A Combined Finite Element-Boundary Element Formulation for Solution of Axially Symmetric Bodies

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Abstract

A new method is presented for the computation of electromagnetic scattering from axially symmetric bodies. To allow the simulation of inhomogeneous cross sections, the method combines the finite element and boundary element techniques. Interior to a fictitious surface enclosing the scattering body, the finite element method is used which results in a sparse submatrix, whereas along the enclosure the Stratton-Chu integral equation is enforced. By choosing the fictitious enclosure to be a right circular cylinder, most of the resulting boundary integrals are convolutional and may therefore be evaluated via the FFT with the system is iteratively solved. In view of the sparse matrix associated with the interior fields, this reduces the storage requirement of the entire system to $O(N)$ making the method attractive for large scale computations. This report describes the details of the corresponding formulation and its numerical implementation.

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Chapter 1

Introduction

A restraining factor in the numerical simulation of three-dimensional structures for electromagnetic scattering computations is the storage requirement associated with the chosen method. For sub-wavelength structures traditional methods [1] have been found to work well. However, for structures spanning several wavelengths, the storage requirement limits the use of these methods.

For the special case of axially symmetric structures or bodies of revolution (BOR), a reduction of the storage requirement is accomplished by reducing the three-dimensional problem to a set of two-dimensional ones. Several moment method codes have been developed for the solution of these ([2] - [7] and others). However, for large structures the required storage of $O(N^2)$, where N denotes the number of unknowns over the BOR cross section, limits their use.

To further reduce the storage requirement, hybrid finite element methods ([8]-[12], etc.) may be used, since the storage associated with the finite element method is $O(N)$ in contrast to the $O(N^2)$ requirement of moment methods. These methods differ from one another primarily by the application of the radiation condition. The most accurate

method enforces the radiation implicitly through an application of the boundary integral equation over a fictitious boundary enclosure [11], and in this case the storage is still $O(N_b^2)$, where N_b is the number of unknowns on the boundary. However, through a judicious choice of the enclosing boundary, the storage requirement can be reduced to $O(N)$. This can be achieved by selecting the enclosing boundary to be rectangular or circular [15], [16], making some of the integrals convolutions which can then be evaluated via the FFT when an iterative solution scheme is employed.

The proposed method combines the finite element and boundary element methods for the solution of inhomogeneous bodies of revolution. The coupled potential equations [10] are discretized via the usual finite element method, and the resulting system is augmented by a discrete representation (via the boundary element method [13]) of the Stratton-Chu equations [14]. By choosing a right circular cylinder to enclose the scatterer, some of the integrals become convolutions and their discrete counterparts are then evaluated via the FFT in conjunction with an iterative solution procedure as was done in the two-dimensional case [15]. With some care, the storage is reduced to $O(N)$.

In this report, we describe the formulation for the proposed finite element - boundary element method as applied to the body of revolution. Some preliminary results are shown to be in reasonable agreement with the method of moments (MOM).

Chapter 2

Analysis

Consider the body of revolution (BOR) illustrated in fig. 2.1. To employ the proposed finite element - boundary element (FE/BE) method, the BOR is tightly enclosed in a fictitious finite length cylinder, which divides the entire space into two regions, i.e. the one enclosed by the cylinder and the other exterior to it. Since the interior region is generally inhomogeneous, the finite element method is suited for formulating the fields of that region, whereas the boundary element method is applicable for the exterior free space region. A usual approach [3] for treating BORs is to introduce a Fourier series (in the azimuthal coordinate ϕ) representation of the fields, reducing the problem to a set of two-dimensional ones. The finite element - boundary element method is then used to compute each modal field and the final result is found by adding the modal fields.

In the following, we present the finite element and boundary element formulations for each mode. First, the finite element formulation is developed.

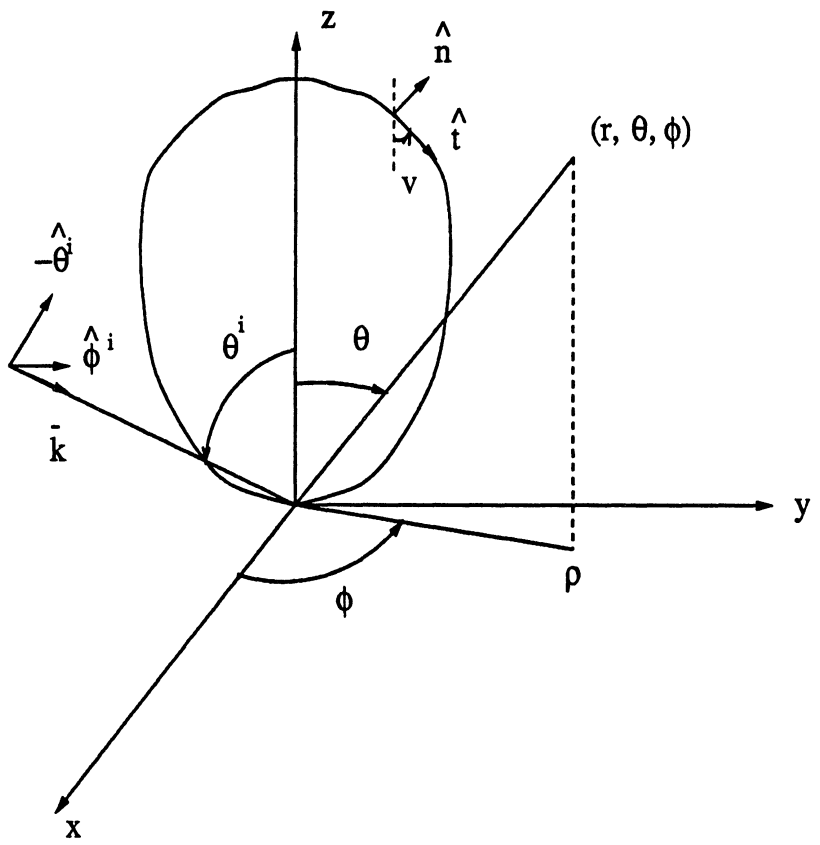

Figure 2.1: General surface of revolution.

2.1 Finite Element Formulation

In this section, we derive the analytical coupled azimuth potential (CAP) equations [17] which are then discretized via the finite element method.

2.1.1 Analytic CAP Formulation

Maxwell's equations in a source free region (a $e^{j\omega t}$ time dependence is assumed and suppressed) are given by

$$\nabla \times \overline{E}(\overline{r}) = -j\omega\mu\overline{H} \tag{2.1}$$

$$\nabla \times \overline{H}(\overline{r}) = j\omega\epsilon\overline{E} \tag{2.2}$$

$$\nabla \cdot \overline{D}(\overline{r}) = 0 \tag{2.3}$$

$$\nabla \cdot \overline{B}(\overline{r}) = 0 \tag{2.4}$$

$$\tag{2.5}$$

For axially symmetric media, the fields may be represented as Fourier series in the cylindrical coordinate ϕ as

$$\overline{E}(\overline{r}) = \sum_{m=-\infty}^{\infty} \overline{e}_m(\rho, z)e^{jm\phi} \tag{2.6}$$

$$\eta\overline{H}(\overline{r}) = \sum_{m=-\infty}^{\infty} \overline{h}_m(\rho, z)e^{jm\phi} \tag{2.7}$$

and when these are substituted into Maxwell's equations (2.1) and (2.2), we obtain

$$\tfrac{1}{R}\left[jmh_{mz} - \tfrac{\partial}{\partial Z}(Rh_{m\phi})\right] = j\epsilon_r e_{m\rho} \tag{2.8}$$

$$\tfrac{1}{R}\left[jme_{mz} - \tfrac{\partial}{\partial Z}(Re_{m\phi})\right] = -j\mu_r h_{m\rho} \tag{2.9}$$

$$R\left[\tfrac{\partial}{\partial Z}h_{m\rho} - \tfrac{\partial}{\partial R}h_{mz}\right] = j\epsilon_r(Re_{m\phi}) \tag{2.10}$$

$$R\left[\frac{\partial}{\partial Z}e_{m\rho} - \frac{\partial}{\partial R}e_{mz}\right] = -j\mu_r(Rh_{m\phi}) \tag{2.11}$$

$$\frac{1}{R}\left[jmh_{m\rho} - \frac{\partial}{\partial R}(Rh_{m\phi})\right] = -j\epsilon_r e_{mz} \tag{2.12}$$

$$\frac{1}{R}\left[jme_{m\rho} - \frac{\partial}{\partial R}(Re_{m\phi})\right] = j\mu_r h_{mz} \tag{2.13}$$

with

$$R = k_0\rho \tag{2.14}$$

$$Z = k_0 z \tag{2.15}$$

to be referred to as normalized coordinates. Substituting h_{mz} of (2.13) into (2.8) gives

$$e_{m\rho} = jf_m\left[m\frac{\partial}{\partial R}(Re_{m\phi}) + R\mu_r\frac{\partial}{\partial Z}(Rh_{m\phi})\right] \tag{2.16}$$

where

$$f_m = \left[R^2\kappa^2 - m^2\right]^{-1} \tag{2.17}$$

$$\kappa^2 = \mu_r\epsilon_r \tag{2.18}$$

Substituting $e_{m\rho}$ of (2.8) into (2.13) we obtain

$$h_{mz} = jf_m\left[m\frac{\partial}{\partial Z}(Rh_{m\phi}) + R\epsilon_r\frac{\partial}{\partial R}(Re_{m\phi})\right] \tag{2.19}$$

Substituting $h_{m\rho}$ of (2.9) into (2.12) yields

$$e_{mz} = jf_m\left[m\frac{\partial}{\partial Z}(Re_{m\phi}) - R\mu_r\frac{\partial}{\partial R}(Rh_{m\phi})\right] \tag{2.20}$$

Substituting e_{mz} of (2.12) into (2.9) yields

$$h_{m\rho} = jf_m\left[m\frac{\partial}{\partial R}(Rh_{m\phi}) - R\epsilon_r\frac{\partial}{\partial Z}(Re_{m\phi})\right] \tag{2.21}$$

Equations (2.16) through (2.21) may be written in compact form as

$$\hat{\phi} \times \overline{e}(R,Z) = jf_m \left[m\hat{\phi} \times \nabla_t \psi_e - \mu_r R \nabla_t \psi_h \right] \tag{2.22}$$

$$\hat{\phi} \times \overline{h}(R,Z) = jf_m \left[m\hat{\phi} \times \nabla_t \psi_h + \epsilon_r R \nabla_t \psi_e \right] \tag{2.23}$$

$$\hat{\phi} \cdot \overline{e}(R,Z) = \psi_e / R \tag{2.24}$$

$$\hat{\phi} \cdot \overline{h}(R,Z) = \psi_h / R \tag{2.25}$$

where

$$\nabla_t = \hat{\rho} \, \tfrac{\partial}{\partial R} + \hat{z} \, \tfrac{\partial}{\partial Z} \tag{2.26}$$

Rewriting (2.10) and (2.11) as

$$R \nabla_t \cdot (\hat{\phi} \times \overline{h}_m) = -j\epsilon_r \psi_e \tag{2.27}$$

$$R \nabla_t \cdot (\hat{\phi} \times \overline{e}_m) = j\mu_r \psi_h \tag{2.28}$$

and then substituting (2.22) and (2.23) into them, we obtain the CAP equations

$$\nabla_t \cdot \left[f_m (\epsilon_r R \nabla_t \psi_e + m\hat{\phi} \times \nabla_t \psi_h \right] + \frac{\epsilon_r \psi_e}{R} = 0 \tag{2.29}$$

$$\nabla_t \cdot \left[f_m (\mu_r R \nabla_t \psi_h - m\hat{\phi} \times \nabla_t \psi_e \right] + \frac{\mu_r \psi_h}{R} = 0. \tag{2.30}$$

This system may be written in operator form as

$$L\psi = 0 \tag{2.31}$$

where

$$L = \begin{bmatrix} \nabla_t \cdot [f_m \epsilon_r R \nabla_t] + \frac{\epsilon_r}{R} & m \nabla_t [f_m \hat{\phi} \times \nabla_t] \\ -m \nabla_t [f_m \hat{\phi} \times \nabla_t] & \nabla_t \cdot [f_m \mu_r R \nabla_t] + \frac{\mu_r}{R} \end{bmatrix} \tag{2.32}$$

and

$$\psi = [\psi_e \quad \psi_h]^T \tag{2.33}$$

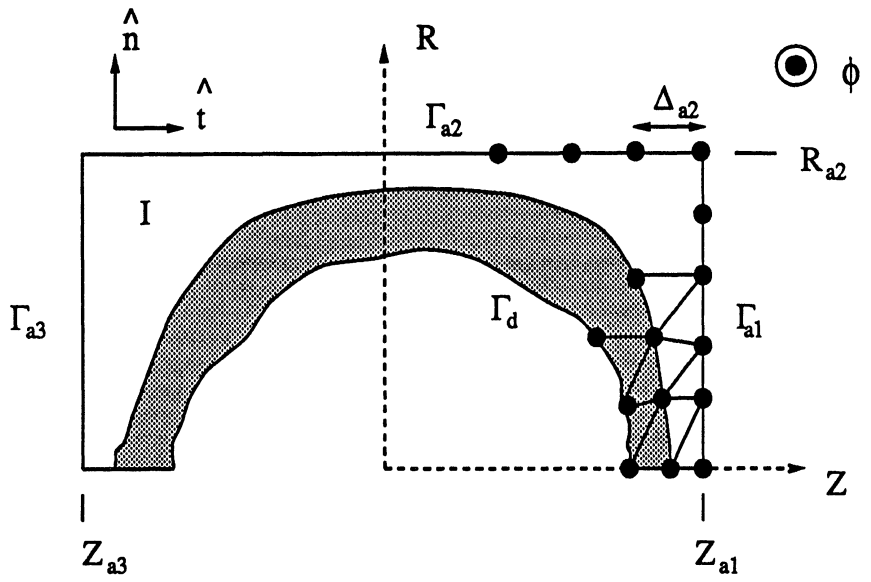

Figure 2.2: Cross section of a generating surface enclosed by the fictitious boundary Γ_a.

2.1.2 Discretization of the CAP Equations

To discretize (2.31), we first enclose the generating contours of BOR in a fictitious boundary Γ_a and the axis of symmetry as shown in Fig. 2.2. The contour Γ_a is chosen to be rectangular in shape thus generating a right circular cylinder. The region interior to Γ_a is divided into N_e linear triangular elements and within each element the corresponding

weighted residual expression is written

$$\iint_{S^e} RN_i^e(R,Z)R_{\Omega^e}dS^e = 0 \tag{2.34}$$

where (RN_i^e) is the weighting function and R_Ω^e is the residual. Further, N_i^e is the usual linear shape function found in any finite element book [18]. Using this definition, (2.29) and (2.30) may be written

$$\iint_{S^e} RN_i^e \left\{ \nabla_t \cdot \left[f_m(\epsilon_r R\nabla_t\psi_e + m\hat{\phi} \times \nabla_t\psi_h \right] + \frac{\epsilon_r\psi_e}{R} \right\} dS^e = 0 \tag{2.35}$$

$$\iint_{S^e} RN_i^e \left\{ \nabla_t \cdot \left[f_m(\mu_r R\nabla_t\psi_h - m\hat{\phi} \times \nabla_t\psi_e \right] + \frac{\mu_r\psi_h}{R} \right\} dS^e = 0 \tag{2.36}$$

and upon using the identity

$$(RN_i^e)\nabla_t \cdot \overline{A}^s = \nabla_t \cdot (RN_i^e\overline{A}^s) - \overline{A}^s \cdot \nabla_t(RN_i^e) \tag{2.37}$$

we obtain

$$\begin{aligned} &\iint_{S^e} RN_i^e \left[\nabla_t \cdot \left\{ RN_i^e f_m \left(\epsilon_r R\nabla_t\psi_e + m\hat{\phi} \times \nabla_t\psi_h \right) \right\} \right. \\ &\left. + \epsilon_r\psi_e N_i^e - f_m \left(\epsilon_r R\nabla_t\psi_e + m\hat{\phi} \times \nabla_t\psi_h \right) \cdot \nabla_t(RN_i^e) \right] dS^e = 0 \end{aligned} \tag{2.38}$$

$$\begin{aligned} &\iint_{S^e} RN_i^e \left[\nabla_t \cdot \left\{ RN_i^e f_m \left(\mu_r R\nabla_t\psi_h - m\hat{\phi} \times \nabla_t\psi_e \right) \right\} \right. \\ &\left. + \mu_r\psi_h N_i^e - f_m \left(\mu_r R\nabla_t\psi_h - m\hat{\phi} \times \nabla_t\psi_e \right) \cdot \nabla_t(RN_i^e) \right] dS^e = 0 \end{aligned} \tag{2.39}$$

Further, by invoking the divergence theorem (2.38) and (2.39) may be written as

$$\begin{aligned} &\iint_{S^e} \left\{ \left[-f_m \left(\epsilon_r R\nabla_t\psi_e + m\hat{\phi} \times \nabla_t\psi_h \right) \right] \cdot \nabla_t(RN_i^e) + \epsilon_r\psi_e N_i^e \right\} dS^e \\ &\qquad + \oint_{C^e} \hat{n} \cdot \left[RN_i^e f_m \left(\epsilon_r R\nabla_t\psi_e + m\hat{\phi} \times \nabla_t\psi_h \right) \right] dl^e = 0 \end{aligned} \tag{2.40}$$

$$\begin{aligned} &\iint_{S^e} \left\{ \left[-f_m \left(\mu_r R\nabla_t\psi_h - m\hat{\phi} \times \nabla_t\psi_e \right) \right] \cdot \nabla_t(RN_i^e) + \mu_r\psi_h N_i^e \right\} dS^e \\ &\qquad + \oint_{C^e} \hat{n} \cdot \left[RN_i^e f_m \left(\mu_r R\nabla_t\psi_h - m\hat{\phi} \times \nabla_t\psi_e \right) \right] dl^e = 0 \end{aligned} \tag{2.41}$$

where $\hat{n}$ is the outward normal along the boundary, C^e, of the eth element. Finally, these may be simplified by making use of (2.22) and (2.23), yielding

$$\iint_{S^e} \left\{ \left[-f_m \left(\epsilon_r R \nabla_t \psi_e + m\hat{\phi} \times \nabla_t \psi_h \right) \right] \cdot \nabla_t (RN_i^e) + \epsilon_r \psi_e N_i^e \right\} dS^e$$
$$- \oint_{C^e} RN_i^e (jh_{mt}) dl^e = 0 \tag{2.42}$$

$$\iint_{S^e} \left\{ \left[-f_m \left(\mu_r R \nabla_t \psi_h - m\hat{\phi} \times \nabla_t \psi_e \right) \right] \cdot \nabla_t (RN_i^e) + \mu_r \psi_h N_i^e \right\} dS^e$$
$$+ \oint_{C^e} RN_i^e (je_{mt}) dl^e = 0 \tag{2.43}$$

where

$$e_{mt} = \hat{t} \cdot \overline{e}_m \tag{2.44}$$

$$h_{mt} = \hat{t} \cdot \overline{h}_m \tag{2.45}$$

with

$$\hat{t} = \hat{n} \times \hat{\phi} \tag{2.46}$$

To form a system of equations over the eth element, the fields are represented as a linear basis expansion as

$$\psi_e(R, Z) = \sum_{j=1}^{3} \psi_{ej}^e RN_j^e(R, Z) \tag{2.47}$$

$$\psi_h(R, Z) = \sum_{j=1}^{3} \psi_{hj}^e RN_j^e(R, Z) \tag{2.48}$$

Substituting these into (2.42) and (2.43) yields

$$\sum_{j=1}^{3} \left[\iint_{S^e} \left\{ R \left[-f_m \epsilon_r \nabla_t (RN_i^e) \cdot \nabla_t (RN_j^e) + \epsilon_r N_i^e N_j^e \right] \psi_{ej}^e \right. \right.$$
$$\left. \left. -m f_m \hat{\phi} \times \nabla_t (RN_j^e) \cdot \nabla_t (RN_i^e) \psi_{hj}^e \right\} dS^e \right]$$

$$- \oint_{C^e} RN_i^e (jh_{mt}) dl^e = 0 \tag{2.49}$$

$$\sum_{j=1}^{3} \left[\iint_{S^e} \left\{ \left[-f_m \mu_r R \nabla_t (RN_i^e) \cdot \nabla_t (RN_j^e) + \mu_r N_i^e N_j^e \right] \psi_{hj}^e + m\hat{\phi} \times \nabla_t (RN_j^e) \cdot \nabla_t (RN_i^e) \psi_{ej}^e \right\} dS^e \right] + \oint_{C^e} RN_i^e (je_{mt}) dl^e = 0 \tag{2.50}$$

To proceed further, it is necessary that we evaluate the integrals over the surface of the element.

Assuming ϵ_r and μ_r are constant within a given element, (2.49) and (2.50)

$$\sum_{j=1}^{3} \left[\epsilon_r^e a_{ij}^e \psi_{ej}^e - b_{ij}^e \psi_{hj}^e \right] - \oint_{C^e} RN_i^e (jh_{mt}) dl^e = 0 \tag{2.51}$$

$$\sum_{j=1}^{3} \left[b_{ij}^e \psi_{ej}^e + \mu_r^e a_{ij}^e \psi_{hj}^e \right] + \oint_{C^e} RN_i^e (je_{mt}) dl^e = 0 \tag{2.52}$$

where

$$a_{ij}^e = \iint_{S^e} R \left[-f_m \nabla_t (RN_i^e) \cdot \nabla_t (RN_j^e) + N_i^e N_j^e \right] dS^e \tag{2.53}$$

$$b_{ij}^e = \iint_{S^e} m f_m \hat{\phi} \times \nabla_t (RN_j^e) \cdot \nabla_t (RN_i^e) dS^e \tag{2.54}$$

Summing over all elements to obtain a solution for the entire problem region, our system becomes

$$\sum_{e=1}^{N_e} \sum_{j=1}^{3} \left[\epsilon_r^e a_{ij}^e \psi_{ej}^e - b_{ij}^e \psi_{hj}^e \right] - \sum_{s=1}^{N_a} c_{is} h_{mts} = 0 \tag{2.55}$$

$$\sum_{e=1}^{N_e} \sum_{j=1}^{3} \left[b_{ij}^e \psi_{ej}^e + \mu_r^e a_{ij}^e \psi_{hj}^e \right] + \sum_{s=1}^{N_a} c_{i1}^s e_{mts} = 0 \tag{2.56}$$

where

$$c_{i1}^s = \int_{\Gamma_a} RN_i^s P_1^s dl^s \tag{2.57}$$

and P_1^s is the pulse function equal to unity in the sth element. Note that in (2.55) and (2.56) the contour integral contribution canceled out except along the boundary Γ_a as shown in Appendix D.

In block matrix form (2.55) and (2.56) may be written

$$\begin{bmatrix} A^{\epsilon}_{aa} & A^{\epsilon}_{aI} & 0 & -B_{aa} & -B_{aI} & -C_{aa} \\ A^{\epsilon}_{Ia} & A^{\epsilon}_{II} & 0 & -B_{Ia} & -B_{II} & 0 \\ B_{aa} & B_{aI} & C_{aa} & A^{\mu}_{aa} & A^{\mu}_{aI} & 0 \\ B_{Ia} & B_{II} & 0 & A^{\mu}_{Ia} & A^{\mu}_{II} & 0 \end{bmatrix} \begin{bmatrix} e_{m\phi a} \\ e_{m\phi I} \\ je_{mt} \\ h_{m\phi a} \\ h_{m\phi I} \\ jh_{mt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.58}$$

in which we have substituted ψ_e and ψ_h with $e_{m\phi}$ and $h_{m\phi}$, respectively, and

$$A^{\epsilon} = \sum_{e=1}^{N_e} \epsilon_r^e a_{ij}^e \tag{2.59}$$

$$A^{\mu} = \sum_{e=1}^{N_e} \mu_r^e a_{ij}^e \tag{2.60}$$

$$B = \sum_{e=1}^{N_e} b_{ij}^e \tag{2.61}$$

$$C = \sum_{s=1}^{N_a} c_{i1}^s \tag{2.62}$$

The subscripts on $A^{\epsilon,\mu}$, B and C refer to the various regions of Ω and its boundary. The elements of (2.59) - (2.62) are derived in Appendix E and are listed as follows

$$a_{ij}^e = [-\alpha_i^e\alpha_j^e Q_{10} - (\beta_i^e\alpha_j^e + \beta_j^e\alpha_i^e)Q_{11} - 2(\gamma_i^e\alpha_j^e + \gamma_j^e\alpha_i^e)Q_{20} - 2(\beta_i^e\gamma_j^e + \beta_j^e\gamma_i^e)Q_{21}$$
$$-\beta_i^e\beta_j^e Q_{12} - (4\gamma_i^e\gamma_j^e + \beta_i^e\beta_j^e)Q_{30}$$
$$+\alpha_i^e\alpha_j^e P_{10} + (\beta_i^e\alpha_j^e + \beta_j^e\alpha_i^e)P_{11} + (\gamma_i^e\alpha_j^e + \gamma_j^e\alpha_i^e)P_{20} + (\beta_i^e\gamma_j^e + \beta_j^e\gamma_i^e)P_{21} + \beta_i^e\beta_j^e P_{12}$$
$$+\gamma_i^e\gamma_j^e P_{30}]\frac{1}{(2\Omega^e)^2} \quad (2.63)$$

and

$$b_{ij}^e = \frac{m}{(2\Omega^e)^2}[(\beta_j^e\alpha_i^e - \beta_i^e\alpha_j^e)Q_{10} + 2(\beta_j^e\gamma_i^e - \beta_i^e\gamma_j^e)Q_{20}] \quad (2.64)$$

where the Ps and Qs are defined in Appendix E. The elements of C are

$$C_{ss} = c_{11}^s$$
$$C_{s+1,s} = c_{21}^s \quad (2.65)$$

where

$$c_{11}^s = c_{21}^s = \frac{R_{a2}\Delta^s}{2} \qquad \text{for } \Gamma_{a2}$$
$$\left.\begin{aligned} c_{11}^s &= \tfrac{1}{2}(\Delta^s \pm R_1^s)(R_1^s + R_2^s) \mp \tfrac{1}{3}(R_2^{s2} + R_1^s R_2^s + R_1^{s2}) \\ c_{21}^s &= \mp[\tfrac{1}{2}R_1^s(R_1^s + R_2^s) - \tfrac{1}{3}(R_1^{s2} + R_1^s R_2^s + R_2^{s2})] \end{aligned}\right\} \quad \left\{\begin{array}{ll} \text{upper sign for} & \Gamma_{a1} \\ \text{lower sign for} & \Gamma_{a3} \end{array}\right. \quad (2.66)$$

To form a complete system, (2.58) must be appended by a discrete version of the boundary integral equation to be discussed next.

2.2 Boundary Integral Formulation

The electric and magnetic fields are represented in the unbounded region by

$$\overline{E}(\overline{r}) = \overline{E}^s(\overline{r}) + \overline{E}^i(\overline{r}) \tag{2.67}$$

$$\overline{H}(\overline{r}) = \overline{H}^s(\overline{r}) + \overline{H}^i(\overline{r}) \tag{2.68}$$

where $\overline{E}^i(\overline{r})$ and $\overline{H}^i(\overline{r})$ are the incident fields and the scattered fields are given by the Stratton-Chu equations [14]

$$\overline{E}^s(\overline{r}) = \oiint_{S'} \left\{ -j\omega\mu \left[\hat{n}' \times \overline{H}(\overline{r}')\right] g(\overline{r},\overline{r}') + \left[\hat{n}' \cdot \overline{E}(\overline{r}')\right] \nabla' g(\overline{r},\overline{r}') + \left[\hat{n}' \times \overline{E}(\overline{r}')\right] \times \nabla' g(\overline{r},\overline{r}') \right\} dS' \tag{2.69}$$

$$\overline{H}^s(\overline{r}) = \oiint_{S'} \left\{ j\omega\epsilon \left[\hat{n}' \times \overline{E}(\overline{r}')\right] g(\overline{r},\overline{r}') + \left[\hat{n}' \cdot \overline{H}(\overline{r}')\right] \nabla' g(\overline{r},\overline{r}') + \left[\hat{n}' \times \overline{H}(\overline{r}')\right] \times \nabla' g(\overline{r},\overline{r}') \right\} dS' \tag{2.70}$$

where $\overline{r}'$ and $\overline{r}$ are the source and observation points, respectively and

$$g(\overline{r},\overline{r}') = \frac{e^{-jk_0|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \tag{2.71}$$

is the free space Green's function. It is convenient for computational purposes to eliminate the presence of the normal field components and after some manipulation we obtain

$$\overline{E}^s(\overline{r}) = \oiint_{S'} \left\{ -jk_0 \left[\hat{n}' \times \eta_0\overline{H}(\overline{r}')\right] g(\overline{r},\overline{r}') + \frac{1}{jk_0} \left[\hat{n}' \cdot \nabla' \times \eta_0\overline{H}(\overline{r}')\right] \nabla' g(\overline{r},\overline{r}') + \left[\hat{n}' \times \overline{E}(\overline{r}')\right] \times \nabla' g(\overline{r},\overline{r}') \right\} dS' \tag{2.72}$$

$$\eta_0\overline{H}^s(\overline{r}) = \oiint_{S'} \left\{ jk_0 \left[\hat{n}' \times \overline{E}(\overline{r}')\right] g(\overline{r},\overline{r}') - \frac{1}{jk_0} \left[\hat{n}' \cdot \nabla \times \overline{E}(\overline{r}')\right] \nabla' g(\overline{r},\overline{r}') + \left[\hat{n}' \times \eta_0\overline{H}(\overline{r}')\right] \times \nabla' g(\overline{r},\overline{r}') \right\} dS' \tag{2.73}$$

For $\overline{r} = \overline{r}'$, the integrals in (2.72) and (2.73) are singular and by removing these singularity, they may be rewritten in terms of principal integrals as

$$\tfrac{1}{2}\overline{E}(\overline{r}) = \overline{E}^i(\overline{r}) + \oiint_{S'} \left\{ -jk_0 \left[\hat{n}' \times \eta_0 \overline{H}(\overline{r}')\right] g(\overline{r},\overline{r}') \right.$$
$$\left. + \frac{1}{jk_0}\left[\hat{n}' \cdot \nabla' \times \eta_0 \overline{H}(\overline{r}')\right] \nabla' g(\overline{r},\overline{r}') + \left[\hat{n}' \times \overline{E}(\overline{r}')\right] \times \nabla' g(\overline{r},\overline{r}') \right\} dS' \qquad (2.74)$$

$$\tfrac{1}{2}\eta_0 \overline{H}(\overline{r}) = \eta_0 \overline{H}^i(\overline{r}) + \oiint_{S'} \left\{ jk_0 \left[\hat{n}' \times \overline{E}(\overline{r}')\right] g(\overline{r},\overline{r}') \right.$$
$$\left. - \frac{1}{jk_0}\left[\hat{n}' \cdot \nabla \times \overline{E}(\overline{r}')\right] \nabla' g(\overline{r},\overline{r}') + \left[\hat{n}' \times \eta_0 \overline{H}(\overline{r}')\right] \times \nabla' g(\overline{r},\overline{r}') \right\} dS' \qquad (2.75)$$

where we have also made use of (2.67) and (2.68). These must now be enforced on the boundary so that they can be coupled with the FEM equations.

Initially, we will allow S' to be a general surface of revolution and will then specialize it to the case of a right circular cylinder. In the next section, we derive the modal boundary integral equations by expressing the fields and the Green's functions as a Fourier series in the cylindrical coordinate ϕ. The resulting modal equations are then discretized and the resulting subsystem is augmented to the finite element system previously derived.

2.2.1 Derivation of the Modal Boundary Integral Equation

Consider the general surface of revolution indicated in fig. 2.1 whose tangential unit vectors are denoted by $\hat{\phi}$ and $\hat{t}$. The angle v is that between the $\hat{t}$ and the z-axis and is negative when $\hat{t}$ points toward the z-axis. Referring to the figure, we may represent the various unit vectors as

$$\hat{n} = \hat{x}\cos v \cos\phi + \hat{y}\cos v \sin\phi - \hat{z}\sin v \qquad (2.76)$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad (2.77)$$

$$\hat{t} = \hat{x}\sin v\cos\phi + \hat{y}\sin v\sin\phi + \hat{z}\cos v \tag{2.78}$$

$$\hat{x} = \hat{t}\sin v\cos\phi + \hat{n}\cos v\cos\phi - \hat{\phi}\sin\phi \tag{2.79}$$

$$\hat{y} = \hat{t}\sin v\sin\phi + \hat{n}\cos v\sin\phi + \hat{\phi}\cos\phi \tag{2.80}$$

$$\hat{z} = \hat{t}\cos v - \hat{n}\sin v \tag{2.81}$$

Expressing the primed unit vectors in terms of the unprimed unit vectors results in

$$\hat{t} = \hat{t}'\left[\sin v'\sin v\cos(\phi-\phi') + \cos v\cos v'\right]$$
$$+\hat{n}'\left[\cos v'\sin v\cos(\phi-\phi') - \cos v\sin v'\right] + \hat{\phi}'\left[\sin v\sin(\phi-\phi')\right] \tag{2.82}$$

$$\hat{n} = \hat{t}'\left[\sin v'\cos v\cos(\phi-\phi') - \sin v\cos v'\right]$$
$$+\hat{n}'\left[\cos v'\cos v\cos(\phi-\phi') + \sin v\sin v'\right] + \hat{\phi}'\left[\cos v\sin(\phi-\phi')\right] \tag{2.83}$$

$$\begin{aligned}\hat{\phi} &= -\hat{t}'\left[\sin v'\sin(\phi-\phi')\right] - \hat{n}'\left[\cos v'\sin(\phi-\phi')\right] + \hat{\phi}'\left[\cos(\phi-\phi')\right] \\ &= -\hat{\rho}'\sin(\phi-\phi') + \hat{\phi}'\cos(\phi-\phi')\end{aligned} \tag{2.84}$$

Taking the ϕ component of (2.74) and noting the identities,

$$\hat{\phi}\cdot(\hat{n}'\times\eta_0\overline{H}) = -\eta_0 H_\phi\sin v'\sin(\phi-\phi') - \eta_0 H_t\cos(\phi-\phi') \tag{2.85}$$

$$\hat{\phi}\cdot\nabla' g = -\hat{\phi}\cdot\nabla g \tag{2.86}$$

$$\hat{n}'\cdot(\nabla'\times\eta_0\overline{H}) = \tfrac{1}{\rho'}\left[-\tfrac{\partial}{\partial t'}(\rho'\eta_0 H_\phi) + \tfrac{\partial}{\partial\phi'}(\eta_0 H_t)\right] \tag{2.87}$$

$$\hat{\phi}\cdot\left[(\hat{n}'\times\overline{E})\times\nabla' g\right] =$$
$$\left[\hat{z}' E_t\sin(\phi-\phi') + \hat{n}' E_\phi\cos(\phi-\phi') + \hat{\phi}' E_\phi\cos v'\sin(\phi-\phi')\right]\cdot\nabla' g \tag{2.88}$$

we may rewrite (2.74) as

$$\frac{1}{2}E_\phi(\overline{r}) = E^i_\phi(\overline{r})$$
$$\oint_\Gamma \int_0^{2\pi} \{jk_0 \left[\eta_0 H_\phi \sin v' \sin(\phi-\phi') + \eta_0 H_t \cos(\phi-\phi')\right] g(\overline{r},\overline{r}')$$
$$+\frac{1}{jk_0\rho'}\left[-\tfrac{\partial}{\partial t'}(\rho'\eta_0 H_\phi) + \tfrac{\partial}{\partial \phi'}(\eta_0 H_t)\right]\hat{\phi}\cdot\nabla g(\overline{r},\overline{r}')$$
$$+\left[\hat{z}' E_t \sin(\phi-\phi') + \hat{n}' E_\phi \cos(\phi-\phi') + \hat{\phi}' E_\phi \cos v' \sin(\phi-\phi')\right]\cdot\nabla' g\Big\}\rho' d\phi' d\Gamma \quad (2.89)$$

Further, by carrying out the derivatives of the Green's functions, we have

$$\frac{1}{2}E_\phi(\overline{r}) = E^i_\phi(\overline{r})$$
$$\oint_\Gamma \int_0^{2\pi} \{jk_0 \left[\eta_0 H_\phi \sin v' \sin(\phi-\phi') + \eta_0 H_t \cos(\phi-\phi')\right] g(\overline{r},\overline{r}')$$
$$+\frac{1}{jk_0}\left[-\tfrac{\partial}{\partial t'}(\rho'\eta_0 H_\phi) + \tfrac{\partial}{\partial \phi'}(\eta_0 H_t)\right]\sin(\phi-\phi')\frac{1}{R_0}\frac{dg}{dR_0}$$
$$+\left(E_t \sin(\phi-\phi') + E_\phi[\rho' \cos v' \cos(\phi-\phi')\right.$$
$$\left.-\rho\cos v' + (z-z')\sin v' \cos(\phi-\phi')]\right)\frac{1}{R_0}\frac{dg}{dR_0}\Big\}\rho' d\phi' d\Gamma \quad (2.90)$$

in which

$$R_0 = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi') + (z-z')^2} \quad (2.91)$$

To generate the corresponding integral equations for the modal components, the fields and Green's function may be expanded as

$$\overline{E}(\overline{r}) = \sum_{m=-\infty}^{\infty} \overline{e}_m(\rho,z)e^{jm\phi} \quad (2.92)$$

$$\eta_0\overline{H}(\overline{r}) = \sum_{m=-\infty}^{\infty} \overline{h}_m(\rho,z)e^{jm\phi} \quad (2.93)$$

$$g^{(k)}(\overline{r},\overline{r}') = \sum_{n=-\infty}^{\infty} g_n^{(k)}(\rho,\rho',z,z')e^{jn(\phi-\phi')} \quad (2.94)$$

where

$$\overline{e}_m(\rho,z)=\frac{1}{2\pi}\fint_{-\pi}^{\pi}\overline{E}(\rho,u,z)e^{-jmu}du \tag{2.95}$$

$$\overline{h}_m(\rho,z)=\frac{\eta_0}{2\pi}\fint_{-\pi}^{\pi}\overline{H}(\rho,u,z)e^{-jmu}du \tag{2.96}$$

$$g_n^{(0)}(\rho,\rho',z,z')=g_n(\rho,\rho',z,z')=\frac{1}{\pi}\fint_0^{\pi}\frac{e^{-jk_0\tilde{R}}}{4\pi\tilde{R}}\cos(nu)du \tag{2.97}$$

$$\begin{aligned}g_n^{(1)}(\rho,\rho',z,z')&=\frac{1}{2}\left[g_{n-1}(\rho,\rho',z,z')+g_{n+1}(\rho,\rho',z,z')\right]\\&=\frac{1}{\pi}\fint_0^{\pi}\cos u\frac{e^{-jk_0\tilde{R}}}{4\pi\tilde{R}}\cos(nu)du\end{aligned} \tag{2.98}$$

$$\begin{aligned}g_n^{(2)}(\rho,\rho',z,z')&=-\frac{j}{2}\left[g_{n-1}(\rho,\rho',z,z')-g_{n+1}(\rho,\rho',z,z')\right]\\&=-\frac{j}{\pi}\fint_0^{\pi}\sin u\frac{e^{-jk_0\tilde{R}}}{4\pi\tilde{R}}\sin(nu)du\end{aligned} \tag{2.99}$$

$$g_n^{(0)'}(\rho,\rho',z,z')=g_n'(\rho,\rho',z,z')=\frac{1}{\pi k_0^2}\fint_0^{\pi}-\frac{1}{\tilde{R}}\frac{dg}{d\tilde{R}}\cos(nu)du \tag{2.100}$$

$$\begin{aligned}g_n^{(1)'}(\rho,\rho',z,z')&=\frac{1}{2}\left[g_{n-1}'(\rho,\rho',z,z')+g_{n+1}'(\rho,\rho',z,z')\right]\\&=-\frac{1}{\pi k_0^2}\fint_0^{\pi}\cos u\frac{1}{\tilde{R}}\frac{dg}{d\tilde{R}}\cos(nu)du\end{aligned} \tag{2.101}$$

$$\begin{aligned}g_n^{(2)'}(\rho,\rho',z,z')&=-\frac{j}{2}\left[g_{n-1}'(\rho,\rho',z,z')-g_{n+1}'(\rho,\rho',z,z')\right]\\&=\frac{j}{\pi k_0^2}\fint_0^{\pi}\sin u\frac{1}{\tilde{R}}\frac{dg}{d\tilde{R}}\sin(nu)du\end{aligned} \tag{2.102}$$

with

$$\tilde{R}=\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos u+(z-z')^2} \tag{2.103}$$

Substituting these into (2.90) yields

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{2}e_{n\phi}(\rho,z)e^{jn\phi}&=\sum_{n=-\infty}^{\infty}e_{n\phi}^{i}(\rho,z)e^{jn\phi}\\+\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}e^{jn\phi}\int_{\Gamma}\fint_0^{2\pi}&\left\{jk_0\left[h_{m\phi}\sin v'g_n^{(2)}+h_{mt}g_n^{(1)}\right]\right.\\&\left.+\frac{1}{jk_0}\left[-\tfrac{\partial}{\partial t'}(\rho'h_{m\phi}+jmh_{mt}\right]k_0^2g_n^{(2)'}\right.\end{aligned}$$

$$-e_{mt}(z-z')k_0^2 g_n^{(2)'} + e_{m\phi}k_0^2(\rho'\cos v' g_n^{(1)'}$$

$$-\rho\cos v' g_n' + (z-z')\sin v' g_n^{(1)'})\Big\} e^{j(m-n)\phi'}\rho' d\phi' d\Gamma \qquad (2.104)$$

and by multiplying each side by $e^{-jp\phi}$ and integrating over $(0, 2\pi)$ to extract the mth modal equation results in

$$\frac{1}{2}e_{m\phi}(\rho,z) = e^i_{m\phi}(\rho,z)$$

$$+2\pi \oint_\Gamma \Big\{ jh_{m\phi}\sin v' g_m^{(2)} - j\,\tfrac{\partial}{\partial t'}(\rho' h_{m\phi} g_m^{(2)'} + (jh_{mt})\left[g_m^{(1)} + jmg_m^{(2)'}\right]$$

$$-e_{mt}(z-z')k_0 g_m^{(2)'} + e_{m\phi}k_0(\rho'\cos v' g_m^{(1)'}$$

$$-\rho\cos v' g_m' + (z-z')\sin v' g_m^{(1)'})\Big\} k_0\rho' d\Gamma \qquad (2.105)$$

after combining terms and where we have used

$$\int_0^{2\pi} e^{j(m-n)\phi'}d\phi' = \begin{cases} 2\pi & m = n \\ 0 & \text{otherwise} \end{cases} \qquad (2.106)$$

For the case in which $\Gamma(=\Gamma_a)$ is the generating cross section of a right circular cylinder (indicated in fig. 2.2) the integral in (2.105) may be written as a sum of three integrals, one over each side of $\Gamma_a(=\sum_{i=1}^3 \Gamma_{a_i})$ as

$$\frac{1}{2}e_{m\phi}(\rho,z) = e^i_{m\phi}(\rho,z)$$

$$+2\pi \int_0^{\rho_2} \Big\{ jh_{m\phi}g_m^{(2)} - j\,\tfrac{\partial}{\partial t'}(\rho' h_{m\phi})g_m^{(2)'} + (jh_{mt})\left[g_m^{(1)} + jmg_m^{(2)'}\right]$$

$$-e_{mt}(z-z_3)k_0 g_m^{(2)'} + e_{m\phi}k_0(z-z_3)g_m^{(1)'}\Big\} k_0\rho' d\rho'$$

$$+2\pi \int_{z_3}^{z_1} \Big\{ -j\,\tfrac{\partial}{\partial t'}(\rho_2 h_{m\phi})g_m^{(2)'} + (jh_{mt})\left[g_m^{(1)} + jmg_m^{(2)'}\right]$$

$$-e_{mt}(z-z')k_0 g_m^{(2)'} + e_{m\phi}k_0(\rho_2 g_m^{(1)'} - \rho g_m')\Big\} k_0\rho_2 dz'$$

$$+2\pi \int_0^{\rho_2} \Big\{ -jh_{m\phi}g_m^{(2)} - j\,\tfrac{\partial}{\partial t'}(\rho' h_{m\phi})g_m^{(2)'} + (jh_{mt})\left[g_m^{(1)} + jmg_m^{(2)'}\right]$$

$$-e_{mt}(z-z')k_0 g_m^{(2)'} - e_{m\phi}k_0(z-z')g_m^{(1)'}\Big\} k_0\rho' d\rho' \qquad (2.107)$$

Introducing the normalized coordinates

$$\begin{aligned} R &= k_0\rho \qquad R' = k_0\rho' \\ Z &= k_0 z \qquad Z' = k_0 z' \ , \\ \tfrac{\partial}{\partial t'} &= k_0 \tfrac{\partial}{\partial T'} \end{aligned} \tag{2.108}$$

(2.107) becomes

$$\begin{aligned} \frac{1}{2} e_{m\phi}(R,Z) &= e^i_{m\phi}(R,Z) \\ &+ \int_0^{R_2} \Big\{ j h_{m\phi} g_m^{(2)} - j \tfrac{\partial}{\partial T'}(R' h_{m\phi}) g_m^{(2)'} + (j h_{mt}) \left[g_m^{(1)} + j m g_m^{(2)'} \right] \\ &\quad + j(j e_{mt})(Z - Z_3) g_m^{(2)'} + e_{m\phi}(Z - Z_3) g_m^{(1)'} \Big\} R' dR' \\ &+ \int_{Z_3}^{Z_1} \Big\{ -j \tfrac{\partial}{\partial T'}(R_2 h_{m\phi}) g_m^{(2)'} + (j h_{mt}) \left[g_m^{(1)} + j m g_m^{(2)'} \right] \\ &\quad + j(j e_{mt})(Z - Z') g_m^{(2)'} + e_{m\phi}(R_2 g_m^{(1)'} - R g_m^{'}) \Big\} R_2 dZ' \\ &+ \int_0^{R_2} \Big\{ -j h_{m\phi} g_m^{(2)} - j \tfrac{\partial}{\partial T'}(R' h_{m\phi}) g_m^{(2)'} + (j h_{mt}) \left[g_m^{(1)} + j m g_m^{(2)'} \right] \\ &\quad + j(j e_{mt})(Z - Z') g_m^{(2)'} - e_{m\phi}(Z - Z') g_m^{(1)'} \Big\} R' dR' \end{aligned} \tag{2.109}$$

This equation and its dual are discretized in the next section.

2.2.2 Discretization of the Modal Boundary Integral Equation

Consider the fig. 2.2 where the rectangular boundary is divided into N_a boundary elements and are equal in length along the Γ_{a2}. Along the boundary, the fields are expanded into pulse basis functions as

$$U(R', Z') = \sum_{j=1}^{N_a} U_{j+\frac{1}{2}} P(R_{j+\frac{1}{2}} - R', Z_{j+\frac{1}{2}} - Z') \tag{2.110}$$

where U represents any one of the components $e_{m\phi}$, $h_{m\phi}$, e_{mt} or h_{mt} and

$$P(R_{j+\frac{1}{2}} - R', Z_{j+\frac{1}{2}} - Z') = \begin{cases} 1 & \text{if } |Z_{j+\frac{1}{2}} - Z'| \le \frac{\Delta_{zj}}{2}, \ |R_{j+\frac{1}{2}} - R'| \le \frac{\Delta_{zj}}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2.111}$$

and

$$Z_{j+\frac{1}{2}} = \frac{Z_{j+1}+Z_j}{2} \quad R_{j+\frac{1}{2}} = \frac{R_{j+1}+R_j}{2} \tag{2.112}$$

Substituting the pulse basis expansion into (2.109) and simplifying yields

$$\begin{aligned}
\frac{1}{2}e_{m\phi}(R,Z) &= e^i_{m\phi}(R,Z) \\
&+ \sum_{j=N_{a1}+N_{a2}+1}^{N_a} \Bigg\{ \{h_{m\phi}\}_{j+\frac{1}{2}} \int_{R_{j+1}}^{R_j} j g_m^{(2)} R' dR' \\
&+ \{(R'h_{m\phi})'\}_{j+\frac{1}{2}} \int_{R_{j+1}}^{R_j} - j g_m^{(2)'} R' dR' + \{j h_{mt}\}_j \int_{R_{j+1}}^{R_j} \left[g_m^{(1)} + jm g_m^{(2)'} \right] R' dR' \\
&+ \{j e_{mt}\}_j \int_{R_{j+1}}^{R_j} j(Z - Z_3) g_m^{(2)'} R' dR' + \{e_{m\phi}\}_{j+\frac{1}{2}} \int_{R_{j+1}}^{R_j} (Z - Z_3) g_m^{(1)'} R' dR' \Bigg\} \\
&+ \sum_{j=N_{a1}+1}^{N_{a1}+N_{a2}} \Bigg\{ \{(R_2 h_{m\phi})'\}_{j+\frac{1}{2}} \int_{Z_{j+1}}^{Z_j} - j g_m^{(2)'} R_2 dZ' \\
&+ \{j h_{mt}\}_j \int_{Z_{j+1}}^{Z_j} \left[g_m^{(1)} + jm g_m^{(2)'} \right] R_2 dZ' \\
&+ \{j e_{mt}\}_j \int_{Z_{j+1}}^{Z_j} j(Z - Z') g_m^{(2)'} R_2 dZ' + \{e_{m\phi}\}_{j+\frac{1}{2}} \int_{Z_{j+1}}^{Z_j} (R_2 g_m^{(1)'} - R g_m') R_2 dZ' \Bigg\} \\
&+ \sum_{j=1}^{N_{a1}} \Bigg\{ \{h_{m\phi}\}_{j+\frac{1}{2}} \int_{R_j}^{R_{j+1}} - j g_m^{(2)} R' dR' \\
&+ \{(R'h_{m\phi})'\}_{j+\frac{1}{2}} \int_{R_j}^{R_{j+1}} - j g_m^{(2)'} R' dR' + \{j h_{mt}\}_j \int_{R_j}^{R_{j+1}} \left[g_m^{(1)} + jm g_m^{(2)'} \right] R' dR' \\
&+ \{j e_{mt}\}_j \int_{R_j}^{R_{j+1}} j(Z - Z_1) g_m^{(2)'} R' dR' - \{e_{m\phi}\}_{j+\frac{1}{2}} \int_{R_j}^{R_{j+1}} (Z - Z_1) g_m^{(1)'} R' dR' \Bigg\}
\end{aligned} \tag{2.113}$$

'roceeding to point-match at the boundary element midpoints, we have

$$\begin{aligned}
\frac{1}{2}e_{m\phi}(R_{i+\frac{1}{2}}, Z_{1_3}) &= e^i_{m\phi}(R_{i+\frac{1}{2}}, Z_{1_3}) \\
&+ \sum_{j=N_{a1}+N_{a2}+1}^{N_a} \Bigg\{ \{h_{m\phi}\}_{j+\frac{1}{2}} \int_{R_{j+1}}^{R_j} j g_m^{(2)} R' dR' \\
&+ \{(R'h_{m\phi})'\}_{j+\frac{1}{2}} \int_{R_{j+1}}^{R_j} - j g_m^{(2)'} R' dR' + \{j h_{mt}\}_j \int_{R_{j+1}}^{R_j} \left[g_m^{(1)} + jm g_m^{(2)'} \right] R' dR'
\end{aligned}$$

$$+\{je_{mt}\}_j \oint_{R_{j+1}}^{R_j} j(Z_{\frac{1}{3}} - Z_3) g_m^{(2)'} R' dR' + \{e_{m\phi}\}_{j+\frac{1}{2}} \oint_{R_{j+1}}^{R_j} (Z_{\frac{1}{3}} - Z_3) g_m^{(1)'} R' dR' \Bigg\}$$

$$+ \sum_{j=N_{a1}+1}^{N_{a1}+N_{a2}} \Bigg\{ \{(R_2 h_{m\phi})'\}_{j+\frac{1}{2}} \oint_{Z_{j+1}}^{Z_j} - j g_m^{(2)'} R_2 dZ'$$

$$+\{jh_{mt}\}_j \oint_{Z_{j+1}}^{Z_j} \left[g_m^{(1)} + j m g_m^{(2)'} \right] R_2 dZ'$$

$$+\{je_{mt}\}_j \oint_{Z_{j+1}}^{Z_j} j(Z_{\frac{1}{3}} - Z') g_m^{(2)'} R_2 dZ' + \{e_{m\phi}\}_{j+\frac{1}{2}} \oint_{Z_{j+1}}^{Z_j} (R_2 g_m^{(1)'} - R_{i+\frac{1}{2}} g_m^{'}) R_2 dZ' \Bigg\}$$

$$+ \sum_{j=1}^{N_{a1}} \Bigg\{ \{h_{m\phi}\}_{j+\frac{1}{2}} \oint_{R_j}^{R_{j+1}} - j g_m^{(2)} R' dR'$$

$$+\{(R' h_{m\phi})'\}_{j+\frac{1}{2}} \oint_{R_j}^{R_{j+1}} - j g_m^{(2)'} R' dR' + \{jh_{mt}\}_j \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + j m g_m^{(2)'} \right] R' dR'$$

$$+\{je_{mt}\}_j \oint_{R_j}^{R_{j+1}} j(Z_{\frac{1}{3}} - Z_1) g_m^{(2)'} R' dR' - \{e_{m\phi}\}_{j+\frac{1}{2}} \oint_{R_j}^{R_{j+1}} (Z_{\frac{1}{3}} - Z_1) g_m^{(1)'} R' dR' \Bigg\} \quad (2.114)$$

for the field points on contours Γ_{a1} and Γ_{a3} and

$$\frac{1}{2} e_{m\phi}(R_2, Z_{i+\frac{1}{2}}) = e^i_{m\phi}(R_2, Z_{i+\frac{1}{2}})$$

$$+ \sum_{j=N_{a1}+N_{a2}+1}^{N_a} \Bigg\{ \{h_{m\phi}\}_{j+\frac{1}{2}} \oint_{R_{j+1}}^{R_j} j g_m^{(2)} R' dR'$$

$$+\{(R' h_{m\phi})'\}_{j+\frac{1}{2}} \oint_{R_{j+1}}^{R_j} - j g_m^{(2)'} R' dR' + \{jh_{mt}\}_j \oint_{R_{j+1}}^{R_j} \left[g_m^{(1)} + j m g_m^{(2)'} \right] R' dR'$$

$$+\{je_{mt}\}_j \oint_{R_{j+1}}^{R_j} j(Z_{i+\frac{1}{2}} - Z_3) g_m^{(2)'} R' dR' + \{e_{m\phi}\}_{j+\frac{1}{2}} \oint_{R_{j+1}}^{R_j} (Z_{i+\frac{1}{2}} - Z_3) g_m^{(1)'} R' dR' \Bigg\}$$

$$+ \sum_{j=N_{a1}+1}^{N_{a1}+N_{a2}} \Bigg\{ \{(R_2 h_{m\phi})'\}_{j+\frac{1}{2}} \oint_{Z_{j+1}}^{Z_j} - j g_m^{(2)'} R_2 dZ'$$

$$+\{jh_{mt}\}_j \oint_{Z_{j+1}}^{Z_j} \left[g_m^{(1)} + j m g_m^{(2)'} \right] R_2 dZ'$$

$$+\{je_{mt}\}_j \oint_{Z_{j+1}}^{Z_j} j(Z_{i+\frac{1}{2}} - Z') g_m^{(2)'} R_2 dZ' + \{e_{m\phi}\}_{j+\frac{1}{2}} \oint_{Z_{j+1}}^{Z_j} (R_2 g_m^{(1)'} - R_2 g_m^{'}) R_2 dZ' \Bigg\}$$

$$+ \sum_{j=1}^{N_{a1}} \Bigg\{ \{h_{m\phi}\}_{j+\frac{1}{2}} \oint_{R_j}^{R_{j+1}} - j g_m^{(2)} R' dR'$$

$$+\{(R' h_{m\phi})'\}_{j+\frac{1}{2}} \oint_{R_j}^{R_{j+1}} - j g_m^{(2)'} R' dR' + \{jh_{mt}\}_j \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + j m g_m^{(2)'} \right] R' dR'$$

$$+\{je_{mt}\}_j \oint_{R_j}^{R_{j+1}} j(Z_{i+\frac{1}{2}} - Z_1) g_m^{(2)'} R' dR' - \{e_{m\phi}\}_{j+\frac{1}{2}} \oint_{R_j}^{R_{j+1}} (Z_{i+\frac{1}{2}} - Z_1) g_m^{(1)'} R' dR' \Big\} \quad (2.115)$$

for the field points on contour Γ_{a2}. The above set of equations may be written more compactly as

$$\tfrac{1}{2}\left[\tfrac{1}{2}I - P^{\phi}\right] C \{e_{m\phi}\}_a - \left[P^t\right] \{je_{mt}\}_a - \left[\tfrac{1}{2}Q^{\phi}C + Q^{\phi'}D\right] \{h_{m\phi}\}_a - \left[Q^t\right] \{jh_{mt}\}_a = \left\{e^i_{m\phi}\right\}_{a\frac{1}{2}} \quad (2.116)$$

where the matrix D arises from the derivative

$$\{R'h_{m\phi}\}'_{j+\frac{1}{2}} = \tfrac{\partial}{\partial T'}(R'h_{m\phi})|_{R'=R_{j+\frac{1}{2}}} = \frac{R_j\{h_{m\phi}\}_j - R_{j+1}\{h_{m\phi}\}_{j+1}}{|R_{j+1} - R_j|} \quad (2.117)$$

and C is a matrix comprised of '1's along the diagonal and superdiagonal. Also, the subscript $\frac{1}{2}$ represents evaluation at the boundary element midpoints. In a parallel fashion, the dual of corresponding (2.116) may be written

$$\tfrac{1}{2}\left[\tfrac{1}{2}I - P^{\phi}\right] C \{h_{m\phi}\}_a - \left[P^t\right] \{jh_{mt}\}_a + \left[\tfrac{1}{2}Q^{\phi}C + Q^{\phi'}D\right] \{e_{m\phi}\}_a + \left[Q^t\right] \{je_{mt}\}_a = \left\{h^i_{m\phi}\right\}_{a\frac{1}{2}} \quad (2.118)$$

The matrices in (2.116) or (2.118) are 3 x 3 in size, each element of which is a matrix corresponding a particular integration and observation (field point) contours. Each element of the submatrices is in Appendix F. For non-self-cell terms, the integrals are evaluated via open formula numerical integration schemes. The self-cell terms are given in detail in Appendix F. The integrals involving g_m are computed via Romberg integration with a specified convergence criterion to ensure accurate evaluation for any mode.

Finally, augmenting finite element system with that formed by (2.116) and (2.118), we derive the system

$$\begin{bmatrix} A^{\epsilon}_{aa} & A^{\epsilon}_{aI} & 0 & -B_{aa} & -B_{aI} & -C_{aa} \\ A^{\epsilon}_{Ia} & A^{\epsilon}_{II} & 0 & -B_{Ia} & -B_{II} & 0 \\ P & 0 & -P^t & -Q & 0 & -Q^t \\ B_{aa} & B_{aI} & C_{aa} & A^{\mu}_{aa} & A^{\mu}_{aI} & 0 \\ B_{Ia} & B_{II} & 0 & A^{\mu}_{Ia} & A^{\mu}_{II} & 0 \\ Q & 0 & Q^t & P & 0 & -P^t \end{bmatrix} \begin{bmatrix} e_{m\phi a} \\ e_{m\phi I} \\ je_{mt} \\ h_{m\phi a} \\ h_{m\phi I} \\ jh_{mt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \{e^i_{m\phi}\}_{a\frac{1}{2}} \\ 0 \\ 0 \\ \{h^i_{m\phi}\}_{a\frac{1}{2}} \end{bmatrix} \tag{2.119}$$

which is to be solved iteratively and where

$$Q = \tfrac{1}{2} Q^{\phi} C + Q^{\phi'} D \tag{2.120}$$

$$P = \tfrac{1}{2}[\tfrac{1}{2} I - P^{\phi}] C \tag{2.121}$$

Chapter 3

Scattered Field Computation

In the far field the scattered fields are given by

$$\overline{E}^s(\overline{r}) = E_\phi(\overline{r})\hat{\phi} + \eta_0 H_\phi(\overline{r})\hat{\theta} \tag{3.1}$$

$$\eta_0\overline{H}^s(\overline{r}) = \eta_0 H_\phi(\overline{r})\hat{\phi} - E_\phi(\overline{r})\hat{\theta} \tag{3.2}$$

We wish to compute the radar cross section given by [19]

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{|\overline{E}^s(\overline{r})|^2}{|\overline{E}^i(\overline{r})|^2} = \lim_{r\to\infty} 4\pi r^2 \frac{|\overline{H}^s(\overline{r})|^2}{|\overline{H}^i(\overline{r})|^2} \tag{3.3}$$

For TM_z polarization we have

$$E^s_\phi(r,\theta,\phi) = 2j\sum_{m=1}^{M} e^s_{m\phi}(r,\theta)\sin(m\phi) \tag{3.4}$$

$$\eta_0 H^s_\phi(r,\theta,\phi) = h^s_0(r,\theta) + 2\sum_{m=1}^{M} h^s_{m\phi}(r,\theta)\cos(m\phi) \tag{3.5}$$

and for TE_z polarization we have

$$E^s_\phi(r,\theta,\phi) = e^s_0(r,\theta) + 2\sum_{m=1}^{M} e^s_{m\phi}(r,\theta)\cos(m\phi) \tag{3.6}$$

$$\eta_0 H^s_\phi(r,\theta,\phi) = 2j\sum_{m=1}^{M} h^s_{m\phi}(r,\theta)\sin(m\phi) \tag{3.7}$$

These combined with a unit amplitude incident field implies that (3.3) becomes

$$\sigma_{TM_z} = \lim_{r\to\infty} 4\pi r^2 \left[\left| 2\sum_{m=1}^{M} e^s_{m\phi}(r,\theta)\sin(m\phi) \right|^2 + \left| h^s_0 + 2\sum_{m=1}^{M} h^s_{m\phi}(r,\theta)\cos(m\phi) \right|^2 \right] \quad (3.8)$$

$$\sigma_{TE_z} = \lim_{r\to\infty} 4\pi r^2 \left[\left| e^s_0 + 2\sum_{m=1}^{M} e^s_{m\phi}(r,\theta)\cos(m\phi) \right|^2 + \left| 2\sum_{m=1}^{M} h^s_{m\phi}(r,\theta)\sin(m\phi) \right|^2 \right] \quad (3.9)$$

We had previously discretized the Stratton-Chu integral equation for field points on the integration contour as given in (2.113). Eliminating the principle value factor for observation not on the integration contour, the corresponding scattered field equation may be written

$$\begin{aligned} e^s_{m\phi}(R,Z) = \int_0^{R_2} &\left\{ jh_{m\phi}g^{(2)}_m - j\,\tfrac{\partial}{\partial T'}(R'h_{m\phi})g^{(2)\prime}_m + (jh_{mt})\left[g^{(1)}_m + jmg^{(2)\prime}_m\right] \right. \\ &\left. +j(je_{mt})(Z-Z_3)g^{(2)\prime}_m + e_{m\phi}(Z-Z_3)g^{(1)\prime}_m \right\} R'dR' \\ + \int_{Z_3}^{Z_1} &\left\{ -j\,\tfrac{\partial}{\partial T'}(R_2h_{m\phi})g^{(2)\prime}_m + (jh_{mt})\left[g^{(1)}_m + jmg^{(2)\prime}_m\right] \right. \\ &\left. +j(je_{mt})(Z-Z')g^{(2)\prime}_m + e_{m\phi}(R_2g^{(1)\prime}_m - Rg'_m) \right\} R_2dZ' \\ + \int_0^{R_2} &\left\{ -jh_{m\phi}g^{(2)}_m - j\,\tfrac{\partial}{\partial T'}(R'h_{m\phi})g^{(2)\prime}_m + (jh_{mt})\left[g^{(1)}_m + jmg^{(2)\prime}_m\right] \right. \\ &\left. +j(je_{mt})(Z-Z_1)g^{(2)\prime}_m - e_{m\phi}(Z-Z_1)g^{(1)\prime}_m \right\} R'dR' \end{aligned} \quad (3.10)$$

We wish to evaluate this expression for large $k_0r = \sqrt{R^2+Z^2}$. For large r

$$\sqrt{R^2+R'^2-2RR'\cos u+(Z-Z')^2} \simeq k_0r - \frac{ZZ'}{k_0r} - \frac{RR'}{k_0r}\cos u \quad (3.11)$$

Thus, we may write (2.97)

$$g_m(R,R',Z,Z') \simeq \frac{1}{\pi}\frac{e^{-jk_0r}}{2k_0r}e^{j\frac{ZZ'}{k_0r}}\int_0^{\pi} e^{j\frac{RR'}{k_0r}\cos u}\cos(mu)du \quad (3.12)$$

Noting that the integral is related to the Bessel function of the first kind, we may write

(3.12) as

$$g_m(R,R',Z,Z') \simeq \frac{e^{-jk_0r}}{2k_0r} e^{jZ'\cos\theta} f_m(R',\theta) \tag{3.13}$$

where we have used

$$j^m J_m(\beta) = \frac{1}{\pi}\int_0^\pi e^{j\beta\cos x}\cos(mx)dx \tag{3.14}$$

$$f_m(R',\theta) = j^m J_m(R'\sin\theta) \tag{3.15}$$

and the fact that

$$R = k_0 r\sin\theta \quad Z = k_0 r\cos\theta \tag{3.16}$$

Likewise (2.98) - (2.102) become

$$g_m^{(1)}(R,R',Z,Z') \simeq \frac{e^{-jk_0r}}{2k_0r} e^{jZ'\cos\theta} f_{cm}(R',\theta) \tag{3.17}$$

$$g_m^{(2)}(R,R',Z,Z') \simeq \frac{e^{-jk_0r}}{2k_0r} e^{jZ'\cos\theta} f_{sm}(R',\theta) \tag{3.18}$$

$$g_m'(R,R',Z,Z') \simeq -\frac{jk_0}{r}\frac{e^{-jk_0r}}{2k_0r} e^{jZ'\cos\theta} f_m(R',\theta) \tag{3.19}$$

$$g_m^{(1)'}(R,R',Z,Z') \simeq -\frac{jk_0}{r}\frac{e^{-jk_0r}}{2k_0r} e^{jZ'\cos\theta} f_{cm}(R',\theta) \tag{3.20}$$

$$g_m^{(2)'}(R,R',Z,Z') \simeq -\frac{jk_0}{r}\frac{e^{-jk_0r}}{2k_0r} e^{jZ'\cos\theta} f_{sm}(R',\theta) \tag{3.21}$$

where

$$f_{cm}(R',\theta) = j^{m-1} J_m'(R'\sin\theta) \tag{3.22}$$

$$f_{sm}(R',\theta) = -mj^m \frac{J_m(R'\sin\theta)}{R'\sin\theta} \tag{3.23}$$

where the prime on J indicates differentiation with respect to the argument. Substituting these expressions into (3.10) results in the expression

$$e^s_{m\phi}(R,Z) = \frac{e^{-jk_0r}}{2k_0r} f_e(m,\theta) \tag{3.24}$$

where

$$f_e(m,\theta) = e^{jZ_3\cos\theta}\int_0^{R_2}\{h_{m\phi}(jf_{sm}) + (jh_{mt})f_{cm} + (je_{mt})\cos\theta f_{sm} - e_{m\phi}\cos\theta(jf_{cm})\}\,R'dR'$$
$$+\int_{Z_3}^{Z_1} e^{jZ'\cos\theta}\{(jh_{mt})f_{cm} + (je_{mt})\cos\theta f_{sm} + e_{m\phi}\sin\theta(jf_m)\}\,R_2dZ'$$
$$+e^{jZ_1\cos\theta}\int_0^{R_2}\{-h_{m\phi}(jf_{sm}) + (jh_{mt})f_{cm} + (je_{mt})\cos\theta f_{sm} + e_{m\phi}\cos\theta(jf_{cm})\}\,R'dR' \qquad (3.25)$$

Using a midpoint integration to compute the integrals, (3.25) becomes

$$f_e(m,\theta) = e^{jZ_3\cos\theta}\sum_{j=N_{a1}+N_{a2}+1}^{N_a}\Big\{\{h_{m\phi}\}_{j+\frac{1}{2}}[jf_{sm}(R_{j+\frac{1}{2}},\theta)] + \{jh_{mt}\}_j f_{cm} + \{je_{mt}\}_j\cos\theta f_{sm} - \{e_{m\phi}\}_{j+\frac{1}{2}}\cos\theta(jf_{cm})\Big\}\,R_{j+\frac{1}{2}}\Delta_j$$
$$+\sum_{j=N_{a1}+1}^{N_{a1}+N_{a2}} e^{jZ_{j+\frac{1}{2}}\cos\theta}\Big\{\{jh_{mt}\}_j f_{cm}(R_2,\theta) + \{je_{mt}\}_j\cos\theta f_{sm} + \{e_{m\phi}\}_{j+\frac{1}{2}}\sin\theta(jf_m)\Big\}\,R_2\Delta_j$$
$$+e^{jZ_1\cos\theta}\sum_{j=1}^{N_{a1}}\Big\{-\{h_{m\phi}\}_{j+\frac{1}{2}}[jf_{sm}(R_{j+\frac{1}{2}},\theta)] + \{jh_{mt}\}_j f_{cm} + \{je_{mt}\}_j\cos\theta f_{sm} + \{e_{m\phi}\}_{j+\frac{1}{2}}\cos\theta(jf_{cm})\Big\}\,R_{j+\frac{1}{2}}\Delta_j \qquad (3.26)$$

Letting $f_h(m,\theta)$ be the dual of (3.25) we may write (3.8) as

$$\frac{\sigma_{TM_z}(\theta,\phi)}{\lambda_0^2} = \frac{1}{4\pi}\left[\left|2\sum_{m=1}^{M} f_e(m,\theta)\sin(m\phi)\right|^2 + \left|f_h(0,\theta) + 2\sum_{m=1}^{M} f_h(m,\theta)\cos(m\phi)\right|^2\right] \qquad (3.27)$$

$$\frac{\sigma_{TE_z}(\theta,\phi)}{\lambda_0^2} = \frac{1}{4\pi}\left[\left|f_e(0,\theta) + 2\sum_{m=1}^{M} f_e(m,\theta)\cos(m\phi)\right|^2 + \left|2\sum_{m=1}^{M} f_h(m,\theta)\sin(m\phi)\right|^2\right] \qquad (3.28)$$

where Δ_j is the length of the jth boundary element.

Chapter 4

Results

The scattering patterns for three structures are shown in figs. 4.1 - 4.6. In each case the bistatic pattern is computed as a function of θ^s with $\phi^i = \phi^s$.

The first body is a conducting right circular cylinder of length 1λ and radius $\frac{\lambda}{10}$. Fig. 4.1 shows both the TE and TM scattering patterns for broadside incidence ($\theta^i = 90°$) for mode 0 and as seen these are in good agreement with the corresponding pattern obtained from the MOM code CICERO [7]. The results for mode 1 are shown in fig. 4.2, while fig. 4.3 shows the sum of modes 0 and 1. Fig. 4.4 depicts the bistatic scattering pattern for the same geometry with axial incidence ($\theta^i = 0$). Only mode 1 yields a non-zero solution in this case and the results are in good agreement with CICERO.

The TE and TM bistatic scattering patterns of a perfectly conducting sphere of radius $\frac{\lambda}{2}$ are shown in fig. 4.5. Again, these are in good agreement with the data obtained from CICERO.

The axial incidence scattering patterns for an ogive of length 1 λ and radius .088 λ are depicted in fig. 4.6 and agree with those generated by CICERO.

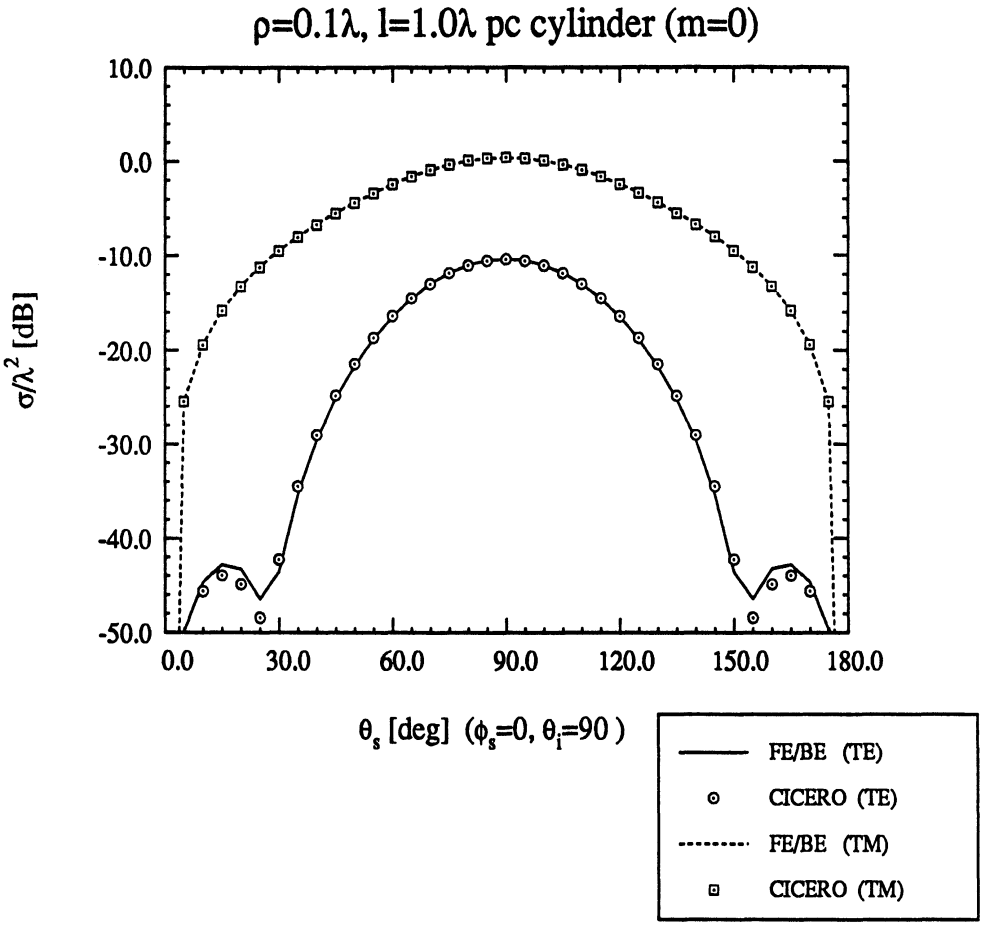

Figure 4.1: Mode 0 TM and TE bistatic scattering pattern from a perfectly conducting circular cylinder of length 1λ and radius 0.1λ for broadside incidence.

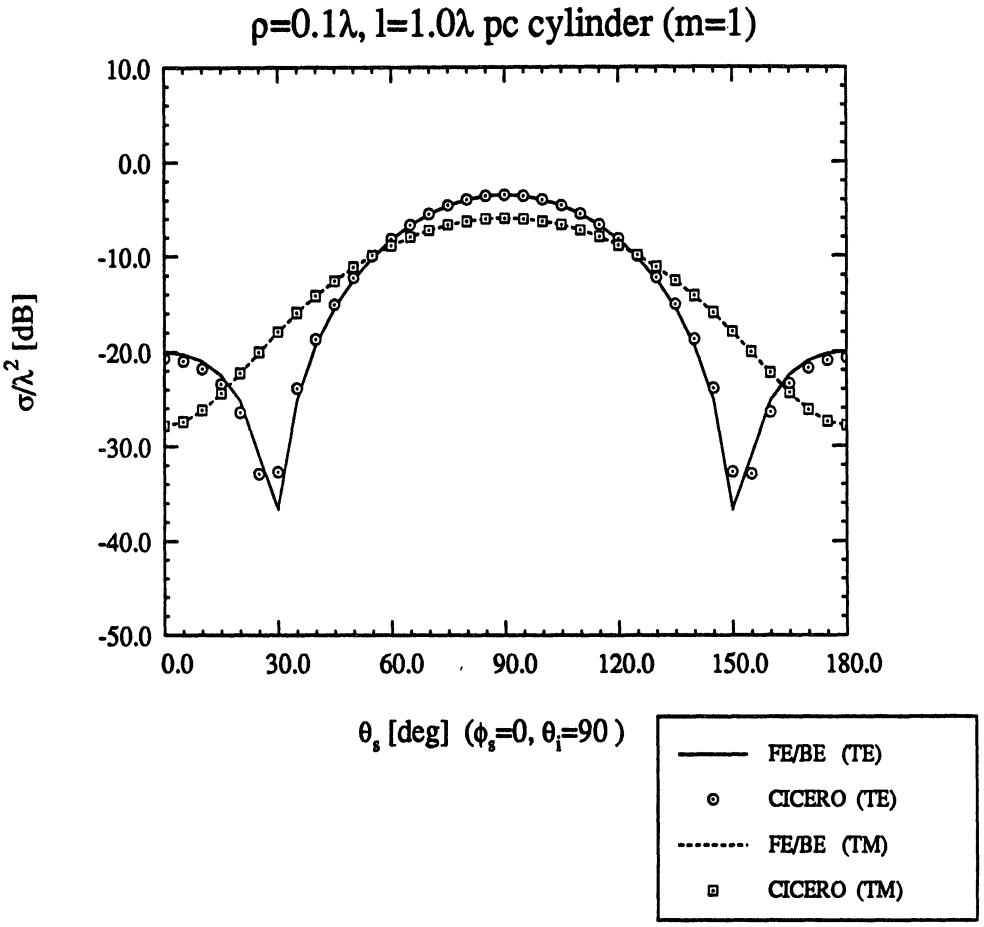

Figure 4.2: Mode 1 TM and TE bistatic scattering pattern from a perfectly conducting circular cylinder of length 1λ and radius 0.1λ for broadside incidence.

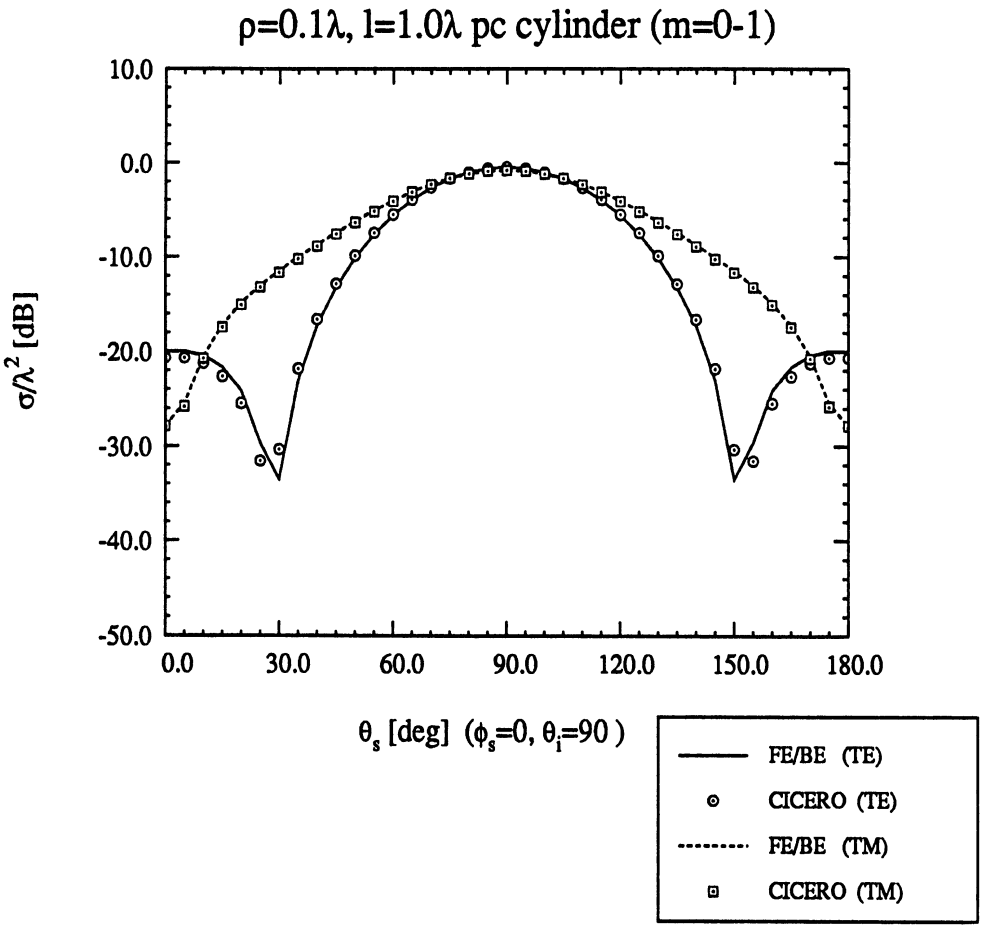

Figure 4.3: Modes 0+1 TM and TE bistatic scattering pattern from a perfectly conducting circular cylinder of length 1λ and radius 0.1λ for broadside incidence.

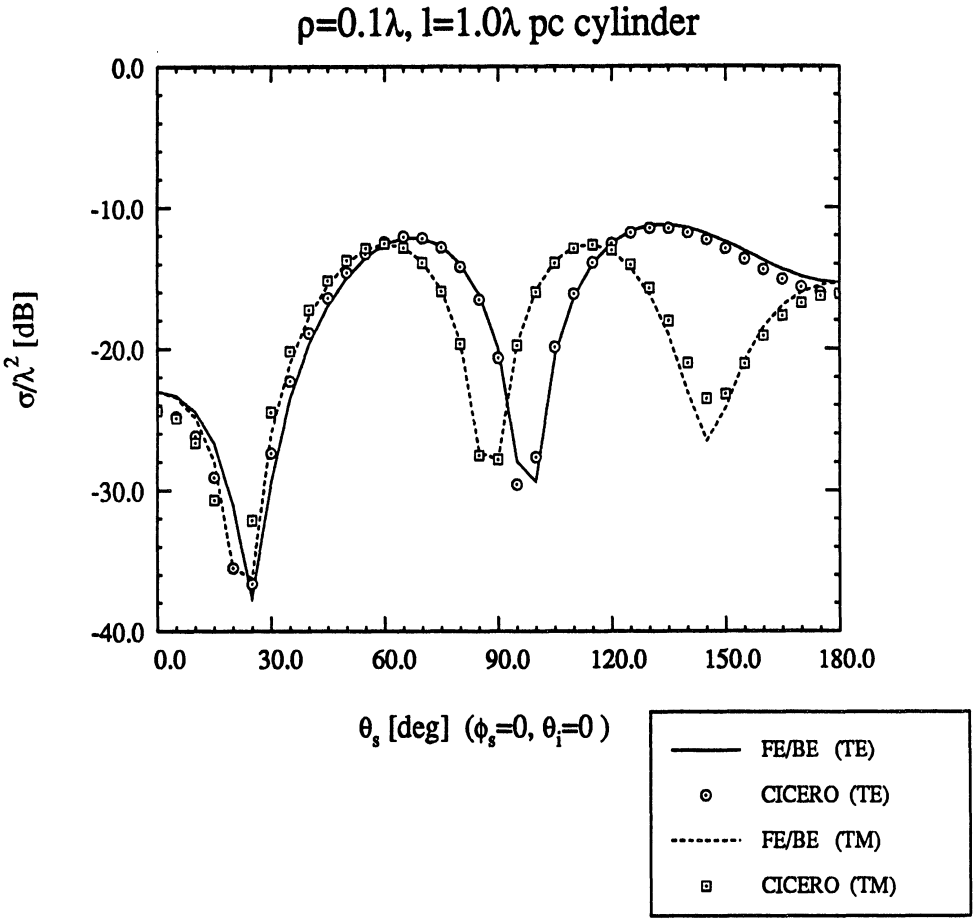

Figure 4.4: TM and TE bistatic scattering pattern from a perfectly conducting circular cylinder of length 1λ and radius 0.1λ for axial incidence.

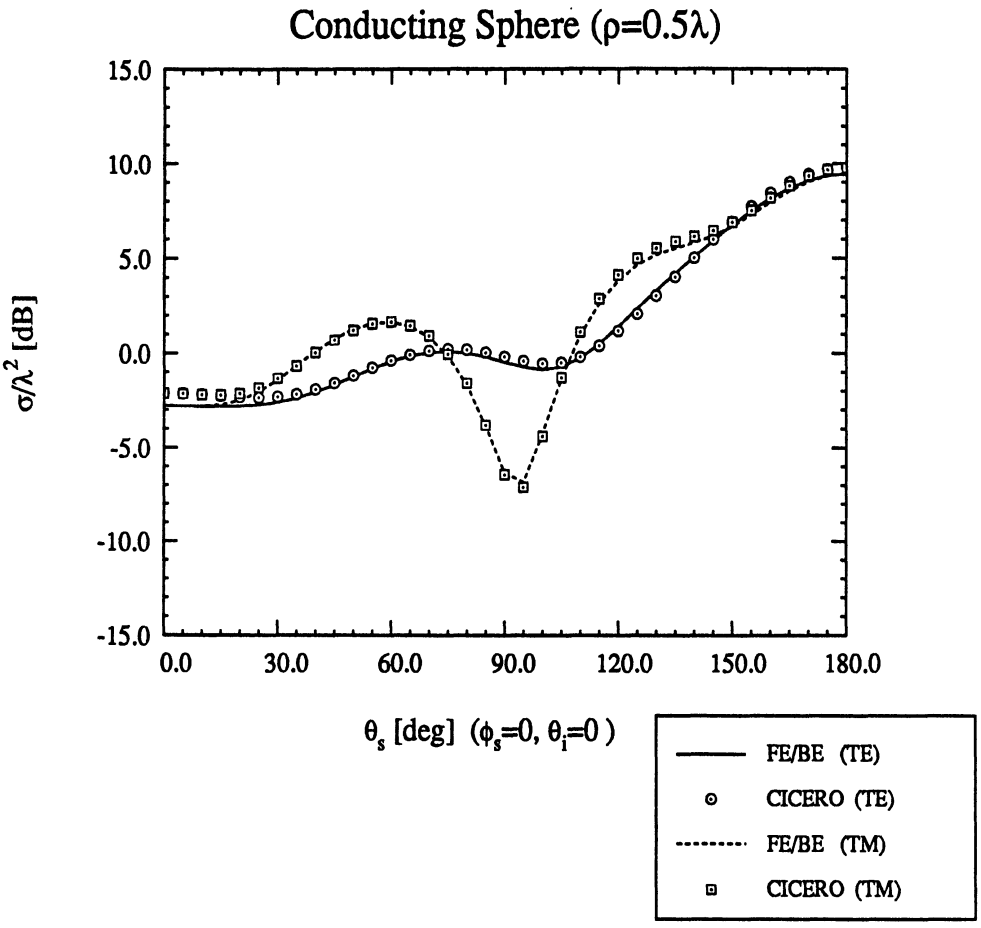

Figure 4.5: TM and TE bistatic scattering pattern from a perfectly conducting sphere of radius $\frac{\lambda}{2}$ for axial incidence.

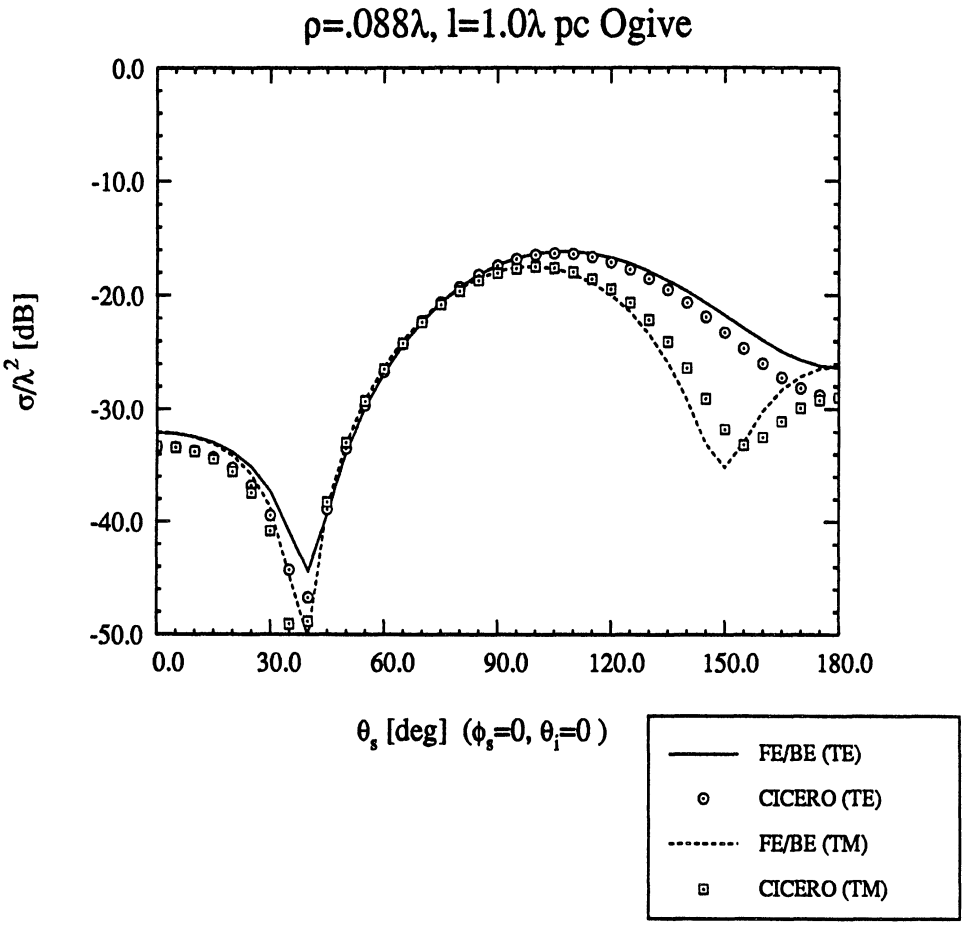

Figure 4.6: TM and TE bistatic scattering pattern from a perfectly conducting ogive length 1λ and maximum radius of 0.088λ for axial incidence.

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Appendix A

Derivation of Modal Incident Field

Consider a field incident at a point $\overline{r} = (r, \phi, z)$ at an angle (θ^i, ϕ^i) (as indicated in fig. 2.1) of the form

$$\overline{\zeta}(\theta^i, \phi^i; \rho, \phi, z) = \hat{\phi}^i e^{-j\overline{k}_0 \cdot \overline{r}} \tag{A.1}$$

$$\overline{\xi}(\theta^i, \phi^i; \rho, \phi, z) = -\hat{\theta}^i e^{-j\overline{k}_0 \cdot \overline{r}} \tag{A.2}$$

where the $\hat{\phi}^i$ direction is perpendicular to the plane of incidence and $\hat{\theta}^i$ direction is in the plane of incidence. Using

$$\hat{r} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta \tag{A.3}$$

$$\hat{z}' = \hat{x} \sin\theta^i \cos\phi^i + \hat{y} \sin\theta^i \sin\phi^i + \hat{z} \cos\theta^i \tag{A.4}$$

the argument of the exponential becomes

$$\overline{k}_0 \cdot \overline{r} = k_0 r(-\hat{z}' \cdot \overline{r}) \tag{A.5}$$

$$= -k_0 \left[\rho \sin\theta^i \cos(\phi - \phi^i) + z \cos\theta^i\right] \tag{A.6}$$

in cylindrical coordinate system. Using these and the fact that

$$\hat{\phi}^i = -\hat{x}\sin\theta^i + \hat{y}\cos\theta^i \tag{A.7}$$

$$\hat{\theta}^i = \hat{x}\cos\theta^i\cos\phi^i + \hat{y}\cos\theta^i\sin\phi^i - \hat{z}\sin\theta^i \tag{A.8}$$

$$\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi \tag{A.9}$$

$$\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi \tag{A.10}$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta \tag{A.11}$$

(A.1) and (A.2) become

$$\overline{\zeta}(\theta^i;\rho,\phi-\phi^i,z) = \left[\hat{\rho}\sin(\phi-\phi^i) + \hat{\phi}\cos(\phi-\phi^i)\right] e^{jk_0[\rho\sin\theta^i\cos(\phi-\phi^i)+z\cos\theta^i]} \tag{A.12}$$

$$\overline{\xi}(\theta^i;\rho,\phi-\phi^i,z) = -\left[\hat{\rho}\cos\theta^i\cos(\phi-\phi^i) - \hat{\phi}\cos\theta^i\sin(\phi-\phi^i) - \hat{z}\sin\theta^i\right] e^{jk_0[\rho\sin\theta^i\cos(\phi-\phi^i)+z\cos\theta^i]} \tag{A.13}$$

The previously derived fields may be expanded into a Fourier series in the parameter $\phi-\phi^i$ by first writing (A.1) and (A.2) as

$$\overline{\zeta}(\theta^i;\rho,\phi-\phi^i,z) = \sum_{m=-\infty}^{\infty} \overline{\zeta}_m(\theta^i;\rho,z)e^{jm(\phi-\phi^i)} \tag{A.14}$$

$$\overline{\xi}(\theta^i;\rho,\phi-\phi^i,z) = \sum_{m=-\infty}^{\infty} \overline{\xi}_m(\theta^i;\rho,z)e^{jm(\phi-\phi^i)} \tag{A.15}$$

and then making the definitions

$$f(\theta^i;\rho,\phi-\phi^i) = e^{jk_0\rho\sin\theta^i\cos(\phi-\phi^i)} \tag{A.16}$$

$$f_c(\theta^i;\rho,\phi-\phi^i) = \cos(\phi-\phi^i)f(\theta^i;\rho,\phi-\phi^i) \tag{A.17}$$

$$f_s(\theta^i;\rho,\phi-\phi^i) = \sin(\phi-\phi^i)f(\theta^i;\rho,\phi-\phi^i) \tag{A.18}$$

Expanding each of these into a Fourier series in $(\phi - \phi^i)$ and using the fact that

$$f(\phi) = f(-\phi) \Longleftrightarrow f_m(u) = f_{-m}(u) \tag{A.19}$$

$$f(\phi) = -f(-\phi) \Longleftrightarrow f_m(u) = -f_{-m}(u) \tag{A.20}$$

we have

$$f(\theta^i; \rho, \phi - \phi^i) = f_0(\theta^i, \rho) + 2\sum_{m=1}^{\infty} f_m(\theta^i, \rho)\cos[m(\phi - \phi^i)] \tag{A.21}$$

$$f_c(\theta^i; \rho, \phi - \phi^i) = f_{c0}(\theta^i, \rho) + 2\sum_{m=1}^{\infty} f_{cm}(\theta^i, \rho)\cos[m(\phi - \phi^i)] \tag{A.22}$$

$$f_s(\theta^i; \rho, \phi - \phi^i) = 2j\sum_{m=1}^{\infty} f_{sm}(\theta^i, \rho)\sin[m(\phi - \phi^i)] \tag{A.23}$$

where

$$f_m(\theta^i, \rho) = \frac{1}{\pi}\int_0^{\pi} e^{jk_0\rho\sin\theta^i\cos u}\cos(mu)du \tag{A.24}$$

$$f_{cm}(\theta^i, \rho) = \frac{1}{\pi}\int_0^{\pi} \cos u e^{jk_0\rho\sin\theta^i\cos u}\cos(mu)du \tag{A.25}$$

$$f_{sm}(\theta^i, \rho) = -\frac{j}{\pi}\int_0^{\pi} \sin u e^{jk_0\rho\sin\theta^i\cos u}\sin(mu)du \tag{A.26}$$

Noting the identities

$$j^m J_m(\beta) = \frac{1}{\pi}\int_0^{\pi} e^{j\beta\cos x}\cos(mx)dx \tag{A.27}$$

$$j^{m-1} J'_m(\beta) = \frac{1}{\pi}\int_0^{\pi} \cos x e^{j\beta\cos x}\cos(mx)dx \tag{A.28}$$

$$-\frac{m}{\beta} j^m J_m(\beta) = -\frac{j}{\pi}\int_0^{\pi} \sin x e^{j\beta\cos x}\sin(mx)dx \tag{A.29}$$

where the last two are derived from the first by differentiation with respect to β and integration by parts respectively, we may write (A.24)-(A.26) as

$$f_m(\theta^i, \rho) = j^m J_m(k_0\rho\sin\theta^i) \tag{A.30}$$

$$f_{cm}(\theta^i,\rho) = j^{m-1}J'_m(k_0\rho\sin\theta^i) \tag{A.31}$$

$$f_{sm}(\theta^i,\rho) = -\frac{m}{k_0\rho\sin\theta^i}f_m(\theta^i,\rho) \tag{A.32}$$

With these, we may proceed to rewrite (A.14) as

$$\overline{\zeta}(\theta^i;\rho,\phi-\phi^i,z) = e^{jk_0z\cos\theta^i}\sum_{m=-\infty}^{\infty}\left[\hat{\rho}f_{sm}(\theta^i,\rho)+\hat{\phi}f_{cm}(\theta^i,\rho)\right]e^{jm(\phi-\phi^i)} \tag{A.33}$$

$$\overline{\xi}(\theta^i;\rho,\phi-\phi^i,z) = -e^{jk_0z\cos\theta^i}$$

$$\sum_{m=-\infty}^{\infty}\left[\hat{\rho}f_{cm}(\theta^i,\rho)\cos\theta^i-\hat{\phi}f_{sm}(\theta^i,\rho)\cos\theta^i-\hat{z}f_m(\theta^i,\rho)\sin\theta^i\right]e^{jm(\phi-\phi^i)} \tag{A.34}$$

or, using (A.19) and (A.20), we have

$$\overline{\zeta}(\theta^i;\rho,\phi-\phi^i,z) = e^{jk_0z\cos\theta^i}\left[\hat{\rho}2j\sum_{m=1}^{\infty}f_{sm}(\theta^i,\rho)\sin[m(\phi-\phi^i)]\right.$$
$$\left.+\hat{\phi}f_{c0}(\theta^i,\rho)+\hat{\phi}2\sum_{m=1}^{\infty}f_{cm}(\theta^i,\rho)\cos[m(\phi-\phi^i)]\right] \tag{A.35}$$

$$\overline{\xi}(\theta^i;\rho,\phi-\phi^i,z) = -e^{jk_0z\cos\theta^i}$$
$$\left\{\hat{\rho}\cos\theta^i\left[f_{c0}(\theta^i,\rho)+2\sum_{m=1}^{\infty}f_{cm}(\theta^i,\rho)\cos[m(\phi-\phi^i)]\right]\right.$$
$$-\hat{\phi}\cos\theta^i\left[2j\sum_{m=1}^{\infty}f_{sm}(\theta^i,\rho)\sin[m(\phi-\phi^i)]\right]$$
$$\left.-\hat{z}\sin\theta^i\left[f_0(\theta^i,\rho)+2\sum_{m=1}^{\infty}f_m(\theta^i,\rho)\cos[m(\phi-\phi^i)]\right]\right\} \tag{A.36}$$

In this work, we will use the ϕ components of each of these equations.

Appendix B

Maxwell's Equations for Axisymmetric Media

The usual Maxwell's equations in a source free region are given by

$$\nabla \times \overline{E}(\overline{r}) = -j\omega\mu\overline{H} \tag{B.1}$$

$$\nabla \times \overline{H}(\overline{r}) = j\omega\epsilon\overline{E} \tag{B.2}$$

$$\nabla \cdot \overline{D}(\overline{r}) = 0 \tag{B.3}$$

$$\nabla \cdot \overline{B}(\overline{r}) = 0 \tag{B.4}$$

In cylindrical coordinates the electric and magnetic fields may be expanded into a Fourier series in ϕ as

$$\overline{E}(\overline{r}) = \sum_{m=-\infty}^{\infty} \overline{e}_m(\rho, z)e^{jm\phi} \tag{B.5}$$

$$\eta\overline{H}(\overline{r}) = \sum_{m=-\infty}^{\infty} \overline{h}_m(\rho, z)e^{jm\phi} \tag{B.6}$$

Substituting these into Maxwell's equations, we obtain

$$\nabla \times \overline{e}_m - \frac{jm}{\rho}\overline{e}_m \times \hat{\phi} = -j\frac{\omega\mu}{\eta}\overline{h}_m \tag{B.7}$$

$$\nabla \times \overline{h}_m + \frac{jm}{\rho}\overline{h}_m \times \hat{\phi} = j\omega\epsilon\eta\overline{e}_m \tag{B.8}$$

Appendix C

Derivation of Boundary Conditions

In this appendix, the axial and perfectly conducting boundary conditions are derived.

C.1 Derivation of Axial Boundary Conditions

Substituting the Fourier series representation of the electric field into the divergence condition we obtain in the normalized cylindrical coordinate system

$$\nabla \cdot (\overline{e}_m e^{jm\phi}) = k_0 e^{jm\phi} [\tfrac{1}{R}\epsilon_r e_{m\rho} + \tfrac{\partial}{\partial R}(\epsilon_r e_{m\rho}) + j\tfrac{m}{R}\epsilon_r e_{m\phi} + \tfrac{\partial}{\partial Z}(\epsilon_r e_{mz})] = 0 \qquad \text{(C.1)}$$

Thus,

$$\epsilon_r e_{m\rho} + R[\tfrac{\partial}{\partial R}(\epsilon_r e_{m\rho}) + \tfrac{\partial}{\partial Z}(\epsilon_r e_{mz})] = -jm\epsilon_r e_{m\phi} \qquad \text{(C.2)}$$

as Morgan had previously derived. Taking the limit of this expression as $R \rightarrow 0^+$, we obtain

$$e_{m\rho} + jme_{m\phi} = 0 \qquad \text{(C.3)}$$

Expanding the derivative w.r.t. R in

$$\tfrac{1}{R}[jme_{m\rho} - \tfrac{\partial}{\partial R}(Re_{m\phi})] = j\mu_r h_{mz} \qquad \text{(C.4)}$$

and taking the limit as $R \rightarrow 0^+$, we obtain

$$e_{m\rho} + \tfrac{j}{m} e_{m\phi} = 0 \tag{C.5}$$

Combining (C.3) and (C.5) and solving for $e_{m\phi}$ we have

$$(m^2 - 1) e_{m\phi}|_{R \rightarrow 0+} = 0 \tag{C.6}$$

In a similar manner we obtain the dual expression

$$(m^2 - 1) h_{m\phi}|_{R \rightarrow 0+} = 0 \tag{C.7}$$

For $m \neq 1$, the following axial condition holds

$$e_{m\phi}|_{R=0+} = h_{m\phi}|_{R=0+} = 0 \quad (m \neq 1) \tag{C.8}$$

To derive the condition for $m = 1$, lets first consider

$$e_{mz} = j f_m \left[m \tfrac{\partial}{\partial Z}(R e_{m\phi}) - R\mu_r \tfrac{\partial}{\partial R}(R h_{m\phi}) \right] \tag{C.9}$$

As $R \rightarrow 0^+$, $e_{mz} \rightarrow 0$ for $m \neq 0$. Differentiating (C.9) with respect to Z we have

$$\begin{aligned} \tfrac{\partial}{\partial Z} e_{mz} = j f_m \Big\{ & mR \tfrac{\partial^2}{\partial Z^2} e_{m\phi} - R\Big[\tfrac{\partial}{\partial Z}\mu_r \left(h_{m\phi} + R \tfrac{\partial}{\partial R} h_{m\phi} \right) \\ & + \mu_r \Big(\tfrac{\partial}{\partial Z} h_{m\phi} + R \tfrac{\partial^2}{\partial R \partial Z} h_{m\phi} \Big) \Big] \Big\} \end{aligned} \tag{C.10}$$

Clearly, as $R \rightarrow 0^+$ $\tfrac{\partial}{\partial Z} e_{mz} = 0$ for $m \neq 0$. Differentiating (C.2) with respect to R after dividing by ϵ_r, we obtain

$$\tfrac{\partial}{\partial R} e_{m\rho} + \tfrac{1}{\epsilon_r} \left[\tfrac{\partial}{\partial R}(\epsilon_r e_{m\rho}) + \tfrac{\partial}{\partial Z}(\epsilon_r e_{mz}) \right] |_{R=0+} = -jm \tfrac{\partial}{\partial R} e_{m\phi}|_{R=0+} \tag{C.11}$$

Accounting for the behavior of e_{mz} and $\tfrac{\partial}{\partial Z} e_{mz}$ (C.11) becomes

$$2 \tfrac{\partial}{\partial R} e_{m\rho} + e_{m\rho} \tfrac{\partial}{\partial R}(\ln \epsilon_r) + jm \tfrac{\partial}{\partial R} e_{m\phi} = 0 \tag{C.12}$$

To find another equation in terms of $\frac{\partial}{\partial R}e_{m\rho}$ and $\frac{\partial}{\partial R}e_{m\phi}$, we multiply (C.4) by R and differentiate it with respect to R to obtain

$$R\ \frac{\partial}{\partial R}(\mu_r h_{mz}) + \mu_r h_{mz} = j\left[2\ \frac{\partial}{\partial R}e_{m\phi} + R\ \frac{\partial^2}{\partial R^2}e_{m\phi} - jm\ \frac{\partial}{\partial R}e_{m\rho}\right] \tag{C.13}$$

Letting $R \rightarrow 0^+$ we obtain

$$2j\ \frac{\partial}{\partial R}e_{m\phi} + m\ \frac{\partial}{\partial R}e_{m\phi} = 0 \tag{C.14}$$

Substituting (C.14) and (C.5) into (C.12) we obtain

$$(4 - m^2)\ \frac{\partial}{\partial R}e_{m\phi} + e_{m\phi}\ \frac{\partial}{\partial R}(\ln \epsilon_r) = 0 \quad R = 0, \quad m \neq 0 \tag{C.15}$$

In an analogous fashion, the dual of (C.15) is given by

$$(4 - m^2)\ \frac{\partial}{\partial R}h_{m\phi} + h_{m\phi}\ \frac{\partial}{\partial R}(\ln \mu_r) = 0 \quad R = 0, \quad m \neq 0 \tag{C.16}$$

For ϵ_r and μ_r constant in R at the axis of symmetry and for $m = 1$, (C.15) and (C.16) reduce to

$$\frac{\partial}{\partial R}e_{m\phi} = 0 \tag{C.17}$$

$$\frac{\partial}{\partial R}h_{m\phi} = 0 \tag{C.18}$$

C.2 Derivation of PEC Boundary Conditions

On a perfect conductor the condition

$$\hat{n} \times \overline{E} = 0 \tag{C.19}$$

Substituting the Fourier series expansion for the field into this boundary condition yields the following condition on each mode

$$\hat{n} \times \overline{e}_m = 0 \tag{C.20}$$

The second Maxwell's equation for the mth mode is given by (see an appendix)

$$\nabla \times \overline{h}_m + \frac{jm}{\rho}\overline{h}_m \times \hat{\phi} = j\omega\epsilon\eta\overline{e}_m \tag{C.21}$$

Crossing this equation with $\hat{n}$ and noting that $\hat{n} \cdot \overline{h}_m = 0$ on the conducting surface, we obtain

$$\hat{n} \times (\nabla \times \overline{h}_m) = 0 \tag{C.22}$$

Carrying out the curl in cylindrical coordates yields

$$\hat{n} \times \left\{ \hat{\rho} \left[\tfrac{1}{\rho} \tfrac{\partial}{\partial \phi} h_{mz} - \tfrac{\partial}{\partial z} h_{m\phi} \right] + \hat{\phi} \left[\tfrac{\partial}{\partial z} h_{m\rho} - \tfrac{\partial}{\partial \rho} h_{mz} \right] \right.$$
$$\left. + \hat{z} \left[\tfrac{\partial}{\partial \rho}(\rho h_{m\phi}) - \tfrac{\partial}{\partial \phi} h_{m\rho} \right] \tfrac{1}{\rho} \right\} = 0 \tag{C.23}$$

Noting the identities

$$\hat{n} \times \hat{\rho} = \hat{\phi}(\hat{n} \cdot \hat{z}) \tag{C.24}$$

$$\hat{n} \times \hat{\phi} = \hat{t} \tag{C.25}$$

$$\hat{n} \times \hat{z} = -\hat{\phi}(\hat{n} \cdot \hat{\rho}) \tag{C.26}$$

we find that the middle term of (C.23) implies

$$\tfrac{\partial}{\partial z} h_{m\rho} = \tfrac{\partial}{\partial \rho} h_{mz} \tag{C.27}$$

and the first and third terms may be written

$$\hat{\phi}(\hat{n} \cdot \hat{z}) \left[\tfrac{1}{\rho} \tfrac{\partial}{\partial \phi} h_{mz} - \tfrac{\partial}{\partial z} h_{m\phi} \right] - \hat{\phi}(\hat{n} \cdot \hat{\rho}) \left[\tfrac{\partial}{\partial \rho}(\rho h_{m\phi}) - \tfrac{\partial}{\partial \phi} h_{m\rho} \right] \tfrac{1}{\rho} = 0 \tag{C.28}$$

Rearranging terms, we have

$$\hat{n} \cdot \left[\nabla_t(\rho h_{m\phi}) - \hat{t} \tfrac{\partial}{\partial \phi} h_{mt} \right] = 0 \tag{C.29}$$

or

$$\frac{\partial}{\partial n}\psi_h = 0 \tag{C.30}$$

and we have used the following

$$\nabla_t = \hat{\rho}\,\frac{\partial}{\partial \rho} + \hat{z}\,\frac{\partial}{\partial z} \tag{C.31}$$

$$\frac{\partial}{\partial n} = \hat{n} \cdot \nabla_t \tag{C.32}$$

$$\psi_h = k_0 \rho h_{m\phi} \tag{C.33}$$

Appendix D

Evaluation of Finite Element Contour Integral

D.1 Contour Integral Evaluation along Conducting Surfaces

It is shown in the appendix that along perfectly conducting surfaces the conditions

$$\psi_e = 0 \tag{D.1}$$

$$\frac{\partial}{\partial n}\psi_h = 0 \tag{D.2}$$

must hold. During the assembly of the finite element equations (i.e., when the summation over all elements is performed), those rows and columns of the finite element matrix corresponding to nodes on the conducting boundary are eliminated. As a result, the corresponding contour integral vanishes along a conducting boundary.

Imposing the condition (D.2) results in the elimination of the associated contour integral since on the conducting surface

$$\hat{n} \cdot (\hat{\phi} \times \nabla_t \psi_e) = \hat{t} \cdot \nabla_t \psi_e = 0 \tag{D.3}$$

(check this stuff)

D.2 Contour Integral Evaluation along the Axis of Symmetry

In the appendix, the axial boundary conditions are derived

$$e_{m\phi} = 0 \tag{D.4}$$

$$h_{m\phi} = 0 \tag{D.5}$$

$$\frac{\partial}{\partial R} e_{m\phi} = 0 \tag{D.6}$$

$$\frac{\partial}{\partial R} h_{m\phi} = 0 \tag{D.7}$$

Conditions (D.4) and (D.5) results in the elimination of the rows and columns of the assembled finite element matrix associated with nodes on the axis.

Alternatively, since $R \rightarrow 0$ all terms in the contour integral are zero by virtue of the chosen weighting function.

(may explore the possiblity of a different weighting function which does not guarantee this)

D.3 Contour Integral Inter-element Connection Cancellation

Since the argument of the contour integrals are tangential fields at the element boundary, they will be continuous between adjacent elements. As a result, the contour integrations along the element intersection cancel.

Appendix E

Evaluation of the Finite Element Matrix Elements

In the evaluation of a_{ij}^e and b_{ij}^e, integrals of the form

$$P_{ab} = \iint_{S^e} R^a Z^b dR dZ \tag{E.1}$$

and

$$Q_{ab} = \iint_{S^e} \frac{R^a Z^b}{R^2\kappa^2 - m^2} dR dZ \tag{E.2}$$

Clearly, Q_{ab} exhibits singularities for real κ. To evaluate this integral, consider an integral of the form

$$I = \iint_{S^e} g(R) Z^b dR dZ \tag{E.3}$$

To evaluate the integral, first transform it to an integration along the element boundary via

$$\overline{v}(R, Z) = -g(R)\frac{Z^{b+1}}{b+1}\hat{\rho} \tag{E.4}$$

Using Stokes' theorem

$$\iint_S (\nabla \times \overline{v}) \cdot d\overline{S} = \oint_{C^e} \overline{v} \cdot d\overline{l} \tag{E.5}$$

and

$$\nabla \times \overline{v} = -\hat{\phi}\, g(R) Z^b \tag{E.6}$$

$$d\overline{S} = \hat{\phi}\, dR dZ \tag{E.7}$$

Inserting these into (E.5) yields

$$\iint_{S^e} g(R) Z^b dR dZ = \frac{1}{b+1} \oint_{\Gamma^e} g(R) Z^{b+1} dR \tag{E.8}$$

Via (E.8), (E.1) and (E.2) become respectively

$$P_{ab} = \frac{1}{b+1} \oint_{\Gamma^e} R^a Z^{b+1} dR \tag{E.9}$$

and

$$Q_{ab} = \frac{1}{b+1} \oint_{\Gamma^e} \frac{R^a Z^{b+1}}{R^2\kappa^2 - m^2} dR \tag{E.10}$$

where the contour integration is taken in a counterclockwise fashion. For linear triangular elements, Γ^e is represented by a summation of three contours, one for each side of the triangle. The variable Z may be thus expressed as

$$Z(R) = v_l R + u_l \tag{E.11}$$

where

$$u_l = \frac{Z_{l+1} - Z_l}{R_{l+1} - R_l} \tag{E.12}$$

$$v_l = Z_l - u_l R_l \tag{E.13}$$

Then Z^{b+1} may be expressed as

$$\begin{aligned} Z^{b+1} = (u + Rv)^{b+1} &= v^{b+1}(1 + \frac{uR}{v})^{b+1} \\ &= v^{b+1} \sum_{p=0}^{b+1} \binom{b+1}{p} \left(\frac{uR}{v}\right)^p \end{aligned} \tag{E.14}$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{E.15}$$

Thus, by writing the integral in (E.10) as a sum of an integral along each side of the triangular element, it be rewritten as

$$Q_{ab} = \frac{1}{b+1}\sum_{l=1}^{3}\sum_{p=0}^{b+1}\binom{b+1}{p}\left(\frac{u_l}{v_l}\right)^p v_l^{b+1}\int_{R_l}^{R_{l+1}}\frac{R^{a+p}}{R^2\kappa^2 - m^2}dR \tag{E.16}$$

Clearly, integrals of the form

$$I(n,m) = \int_{R_l}^{R_{l+1}}\frac{R^n}{R^2\kappa^2 - m^2}dR \tag{E.17}$$

for $n = 0, 1, ..., a+b+1$ must be solved. For $n = 0$

$$I(0,m) = \begin{cases} -\frac{1}{\kappa^2 R}\Big|_{R_l}^{R_{l+1}} & m = 0 \\ \frac{1}{2m\kappa}\left[\ln(m - R\kappa) - \ln(m + R\kappa)\right]\Big|_{R_l}^{R_{l+1}} & m > 0 \end{cases} \tag{E.18}$$

For $n = 1$ it is easily shown that

$$I(1,m) = \frac{1}{2\kappa^2}\left[\ln(m - R\kappa) + \ln(m + R\kappa)\right]\Big|_{R_l}^{R_{l+1}} \quad m > 0 \tag{E.19}$$

Using the definition of the principle branch of the natural logarithm in the equations above guarantees that the singularity is properly handled. For values of $n > 1$, the recursive formula [17]

$$I(n,m) = I(n,0) + \frac{m^2}{\kappa^2}I(n-2,m) \tag{E.20}$$

is used. Thus, (E.16) may be written in terms of I as

$$Q_{ab} = \frac{1}{b+1}\sum_{l=1}^{3}\sum_{p=0}^{b+1}\binom{b+1}{p}\left(\frac{u_l}{v_l}\right)^p v_l^{b+1} I(a+p,m) \tag{E.21}$$

In a similar fashion,

$$P_{ab} = \frac{1}{b+1}\sum_{l=1}^{3}\sum_{p=0}^{b+1}\binom{b+1}{p}\left(\frac{u_l}{v_l}\right)^p v_l^{b+1}\left.\frac{R^{a+p+1}}{a+p+1}\right|_{R_l}^{R_{l+1}} \tag{E.22}$$

The shape function is written in expanded form as

$$N_i^e(R,Z) = \frac{1}{2\Omega^e}(\alpha_i^e + \beta_i^e Z + \gamma_i^e R) \tag{E.23}$$

where

$$\Omega^e = \frac{1}{2}(\beta_i^e\gamma_i^e - \beta_j^e\gamma_i^e \tag{E.24}$$

$$\alpha_i^e = Z_j^e R_k^e - Z_k^e R_j^e \tag{E.25}$$

$$\beta_i^e = R_j^e - R_k^e \tag{E.26}$$

$$\gamma_i^e = Z_k^e - Z_j^e \tag{E.27}$$

We had derived in section 2.1

$$a_{ij}^e = \iint_{S^e} R\left[-f_m\epsilon_r\nabla_t(RN_i^e)\cdot\nabla_t(RN_j^e) + \epsilon_r N_i^e N_j^e\right]dS^e \tag{E.28}$$

Noting that

$$\nabla_t(RN_i^e) = \hat{\rho}\left[N_i^e + \frac{R\gamma_i^e}{2\Omega^e}\right] + \hat{z}\frac{R\beta_i^e}{2\Omega^e} \tag{E.29}$$

$$N_i^e N_j^e = \frac{1}{(2\Omega^e)^2}\left[\alpha_i^e\alpha_j^e + Z(\beta_i^e\alpha_j^e + \beta_j^e\alpha_i^e) + R(\gamma_i^e\alpha_j^e + \gamma_j^e\alpha_i^e)\right.$$
$$\left. + RZ(\beta_i^e\gamma_j^e + \beta_j^e\gamma_i^e) + Z^2\beta_i^e\beta_j^e + R^2\gamma_i^e\gamma_j^e\right] \tag{E.30}$$

Substituting these into (E.28) and reducing we obtain the desired result

$$a_{ij}^e = [-\alpha_i^e\alpha_j^e Q_{10} - (\beta_i^e\alpha_j^e + \beta_j^e\alpha_i^e)Q_{11} - 2(\gamma_i^e\alpha_j^e + \gamma_j^e\alpha_i^e)Q_{20}$$
$$-2(\beta_i^e\gamma_j^e + \beta_j^e\gamma_i^e)Q_{21} - \beta_i^e\beta_j^e Q_{12} - (4\gamma_i^e\gamma_j^e + \beta_i^e\beta_j^e)Q_{30}$$

$$+\alpha_i^e\alpha_j^e P_{10} + (\beta_i^e\alpha_j^e + \beta_j^e\alpha_i^e)P_{11} + (\gamma_i^e\alpha_j^e + \gamma_j^e\alpha_i^e)P_{20} + (\beta_i^e\gamma_j^e + \beta_j^e\gamma_i^e)P_{21}$$

$$+\beta_i^e\beta_j^e P_{12} + \gamma_i^e\gamma_j^e P_{30}]\frac{1}{(2\Omega^e)^2} \qquad \text{(E.31)}$$

In a similar manner, we may write

$$b_{ij}^e = \iint_{S^e} m f_m \hat{\phi} \times \nabla_t(RN_j^e) \cdot \nabla_t(RN_i^e) dS^e \qquad \text{(E.32)}$$

as

$$b_{ij}^e = \frac{m}{2\Omega^e} \iint_{S^e} f_m^e \left[\beta_j^e \left(N_i^e + \frac{R\gamma_i^e}{2\Omega^e}\right) - \beta_i^e \left(N_j^e + \frac{R\gamma_j^e}{2\Omega^e}\right)\right] dS^e \qquad \text{(E.33)}$$

and likewise as

$$b_{ij}^e = \frac{m}{(2\Omega^e)^2}[(\beta_j^e\alpha_i^e - \beta_i^e\alpha_j^e)Q_{10} + 2(\beta_j^e\gamma_i^e - \beta_i^e\gamma_j^e)Q_{20}] \qquad \text{(E.34)}$$

Appendix F

Boundary Integral Matrix Elements

In this appendix, the elements for the discrete boundary integral system are presented.

F.1 Elements of P^{ϕ}

$$\left[P^{\phi}_{11}\right]_{ij} = 0 \tag{F.1}$$

$$\left[P^{\phi}_{12}\right]_{ij} = \int_{Z_{j+1}}^{Z_j} \left[R_2 g_m^{(1)'} - R_{i+\frac{1}{2}} g'_m(R_{i+\frac{1}{2}}, R_2, Z_1, Z')\right] R_2 dZ' \tag{F.2}$$

$$\left[P^{\phi}_{13}\right]_{ij} = (Z_1 - Z_3) \int_{R_{j+1}}^{R_j} g_m^{(1)'}(R_{i+\frac{1}{2}}, R', Z_1, Z_3) R' dR' \tag{F.3}$$

$$\left[P^{\phi}_{21}\right]_{ij} = (Z_{i+\frac{1}{2}} - Z_1) \int_{R_j}^{R_{j+1}} - g_m^{(1)'}(R_2, R', Z_{i+\frac{1}{2}}, Z_1) R' dR' \tag{F.4}$$

$$\left[P^{\phi}_{22}\right]_{ij} = \int_{Z_{j+1}}^{Z_j} \left[R_2 g_m^{(1)'} - R_2 g'_m(R_2, R_2, Z_{i+\frac{1}{2}}, Z')\right] R_2 dZ' \tag{F.5}$$

$$\left[P^{\phi}_{23}\right]_{ij} = (Z_{i+\frac{1}{2}} - Z_3) \int_{R_{j+1}}^{R_j} g_m^{(1)'}(R_2, R', Z_{i+\frac{1}{2}}, Z_3) R' dR' \tag{F.6}$$

$$\left[P^{\phi}_{31}\right]_{ij} = (Z_3 - Z_1) \int_{R_j}^{R_{j+1}} - g_m^{(1)'}(R_{i+\frac{1}{2}}, R', Z_3, Z_1) R' dR' \tag{F.7}$$

$$\left[P_{32}^{\phi}\right]_{ij} = \oint_{Z_{j+1}}^{Z_j} \left[R_2 g_m^{(1)\prime} - R_{i+\frac{1}{2}} g_m'(R_{i+\frac{1}{2}}, R_2, Z_3, Z')\right] R_2 dZ' \quad \text{(F.8)}$$

$$\left[P_{11}^{\phi}\right]_{ij} = 0 \quad \text{(F.9)}$$

F.2 Elements of P^t

$$\left[P_{11}^{t}\right]_{ij} = 0 \quad \text{(F.10)}$$

$$\left[P_{12}^{t}\right]_{ij} = \int_{Z_{j+1}}^{Z_j} j(Z_1 - Z') g_m^{(2)\prime}(R_{i+\frac{1}{2}}, R_2, Z_1, Z') R_2 dZ' \quad \text{(F.11)}$$

$$\left[P_{13}^{t}\right]_{ij} = (Z_1 - Z_3) \int_{R_{j+1}}^{R_j} j g_m^{(2)\prime}(R_{i+\frac{1}{2}}, R', Z_1, Z_3) R' dR' \quad \text{(F.12)}$$

$$\left[P_{21}^{t}\right]_{ij} = (Z_{i+\frac{1}{2}} - Z_1) \int_{R_j}^{R_{j+1}} j g_m^{(2)\prime}(R_2, R', Z_{i+\frac{1}{2}}, Z_1) R' dR' \quad \text{(F.13)}$$

$$\left[P_{22}^{t}\right]_{ij} = \int_{Z_{j+1}}^{Z_j} j(Z_{i+\frac{1}{2}} - Z') g_m^{(2)\prime}(R_2, R_2, Z_{i+\frac{1}{2}}, Z') R_2 dZ' \quad \text{(F.14)}$$

$$\left[P_{23}^{t}\right]_{ij} = (Z_{i+\frac{1}{2}} - Z_3) \int_{R_{j+1}}^{R_j} j g_m^{(2)\prime}(R_2, R', Z_{i+\frac{1}{2}}, Z_3) R' dR' \quad \text{(F.15)}$$

$$\left[P_{31}^{t}\right]_{ij} = (Z_3 - Z_1) \int_{R_j}^{R_{j+1}} j g_m^{(2)\prime}(R_{i+\frac{1}{2}}, R', Z_3, Z_1) R' dR' \quad \text{(F.16)}$$

$$\left[P_{32}^{t}\right]_{ij} = \int_{Z_{j+1}}^{Z_j} j(Z_3 - Z') g_m^{(2)\prime}(R_{i+\frac{1}{2}}, R_2, Z_3, Z') R_2 dZ' \quad \text{(F.17)}$$

$$\left[P_{33}^{t}\right]_{ij} = 0 \quad \text{(F.18)}$$

F.3 Elements of Q^{ϕ}

$$\left[Q_{11}^{\phi}\right]_{ij} = \oint_{R_j}^{R_{j+1}} - j g_m^{(2)}(R_{i+\frac{1}{2}}, R', Z_1, Z_1) R' dR' \quad \text{(F.19)}$$

$$\left[Q_{12}^{\phi}\right]_{ij} = 0 \quad \text{(F.20)}$$

$$\left[Q_{13}^{\phi}\right]_{ij} = \oint_{R_j}^{R_{j+1}} j g_m^{(2)}(R_{i+\frac{1}{2}}, R', Z_1, Z_3) R' dR' \quad \text{(F.21)}$$

$$\left[Q_{21}^{\phi}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)}(R_2, R', Z_{i+\frac{1}{2}}, Z_1)R'dR' \quad \text{(F.22)}$$

$$\left[Q_{22}^{\phi}\right]_{ij} = 0 \quad \text{(F.23)}$$

$$\left[Q_{23}^{\phi}\right]_{ij} = \oint_{R_j}^{R_{j+1}} jg_m^{(2)}(R_2, R', Z_{i+\frac{1}{2}}, Z_3)R'dR' \quad \text{(F.24)}$$

$$\left[Q_{31}^{\phi}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)}(R_{i+\frac{1}{2}}, R', Z_3, Z_1)R'dR' \quad \text{(F.25)}$$

$$\left[Q_{32}^{\phi}\right]_{ij} = 0 \quad \text{(F.26)}$$

$$\left[Q_{33}^{\phi}\right]_{ij} = \oint_{R_j}^{R_{j+1}} jg_m^{(2)}(R_{i+\frac{1}{2}}, R', Z_3, Z_3)R'dR' \quad \text{(F.27)}$$

F.4 Elements of $Q^{\phi'}$

$$\left[Q_{11}^{\phi'}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)'}(R_{i+\frac{1}{2}}, R', Z_1, Z_1)R'dR' \quad \text{(F.28)}$$

$$\left[Q_{12}^{\phi'}\right]_{ij} = \oint_{Z_{j+1}}^{Z_j} -jg_m^{(2)'}(R_{i+\frac{1}{2}}, R_2, Z_1, Z')R'dZ' \quad \text{(F.29)}$$

$$\left[Q_{13}^{\phi'}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)'}(R_{i+\frac{1}{2}}, R', Z_1, Z_3)R'dR' \quad \text{(F.30)}$$

$$\left[Q_{21}^{\phi'}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)'}(R_2, R', Z_{i+\frac{1}{2}}, Z_1)R'dR' \quad \text{(F.31)}$$

$$\left[Q_{22}^{\phi'}\right]_{ij} = \oint_{Z_{j+1}}^{Z_j} -jg_m^{(2)'}(R_2, R_2, Z_{i+\frac{1}{2}}, Z')R'dZ' \quad \text{(F.32)}$$

$$\left[Q_{23}^{\phi'}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)'}(R_2, R', Z_{i+\frac{1}{2}}, Z_3)R'dR' \quad \text{(F.33)}$$

$$\left[Q_{31}^{\phi'}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)'}(R_{i+\frac{1}{2}}, R', Z_3, Z_1)R'dR' \quad \text{(F.34)}$$

$$\left[Q_{32}^{\phi'}\right]_{ij} = \oint_{Z_{j+1}}^{Z_j} -jg_m^{(2)'}(R_{i+\frac{1}{2}}, R_2, Z_3, Z')R'dZ' \quad \text{(F.35)}$$

$$\left[Q_{33}^{\phi'}\right]_{ij} = \oint_{R_j}^{R_{j+1}} -jg_m^{(2)'}(R_{i+\frac{1}{2}}, R', Z_3, Z_3)R'dR' \quad \text{(F.36)}$$

F.5 Elements of Q^t

$$\left[Q^t_{11}\right]_{ij} = \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_{i+\frac{1}{2}}, R', Z_1, Z_1)\right] R'dR' \tag{F.37}$$

$$\left[Q^t_{12}\right]_{ij} = \oint_{Z_{j+1}}^{Z_j} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_{i+\frac{1}{2}}, R_2, Z_1, Z')\right] R'dZ' \tag{F.38}$$

$$\left[Q^t_{13}\right]_{ij} = \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_{i+\frac{1}{2}}, R', Z_1, Z_3)\right] R'dR' \tag{F.39}$$

$$\left[Q^t_{21}\right]_{ij} = \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_2, R', Z_{i+\frac{1}{2}}, Z_1)\right] R'dR' \tag{F.40}$$

$$\left[Q^t_{22}\right]_{ij} = \oint_{Z_{j+1}}^{Z_j} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_2, R_2, Z_{i+\frac{1}{2}}, Z')\right] R'dZ' \tag{F.41}$$

$$\left[Q^t_{23}\right]_{ij} = \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_2, R', Z_{i+\frac{1}{2}}, Z_3)\right] R'dR' \tag{F.42}$$

$$\left[Q^t_{31}\right]_{ij} = \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_{i+\frac{1}{2}}, R', Z_3, Z_1)\right] R'dR' \tag{F.43}$$

$$\left[Q^t_{32}\right]_{ij} = \oint_{Z_{j+1}}^{Z_j} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_{i+\frac{1}{2}}, R_2, Z_3, Z')\right] R'dZ' \tag{F.44}$$

$$\left[Q^t_{33}\right]_{ij} = \oint_{R_j}^{R_{j+1}} \left[g_m^{(1)} + jmg_m^{(2)\prime}(R_{i+\frac{1}{2}}, R', Z_3, Z_3)\right] R'dR' \tag{F.45}$$

F.6 Self-Cell Evaluation

The integrals in the matrix elements $[P^{\phi}_{22}]_{ii}$, $[Q^{\phi'}_{kk}]_{ii}$ and $[Q^t_{kk}]_{ii}$ contain integrable singularities. They could be integrated numerically without modification as long as the singularity point is avoided, but costs excessive computation time. To avoid the resulting excessive computation time and innacuracies, the integrals are evaluated as in [5].

For self-cell integrals involving g_m, Glisson gives

$$\oint_{l_1}^{l_2} g_m(R_{i+\frac{1}{2}}, R', Z_{i+\frac{1}{2}}, Z')R'dl'$$

$$= \frac{1}{2\pi} \int_{l_1}^{l_2} \int_0^{\pi} \left[\frac{e^{-j\tilde{R}_0}}{\tilde{R}_0} \cos(mu) R' - \frac{R_{i+\frac{1}{2}}}{\tilde{R}_0} \right] du dl'$$
$$+ I(R_{i+\frac{1}{2}}, l_{i+\frac{1}{2}}, l_1, l_2) \tag{F.46}$$

where

$$I(R, l, l_1, l_2) = \frac{R_{i+\frac{1}{2}}}{\pi} \int_{l_1}^{l_2} \left[\frac{1}{\mathcal{R}_2} K\left(\frac{2\sqrt{R' R_{i+\frac{1}{2}}}}{\mathcal{R}_2} \right) + \frac{\ln \mathcal{R}_1}{2R_{i+\frac{1}{2}}} \right] dl'$$
$$+ \frac{1}{2\pi} [(l_2 - l_1) - (l_2 - l) \ln(l_2 - l) - (l - l_1) \ln(l - l_1)] \tag{F.47}$$

and where K is the complete elliptical integral of the first kind, l may be either Z or R and

$$\mathcal{R}_1 = \sqrt{(R_{i+\frac{1}{2}} - R')^2 + (Z_{i+\frac{1}{2}} - Z')^2} \tag{F.48}$$

$$\mathcal{R}_2 = \sqrt{(R_{i+\frac{1}{2}} + R')^2 + (Z_{i+\frac{1}{2}} - Z')^2} \tag{F.49}$$

Also,

$$\tilde{R}_0 = \sqrt{R_{i+\frac{1}{2}}^2 + R'^2 - 2R_{i+\frac{1}{2}} R' \cos u + (Z_{i+\frac{1}{2}} - Z')^2} \tag{F.50}$$

$$\tag{F.51}$$

The first and second integrals of (F.46) may be computed using an open interval numerical scheme that also avoids the midpoint.

The integral expression for the self-cell of P_{22}^{ϕ} may be rewritten as

$$\left[P_{22}^{\phi}\right]_{ii} = \int_{Z_{i+1}}^{Z_i} \left[-\frac{1}{\pi} \int_0^{\pi} \cos u \frac{(1 + j\tilde{R}_0)}{\tilde{R}_0^2} \frac{e^{-j\tilde{R}_0}}{2\tilde{R}_0} \cos(mu) du \right.$$
$$\left. \frac{1}{\pi} \int_0^{\pi} \frac{(1 + j\tilde{R}_0)}{\tilde{R}_0^2} \frac{e^{-j\tilde{R}_0}}{2\tilde{R}_0} \cos(mu) du \right] R_2^2 dZ'$$
$$= \int_{Z_{i+1}}^{Z_i} \left[\frac{1}{\pi} \int_0^{\pi} \frac{(1 + j\tilde{R}_0)}{\tilde{R}_0^2} \frac{e^{-j\tilde{R}_0}}{\tilde{R}_0} \sin^2(\frac{u}{2}) \cos(mu) du \right] R_2^2 dZ' \tag{F.52}$$

where we have used the identity

$$1 - \cos u = 2\sin^2(\tfrac{u}{2}) \tag{F.53}$$

The solution to (F.52) is

$$\begin{aligned}\left[P_{22}^{\phi}\right]_{ii} = \oint_{Z_{i+1}}^{Z_i} \frac{1}{\pi} \int_0^{\pi} \Bigg[& \frac{(1+j\tilde{R}_0)}{\tilde{R}_0^2} \frac{e^{-j\tilde{R}_0}}{\tilde{R}_0} \cos(mu)\sin^2(\tfrac{u}{2}) \\ & - \frac{u^2}{4\left[R_2^2 u^2 + (Z_{i+\frac{1}{2}} - Z')^2\right]^{3/2}} \Bigg] du R_2^2 dZ' \\ & + \frac{R_2^2}{4\pi} I'(R_2, Z_{i+\frac{1}{2}}, Z_{i+1}, Z_i)\end{aligned} \tag{F.54}$$

where

$$\begin{aligned} I'(R,l,l_1,l_2) &= \oint_{l_1}^{l_2} \int_0^{\pi} \frac{u^2}{[R^2u^2 + (l-l')^2]^{3/2}} du dl' \\ &= \frac{1}{R^3}\Big\{(l-l_1)\ln\left[R\pi + \sqrt{(l-l_1)^2 + R^2\pi^2}\right] \\ &\quad + (l_2 - l)\ln\left[R\pi + \sqrt{(l_2-l)^2 + R^2\pi^2}\right] \\ &\quad - (l-l_1)\ln(l-l_1) - (l_2-l)\ln(l_2-l)\Big\} \end{aligned} \tag{F.55}$$

In the same manner we have

$$\begin{aligned}\left[Q_{aa}^{\phi'}\right]_{ii} = -j \oint_{l_1}^{l_2} \frac{j}{\pi} \int_0^{\pi} \Bigg[& \frac{(1+j\tilde{R}_0)}{\tilde{R}_0^2} \frac{e^{-j\tilde{R}_0}}{2\tilde{R}_0} \sin(mu)\sin(u)R' \\ & - \frac{mu^2 R_{i+\frac{1}{2}}}{2\left[R_{i+\frac{1}{2}}^2 u^2 + (R_{i+\frac{1}{2}} - R')^2\right]^{3/2}} \Bigg] du R' dl' \\ & + \frac{mR_{i+\frac{1}{2}}}{2\pi} I'(R_{i+\frac{1}{2}}, R_{i+\frac{1}{2}}, R_i, R_{i+1})\end{aligned}$$

where

$$\begin{aligned} l_1 &= R_i \quad l_2 = R_{i+1} \quad l_{i+\frac{1}{2}} = R_{i+\frac{1}{2}} \quad l' = R' \quad \text{for } a = 1 \\ l_1 &= Z_{i+1} \quad l_2 = Z_i \quad l_{i+\frac{1}{2}} = Z_{i+\frac{1}{2}} \quad l' = Z' \quad \text{for } a = 2 \\ l_1 &= R_{i+1} \quad l_2 = R_i \quad l_{i+\frac{1}{2}} = R_{i+\frac{1}{2}} \quad l' = R' \quad \text{for } a = 3 \end{aligned} \tag{F.57}$$

Finally, we treat each term in $[Q^t_{aa}]_{ii}$ seperately and obtain

$$\begin{aligned} \left[Q^t_{aa}\right]_{ii} &= \int_{l_1}^{l_2} \left[g_m^{(1)} + jmg_m^{(2)'}(R_{i+\frac{1}{2}}, R', Z_{i+\frac{1}{2}}, Z')\right] R'dR' \\ &= \frac{1}{2\pi} \int_{l_1}^{l_2} \int_0^{\pi} \left[\frac{e^{-j\tilde{R}_0}}{\tilde{R}_0} \cos u \cos(mu) R' - \frac{R_{i+\frac{1}{2}}}{\tilde{R}_0}\right] dudl' \\ &\qquad + I(R, l, l_1, l_2) \\ &\quad + jm \left\{ \int_{l_1}^{l_2} \frac{j}{\pi} \int_0^{\pi} \left[\frac{(1 + j\tilde{R}_0)}{\tilde{R}_0^2} \frac{e^{-j\tilde{R}_0}}{2\tilde{R}_0} \sin(mu) \sin u R' \right.\right. \\ &\qquad \left. - \frac{mu^2 R_{i+\frac{1}{2}}}{2\left[R^2_{i+\frac{1}{2}} u^2 + (l_{i+\frac{1}{2}} - l')^2\right]^{3/2}}\right] duR'dl' \\ &\qquad \left. + \frac{jmR_{i+\frac{1}{2}}}{2\pi} I'(R_{i+\frac{1}{2}}, l_{i+\frac{1}{2}}, l_1, l_2)\right\} \end{aligned} \tag{F.58}$$

where (F.57) is used to determine the expression for each value of a.

The self cells involved in the other matrices contain non-singular integrands and may thus be integrated numerically without modification.