SUPERCONNECTIONS
AND HIGHER INDEX THEORY

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Abstract

Let $M$ be a smooth closed spin manifold. The higher index theorem, as given for example in Proposition 6.3 of [CM], computes the pairing between the group cohomology of $\pi_1(M)$ and the Chern character of the “higher” index of a Dirac-type operator on $M$. Using superconnections, we give a heat equation proof of this theorem on the level of differential forms on a noncommutative base space. As a consequence, we obtain a new proof of the Novikov conjecture for hyperbolic groups.

I. Introduction

Let $M$ be a smooth closed connected spin manifold. Let $V$ be a Hermitian vector bundle on $M$. If $M$ is even-dimensional, the Atiyah-Singer index theorem identifies the topological expression $\int_M \hat{\mathcal{A}}(M) \wedge \text{Ch}(V)$ with the index of the Dirac-type operator acting on $L^2$-sections of the bundle $S(M) \otimes V$, where $S(M)$ is the spinor bundle on $M$ [AS1].

When $M$ is not simply-connected, one can refine the index theorem to take the fundamental group into account [Co3, Kas, Mo]. Let $\Gamma$ denote the fundamental group of $M$. Let $\nu : M \to \text{BO}$ be the classifying map for the universal cover $\widetilde{M}$ of $M$. For $[\eta] \in H^*(\text{BO}; \mathbb{C})$, higher index theory attempts to identify $\int_M \hat{\mathcal{A}}(M) \wedge \text{Ch}(V) \wedge \nu^*[\eta]$ with an analytic expression. The main topological and geometric applications of higher index theory are to Novikov's conjecture on homotopy-invariants of non-simply-connected manifolds [No], and to questions of the existence of positive-scalar-curvature metrics on $M$ [Ro].

In order to motivate the statement of the higher index theorem, let us first recall how Lusztig used the index theorem for families of operators
to prove a higher index theorem in the case of $\Gamma = \mathbb{Z}^k$ [Lu]. Let $T^k = \text{Hom}(\Gamma, U(1))$ be the dual group to $\Gamma$ and let $L_\theta$ be the flat unitary line bundle over $M$ whose holonomy is specified by $\theta \in T^k$. Consider the product fibration $M \to M \times T^k \to T^k$. Suppose for simplicity that $M$ is even-dimensional; then there is a bundle $\mathcal{H}$ over $T^k$ of $\mathbb{Z}_2$-graded Hilbert spaces, where $\mathcal{H}_\theta$, the fiber over $\theta \in T^k$, consists of the $L^2$-sections of $S(M) \otimes V \otimes L_\theta$. There is also a family $Q$ of vertical Dirac-type operators parametrized by $T^k$, where $Q_\theta$ acts on $\mathcal{H}_\theta$. The analytic index $\text{Index}(Q)$ of the family of elliptic operators, as defined in [AS2], lies in $K^0(T^k)$. An element $[\eta]$ of the group cohomology $H^\ell(T^k; \mathbb{C})$ gives a homology class $[\eta] \in H^\ell(T^k; \mathbb{C})$, against which the Chern character $\text{Ch}(\text{Index}(Q)) \in H^*(T^k; \mathbb{C})$ can be paired. The families index theorem [AS2] then implies

$$\int_{[\eta]} \text{Ch}(\text{Index}(Q)) = \text{const.}(1) \int_M \hat{A}(M) \wedge \text{Ch}(V) \wedge \nu^*[\eta],$$

(*)
giving the desired analytic interpretation of the right-hand-side. The purpose of [Lu] was to apply (*) to the Novikov conjecture.

In order to extend these methods to nonabelian $\Gamma$, let us note some algebraic properties of the above construction. The algebra of continuous functions $C(T^k)$ acts on the vector space $C(\mathcal{H})$ of continuous sections of $\mathcal{H}$ by multiplication. Upon performing Fourier transform over $T^k$, $C(\mathcal{H})$ maps to a certain subspace of the $L^2$-sections of the pullback bundle $S(\tilde{M}) \otimes \tilde{V}$ on $\tilde{M}$, this subspace thus being a $C(T^k)$-Hilbert module in the sense of [Kas]. The generalization of Lusztig's method to nonabelian $\Gamma$ is based on a "fibration" $M \to P \to B$ which exists only morally, where $B$ is a noncommutative space whose "algebra of continuous functions" is taken to be the algebra $A = C^*_\Gamma \Gamma$, the reduced group $C^*$-algebra [Co3]. (When $\Gamma = \mathbb{Z}^k$, $A \cong C(T^k)$.) Mishchenko and Kasparov define a Hilbert $A$-module of $L^2$-sections of $S(\tilde{M}) \otimes \tilde{V}$, upon which a Dirac-type operator $\tilde{D}$ acts. The analytic index of $\tilde{D}$ lies in "$K^0(B)$", or more precisely in $K_0(A)$ [Mi, Kas]. The Mishchenko-Fomenko index theorem identifies the analytic index with a topological index [MF].

In order to pair these indices with the group cohomology of $\Gamma$, one needs additional structure on $B$. Let $\mathcal{B}^\infty$ be a dense subalgebra of $A$ containing $C\Gamma$ which is stable under the holomorphic functional calculus of $A$ [Co1]. (For example, if $\Gamma = \mathbb{Z}^k$, one can take $\mathcal{B}^\infty$ to be $C^\infty(T^k)$.) Then $K_0(A) \cong K_0(\mathcal{B}^\infty)$. One can think of the image of $\text{Index}(\tilde{D})$ under this isomorphism as being a "smoothing" of $\text{Index}(\tilde{D})$. 
One can then use the fact that $K_0(\mathcal{B}^\infty)$ pairs with the cyclic cohomology $HC^*(\mathcal{B}^\infty)$ of $\mathcal{B}^\infty$ [Co1] to extract numbers from $\text{Index}(\tilde{D})$. In loose but more familiar terms, the Chern character $\text{Ch}(\text{Index}(\tilde{D}))$ lies in the "cohomology" of $B$. More precisely, it lies in the cyclic homology group $HC_*(\mathcal{B}^\infty)$ [Co1, Ka]. One then wants to define a "homology class" of $B$ which one can pair with $\text{Ch}(\text{Index}(\tilde{D}))$. The correct notion of homology for $B$ is given by the (periodic) cyclic cohomology of $\mathcal{B}^\infty$. In particular, given a group cocycle $\eta \in Z^l(\Gamma; \mathbb{C})$, one obtains an cyclic cocycle $\tau_\eta \in ZC^l(\mathbb{C}\Gamma)$ (eqn. (62)). If $\tau_\eta$ extends to an element of $ZC^l(\mathcal{B}^\infty)$ then Proposition 6.3 of [CM] gives

$$\left\langle \text{Ch}(\text{Index}(\tilde{D})), \tau_\eta \right\rangle = \text{const.}(l) \int_M \hat{A}(M) \wedge \text{Ch}(V) \wedge \nu^*[\eta] .$$

The special case when $l = 0$ is the $L^2$-index theorem [At].

An equivalent and more concrete description of the above "fibration" is given by a vector bundle $\mathcal{E}$ over $M$ whose fibers are finitely-generated right projective $\mathcal{B}$-modules for an appropriate algebra $\mathcal{B}$ [Mi]. We will use this latter description in making things precise, although we will move back and forth freely between the two pictures.

In another direction, using Quillen's theory of superconnections [Q], Bismut gave a heat equation proof of the Atiyah-Singer families index theorem on the level of differential forms on the base space [Bi]. Equation (*) is a consequence.

Analogously, we wish to give a heat equation proof of (**). Our original purpose was to study higher versions of spectral invariants, such as the eta invariant [Lo1]. These higher eta invariants should enter into a higher index theorem for manifolds with boundary. However, it turned out to be necessary to first understand the case of closed manifolds, i.e. equation (**), in terms of superconnections. This is what we present here.

As in [Bi], we wish to produce an explicit differential form on $B$ which represents $\text{Ch}(\text{Index}(\tilde{D}))$. First, one needs to know what a form on the noncommutative space $B$ should mean. A differential complex $\Omega_*(\mathcal{B})$ was defined in [Ka], and its homology can be identified with a subspace of the cyclic homology of the relevant algebra $\mathcal{B}$. In Section II we briefly review this theory. In this section we also consider integral operators on sections of $\mathcal{E}$ and define their traces and supertraces.

In the case at hand, the relevant vector bundles $\mathcal{E}$ come from a flat $\mathcal{B}$-bundle over $M$. There is some choice in exactly which subalgebra $\mathcal{B}$ of $\Lambda$ is
taken. In Section III we consider a subalgebra \( \mathfrak{B}^\omega \) of \( \Lambda \) consisting of elements whose coefficients decay faster than any exponential in a word-length metric. If \( \Gamma = \mathbb{Z} \) then \( \mathfrak{B}^\omega \) is isomorphic to the restrictions of holomorphic functions on \( \mathbb{C} - 0 \) to the unit circle, and so \( \mathfrak{B}^\omega \) is like an algebra of "analytic" functions on \( B \). (The technical reason for the appearance of this algebra is the existence of finite-propagation-speed estimates for heat kernels on \( \widetilde{M} \).) The smooth sections \( \Gamma^\infty(\mathcal{E}^\omega) \) of the corresponding vector bundle \( \mathcal{E}^\omega \) are shown to correspond to smooth sections of \( S(\widetilde{M}) \otimes V \) with rapid decay. Using this description, we make the trace of Section II more explicit.

By construction, the vector space of smooth sections of \( \mathcal{E}^\omega \) is a right \( \mathfrak{B}^\omega \)-module. Let \( \nabla : \Gamma^\infty(\mathcal{E}^\omega) \to \Gamma^\infty(\mathcal{E}^\omega \otimes_{\mathfrak{B}^\omega} \Omega_1(\mathfrak{B}^\omega)) \) be a connection on \( \mathcal{E}^\omega \). This is, in a sense, a connection in the vertical direction of \( \mathcal{E}^\omega \), when thought of as a vector bundle over \( M \). Let \( Q \) be the Dirac-type operator on \( \Gamma^\infty(\mathcal{E}^\omega) \). Applying Quillen's formalism \([Q]\), for any \( \beta, s > 0 \), the Chern character of \( \mathcal{E}^\omega \) is defined to be

\[
\text{ch}_{\beta,s}(\mathcal{E}^\omega) = \text{STR} \exp \left(-\beta(\nabla + sQ)^2\right) \in \overline{\Omega}_*(\mathfrak{B}^\omega). \tag{***}
\]

To make this expression useful, one needs an explicit description of a connection on \( \mathcal{E}^\omega \). In Section IV we show that the simplest such connection comes from a function \( h \in C^\infty_0(\widetilde{M}) \) with the property that the sum of the translates of \( h \) is 1. Then (*** is a well-defined closed element of \( \overline{\Omega}_*(\mathfrak{B}^\omega) \), and its homology class is independent of \( s \).

Given a group cocycle \( \eta \in Z^l(\Gamma; \mathbb{C}) \), if the corresponding cyclic cocycle \( \tau_\eta \in ZC^l(\mathbb{C} \Gamma) \) extends to an element of \( ZC^l(\mathfrak{B}^\omega) \) then the pairing

\[
\langle \text{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta \rangle \in \mathbb{C} \tag{****}
\]

is well-defined and independent of \( s \). As usual with heat equation approaches to index theory, the \( s \to 0 \) limit of (**** becomes the integral of a local expression on \( M \). In Section V we compute this limit. (The local analysis is easier than in [Bi], as there is no need to use a Levi-Civita superconnection.) The limit must involve \( \nu^*\eta \), and it may seem strange that this could become a local expression on \( M \), but this is where the choice of \( h \) enters. In Proposition 12 we find

\[
\lim_{s \to 0} \langle \text{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta \rangle = \beta^{1/2}/(!!) \int_M \widetilde{A}(M) \wedge \text{Ch}(V) \wedge \omega,
\]
where $\omega$ is a closed $l$-form on $M$ whose pullback to $\tilde{M}$ is given by

$$
\pi^*\omega = \sum R^*_{g_1} dh \wedge \cdots \wedge R^*_{g_l} dh \; \eta(e, g_1, \ldots, g_l) \in \Lambda^l(\tilde{M}) .
$$

We then show that $\omega$ represents $\nu^*[\eta] \in H^l(M; \mathbb{C})$.

It remains to show that

$$
\langle ch_{\beta,s}(E^\omega), \tau_\eta \rangle = \left\langle Ch_{\beta}(\text{Index}(\tilde{D})), \tau_\eta \right\rangle .
$$

(****)

For this, we find it necessary to work with the algebra $\mathcal{B}^\infty$ and assume that $\tau_\eta$ extends to a cyclic cocycle of $\mathcal{B}^\infty$. In Section VI we sketch a proof of (****). We reduce to the case of invertible $\tilde{D}$, and then use a trick of [Bi] to show the equality. This completes the proof of (**).

One application of (**) is to the Novikov conjecture. Taking $\tilde{D}$ to be the signature operator, the right-hand-side of (**) becomes $\text{const.}(l) \int_M L(M) \wedge \nu^*[\eta]$, where $L(M) \in H^*(M; \mathbb{C})$ is the Hirzebruch $L$-polynomial. The Novikov conjecture states that this "higher" signature is an (orientation-preserving) homotopy invariant of $M$. One can show that $\text{Index}(\tilde{D}) \in K_0(\Lambda)$ is a homotopy invariant of $M$ [Mi, Kas, HS]. If the group $\Gamma$ is such that one can apply (**) then the validity of the Novikov conjecture follows. In particular, in [CM] it was shown that if $\Gamma$ is hyperbolic in the sense of Gromov [GH] then (**) applies. Thus our proof of (**) gives a new proof of the validity of the Novikov conjecture for hyperbolic groups. One can also apply (**) to find obstructions to the existence of positive-scalar-curvature metrics on $M$ [Ro]. If one takes $\tilde{D}$ to be the pure Dirac operator then if $M$ has positive scalar curvature, $\text{Index}(\tilde{D})$ vanishes. Thus if the group $\Gamma$ is such that one can apply (**) , $\int_M \tilde{A}(M) \wedge \nu^*[\eta]$ is an obstruction to the existence of a positive-scalar-curvature metric on $M$.

In [Lo1] a bivariant Chern character was proposed in the case of finitely-generated projective modules. The obstacle to defining a bivariant Chern character for more general projective modules was the lack of a good trace theory for Hilbert modules. In the present case there is such a trace. The smooth sections of $E^\infty = E^\omega \otimes_{\mathcal{B}^\infty} \mathcal{B}^\infty$ form a $(C^\infty(M), \mathcal{B}^\infty)$-bivariant module, and the pairing $\langle ch_{\beta,s}, \tau_\eta \rangle$ of the bivariant Chern character with $\tau_\eta$ is a cocycle in the space $C^*_e(C^\infty(M))$ of entire cyclic cochains [Co2]. In Section VII we compute the $s \to 0$ limit of $\langle ch_{\beta,s}, \tau_\eta \rangle$.

Heat equation methods were also used in the paper of Connes and Moscovici [CM] to attack the Novikov conjecture, and it is worth comparing
the two approaches. One difference is that we use heat kernels to form the Chern character of a superconnection as in (**), whereas in [CM] the heat kernels are used to form an idempotent matrix over an algebra of smoothing operators [CM, Section 2]. Theorem 5.4 of [CM] is similar to our Corollary 2, but is stronger in that it is a statement about \( \mathcal{C} \Gamma \), whereas Corollary 2 is a statement about \( \mathcal{B}^\omega \). We believe that there is some point to taking a superconnection approach to these questions, as there should be interesting extensions.

This paper is an extension of [Lo1], in which the finite-dimensional analog was worked out. An exposition of the Mischenko-Fomenko theorem and related results appears in [Hi].

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II. Algebraic Preliminaries

Let \( \mathcal{B} \) be a Fréchet locally \( m \)-convex algebra with unit, i.e. the projective limit of a sequence of Banach algebras with unit [Mal]. We first define a graded differential algebra (GDA) \( \Omega_*(\mathcal{B}) \). This will be an appropriate completion of

\[
\Omega_*(\mathcal{B}) = \bigoplus_{k=0}^{\infty} \Omega_k(\mathcal{B}),
\]

the universal GDA of \( \mathcal{B} \) [Co1, Ka]. As a vector space, \( \Omega_k(\mathcal{B}) \) is given by

\[
\Omega_k(\mathcal{B}) = \mathcal{B} \otimes \left( \otimes^k \left( \mathcal{B}/\mathbb{C} \right) \right).
\]

As a GDA, \( \Omega_*(\mathcal{B}) \) is generated by \( \mathcal{B} \) and \( d\mathcal{B} \) with the relations

\[
d1 = 0, \quad d^2 = 0, \quad d(\omega_k \omega_\ell) = (d\omega_k)\omega_\ell + (-1)^k \omega_k (d\omega_\ell)
\]

for \( \omega_k \in \Omega_k(\mathcal{B}) \), \( \omega_\ell \in \Omega_\ell(\mathcal{B}) \). It will be convenient to write an element \( \omega_k \) of \( \Omega_k(\mathcal{B}) \) as a finite sum \( \sum b_0 db_1 \ldots db_k \). Recall that the homology of the differential complex \( \Omega_*(\mathcal{B}) = \Omega_*(\mathcal{B})/[\Omega_*(\mathcal{B}), \Omega_*(\mathcal{B})] \) is isomorphic to a subspace of the reduced cyclic homology of \( \mathcal{B} \) [Ka]. (This statement must be modified in degree zero, for which we refer to [Ka].)

Let \( \Theta_*(\mathcal{B}) \) denote the GDA

\[
\Theta_*(\mathcal{B}) = \bigoplus_{k=0}^{\infty} (\otimes^{k+1}\mathcal{B}),
\]
with the product given by
\[(b_0 \otimes b_1 \otimes \ldots \otimes b_k)(c_0 \otimes c_1 \otimes \ldots \otimes c_\ell) = b_0 \otimes b_1 \otimes \ldots \otimes b_k c_0 \otimes c_1 \otimes \ldots \otimes c_\ell\] (5)
and the differential given by
\[d(b_0 \otimes b_1 \otimes \ldots \otimes b_k) = 1 \otimes b_0 \otimes b_1 \otimes \ldots \otimes b_k - b_0 \otimes 1 \otimes b_1 \otimes \ldots \otimes b_k + \ldots + (-1)^{k+1} b_0 \otimes b_1 \otimes \ldots \otimes b_k \otimes 1 .\] (6)

Give $\Theta_k(\mathcal{B})$ the projective tensor product topology, with closure $\Theta_k(\mathcal{B})$. Let
\[\hat{\Theta}_\ast(\mathcal{B}) = \prod_{k=0}^{\infty} \Theta_k(\mathcal{B})\] (7)
denote the completion of $\Theta_\ast(\mathcal{B})$ in the product topology.

**Proposition 1.** $\hat{\Theta}_\ast(\mathcal{B})$ is a Fréchet GDA.

There is a natural embedding $\epsilon$ of $\Omega_\ast(\mathcal{B})$, as a graded differential algebra, in $\hat{\Theta}_\ast(\mathcal{B})$, with
\[\epsilon(b) = b , \quad \epsilon(db) = 1 \otimes b - b \otimes 1 .\] (8)

Let $\hat{\Omega}_\ast(\mathcal{B})$ denote the closure of $\epsilon(\Omega_\ast(\mathcal{B}))$ in $\hat{\Theta}_\ast(\mathcal{B})$.

**Corollary 1.** $\hat{\Omega}_\ast(\mathcal{B})$ is a Fréchet GDA.

Define $\bar{\Omega}_\ast(\mathcal{B})$ to be $\hat{\Omega}_\ast(\mathcal{B})/[\hat{\Omega}_\ast(\mathcal{B}), \hat{\Omega}_\ast(\mathcal{B})]$. Let $\bar{\mathcal{H}}_\ast(\mathcal{B})$ denote the homology of the differential complex $\bar{\Omega}_\ast(\mathcal{B})$.

Let $\mathcal{E}$ be a Fréchet space which is a (continuous) right $\mathcal{B}$-module. If $\mathcal{F}$ is a Fréchet space which is a (continuous) left $\mathcal{B}$-module, let $\mathcal{E} \hat{\otimes} \mathcal{F}$ be the projective topological tensor product of $\mathcal{E}$ and $\mathcal{F}$. Let $\mathcal{H}$ be the closure in $\mathcal{E} \hat{\otimes} \mathcal{F}$ of
\[\text{span}\{eb \otimes f - e \otimes bf : e \in \mathcal{E} , f \in \mathcal{F} , b \in \mathcal{B}\} .\] (9)

We put $\mathcal{E} \hat{\otimes}_B \mathcal{F}$ to be the Fréchet space $(\mathcal{E} \hat{\otimes} \mathcal{F})/\mathcal{H}$.

With this definition, $\mathcal{E} \hat{\otimes}_B \Omega_k(\mathcal{B})$ is isomorphic to the closure of the algebraic tensor product $\mathcal{E} \otimes_B \Omega_k(\mathcal{B}) \subset \mathcal{E} \otimes_B (\otimes^{k+1} \mathcal{B}) = \mathcal{E} \otimes (\otimes^k \mathcal{B})$ in $\mathcal{E} \hat{\otimes} (\otimes^k(\mathcal{B}))$, where the latter has the projective tensor product topology.
For the rest of this section, we assume that $\mathcal{E}$ is a finitely generated right projective $\mathcal{B}$-module. Let $\mathcal{F}$ be a Fréchet $\mathcal{B}$-bimodule. Then there is a trace

$$\text{Tr} : \text{Hom}_\mathcal{B}(\mathcal{E}, \mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F}) \to \mathcal{F}/[\mathcal{B}, \mathcal{F}] \, .$$

To define $\text{Tr}$, write $\mathcal{E}$ as $e\mathcal{B}^n$, with $e$ a projector in $M_n(\mathcal{B})$. Then an operator $T \in \text{Hom}_\mathcal{B}(\mathcal{E}, \mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F}) = \text{Hom}_\mathcal{B}(e\mathcal{B}^n, e\mathcal{F}^n)$ can be represented by a matrix $T \in M_n(\mathcal{F})$ satisfying $eT = Te = T$. Put

$$\text{Tr}(T) = \sum_{i=1}^n T_{ii} \left( \bmod [\mathcal{B}, \mathcal{F}] \right) \, .$$

This is independent of the choices made. (We quotient by the closure of $[\mathcal{B}, \mathcal{F}]$ to ensure that the trace takes value in a Fréchet space.)

**Lemma 1.** Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are finitely generated right projective $\mathcal{B}$-modules and $\mathcal{F}$ is a Fréchet algebra containing $\mathcal{B}$. Given $T \in \text{Hom}_\mathcal{B}(\mathcal{E}, \mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})$ and $T' \in \text{Hom}_\mathcal{B}(\mathcal{E}', \mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})$, let $T'T \in \text{Hom}_\mathcal{B}(\mathcal{E}, \mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})$ and $TT' \in \text{Hom}_\mathcal{B}(\mathcal{E}', \mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})$ be the induced products. Then $\text{Tr}(T'T) = \text{Tr}(TT') \in \mathcal{F}/[\mathcal{F}, \mathcal{F}]$.

We omit the proof.

In the case that $\mathcal{E}$ is $\mathbb{Z}_2$-graded by an operator $\Gamma_\mathcal{E} \in \text{End}_\mathcal{B}(\mathcal{E})$ satisfying $\Gamma_\mathcal{E}^2 = 1$, we can extend the trace to a supertrace by $\text{Tr}_s(T) = \text{Tr}(\Gamma_\mathcal{E}T)$.

Let $M$ be a closed connected oriented smooth Riemannian manifold. Let $\mathcal{E}$ be a smooth $\mathcal{B}$-vector bundle on $M$ with fibers isomorphic to $\mathcal{E}$. This means that if $\mathcal{E}$ is defined using charts $\{U_\alpha\}$, then a transition function is a smooth map $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{End}_\mathcal{B}(\mathcal{E})$. We will denote the fiber over $m \in M$ by $\mathcal{E}_m$. If $\mathcal{F}$ is a Fréchet algebra containing $\mathcal{B}$, let $\mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F}$ denote the $\mathcal{B}$-vector bundle with fibers $(\mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})_m = \mathcal{E}_m \hat{\otimes}_\mathcal{B} \mathcal{F}$ and transition functions $\phi_{\alpha\beta} \hat{\otimes}_\mathcal{B} \text{Id}_\mathcal{F} \in \text{End}_\mathcal{B}(\mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})$. Let $\Gamma^\infty(\mathcal{E})$ denote the right $\mathcal{B}$-module of smooth sections of $\mathcal{E}$.

**Definition:** Let $\text{Hom}_\mathcal{B}^\infty(\mathcal{E}, \mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})$ be the algebra of integral operators $T : \Gamma^\infty(\mathcal{E}) \to \Gamma^\infty(\mathcal{E} \hat{\otimes}_\mathcal{B} \mathcal{F})$ with smooth kernels $T(m_1, m_2) \in \text{Hom}_\mathcal{B}(\mathcal{E}_m, \mathcal{E}_m \hat{\otimes}_\mathcal{B} \mathcal{F})$. That is, for $s \in \Gamma^\infty(\mathcal{E})$,

$$\left(Ts\right)(m_1) = \int_M T(m_1, m_2)s(m_2)\text{dvol}(m_2) \in \mathcal{E}_{m_1} \hat{\otimes}_\mathcal{B} \mathcal{F} \, .$$

(12)
**DEFINITION:** For $T \in \text{Hom}_\mathcal{F}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{B}} \mathcal{F})$,

$$\text{TR}(T) = \int_M \text{Tr} (T(m, m)) \, d\text{vol}(m) \in \mathcal{F}/[\mathcal{F}, \mathcal{F}] .$$ (13)

**PROPOSITION 2.** TR is a trace.

**Proof:** We have

$$(TT')(m, m') = \int_M T(m, m'') T'(m'', m') \, d\text{vol}(m'').$$ (14)

Then

$$\text{TR}(TT') = \int_M \text{Tr} (T(m, m'')T'(m'', m)) \, d\text{vol}(m'') = \int_M \text{Tr} (T'(m'', m)T(m, m'')) \, d\text{vol}(m) \, d\text{vol}(m'') = \text{TR}(T'T) .$$ (15)

If the fibers of $\mathcal{E}$ are $\mathbb{Z}_2$-graded, we can extend TR to a supertrace STR on $\text{Hom}_\mathcal{F}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{B}} \mathcal{F})$ by

$$\text{STR}(T) = \int_M \text{Tr}_s (T(m, m)) \, d\text{vol}(m) \in \mathcal{F}/[\mathcal{F}, \mathcal{F}] .$$ (16)

### III. $\mathcal{B}^\omega$-Bundles

Let $\Gamma$ be a finitely-generated discrete group and let $\| \cdot \|$ be a right-invariant word-length metric on $\Gamma$. For $q \in \mathbb{Z}$, define the Hilbert space

$$\ell^{2,q}(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} : |f|_q^2 = \sum_g \exp (2q\|g\|)|f(g)|^2 < \infty \right\}$$ (17)

and let $\mathcal{B}^\omega$ be the vector space

$$\mathcal{B}^\omega = \bigcap_q \ell^{2,q}(\Gamma) .$$ (18)

**LEMMA 2.**

$$\mathcal{B}^\omega = \left\{ f : \Gamma \to \mathbb{C} : \text{for all } q \in \mathbb{Z}, \sup_g \left( \exp (q\|g\|)|f(g)| \right) < \infty \right\} .$$
Proof: If \( f \in \mathcal{B}^\omega \) then for all \( q \in \mathbb{Z} \), \( \exp(2q\|g\|)|f(g)|^2 \) is bounded in \( g \), and so \( \exp(q\|g\||f(g)| \) is bounded in \( g \). Suppose that \( f : \Gamma \to \mathbb{C} \) is such that for all \( r \in \mathbb{Z} \),

\[
\sup_g (\exp(r\|g\|)|f(g)|) = C_r < \infty.
\]

Then \( \sum_g \exp(2q\|g\|)|f(g)|^2 \leq C_r^2 \sum_g \exp(2(q - r)\|g\|) \). As \( \Gamma \) has at most exponential growth, by taking \( r \) large enough we can ensure that the last sum is finite. \( \square \)

**Proposition 3.** \( \mathcal{B}^\omega \) is independent of the choice of \( \| \circ \| \), and is an algebra with unit under convolution.

**Proof:** As all word-length metrics are quasi-isometric [GH], the independence follows. If \( T \in \mathcal{B}^\omega \) and \( f \in \ell^2,q(\Gamma) \), we will show that

\[
|T * f|_q \leq \text{const.}(q,T)|f|_q. \tag{19}
\]

If we then take both \( T \) and \( f \) in \( \mathcal{B}^\omega \), the proposition will follow. Let \( f_h \) denote \( f(h) \). Then

\[
\left| \sum_h T_{gh^{-1}} f_h \right|^2 \leq \left( \sum_h \exp(-q\|h\|)|T_{gh^{-1}}|^{1/2}|T_{gh^{-1}}|^{1/2} \exp(q\|h\||f_h|) \right)^2 \leq \left( \sum_h \exp(-2q\|h\|)|T_{gh^{-1}}| \right) \left( \sum_{h'} |T_{gh'}^{-1}| \exp(2q\|h'\||f_{h'}|^2) \right). \tag{20}
\]

Thus

\[
|T * f|_q^2 = \sum_g \exp(2q\|g\|) \left| \sum_h T_{gh^{-1}} f_h \right|^2 \leq \sum_g \left( \sum_h \exp(2q(\|g\| - \|h\|)) |T_{gh^{-1}}| \right) \left( \sum_{h'} |T_{gh'^{-1}}| \exp(2q\|h'\||f_{h'}|^2) \right) \leq \sum_g \left( \sum_h \exp(2q\|gh^{-1}\|)|T_{gh^{-1}}| \right) \left( \sum_{h'} |T_{gh'^{-1}}| \exp(2q\|h'\||f_{h'}|^2) \right) \leq \sum_k \exp(2q\|k\||T_k|) \left( \sum_{\ell} |T_{\ell}| \right) \left( \sum_{h'} \exp(2q\|h'\||f_{h'}|^2) \right) = \left( \sum_k \exp(2q\|k\||T_k|) \right) \left( \sum_{\ell} |T_{\ell}| \right) |f|_q^2. \tag{21}
\]

\( \square \)
Let $\Lambda$ denote the reduced group $C^*$-algebra of $\Gamma$, namely the completion of $C\Gamma$ with respect to the operator norm on $B(\ell^2(\Gamma))$, where $C\Gamma$ acts on $\ell^2(\Gamma)$ by convolution.

There is a Fréchet topology on $\mathcal{B}^\omega$ coming from its definition as a projective limit of Hilbert spaces. There is also a description of $\mathcal{B}^\omega$ as a Fréchet locally $m$-convex algebra. Namely, put

$$ \mathcal{P} = \{ T \in \Lambda : \text{for all } q \in \mathbb{Z}, T \text{ acts as a bounded operator by convolution on } \ell^{2,q}(\Gamma) \}. $$

By its definition, $\mathcal{P}$ is equipped with a sequence of norms.

**Proposition 4.** As topological vector spaces, $\mathcal{B}^\omega = \mathcal{P}$.

**Proof:** By the proof of Proposition 3, $\mathcal{B}^\omega$ injects continuously into $\mathcal{P}$. Applying an element $T$ of $\mathcal{P}$ to the element $e \in \bigcap_q \ell^{2,q}(\Gamma)$ gives a continuous injection of $\mathcal{P}$ into $\mathcal{B}^\omega$. These two maps are clearly inverses of each other. 

It follows that $\mathcal{B}^\omega$ has a holomorphic functional calculus.

**Note.** $\mathcal{B}^\omega$ is generally not holomorphically closed in $\Lambda$. For example, if $\Gamma = \mathbb{Z}$ then an element $T$ of $\mathcal{B}^\omega$ can be identified with its Fourier transform $T = \sum T_g z^g$, a holomorphic function on $\mathbb{C} - 0$. This identification gives $\mathcal{B}^\omega \cong H(\mathbb{C} - 0)$. On the other hand, in this case $\Lambda \cong C(S^1)$. Taking for example $T = z \in H(\mathbb{C} - 0) \subset C(S^1)$, the spectrum of $T$ in $C(S^1)$ consists of the unit circle. If $f$ is the holomorphic function defined on a neighborhood of the unit circle by $f(w) = (w - 2)^{-1}$, $f(T)$ is well-defined in $C(S^1)$, but does not lie in $H(\mathbb{C} - 0)$.

Let $\Gamma$ denote the fundamental group of $M$. Let $\widetilde{M}$ denote the universal cover of $M$, on which $g \in \Gamma$ acts on the right by $R_g \in \text{Diff}(\widetilde{M})$. Denote the covering map by $\pi : \widetilde{M} \to M$. As $\Gamma$ acts on $\mathcal{B}^\omega$ on the left, we can form $\widetilde{M} \times_\Gamma \mathcal{B}^\omega$, a flat $\mathcal{B}^\omega$-bundle over $M$. Let $E$ be a Hermitian vector bundle with Hermitian connection on $M$ and let $\widetilde{E}$ be the pullback of $E$ to $\widetilde{M}$, with the pulled-back connection. Let $R^*_g \in \text{Aut}(\widetilde{E})$ denote the action of $g \in \Gamma$ on $\widetilde{E}$.

**Definition:** $\mathcal{E}^\omega = (\widetilde{M} \times_\Gamma \mathcal{B}^\omega) \otimes E$, a $\mathcal{B}^\omega$-bundle over $M$.

Fix a base point $x_0 \in \widetilde{M}$.
PROPOSITION 5. There is an isomorphism

\[ L : \Gamma^\infty(\mathcal{E}^\omega) \to \left\{ f \in C^\infty(\tilde{M}, \tilde{E}) : \text{for all } q \in \mathcal{I} \text{ and all multi-indices } \alpha, \right. \]

\[ \sup_x (\exp(qd(x_0, x))|\nabla^\alpha f(x)|) < \infty. \]

**Proof:** By the construction of \( \mathcal{E}^\omega \), \( \Gamma^\infty(\mathcal{E}^\omega) \) consists of the \( \Gamma \)-equivariant elements of \( C^\infty(\tilde{M}, \tilde{E} \otimes \mathcal{B}^\omega) \). Writing \( s \in \Gamma^\infty(\mathcal{E}^\omega) \) as \( \sum g_s g \) with \( g_s \in C^\infty(\tilde{M}, \tilde{E}) \), the equivariance means that

\[ R^*_\gamma s = s \text{ for all } \gamma \in \Gamma. \quad (23) \]

This becomes \( \sum g_s (R^*_\gamma g) = \sum g_s g \gamma g \), and so \( R^*_\gamma g_s = g_{\gamma g} \) for all \( \gamma, g \in \Gamma \). Thus \( g_s = R^*_g s_1 \), and so \( s = \sum g_s (R^*_g s_1) g \).

Let \( L \) be the map which takes \( s \) to \( s_1 \). We will show that \( L \) is the desired isomorphism. First, if \( \tilde{m} \in \tilde{M} \) then

\[ s(\tilde{m}) = \sum g_s (R^*_g s_1)(\tilde{m}) g \in \tilde{E}_m \otimes \mathcal{B}^\omega. \quad (24) \]

Thus for all \( q \in \mathcal{I} \), \( \sup g_s (\exp(q||g||)|s_1(\tilde{m} g)|) < \infty \). By the smoothness of \( s \), we have such an estimate uniformly for \( \tilde{m} \) lying within a fundamental domain of \( \tilde{M} \) containing \( x_0 \). As \( \tilde{M} \) is quasi-isometric to \( \Gamma [GH] \), there are constants \( A > 0 \) and \( B \geq 0 \) such that for all \( x \in \tilde{M} \) and \( g \in \Gamma \),

\[ A^{-1}||g|| - B \leq d(xg^{-1}, x) \leq A||g|| + B. \quad (25) \]

Then

\[ \exp(qd(x_0, x))|s_1(x)| \leq \exp(qd(x_0, xg^{-1})) \exp(qd(xg^{-1}, x))|s_1(xg^{-1}g)| \leq \text{const.} \exp(qd(x_0, xg^{-1})) \exp(qA||g||)|s_1(xg^{-1}g)|. \quad (26) \]

By choosing \( g \) so that \( xg^{-1} \) lies within a fundamental domain containing \( x_0 \), we obtain from (26) that \( \exp(qd(x_0, x))|s_1(x)| \) is uniformly bounded in \( x \). The same argument applies to the covariant derivatives of \( s_1 \).

Now suppose that \( f \in C^\infty(\tilde{M}, \tilde{E}) \) is such that for all \( q \in \mathcal{I} \) and all multi-indices \( \alpha \),

\[ \sup_x (\exp(qd(x_0, x))|\nabla^\alpha f(x)|) < \infty. \quad (27) \]
Put $L'(f) = \sum_{g}(R_{g}^{*}f) g$. We must show that $L'(f) \in \Gamma^{\infty}(\mathcal{E}^\omega)$. It will then follow that $L'$ is an inverse to $L$.

By construction, $L'(f)$ is $\Gamma$-equivariant. Let $\{V_{\alpha}\}$ be a collection of charts on $M$ over which $E$ is trivialized. Then we can reduce to the case that $E$ is a trivial $\mathbb{C}$-bundle and $f \in C^{\infty}(V_{\alpha} \times \Gamma, \mathbb{C})$, with the above decay conditions. It is enough to show that when restricted to $V_{\alpha} \times \{e\}$, $\sum_{g}(R_{g}^{*}f)g$ represents a smooth map from $V_{\alpha}$ to $\mathfrak{B}^\omega$. For $\tilde{m} \in V_{\alpha} \times \{e\}$,

$$\left( \sum_{g}(R_{g}^{*}f)g \right)(\tilde{m}) = \sum_{g}f(\tilde{m}g)g,$$

and so for all $q \in \mathbb{Z}$,

$$\exp (q \|g\|) |f(\tilde{m}g)| \leq \text{const.} \exp (qA \text{d}(\tilde{m}, \tilde{m}g)) |f(\tilde{m}g)| \leq \text{const.} \exp (qA \text{d}(\tilde{m}, x_{0})) \exp (qA \text{d}(x_{0}, \tilde{m}g)) |f(\tilde{m}g)| \leq \text{const.} \sup_{x} \left( \exp (qA \text{d}(x_{0}, x)) |f(x)| \right) < \infty. \quad (29)$$

Thus $\sum_{g}(R_{g}^{*}f)g$ is a map from $V_{\alpha}$ to $\mathfrak{B}^\omega$. Doing the same estimates using covariant derivatives gives the smoothness.

**PROPOSITION 6.** The algebra $\text{End}_{\mathfrak{B}^\omega}(\mathcal{E}^\omega) = \text{Hom}_{\mathfrak{B}^\omega}(\mathcal{E}^\omega, \mathcal{E}^\omega)$ is isomorphic to the algebra of $\Gamma$-invariant integral operators $T$ on $L^{2}(\widetilde{M}, E)$ with smooth kernels $T(x, y) \in \text{Hom}(\widetilde{E}_{y}, \widetilde{E}_{x})$ such that for all $q \in \mathbb{Z}$ and multi-indices $\alpha$ and $\beta$,

$$\sup_{x, y} \left( \exp (qd(x, y)) \left\| \nabla_{x}^{\alpha} \nabla_{y}^{\beta} T(x, y) \right\| \right) < \infty.$$

We omit the proof, which is similar to that of Proposition 5.

Let $\phi \in C^{\infty}_{0}(\widetilde{M})$ be such that

$$\sum_{g}R_{g}^{*}\phi = 1. \quad (30)$$

Let $\text{tr}$ denote the local trace on $\text{End}(\widetilde{E}_{x})$. 
Note. We now have defined three traces: \( \text{tr} \) is the trace on \( \text{End}(E_x) \), \( \text{Tr} \) is the trace on \( \text{End}_{\mathcal{B}}(\mathcal{E}^w_m) \) and \( \text{TR} \) is the trace on \( \text{End}_{\mathcal{B}}(\mathcal{E}^w) \). If \( E \) is \( \mathbb{Z}_2 \)-graded, the corresponding supertraces are denoted \( \text{tr}_s, \text{Tr}_s \) and \( \text{STR} \).

**Proposition 7.** Representing an element \( T \in \text{End}_{\mathcal{B}}(\mathcal{E}^w) \) by an operator \( \tilde{T} \in B(L^2(M, \tilde{E})) \) as in Proposition 6, its trace is given by

\[
\text{TR}(T) = \sum_g \left[ \int_{\tilde{M}} \phi(x) \text{tr} ((R^*_g \tilde{T})(x, x)) d\text{vol}(x) \right] g \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) \quad (31)
\]

\[
= \sum_g \left[ \int_{\tilde{M}} \phi(x) \text{tr} (\tilde{T}(x, x)) d\text{vol}(x) \right] g \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) . \quad (32)
\]

**Proof:** The proof is a matter of unraveling the isomorphisms of Propositions 5 and 6. Let \( \{V_\alpha\} \) be a collection of charts on \( M \) over which \( E \) is trivialized. Then we can reduce to the case that \( E \) is a trivial \( \mathbb{C} \)-bundle. We have \( \pi^{-1}(V_\alpha) \cong V_\alpha \times \Gamma \). For \( \tilde{m}_1, \tilde{m}_2 \in V_\alpha \times \{e\} \), we can use isomorphisms to represent

\[
T(m_1, m_2) \in \text{Hom}_{\mathcal{B}}(\mathcal{E}^w_{m_1}, \mathcal{E}^w_{m_2}) \cong \text{Hom}_{\mathcal{B}}(\mathcal{B}^w, \mathcal{B}^w) \cong \mathcal{B}^w \quad (33)
\]

by \( \sum_g \tilde{T}(\tilde{m}_1g, \tilde{m}_2g) \). Then

\[
\int_{V_\alpha} \text{Tr} (T(m, m)) d\text{vol}(m) = \\
\int_{V_\alpha} \sum_g \tilde{T}(mg, m)g d\text{vol}(m) \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) = \\
\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma) \tilde{T}(mg\gamma, m\gamma)g d\text{vol}(m) \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) = \\
\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma) \tilde{T}(m\gamma\gamma^{-1}g\gamma, m\gamma)g d\text{vol}(m) \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) = \\
\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma) \tilde{T}(m\gamma g, m\gamma) \gamma g\gamma^{-1} d\text{vol}(m) \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) = \\
\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma) \tilde{T}(m\gamma g, m\gamma)(g + [\gamma g, \gamma^{-1}]) d\text{vol}(m) \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) = \\
\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma) \tilde{T}(m\gamma g, m\gamma)g d\text{vol}(m) \quad (\text{mod } [\mathcal{B}^w, \mathcal{B}^w]) =
\]
\[
\int_{\pi^{-1}(V_{\alpha})} \sum_g \phi(x) \tilde{T}(xg, x) g \, d\text{vol}(x) \pmod{[\mathcal{B}^\omega, \mathcal{B}^\omega]}.
\] (34)

Using a partition of unity subordinate to \(\{V_{\alpha}\}\) and adding the contributions of the various charts gives (31).

We now give the extension of the previous propositions to form-valued sections of \(\mathcal{E}^\omega\). With the notation of Section II, put \(\mathfrak{F}^\omega = \hat{\Omega}_*(\mathcal{B}^\omega)\). As in Proposition 5, we can represent an element \(f\) of \(\Gamma^\infty(\mathcal{E}^\omega \otimes_{\mathcal{B}^\omega} \mathfrak{F}^\omega)\) of degree \(k\) as \(\sum f_{g_1 \ldots g_k} d_{g_1} \ldots d_{g_k}\), with each \(f_{g_1 \ldots g_k} \in C^\infty(M, \tilde{E})\) a smooth rapidly decreasing section of \(\tilde{E}\). As in Proposition 6, we can represent an element \(K\) of \(\text{Hom}_{\mathfrak{F}^\omega}(\mathcal{E}^\omega, \mathcal{E}^\omega \otimes_{\mathcal{B}^\omega} \mathfrak{F}^\omega)\) of degree \(k\) by smooth rapidly decreasing kernels \(K_{g_1 \ldots g_k}(x, y) \in \text{Hom}(\tilde{E}_y, \tilde{E}_x)\) such that \(K = \sum K_{g_1 \ldots g_k} d_{g_1} \ldots d_{g_k}\) is \(\Gamma\)-invariant. Then for \(f \in \Gamma^\infty(\mathcal{E}^\omega)\) we have

\[
(Kf)(x) = \sum \int_{M} K_{g_1 \ldots g_k}(x, y) f(y) \, d\text{vol}(y) d_{g_1} \ldots d_{g_k}.
\] (35)

As in Proposition 7, we have

\[
\text{TR}(K) = \sum \int_{M} \phi(x) \text{tr} (K_{g_1 \ldots g_k}(xg_0, x)) \, d\text{vol}(x) g_0 d_{g_1} \ldots d_{g_k} \pmod{[\hat{\Omega}_*(\mathcal{B}^\omega), \hat{\Omega}_*(\mathcal{B}^\omega)]}.
\] (36)

IV. The Chern Character

Now suppose in addition that \(M^n\) is even-dimensional and spin. Let \(S\) be the \(\mathbb{Z}_2\)-graded spinor bundle on \(M\), with the Levi-Civita connection, and let \(V\) be a Hermitian bundle on \(M\) with Hermitian connection. Take \(E\) to be \(S \otimes V\). Let \(Q\) denote the self-adjoint extension of the Dirac-type operator acting on \(C^\infty_0(M, \tilde{E})\) [At]. In terms of a local framing of the tangent bundle,

\[
Q = -i \sum_{\mu=1}^n \gamma^\mu D_\mu,
\] (37)

with the Dirac matrices \(\{\gamma^\mu\}_{\mu=1}^n\) satisfying

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu}.
\] (38)

PROPOSITION 8. For \(T > 0\), \(e^{-TQ^2} \in \text{End}_{\mathfrak{F}^\omega}(\mathcal{E}^\omega)\).
Proof: First, \( e^{-TQ^2} \) is a \( \Gamma \)-invariant operator. By elliptic regularity, \( e^{-TQ^2}(x, y) \) is smooth. Put \( N = [n/4] + 1 \). Let \( \epsilon \) be a fixed sufficiently small number. If \( d(x, y) > \epsilon \), put \( R = d(x, y) - \epsilon \). By the finite-propagation-speed estimates of [CGT], we have the estimate [Lo2]

\[
\left| (Q^{2k} e^{-TQ^2} Q^{2\ell})(x, y) \right| \leq \text{const.} \frac{R^2}{T} \left( R^{-2(k+\ell)} + R^{-2(k+\ell)-4N} + R^{2(k+\ell)T-2(k+\ell)-4N} e^{-R^2/4T} \right).
\]

The requisite bounds on the covariant derivatives of \( e^{-TQ^2}(x, y) \) follow by standard methods. Then the proposition follows from Proposition 6. \( \Box \)

Note. In the “fibration” picture, the fact that \( e^{-TQ^2} \) commutes with \( \mathfrak{B}^\omega \) means that it corresponds to a family of vertical operators.

Let \( h \in C_0^\infty(\widetilde{M}) \) be such that

\[
\sum_g R^*_g h = 1.
\]

Given \( f \in \Gamma^\infty(\mathcal{E}^\omega) \), considering it as an element of \( C^\infty(\widetilde{M}, \widetilde{E}) \) by Proposition 5, define its covariant derivative to be

\[
\nabla_g f = h R^*_g f \in C^\infty(\widetilde{M}, \widetilde{E}).
\]

Note that \( C^\infty(M) \) acts on sections of \( \Gamma^\infty(\mathcal{E}^\omega \otimes_\mathfrak{B}^\omega \mathfrak{F}^\omega) \) by multiplication.

**Proposition 9.**

\[
\nabla f = \sum_g \nabla_g f \otimes_\mathfrak{B}^\omega dg
\]

defines a connection

\[
\nabla : \Gamma^\infty(\mathcal{E}^\omega) \to \Gamma^\infty(\mathcal{E}^\omega \otimes_\mathfrak{B}^\omega \tilde{\Omega}_1(\mathfrak{B}^\omega))
\]

which commutes with the action of \( C^\infty(M) \).
Proof: We first show that $\nabla$ formally commutes with the action of $C^\infty(M)$. Given $\alpha \in C^\infty(M)$, $\alpha$ acts on $C^\infty(M, E)$ by multiplication by $\pi^*(\alpha)$. Then

$$\nabla(\alpha \cdot f) = \nabla(\pi^*(\alpha)f) = \sum_g h \, R_g^* (\pi^*(\alpha)f) \hat{\otimes}_{\mathcal{B}^\omega} dg =$$

$$\sum_g h \, \pi^*(\alpha) R_g^* f \hat{\otimes}_{\mathcal{B}^\omega} dg = \alpha \cdot \nabla f .$$

Thus $\nabla$ acts fiberwise on the vector bundle $\mathcal{E}^\omega$. To make this explicit, as in the proof of Proposition 5 we can consider the element $s$ of $\Gamma^\infty(\mathcal{E}^\omega)$ corresponding to $f$ to be a sum $s = \sum s_g g$, where $s_g \in C^\infty(M, E)$ and $s_g = R_g^* f$. Then $\nabla s$ becomes

$$\sum_{g,k} R_g^* (hR_k^* f) g dk = \sum_{g,k} R_g^* h(R_k^* f) g dk = \sum_{g,k} R_g^* h s_{g,k} g dk .$$

Applied to a point $\tilde{m} \in \tilde{M}$, we have

$$\nabla \left( \sum_g s_g(\tilde{m}) g \right) = \sum_{g,k} h(\tilde{m}g) s_{g,k}(\tilde{m}) g dk .$$

Let

$$\nabla_m : \mathcal{E}_m^\omega \to \mathcal{E}_m^\omega \hat{\otimes}_{\mathcal{B}^\omega} \hat{\Omega}_1(\mathcal{B}^\omega)$$

be the restriction of $\nabla$ to the fiber $\mathcal{E}_m^\omega \cong E_m \otimes \mathcal{B}^\omega$ over $m = \pi(\tilde{m})$. Then $\nabla_m$ can be represented by

$$\nabla_m \left( \sum_g t_g g \right) = \sum_{g,k} h(\tilde{m}g) t_{g,k} g dk ,$$

where $t_g \in \tilde{E}_m^\omega$.

By hypothesis, $t = \sum g t_g g \in \mathcal{E}_m^\omega \cong E_m \otimes \mathcal{B}^\omega$. We must show that $\nabla_m(t)$ is in $\mathcal{E}_m^\omega \hat{\otimes}_{\mathcal{B}^\omega} \hat{\Omega}_1(\mathcal{B}^\omega) \cong E_m \otimes \hat{\Omega}_1(\mathcal{B}^\omega)$.

As in Section II, let us think of $E_m \otimes \hat{\Omega}_1(\mathcal{B}^\omega)$ as embedded in $E_m \otimes \mathcal{B}^\omega \hat{\otimes} \mathcal{B}^\omega$. Then $\nabla_m(t)$ is formally represented as

$$\nabla_m(t) = \sum_{g,k} h(\tilde{m}g) t_{g,k} g(1 \otimes k - k \otimes 1) =$$
\[ \sum_{g,k} h(\tilde{m} g) \, t_{g k} \otimes k - \sum_{g,k} h(\tilde{m} g) \, t_{g k} \, k \otimes 1 = \]
\[ \sum_{g,k} h(\tilde{m} g) \, t_{g k} \otimes k - \sum_{g,k} h(\tilde{m} g) \, t_{k} \, k \otimes 1 = \]
\[ \left( \sum_{g} h(\tilde{m} g) g \otimes \sum_{k} t_{g k} \, k \right) - t \otimes 1 = \]
\[ \left( \sum_{g} h(\tilde{m} g) g \otimes (g^{-1} t) \right) - t \otimes 1 . \quad (48) \]

As \( h \) has compact support, the \( g \)-sum in \( \sum_{g} h(\tilde{m} g) g \otimes (g^{-1} t) \) is finite, and it follows that (48) makes sense in \( E_m \otimes \mathcal{B}^\omega \otimes \mathcal{B}^\omega \).

We now show that \( \nabla_m \) is a connection. If \( \gamma \in \Gamma \),
\[ \nabla_m(t \gamma) = \nabla_m \left( \sum_{g} t_{g} \, g \gamma \right) = \nabla_m \left( \sum_{g} t_{g \gamma^{-1}} \, g \right) = \]
\[ \sum_{g,k} h(\tilde{m} g) \, t_{g k \gamma^{-1}} \, g d k = \sum_{g,k} h(\tilde{m} g) \, t_{g k} \, g d(k \gamma) = \]
\[ \sum_{g,k} h(\tilde{m} g) \, t_{g k} \, g(d \gamma) + \sum_{g,k} h(\tilde{m} g) \, t_{g k} \, g k d \gamma = \]
\[ \nabla_m(t) \gamma + \sum_{g,k} h(\tilde{m} g) \, t_{k} \, k d \gamma = \nabla_m(t) \gamma + t d \gamma . \quad (49) \]

Then
\[ \nabla_m(t b) = (\nabla_m t) b + t \hat{\otimes} \mathcal{B}^\omega db \quad (50) \]
for any \( b \in \mathcal{B}^\omega \).

As \( h \) is smooth, it follows that \( \nabla \) is also a connection. \( \square \)

**Note.** There is a strong relationship between the connections \( \nabla \) considered here and the partially flat connections of [Ka, Chapitre 4].

Define the superconnection
\[ D_s = \nabla + \sigma Q \in \text{Hom}^\infty \left( \mathcal{E}^\omega, \mathcal{E}^\omega \hat{\otimes} \tilde{\Omega}_{\bullet}(\mathcal{B}^\omega) \right) . \quad (51) \]

Then \( D_s^2 \in \text{Hom}_{\mathcal{B}^\omega} \left( \mathcal{E}^\omega, \mathcal{E}^\omega \hat{\otimes} \tilde{\Omega}_{\bullet}(\mathcal{B}^\omega) \right) \) is given by
\[ D_s^2 = s^2 Q^2 + s (\nabla Q + Q \nabla) + \nabla^2 . \quad (52) \]
Here $\nabla Q + Q\nabla$ is given explicitly by

$$
(\nabla Q + Q\nabla)(f) = \sum_g (\partial h) R^*_g f \otimes_{\mathcal{M}} dg ,
$$

where $f \in C^\infty(\tilde{M}, \tilde{E})$ and

$$
\partial h = [Q, h] = -i \sum_\mu \gamma^\mu \partial_\mu h ,
$$

and $\nabla^2$ is given by

$$
\nabla^2(f) = \sum_g \sum_{g'} h R^*_g h R^*_{g'} f \otimes_{\mathcal{M}} dg dg' .
$$

Put

$$
\mathcal{P} = -(s(\nabla Q + Q\nabla) + \nabla^2) ,
$$

and for $\beta > 0$ define

$$
\exp(-\beta D^2_s) \in \text{Hom}_{\mathcal{M}}(\mathcal{E}^\omega, \mathcal{E}^\omega \otimes_{\mathcal{M}} \Omega_* (\mathcal{V}^\omega))
$$

to be

$$
\exp(-\beta D^2_s) = \exp(-\beta s^2 Q^2) + \int_0^\beta \exp(-u_1 s^2 Q^2) \mathcal{P} \exp(\beta - u_1 s^2 Q^2) du_1 + \int_0^\beta \int_0^{u_1} \exp(-u_1 s^2 Q^2) \mathcal{P} \exp(-u_2 s^2 Q^2) \mathcal{P} \exp(\beta - u_1 - u_2 s^2 Q^2) du_2 du_1 + \ldots
$$

As only a finite number of terms of the expansion of (58) contribute to the degree-$k$ component of $\exp(-\beta D^2_s)$, it is clear that (58) converges.

**Definition:** For $s > 0$, the Chern character $\text{ch}_{\beta, s}(\mathcal{E}^\omega) \in \Omega_{\text{even}}(\mathcal{V}^\omega)$ is given by

$$
\text{ch}_{\beta, s}(\mathcal{E}^\omega) = \text{STR} \exp(-\beta D^2_s) .
$$

**Proposition 10.** $\text{ch}_{\beta, s}(\mathcal{E}^\omega)$ is closed.

We omit the proof, which is straightforward.

**Proposition 11.** The class of $\text{ch}_{\beta, s}(\mathcal{E}^\omega)$ in $\overline{H}_*(\mathcal{V}^\omega)$ is independent of $s \in (0, \infty)$. 
Proof: Formally,

\[
\frac{d}{ds} \text{ch}\beta,s(\mathcal{E}^\omega) = d(-\beta \text{ STRQ} e^{-\beta D_s^2}).
\]

(60)

It is straightforward to check that this equation is valid. Then if \( s_1, s_2 \in (0, \infty) \),

\[
\text{ch}\beta,s_1(\mathcal{E}^\omega) - \text{ch}\beta,s_2(\mathcal{E}^\omega) = d\left( -\beta \int_{s_2}^{s_1} \text{ STRQ} e^{-\beta D_s^2} ds \right).
\]

(61)

\[
\square
\]

Let \( \eta \) be an antisymmetric left-invariant (unnormalized) group \( k \)-cocycle. Then \( \eta \) defines a cyclic \( k \)-cocycle \( \tau_\eta \) on \( \mathfrak{G} \Gamma \) by

\[
\tau_\eta(g_0, \ldots, g_k) = \eta(g_0, g_0 g_1, g_0 g_1 g_2, \ldots, g_0 g_1 \ldots g_k) \text{ if } g_0 g_1 \ldots g_k = e
\]

(62)

\[
\tau_\eta(g_0, \ldots, g_k) = 0 \text{ if } g_0 g_1 \ldots g_k \neq e \quad [\text{Co1}].
\]

Suppose that there are constants \( C \) and \( D \) so that

\[
|\tau_\eta(g_0, \ldots, g_k)| \leq C \exp \left(D\left(\|g_0\| + \ldots + \|g_k\|\right)\right).
\]

(63)

Then \( \tau_\eta \) extends to a \( k \)-cocycle on \( \mathfrak{B}^\omega \) and so can be paired with \( \text{ch}\beta,s \). By Proposition 11, the pairing \( \langle \text{ch}\beta,s(\mathcal{E}^\omega), \tau_\eta \rangle \) is independent of \( s \).

V. Small-Time Limit

**Proposition 12.**

\[
\lim_{s \to 0} \langle \text{ch}\beta,s(\mathcal{E}^\omega), \tau_\eta \rangle = \beta^{k/2}/(k!) \int_M \widehat{\text{A}}(M) \wedge \text{Ch}(V) \wedge \omega,
\]

(64)

where \( \omega \) is the closed \( k \)-form on \( M \) given by

\[
\pi^* \omega = \sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh \ \eta(e, g_1, \ldots, g_k) \in \Lambda^k(\widehat{M}).
\]

(65)
Proof: First, let us consider the contribution to \( (\text{ch}_{\beta,s}, \tau_\eta) \) coming from the term

\[
(-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \exp(-u_1 s^2 Q^2)s(\nabla Q + Q \nabla)\exp(-u_2 s^2 Q^2)s(\nabla Q + Q \nabla)\cdots \exp(-\sum_{j=0}^k u_k s^2 Q^2)du_k \cdots du_1
\]

of \( \exp(-\beta D_s^2) \). Written out explicitly, this will be

\[
\sum (-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \int_M \phi(x_0) \text{tr}_s \left[ R_{g_0}^* \exp(-u_1 s^2 Q^2)s(\partial h) R_{g_1}^* \right. \\
\left. \exp(-u_2 s^2 Q^2) \cdots \exp(-\sum_{j=0}^k u_k s^2 Q^2) \right] (x_0, x_0) \]

\[
dvol(x_0)du_k \cdots du_1 \tau_\eta(g_0, \ldots, g_k) =
\]

\[
\sum (-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \int_M \phi(x_0) \text{tr}_s \left[\exp(-u_1 s^2 Q^2) R_{g_0}^* (\partial h) \cdots \exp(-\sum_{j=0}^k u_k s^2 Q^2) R_{g_0 g_1}^* (\partial h) \cdots \right. \\
\left. \cdots (\partial h) \right] (x_0, x_0) dvol(x_0)du_k \cdots du_1 \tau_\eta(g_0, \ldots, g_k) =
\]

\[
\sum (-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \int_M \phi(x_0) \text{tr}_s \left[ \exp(-u_1 s^2 Q^2)(x_0, x_1) \right. \\
\left. \cdots \exp(-\sum_{j=0}^k u_k s^2 Q^2)(x_k g_0 g_1 \ldots g_k - 1) \right] dvol(x_k) \cdots dvol(x_0) \]

Because for small \( s \) the heat kernels are concentrated near the diagonal, the only terms which will survive in the \( s \to 0 \) limit will have \( g_0 \cdots g_k = e \). Furthermore, the \( s \to 0 \) limit reduces to a question of local asymptotics on \( \tilde{M} \). By the Getzler calculus [G], (69) equals \( (2\pi)^{-n} \int_{\tilde{T} \tilde{M}} \text{tr}_s(\sigma P)_{s^{-1}} dx d\xi \), where \( P \) denotes the operator appearing in (69), \( \sigma P \) is its symbol in the Getzler calculus and \( (\sigma P)_{s^{-1}} \) is the rescaled symbol. A straightforward
calculation gives that in the limit $s \to 0$, this becomes
\[
\sum (-1)^k \beta^{-k/2} \left( \int_0^\beta \cdots \int_0^{u_{k-1}} du_k \cdots du_1 \right) \int_{\widetilde{M}} \phi(x) \tilde{A}(x) \land \text{Ch}(\tilde{V})(x) \land dh(x_{g_0}) \land \cdots \land dh(x_{g_0 \cdots g_{k-1}}) \eta(g_0, g_0 g_1, g_0 g_1 g_2, \ldots, g_0 g_1 \cdots g_{k-1}, e) = \]
(70)

\[
\sum (-1)^k \beta^{-k/2} / (k!) \int_{\widetilde{M}} \phi \tilde{A}(\widetilde{M}) \land \text{Ch}(\tilde{V}) \land R^*_{g_0} dh \land \cdots \land R^*_{g_0 \cdots g_{k-1}} dh \eta(g_0, g_0 g_1, g_0 g_1 g_2, \ldots, g_0 \cdots g_{k-1}, e) = \]
(71)

\[
\beta^{-k/2} / (k!) \int_{\widetilde{M}} \phi \tilde{A}(\widetilde{M}) \land \text{Ch}(\tilde{V}) \land \tilde{\omega}, \]
(72)

where $\tilde{\omega} \in \Lambda^k(\widetilde{M})$ is given by
\[
\tilde{\omega} = \sum R^*_{g_1} dh \land \cdots \land R^*_{g_k} dh \eta(e, g_1, \ldots, g_k). \]
(73)

Now let us consider the contribution to $(\text{ch}_{\beta,s}, \tau_\eta)$ coming from a term of $\exp(-\beta D^2_s)$ which contains a $\nabla^2$, such as, for example,

\[
(-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \exp(-u_1 s^2 Q^2) \nabla^2 \exp(-u_2 s^2 Q^2) s(\nabla Q + Q \nabla) \ldots \nabla^2 \exp(-u_k s^2 Q^2) \exp(-u_1 - \ldots - u_k) s^2 Q^2) du_k \ldots du_1. \]
(74)

Written out explicitly, this gives
\[
\sum (-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \int_{\tilde{M}} \cdots \int_{\tilde{M}} \phi(x_0) \text{tr}_s \left[ \exp(-u_1 s^2 Q^2)(x_0, x_1) \right. \\
\left. h(x_1 g_0) h(x_1 g_0 g_1) \exp(-u_2 s^2 Q^2)(x_1, x_2) s(\partial h)(x_2 g_0 g_1 g'_1) \right. \\
s(\partial h)(x_k g_0 g_1 g'_1 g_2 \cdots g_{k-1}) \exp(-\beta - u_1 - \ldots - u_k) s^2 Q^2) \\
\left. (x_k g_0 g_1 g'_1 g_2 \cdots g_k, x_0) \right] \text{dvol}(x_k) \\
\text{dvol}(x_0) du_k \ldots du_1 \tau_\eta(g_0, g_1, g'_1, g_2, \ldots, g_k). \]
(75)

By the Getzler calculus, in the $s \to 0$ limit, (75) becomes
\[
\sum (-1)^k \beta^{-(k-1)/2} \left( \int_0^\beta \cdots \int_0^{u_{k-1}} du_k \cdots du_1 \right) \int_{\tilde{M}} \phi(x) \tilde{A}(x) \text{Ch}(\tilde{V})(x) h(x_0) h(x_0 g_1) dh(x_0 g_1 g'_1) \land \cdots \land dh(x_0 g_1 g'_1 g_2 \cdots g_{k-1}) \eta(g_0, g_0 g_1, g_0 g_1 g'_1, \ldots, g_0 g_1 g'_1 g_2 \cdots g_{k-1}, e) = \]
(76)
\[\sum (-1)^k \beta^{(k+1)/2}/(k!)\]
\[\int_{\widetilde{M}} \phi \, \widetilde{A}(\widetilde{M}) \wedge \text{Ch}(\widetilde{V}) \wedge R_{g_0}^* h \wedge R_{g_0 g_1}^* h \wedge R_{g_0 g_1 g_2}^* dh \wedge \ldots \]
\[\wedge R_{g_0 g_1 g_2 \ldots g_{k-1}}^* dh \eta(g_0, g_0 g_1, g_0 g_1 g_2', \ldots, g_0 g_1 g_2' \ldots g_{k-1}, e) = (77)\]
\[\pm \beta^{(k+1)/2}/(k!) \int_{\widetilde{M}} \phi \, \widetilde{A}(\widetilde{M}) \wedge \text{Ch}(\widetilde{V}) \wedge \widetilde{\omega}', \quad (78)\]

where \(\widetilde{\omega}' \in \Lambda^k(\widetilde{M})\) is given by

\[\widetilde{\omega}' = \sum R_{g_1}^* h \wedge R_{g_1'}^* h \wedge \ldots \wedge R_{g_k}^* dh \eta(e, g_1, g_1', g_2, \ldots, g_k). \quad (79)\]

As \(\eta(e, g_1, g_1', g_2, \ldots, g_k)\) is antisymmetric in \(g_1\) and \(g_1'\), it follows that \(\widetilde{\omega}'\) vanishes. The same argument shows that all of the terms involving \(\nabla^2\) vanish.

**Lemma 3.** The form \(\widetilde{\omega}\) of (73) is a closed \(\Gamma\)-invariant form on \(\widetilde{M}\).

**Proof:** \(\widetilde{\omega}\) is clearly closed. For all \(\gamma \in \Gamma\), we have

\[R_{\gamma}^* \widetilde{\omega} = \sum R_{\gamma g_1}^* dh \wedge \ldots \wedge R_{\gamma g_k}^* dh \eta(e, g_1, \ldots, g_k) = \]
\[\sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh \eta(e, \gamma^{-1} g_1, \ldots, \gamma^{-1} g_k) = \]
\[\sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh \eta(\gamma, g_1, \ldots, g_k). \quad (80)\]

From the cocycle condition, this equals

\[\sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh \left[ \eta(e, g_1, \ldots, g_k) - \eta(e, \gamma, g_2, \ldots, g_k) + \ldots + \right. \]
\[\left. (-1)^k \eta(e, \gamma, g_1, \ldots, g_{k-1}) \right]. \quad (81)\]

But for all \(r\),

\[\sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh \eta(e, \gamma, g_1, \ldots, \hat{g}_r, \ldots, g_k) = \]
\[\pm \left( \sum R_{g_r}^* dh \right) \wedge \sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_{r-1}}^* dh \wedge R_{g_{r+1}}^* dh \wedge \]
\[\ldots \wedge R_{g_k}^* dh \eta(e, \gamma, g_1, \ldots, \hat{g}_r, \ldots, g_k) \]
\[\text{and} \]
\[\sum_{g_r} R_{g_r}^* dh = d\left( \sum_{g_r} R_{g_r}^* h \right) = d(1) = 0. \quad (82)\]
Thus only the first term of (81) contributes, and so

\[ R^*_g \omega = \sum R^*_g dh \wedge \ldots \wedge R^*_g dh \eta(e, g_1, \ldots, g_k) = \tilde{\omega}. \quad (84) \]

End of Proof of Proposition 12: From Lemma 3, there is a closed form \( \omega \) on \( M \) such that \( \tilde{\omega} = \pi^*(\omega) \). Then

\[ \int_{\tilde{M}} \phi \hat{A}(\tilde{M}) \wedge \text{Ch}(\tilde{V}) \wedge \tilde{\omega} = \int_M \hat{A}(M) \wedge \text{Ch}(V) \wedge \omega. \quad (85) \]

We now wish to show that the cohomology class of the closed form \( \tilde{\omega} \) is the pullback to \( M \) of the cohomology class \( [\eta] \) on \( B(\Gamma) \). To do so, it is convenient to first relax the smoothness conditions on \( \tilde{\omega} \).

Let \( h \) be a Lipschitz function on \( \tilde{M} \) of compact support with

\[ \sum_g R^*_g h = 1. \quad (86) \]

As the distributional derivatives of a Lipschitz function are \( L^\infty \)-functions, it makes sense to define \( \tilde{\omega}_h \) by

\[ \tilde{\omega}_h = \sum R^*_g dh \wedge \ldots \wedge R^*_g dh \eta(e, g_1, \ldots, g_k), \quad (87) \]

a closed \( \Gamma \)-invariant \( L^\infty \) \( k \)-form on \( \tilde{M} \), and let \( \omega_h \in \Lambda^k(M) \) be such that \( \pi^* \omega_h = \tilde{\omega}_h \). It is known that one can compute the de Rham cohomology of \( M \) using flat forms (i.e. \( L^\infty \)-forms \( \tau \) such that \( d\tau \) is also \( L^\infty \)) [Te].

**Lemma 4.** The cohomology class of \( \omega_h \) is independent of \( h \).

**Proof:** Let \( h' \) be another choice for \( h \). Then

\[ \tilde{\omega}_h - \tilde{\omega}_{h'} = \sum [R^*_g dh(h - h') \wedge \ldots \wedge R^*_g dh + \ldots + R^*_g dh' \wedge \ldots \wedge R^*_g dh(h - h')] \eta(e, g_1, \ldots, g_k). \quad (88) \]

Put

\[ \tilde{\sigma}_r = \sum R^*_g dh' \wedge \ldots \wedge R^*_g (h - h') \wedge \ldots \wedge R^*_g dh \eta(e, g_1, \ldots, g_k), \quad (89) \]
a flat \((k - 1)\)-form on \(\widetilde{M}\). Then

\[\tilde{\omega}_h - \tilde{\omega}_{h'} = d\left(\sum_{r=1}^{k} (-1)^{r+1}\tilde{\sigma}_r\right) .\] (90)

Furthermore, for all \(\gamma \in \Gamma\),

\[R^*_\gamma \tilde{\sigma}_r = \sum R^*_{g_1} dh' \wedge \ldots \wedge R^*_{g_r} (h - h') \wedge \ldots \wedge R^*_{g_k} dh \eta(\gamma, g_1, \ldots, g_k) = \sum R^*_{g_1} dh' \wedge \ldots \wedge R^*_{g_r} (h - h') \wedge \ldots \wedge R^*_{g_k} dh [\eta(e, g_1, \ldots, g_k) - \eta(e, \gamma, g_2, \ldots, g_k) + \ldots + (-1)^k \eta(e, \gamma, g_1, \ldots, g_{k-1})] .\] (91)

As

\[\sum_{g} R^*_g dh = \sum_{g} R^*_g dh' = \sum_{g} R^*_g (h - h') = 0 ,\] (92)

it follows that \(\tilde{\sigma}_r\) is \(\Gamma\)-invariant. Then \(\omega - \omega' = d\sigma\), where \(\sigma \in \Lambda^{k-1}(M)\) is such that

\[\pi^* \sigma = \sum_{r=1}^{k} (-1)^{r+1}\tilde{\sigma}_r .\] (93)

Let \(X\) be the simplicial complex whose ordered cochain complex is the standard complex of \(\Gamma [\mathrm{Br}]\). The \(k\)-simplices of \(X\) are \((k + 1)\)-tuples of distinct elements of \(\Gamma\). We will take \(\Gamma\) to act on the right on \(X\). Then the simplicial complex \(X/\Gamma\) is a model for \(B\Gamma\). For a vertex \(v\), let \(b_v\) denote the barycentric coordinate (on a simplex containing \(v\)) corresponding to \(v\). Let \(j\) be the continuous piecewise linear function on \(X\) given by

\[j(x) = 0 \text{ if } x \in [g_0, \ldots, g_k] \text{ and } g_0 \neq e, \ldots, g_k \neq e\]
\[b_e \text{ if } x \in [g_0, \ldots, g_k] \text{ and } g_i = e \text{ for some } i .\] (94)

**Lemma 5.** \(\sum_{g} R^*_g j = 1.\)
Proof: Suppose that $x \in [g_0, \ldots, g_k]$. Then

$$
\sum_g (R^*_{gj})(x) = \sum_g j(xg) = \sum_{i=0}^k j(xg_i^{-1}) = \sum_{i=0}^k b_e(xg_i^{-1}) = \sum_{i=0}^k b_{gi}(x) = 1.
$$

(95)

Let $\tilde{\omega}_j$ be the polynomial form on $X$, with coefficients in $C$, given by

$$
\tilde{\omega}_j = \sum R^*_g dj \wedge \ldots \wedge R^*_g dj \eta(e, g_1, \ldots, g_k).
$$

(96)

Let $\omega_j$ be the polynomial form on $X/\Gamma$ such that $\tilde{\omega}_j = \pi^*\omega_j$.

We define a $k$-cocycle $\tilde{\eta} \in C^k(X; C)$ by putting

$$
\langle \tilde{\eta}, [\gamma_0, \gamma_1, \ldots, \gamma_k] \rangle = \eta(\gamma_0^{-1}, \ldots, \gamma_k^{-1}).
$$

(97)

By the left invariance of the group cocycle $\eta$, $\tilde{\eta}$ is right-invariant on $X$. With abuse of notation, let $\eta$ denote the corresponding simplicial cocycle on $X/\Gamma$.

**Proposition 13.** As elements of $H^k(X/\Gamma; C)$, $[\omega_j] = [\eta]$.

**Proof:** Let $A$ denote the de Rham map from polynomial forms on $X$ to $C^*(X)$. Then

(98)

$$
(A\omega_j)[\gamma_0, \ldots, \gamma_k] =
\sum \eta(e, g_1, \ldots, g_k) \langle R^*_g dj \wedge \ldots \wedge R^*_g dj, [\gamma_0, \ldots, \gamma_k] \rangle =
\sum \eta(e, \gamma_{i_1}^{-1}, \ldots, \gamma_{i_k}^{-1}) \langle R_{\gamma_{i_1}^{-1}} dj \wedge \ldots \wedge R_{\gamma_{i_k}^{-1}} dj, [\gamma_0, \ldots, \gamma_k] \rangle,
$$

where $i_1, \ldots, i_k \in \{0, 1, \ldots, k\}$. Now (98) equals

$$
\sum \eta(e, \gamma_{i_1}^{-1}, \ldots, \gamma_{i_k}^{-1}) \langle db_{\gamma_{i_1}} \wedge \ldots \wedge db_{\gamma_{i_k}}, [\gamma_0, \ldots, \gamma_k] \rangle.
$$

(99)

A simple calculation gives that (99) in turn equals

$$
\sum_{r=0}^k (-1)^{r+1} \eta(e, \gamma_0^{-1}, \ldots, \gamma_r^{-1}, \ldots, \gamma_k^{-1}) = \eta(\gamma_0^{-1}, \ldots, \gamma_k^{-1}).
$$

(100)

Thus $A(\omega_j)$ is the cochain $\eta$. As the de Rham map is an isomorphism on complex cohomology [GM], the proposition follows.

Let $\nu$ be the canonical (up to homotopy) map $\nu : M \to BG$ classifying the universal cover $\widetilde{M}$, with lift $\widetilde{\nu} : \widetilde{M} \to E\Gamma$.

**Proposition 14.** As elements of $H^*(M, C)$, $[\omega] = \nu^*([\eta])$. 
Proof: Let us triangulate $M$. Upon subdivision, we can homotop $\nu$ to be a simplicial map. Then with $h = \nu^* j$, we have $\omega_h = \nu^* \omega_j$. Thus as elements of $H^*(M, \mathbb{C})$,

$$[\omega_h] = [\nu^* \omega_j] = \nu^* [\omega_j] = \nu^* [\eta] .$$  \hspace{1cm} (101)

By Lemma 4, $[\omega_h]$ is independent of the particular choice of $h$, and the proposition follows. \hfill \Box

Corollary 2. For all $s > 0$,

$$\langle \text{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta \rangle = \beta^{k/2}/(k!) \int_M \mathcal{A}(M) \wedge \text{Ch}(V) \wedge \nu^*[\eta] .$$  \hspace{1cm} (102)

Note. One can equally well pair $\text{ch}_{\beta,s}(\mathcal{E}^\omega)$ with any element of $HC^s(\mathcal{B}^\omega)$. Modulo growth conditions, there is a way of producing an element $\tau \in HC^k(\mathcal{B}^\omega)$ from a conjugacy class $\langle x \rangle$ of $\Gamma$ and a $k$-cocycle of the group $\Gamma_x/\{x\}$, where $\Gamma_x$ is the centralizer of $x$ in $\Gamma$ and $\{x\}$ is the subgroup generated by $x$ [Bu]. (The cocycle (62) comes from the special case when $\langle x \rangle = \langle e \rangle$.) However, the cyclic cohomology classes corresponding to $\langle x \rangle \neq \langle e \rangle$ will pair with $\text{ch}_{\beta,s}(\mathcal{E}^\omega)$ to give zero. The reason is that a cyclic $k$-cocycle $\tau$ based on $\langle x \rangle$ will have $\tau(g_0, \ldots, g_k) = 0$ if $g_0 g_1 \ldots g_k \not\in \langle x \rangle$. However, by the proof of Proposition 12, in the $s \to 0$ limit one sees that the terms with $g_0 g_1 \ldots g_k \neq e$ do not contribute to $\langle \text{ch}_{\beta,s}(\mathcal{E}^\omega), \tau \rangle$.

VI. Reduction to the Index Bundle

We first review some of the results of [MF]. Recall that $\Lambda$ is the reduced group $C^*$-algebra of $\Gamma$. Let $\mathcal{E}$ denote the $\mathbb{Z}_2$-graded $\Lambda$-bundle over $M$ given by $\mathcal{E} = (\tilde{M} \times_{\Gamma} \Lambda) \otimes E$. The $L^2$-sections $\Gamma^0(\mathcal{E})$ of $\mathcal{E}$ form a right $\Lambda$-Hilbert module. The Dirac-type operator $\tilde{D}$ is an odd densely-defined unbounded operator on $\Gamma^0(\mathcal{E})$. One can find finitely-generated right projective $\Lambda$-Hilbert submodules $F^{\pm}$ of $\Gamma^0(\mathcal{E}^{\pm})$ and complementary $\Lambda$-Hilbert modules $G^{\pm} \subset \Gamma^0(\mathcal{E}^{\pm})$ such that $\tilde{D}$ is diagonal with respect to the decomposition $\Gamma^0(\mathcal{E}^{\pm}) = G^{\pm} \oplus F^{\pm}$, and writing $\tilde{D} = \tilde{D}_G \oplus \tilde{D}_F$, in addition $\tilde{D}_G : G^{\pm} \to G^{\mp}$ is invertible. By definition, the index of $\tilde{D}$ is

$$\text{Index}(\tilde{D}) \equiv [F^+] - [F^-] \in K_0(\Lambda) ;$$  \hspace{1cm} (103)

this is independent of the choice of $F^{\pm}$.
Now suppose that $\mathcal{B}^\infty$ is a densely-defined subalgebra of $\Lambda$ which is stable with respect to the holomorphic functional calculus on $\Lambda$, and $\mathcal{B}^\omega \subset \mathcal{B}^\infty \subset \Lambda$. A standard result in $K$-theory is that $K_0(\Lambda) \cong K_0(\mathcal{B}^\infty)$ [Bo, Appendix]. There is a Chern character $\text{Ch}_\beta$ from $K_0(\mathcal{B}^\infty)$ to $HC_*(\mathcal{B}^\infty)$, the reduced cyclic homology of $\mathcal{B}^\infty$ [Ka]. Let $\eta$ be a group $k$-cocycle on $G\Gamma$ which extends to an element $\tau_\eta$ of the cyclic cohomology of $\mathcal{B}^\infty$. By the explicit formula (62), $\tau_\eta$ is a reduced cyclic cohomology class if $k > 0$.

We will sketch a proof of the following proposition. Many of the details are as in [Bi].

**Proposition 15.**

$$\langle \text{Ch}_\beta(\text{Index}(\tilde{D})), \tau_\eta \rangle = \beta^{k/2}/(k!) \int_M \tilde{A}(M) \wedge \text{Ch}(V) \wedge \nu^*([\eta]).$$

**Proof:** Define $E^\infty$ to be $(\tilde{M} \times_\Gamma \mathcal{B}^\infty) \otimes E$. An examination of the proof of [MF] shows that $F^\pm$ and $G^\pm$ can be chosen to be of the form $F^\pm = \mathcal{F}^\pm \otimes \mathcal{B}^\infty \Lambda$ and $G^\pm = \mathcal{G}^\pm \otimes \mathcal{B}^\infty \Lambda$, where $\mathcal{F}^\pm$ and $\mathcal{G}^\pm$ are subspaces of $\Gamma^\infty(E^\infty)$. (This uses the fact that $\mathcal{B}^\infty$ is stable with respect to the holomorphic functional calculus in $\Lambda$.) Write $\tilde{D}_{F^\pm}$ and $\tilde{D}_{G^\pm}$ for the restrictions of $\tilde{D}$ to $F^\pm$ and $G^\pm$ respectively. Put

$$\mathcal{H}^\pm = \mathcal{G}^\pm \oplus F^\pm \oplus F^\mp.$$  

(104)

For $\alpha \in \mathbb{C}$, define $R^\pm_\alpha : \mathcal{H}^\pm \rightarrow \mathcal{H}^\mp$ by

$$R^\pm_\alpha = \begin{pmatrix} \tilde{D}_{G^\pm} & 0 & 0 \\ 0 & \tilde{D}_{F^\pm} & \alpha \\ 0 & \alpha & 0 \end{pmatrix}.$$  

(105)

We have that $\tilde{D}_{G^\pm}$ is invertible. Put

$$S^\pm_\alpha = \begin{pmatrix} \tilde{D}_{F^\pm} & \alpha \\ \alpha & 0 \end{pmatrix}.$$  

(106)

and let

$$S^\pm_\alpha \otimes \mathcal{B}^\infty \Lambda : F^\pm \oplus F^\mp \rightarrow F^\mp \oplus F^\pm$$  

(107)

be the extension to a bounded operator on finitely-generated Hilbert $\Lambda$-modules. As $\tilde{D}_F$ is a bounded operator, it follows that $S^\pm_\alpha \otimes \mathcal{B}^\infty \Lambda$ is invertible for $\alpha$ large. Then the fact that $\mathcal{B}^\infty$ is stable under the holomorphic
functional calculus in $\Lambda$ implies that $S^\pm_\alpha$ is also invertible for $\alpha$ large. Thus $R_\alpha^\pm$ is invertible for $\alpha$ large. We define $\exp(-TR^2_\alpha)$ by the Duhamel expansion in $\alpha$. As $R_\alpha$ differs from $\widetilde{D} \oplus 0$ by a finite-rank operator in the sense of [Kas], there is no problem in showing that $\exp(-TR^2_\alpha)$ is well-defined.

Extend the $\mathcal{B}^\omega$-connection $\nabla$ on $\mathcal{E}^\omega$ to a $\mathcal{B}^\infty$-connection on

$$\mathcal{E}^\infty = \mathcal{E}^\omega \otimes_{\mathcal{B}^\omega} \mathcal{B}^\infty. \tag{108}$$

Let $\nabla_\mathcal{F}$ be a $\mathcal{B}^\infty$-connection on $\mathcal{F}$ and let

$$\nabla' = \nabla \oplus \nabla_\mathcal{F} \tag{109}$$

be the sum connection on $\mathcal{H}$. Define the Chern character

$$c_{\beta,s,\alpha}(\mathcal{H}) = \text{STR} \exp \left( -\beta(\nabla' + sR^2_\alpha) \right) \in \widetilde{\Omega}(\mathcal{B}^\infty) \tag{110}$$

by a Duhamel expansion in $\nabla'$. For $\alpha = 0$, we have

$$c_{\beta,s,0}(\mathcal{H}) = c_{\beta,s}(\mathcal{E}^\infty) - \text{STR} \exp(-\beta\nabla^2_\mathcal{F}) \tag{111}$$

Now $\text{STR} \exp(-\beta\nabla^2_\mathcal{F}) \in \widetilde{\Omega}(\mathcal{B}^\infty)$ represents $\text{Ch}_\beta([\mathcal{F}])$ [Ka]. If we can show that $c_{\beta,s,0}(\mathcal{H})$ is zero in $H_\ast(\mathcal{B}^\infty)$ then we will have that as classes in $H_\ast(\mathcal{B}^\infty)$,

$$c_{\beta,s}(\mathcal{E}^\infty) = \text{STR} \exp(-\beta\nabla^2_\mathcal{F}) = \text{Ch}_\beta([\mathcal{F}]) = \text{Ch}_\beta(\text{Index}(\tilde{D})) \tag{112}$$

and the proposition will follow.

A standard homotopy argument shows that the class of $c_{\beta,s,\alpha}(\mathcal{H})$ in $H_\ast(\mathcal{B}^\infty)$ is independent of $\alpha$. Take $\alpha$ large enough that $R_\alpha$ is invertible.

We define a pseudodifferential calculus as in [MF], except that the symbol $a(m, \xi)$ will take value in $\text{End}_{\mathcal{B}^\infty}(\mathcal{E}^\infty_m)$. Then $R_\alpha$ is an elliptic first-order $\psi$do. (In terms of the "fibration" picture, it corresponds to a smooth family of elliptic first-order vertical $\psi$do's.) As in the usual calculus of $\psi$do's, $R_\alpha$ has a parametrix $P_\alpha$, an order -1 $\psi$do, such that

$$I - R_\alpha P_\alpha = K_{1\alpha} \quad \text{and} \quad I - P_\alpha R_\alpha = K_{2\alpha}, \tag{113}$$

where $K_{1\alpha}$ and $K_{2\alpha}$ are smoothing operators. It follows that

$$(R_\alpha)^{-1} = P_\alpha + K_{2\alpha}(R_\alpha)^{-1} \tag{114}$$
is also an order -1 operator.

Define a connection $\nabla''_\mathcal{H}^-$ on $\mathcal{H}^-$ by

$$\nabla''_\mathcal{H}^- = (R^-_\alpha)^{-1} \nabla'_{\mathcal{H}^+} R^-_\alpha$$

and define $\nabla''$ to be $\nabla'_{\mathcal{H}^+} \oplus \nabla''_\mathcal{H}^-$. Then

$$\nabla''_{\mathcal{H}^-} - \nabla'_{\mathcal{H}^-} = 0$$

and

$$\nabla''_{\mathcal{H}^-} - \nabla'_{\mathcal{H}^-} = (R^-_\alpha)^{-1} \left( \nabla'_{\mathcal{H}^+} R^-_\alpha - R^-_\alpha \nabla'_{\mathcal{H}^+} \right)$$

is an order -1 operator. We have a homotopy of connections on $\mathcal{H}$ from $\nabla'$ to $\nabla''$ given by $\nabla' + u(\nabla'' - \nabla')$, $u \in [0, 1]$. It follows as in [Bi, Prop. 2.10] that $\text{ch}_{\beta,s,\alpha}(\mathcal{H}) = \text{STR exp} \left( - \beta (\nabla' + s R^-_\alpha)^2 \right)$ represents the same class in $\overline{\text{H}}_*(\mathfrak{g}^\infty)$ as $\text{STR exp} \left( - \beta (\nabla'' + s R^-_\alpha)^2 \right)$.

We claim that if $\text{STR exp} \left( - \beta (\nabla'' + s R^-_\alpha)^2 \right)$ is expanded in $\nabla''$, the terms vanish algebraically. To see this formally, write $\nabla'' + s R^-_\alpha$ in terms of the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ as

$$\nabla'' + s R^-_\alpha = \begin{pmatrix} \nabla'_{\mathcal{H}^+} + s R^-_\alpha \\ s R^+_\alpha \end{pmatrix} = \begin{pmatrix} \nabla'_{\mathcal{H}^+} - s R^-_\alpha & I \\ s^2 R^-_\alpha R^+_\alpha & s R^-_\alpha \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

and so formally,

$$\text{STR exp} \left( - \beta (\nabla'' + s R^-_\alpha)^2 \right) = \text{STR exp} \left( - \beta \begin{pmatrix} \nabla'_{\mathcal{H}^+} & I \\ s^2 R^-_\alpha R^+_\alpha & \nabla'_{\mathcal{H}^+} \end{pmatrix} \right)^2 \in \overline{\text{H}}_* (\mathfrak{g}^\infty) .$$

However, expanding (119) in $\nabla'_{\mathcal{H}^+}$, one finds that (119) vanishes for algebraic reasons.

(To see this last point, consider an analogous statement in the finite-dimensional case. For $A, B \in M_N(\mathbb{C})$ put

$$M = \begin{pmatrix} A & I \\ B & A \end{pmatrix} \in M_{2N}(\mathbb{C}) .$$
Then \( \det(M) = \det(A^2 - B) \) and if \( A^2 - B \) is invertible,
\[
M^{-1} = \begin{pmatrix}
(A^2 - B)^{-1} A & -(A^2 - B)^{-1} \\
I - A(A^2 - B)^{-1} A & A(A^2 - B)^{-1}
\end{pmatrix}.
\] (121)

Thus \( \text{Str} M^{-1} = 0 \). If \( \lambda \not\in \text{Spec}(M) \), by changing \( A \) to \( A - \lambda I \), we obtain that \( \text{Str}(M - \lambda I)^{-1} = 0 \). Then by the functional calculus, if \( f \) is a holomorphic function in a neighborhood of \( \text{Spec}(M) \), \( \text{Str} f(M) = 0 \).

This formal argument can be made rigorous as in [Bi, Prop. 2.17].

**Note.** If \( M \) is odd-dimensional then one can use Quillen's formalism [Q] to define the odd Chern character
\[
\text{ch}_{\beta}(C^\infty) = \text{Tr}_\sigma \exp \left( -\beta(\nabla + sQ\sigma)^2 \right) \in \Omega_{\text{odd}}(\mathcal{B}^\infty).
\] (122)

The operator \( \tilde{D} \) gives an element \( \text{Index}(\tilde{D}) \) of \( K_1(\mathcal{B}^\infty) \) [Kas]. Using a suspension argument as in [BF], one can show that Proposition 15 also holds in the odd case.

**Corollary 3 [CM].** If \( \Gamma \) is a hyperbolic group in the sense of Gromov [GH] then for all \([\eta] \in H^*(\Gamma; \mathbb{C})\), the higher-signature \( \int_M L(M) \wedge \nu^*([\eta]) \) is an (orientation-preserving) homotopy invariant of \( M \).

**Proof:** Let \( \mathcal{B}^\infty \) be the algebra
\[
\mathcal{B}^\infty = \{ A \in \Lambda : \tilde{\partial}^k(A) \text{ is bounded for all } k \in \mathbb{N} \},
\] (123)

where \( \tilde{\partial} \) is the operator of [CM, p. 383]. By [CM, p. 385], if \([\eta] \in H^*(\Gamma; \mathbb{C})\) then \([\eta]\) can be represented by a group cocycle \( \eta \) such that \( \tau_\eta \) extends to a cyclic cocycle on \( \mathcal{B}^\infty \). Letting \( \tilde{D} \) be the signature operator, the result of Mishchenko and Kasparov [Mi, Kas, HS] on the homotopy invariance of
\[
\text{Index}(\tilde{D}) \in K_0(\Lambda) \cong K_0(\mathcal{B}^\infty)
\] (124)

along with Corollary 2 implies the result. (As usual when dealing with the signature operator, it is irrelevant whether or not \( M \) is spin.) \( \square \)
VII. Bivariant Extension

Let $\mathfrak{A}$ be the $C^*$-algebra $C(M)$. Then $(\Gamma^0(\mathcal{E}), \tilde{D})$ forms an unbounded $(\mathfrak{A}, \Lambda)$ Kasparov module, and so gives an element of $KK(\mathfrak{A}, \Lambda)$ [BJ]. A bivariant Chern character $\text{ch}_{\beta, s}$ was defined in [Lo1] in the case of finite-dimensional projective modules, and it was indicated that the bivariant Chern character should be well-defined whenever there is a good notion of trace on the Hilbert modules. Such is the case here. The bivariant Chern character is a combination of Quillen's superconnection Chern character [Q] and the entire cyclic cocycle of [JLO]. In the setup of Section IV, given $\eta \in Z^k(T; \mathbb{C})$ such that $\tau_\eta$ pairs with $\mathcal{B}^\infty$, there is a corresponding entire cyclic cocycle $\langle \text{ch}_{\beta, s}, \tau_\eta \rangle \in C^*_e(C^\infty(M))$. It is given explicitly as follows:

**DEFINITION:** For $a_0, \ldots, a_m \in C^\infty(M)$,

$$
\langle \text{ch}_{\beta, s}, \tau_\eta \rangle (a_0, \ldots, a_m) = \beta^{-m/2} \left\langle \int_0^\beta \cdots \int_0^{u_m - 1} \text{STRA}_0 \exp(-u_1 D_s^2) [D_s, a_1] \exp(-u_2 D_s^2)[D_s, a_2] \cdots [D_s, a_m] \exp \left( - (\beta - u_1 - \cdots - u_m) D_s^2 \right) 
\frac{du_m \cdots du_1}{\eta} \right]\ .
$$

(Note that the $\langle \text{ch}_{\beta, s}, \tau_\eta \rangle (1)$ of equation (125) equals the $\langle \text{ch}_{\beta, s}, \tau_\eta \rangle$ of Proposition 12.)

As before, the class of $\langle \text{ch}_{\beta, s}, \tau_\eta \rangle$ in $H^*_e(C^\infty(M))$ is independent of $s$. As in Section V, we can take the $s \to 0$ limit to obtain that $\langle \text{ch}_{\beta, s}, \tau_\eta \rangle$ is cohomologous to the entire cyclic cocycle $\langle \text{ch}_{\beta, 0}, \tau_\eta \rangle$ given by

$$
\langle \text{ch}_{\beta, 0}, \tau_\eta \rangle (a_0, \ldots, a_m) = \beta^{k/2} / (k! m!) \int_M \widetilde{A}(M) \wedge \text{Ch}(V) \wedge 
\omega \wedge a_0 da_1 \wedge da_2 \wedge \ldots \wedge da_m .
$$

Here $\omega$ is the differential form of (65).

If $W \in K^0(M)$ is represented by a projection $p \in M_r(C^\infty(M))$, let $\text{Ch}_*(p)$ be the entire cyclic cycle of [GS]. Then we obtain that $\langle \text{ch}_{\beta, s}, \tau_\eta \rangle (\text{Ch}_*(p))$ is proportionate to $\int_M \widetilde{A}(M) \wedge \text{Ch}(V) \wedge \omega \wedge \text{Ch}(W)$. Note that in the case of the signature operator, the entire cyclic cohomology class of $\langle \text{ch}_{\beta, s}, \tau_\eta \rangle$ is not a homotopy invariant, as otherwise one could take $[\eta]$ to be a 0-group cocycle and conclude that the rational $L$-class is a homotopy invariant, which is false.
References

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