |w| \le r < 1, |z| \le 1+\delta\} \cup \{(z,w); 1 \le |z| \le 1+\delta, |w| \le 1\}$. We can also assume that on a fixed neighborhood of $\hat{H} = \{(z,w); |z| \le 1+\delta, |w| \le 1\}$ there is a sequence of $\mathcal{C}^2$ functions $\phi_n \searrow \phi$ on $\mathrm{Supp}(S)$, $i\partial\overline{\partial}\phi_n \wedge S \ge 0$. We can assume $\phi = 0$ on $K(\phi)$. Then $\phi < 0$ on $H \cap \mathrm{Supp}(S)$ and $\phi(z_0, w_0) = 0$.

Let $X := \{|z| \le 1, r < |w| \le 1\}$. Define the function u by

$$u(z,w) := \eta b\big(1 - |z|^2\big) - \tfrac{\epsilon}{|w|^2} + \delta$$

where $\eta \ll \epsilon < \delta \ll 1$. We will choose the constants so that $\phi < u$ on $\partial X \cap \mathrm{Supp}(S)$. First observe that if δ is small enough, then automatically $\phi < u$ on all of $\partial X \cap \mathrm{Supp}(S)$ except possibly where $|w| = 1$ and $|z| < 1$. Fix any such δ. Let $\epsilon < \delta$ be chosen big enough that $-\frac{\epsilon}{|w_0|^2} + \delta < 0$. Since $\epsilon < \delta$ and $\phi \le 0$ on $|w| = 1$ and $|z| < 1$, $(z,w) \in \mathrm{Supp}(S)$, $\phi < u$ on all of $\partial X \cap \mathrm{Supp}(S)$ if we choose $\eta = 0$. To finish the choice of constants, η, ϵ, δ, choose $\eta > 0$ small enough that $\phi < u$ still on $\partial X \cap \mathrm{Supp}(S)$ and in, addition $u(z_0, w_0) < 0$.

Next choose n large enough so that $\phi_n < u$ on $\partial X \cap \mathrm{Supp}(S)$. Then if we add a strictly positive constant c to u we can assume that

$$\phi_n < u + c \text{ on } \partial X \cap \mathrm{Supp}(S)$$

$$\phi_n \le u + c \text{ on } X \cap \mathrm{Supp}(S)$$

$$\phi_n(z_1, w_1) = u(z_1, w_1) + c\,,\ (z_1, w_1) \in X \cap \mathrm{Supp}(S)\,.$$

Now, $-u$ is plurisubharmonic, so $i\partial\overline{\partial}(-u) \wedge S \ge 0$. Hence $\phi_n - u$ is S-plurisubharmonic so this contradicts the local maximum modulus principle for S-plurisubharmonic functions.

If ϕ is S-pluriharmonic, then $K(\phi)$ and $K(-\phi)$ intersect, hence ϕ is constant. □

PROPOSITION 7.10. *If $v \in \mathcal{P}_-^{(k)}(M)$ then v satisfies the local maximum principle.*

Proof. This follows, since the Hartogs figure argument is local. In fact, let K denote the compact set at which the maximum is reached. Let $p \in \partial K$ and use a Hartogs figure there. □

There are positive closed currents T on $\mathbb{P}^2$ of the form $T = i\partial u \wedge \overline{\partial u}$, u continuous except at one point and such that $\int T \wedge T \neq 0$, for example, $u = \log^+ |z|$ in $\mathbb{C}^2$ if $[z : w : t]$ are the homogeneous coordinates in $\mathbb{P}^2$.

Proposition 7.11. *Consider on* $\mathbb{P}^2$ *the cone* $C = \{T \geq 0, i\partial\overline{\partial}T = 0, \int T \wedge \omega = 1\}$. *Then* $\inf_{T \in C} \int T \wedge T \leq 1 - \frac{1}{2\pi^2}$.

Proof. Let $u(|z|^2), v(|w|^2)$ be $\mathcal{C}^\infty$ real valued functions with support in the unit interval. Define $\psi(z,w) := u(|z|^2) + iv(|w|^2)$. Let $T := i\partial\psi \wedge \overline{\partial\psi}$ on $\mathbb{C}^2$. Then $T \geq 0$ and $T \wedge T = 0$. Moreover, T is pluriharmonic on $\mathbb{C}^2$.

We want to decompose T as in Proposition 2.2.

$$\begin{aligned} i\partial\psi \wedge \overline{\partial\psi} &= i(u'\overline{z}dz + iv'\overline{w}dw) \wedge (u'zd\overline{z} - iv'wd\overline{w}) \\ &= i(u')^2 z\overline{z}dz \wedge d\overline{z} + i(v')^2 w\overline{w}dw \wedge d\overline{w} \\ &\quad + u'v'z\overline{w}d\overline{z} \wedge dw + u'v'\overline{z}wdz \wedge d\overline{w}\,. \end{aligned}$$

Let

$$\begin{aligned} U(z) &:= \tfrac{i}{\pi} \int \log|z-x| \big(u'(x\overline{x})\big)^2 x\overline{x}dx \wedge d\overline{x} \\ V(w) &:= \tfrac{i}{\pi} \int \log|w-y| \big(v'(y\overline{y})\big)^2 y\overline{y}dy \wedge d\overline{y}\,. \end{aligned}$$

Then

$$\begin{aligned} \partial\psi \wedge \overline{\partial\psi} &= i\partial\overline{\partial}U(z) + i\partial\overline{\partial}V(w) + \overline{\partial}(uv'\overline{w}dw) - uv''w\overline{w}d\overline{w} \wedge dw \\ &\quad - uv'd\overline{w} \wedge dw + \partial(uv'wd\overline{w}) - uv''w\overline{w}dw \wedge d\overline{w} - uv'dw \wedge d\overline{w}\,. \end{aligned}$$

Hence,

Lemma 7.12. *On* $\mathbb{C}^2$,

$$\begin{aligned} T &:= i\partial\psi \wedge \overline{\partial\psi} \\ &= i\partial\overline{\partial}U(z) + i\partial\overline{\partial}V(w) + \overline{\partial}(uv'\overline{w}dw) + \partial(uv'wd\overline{w})\,. \end{aligned}$$

Let $A := i/\pi \int (u')^2|z|^2 dz \wedge d\overline{z}$, $B := i/\pi \int (v')^2|w|^2 dw \wedge d\overline{w}$. Then $U(z) = A\log|z|$, $|z| > 1$, $V(w) = B\log|w|$, $|w| > 1$. We decompose T further: Let $h := U(z) + V(w) - \frac{1}{2}(A+B)\log(1+|z|^2+|w|^2)$. Then

Lemma 7.13. *On* $\mathbb{C}^2$,

$$\begin{aligned} T &= i\partial\overline{\partial}h(z,w) + \tfrac{2\pi}{2}(A+B)\omega + \overline{\partial}(uv'\overline{w}dw) + \partial(uv'wd\overline{w}) \\ \omega &:= \tfrac{i}{2\pi}\partial\overline{\partial}\log\big(1+|z|^2+|w|^2\big)\,, \text{ the Kähler form}\,. \end{aligned}$$

We extend T to $\mathbb{P}^2$ as $\tilde{T}$, the trivial extension. We need to know that T has finite mass near $[0:1:0]$ and $[1:0:0]$ for $\tilde{T}$ to be well defined. To extend first across the line at infinity, $t = 0$ away from $[0:1:0]$ and $[1:0:0]$, we extend the three parts individually. First ω extends as the Kähler form, also called ω. The form $uv'wd\overline{w}$ has compact support and extends as S trivially. Next we investigate h near $[0:1:0]$. We calculate in local coordinates, $[z:w:1] = [Z:1:t]$, to get $h(z,w) = \tilde{h}(Z,t) = U(Z/t) + V(1/t) - \frac{1}{2}(A+B)\log(1+|Z/t|^2+|1/t|^2)$. When $|Z/t| > 1$ we have $U(Z/t) = A\log|Z/t|$, $V(1/t) = B\log|1/t|$. So $\tilde h(Z,t) = A\log|Z| - A\log|t| - B\log|t| - \frac{1}{2}(A+B)\log(1+|Z|^2+|t|^2) + \frac{1}{2}(A+B)\log|t|^2$, $\tilde h(Z,t) = A\log|Z| - \frac{1}{2}\log(1+|t|^2+|Z|^2)$. Hence $\tilde h(Z,t)$ extends smoothly across $\eta = 0$ except possibly at $[0:1:0]$ and $[1:0:0]$. In particular $i\partial\overline{\partial}\tilde h$ extends trivially at $\eta = 0$ except at $[1:0:0]$ and $[0:1:0]$. Next we calculate in a neighborhood of $[0:1:0]$. We get

$$\tilde h(Z,t) = U(Z/t) + A\log|t| - \frac{A+B}{2}\log\left(1+|t|^2+|Z|^2\right).$$

The function $U(Z/t) + A\log|t| =: \phi(Z,t)$ is plurisubharmonic when $t \neq 0$ and equals $A\log|Z|$ when $|Z/t| > 1$ or $t = 0$, $Z \neq 0$. So $\phi(Z,t)$ is plurisubharmonic away from the origin. Hence ϕ has a well-defined plurisubharmonic extension through $(0,0)$ by setting $\phi(0,0) = -\infty$. It follows that $\tilde h$ is a global quasi-plurisubharmonic function on $\mathbb{P}^2$ with poles at $[0:1:0], [1:0:0]$. Hence,

Lemma 7.14. *The trivial extension $\tilde T$ is given by $\tilde T = \pi(A+B)\omega + \partial S + \overline{\partial S} + i\partial\overline{\partial}\tilde h$, $S = uv'wd\overline{w}$.*

It is easy to check that $\tilde T$ is pluriharmonic on $\mathbb{P}^2$.

End of proof of Proposition 7.11. This follows since $\partial\overline{\partial}\tilde h$ has no mass on the line at infinity. Hence $\int \overline{\partial}S \wedge \partial\overline{S} = AB$, $\int T \wedge T = (A+B)^2\pi^2 - 2AB$ and $\int T \wedge \omega = (A+B)\pi$ so if we let $T_1 = T/\int T\wedge\omega$ we find

$$\int T_1 \wedge T_1 = 1 - \frac{2AB}{\left(\frac{A+B}{2}\right)^2 4\pi^2}.$$

The minimum is reached for $A = B$ and equals $1 - \frac{1}{2\pi^2}$. □

Proposition 7.15. *Let M be a complex surface and $\rho \in \mathcal{C}^2_{\mathbb{R}}(M)$. Assume that $\partial\rho$ is non-vanishing on $X = \{\rho = 0\}$. Assume $(i\partial\overline{\partial}\rho)^2 = 0$ on X and $i\partial\overline{\partial}\rho \wedge \partial\rho \wedge \overline{\partial}\rho = \mathcal{O}(\rho^2)$. Then $T = i\delta_{\{\rho=0\}}\partial\rho \wedge \overline{\partial}\rho$ is a smooth positive harmonic current. Moreover $T \wedge T = 0$.*

Proof. Choose a $\chi \in \mathcal{C}^\infty_0(-1,1)$, $\chi \geq 0$, $\int \chi = 1$. Let

$$T_\epsilon = i\tfrac{1}{\epsilon}\chi\left(\tfrac{\rho}{\epsilon}\right)\partial\rho \wedge \overline{\partial}\rho.$$

Then $T_\epsilon \to T$ and we have

$$i\partial\overline{\partial}T_\epsilon = -\tfrac{1}{\epsilon^2}\chi'\left(\tfrac{\rho}{\epsilon}\right)\partial\rho \wedge \overline{\partial}\partial\rho \wedge \overline{\partial}\rho - \tfrac{1}{\epsilon}\chi\left(\tfrac{\rho}{\epsilon}\right)\overline{\partial}\partial\rho \wedge \partial\overline{\partial}\rho.$$

Clearly then $i\partial\overline{\partial}T_\epsilon \to 0$. □

If a hypersurface X is foliated by Riemann surfaces, i.e. X is Levi flat, then one does not have necessarily that $(i\partial\overline{\partial}\rho)^2 = 0$ on X for a defining function ρ.

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