*Abstract*-The theoretical analysis of an enclosed air-bridge is presented. This includes a derivation of the Green's function in the x and z directions which is used to find the current distribution on the conducting strips of the air-bridge. Boundary conditions at the interfaces are applied and the numerical technique Method of Moments is used to solve the integral equation for the unknown current. Upon derivation of the current distribution on the conductors, an ideal transmission line model is applied to obtain the scattering parameters of the structure.

# Contents

# 1 Introduction

Millimeter wave technology concerns itself with that portion of the electromagnetic spectrum between 0.3 Ghz and 300 Ghz, corresponding to wavelengths of 1000 mm to 1mm. Effective quasi-static techniques have been developed for the lower frequencies (0.3 Ghz to 3 Ghz) but for the higher frequency part of the spectrum, a full-wave analysis must be employed.

Millimeter and microwave systems may be overshadowed by infarred and optical systems but limitations to the latter, in particular their disadvantages in fog, dust, rain, and nighttime viewing support further development of the former. As with many technologies, the number of applications will increase with the passage of time.

The typical millimeter microwave integrated circuit contains associated active and passive elements interconnected by transmission lines. In integrating these components together, various discontinuities arise where evanescent fields and surface waves play an important role in their operation. Work has been done to model these discontinuities with lumped elements but the numerical techniques used to derive equivalent circuits are either frequency bound or dependent.

Here we present a full-wave analysis, which in not frequency bound or dependent, to analyze an air-bridge structure. The resultant expressions and methods used are general enough to be applied to an array of three dimensional problems.

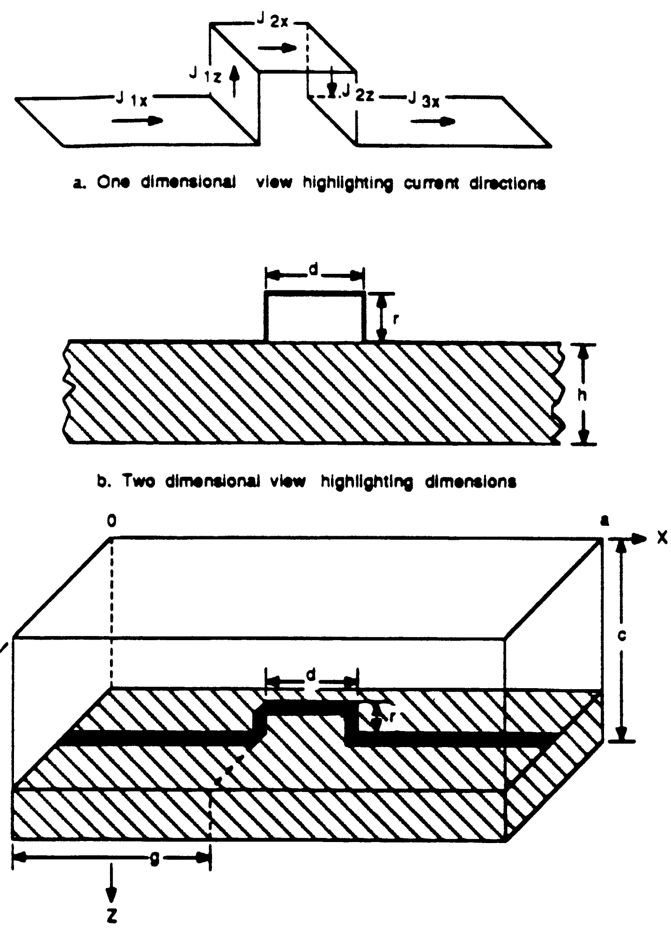



Figure 1: Air-bridge in enclosed microstrip.

## 2 Evaluation of the unknown current

In our problem the current distribution on the structure of interest must be determined accurately. Then by the use of an ideal transmission line model, the scattering parameters can be evaluated. In obtaining the current distribution, Pocktnigton's integral equation is solved numerically.

The formulation of the Pocktnigton's integral equation and the solution for the given structure are presented in the following work. Our structure under study is shown in Fig.1.

## 2.1 Formulation of the Integral Equation

Through the manipulation of Maxwell's equations [1]

$$\overline{\nabla} \times \overline{E} = -j\omega\mu\overline{H} \tag{1}$$

$$\overline{\nabla} \times \overline{H} = j\omega\mu\overline{E} + \overline{J} \tag{2}$$

$$\overline{\nabla} \cdot \epsilon\overline{E} = \rho \tag{3}$$

$$\overline{\nabla} \cdot \mu\overline{H} = 0 \tag{4}$$

along with the representation of the magnetic vector potential $\overline{A}$

$$\overline{H} = \frac{\overline{\nabla} \times \overline{A}}{\mu} \tag{5}$$

one arrives at an expression relating current and the magnetic vector potential

$$\nabla^2\overline{A} + k^2\overline{A} = -\mu\overline{J}. \tag{6}$$

When the current $\overline{J}$ is represented by a dirac delta function in equation (6), the Green's function becomes a solution as shown by the equation

$$\nabla^2\overline{\overline{G}} + k^2\overline{\overline{G}} = -\mu\overline{\overline{I}}\delta(\overline{r} - \overline{r}'). \tag{7}$$

To obtain a unique solution that applies to the specific geometry as shown in Fig.1, one must apply the characteristic boundary conditions of the structure. Note that we have introduced the dyadic form as we need to be able to describe fields which are produced by a current of arbitrary orientation.

[1] throughout this report an $e^{j\omega t}$ time convention is assumed and suppressed

For the case of a single x-directed current, the unit dyadic $\overline{\overline{I}}$ takes on the form;

$$\overline{\overline{I}} = \hat{x}\hat{x}. \tag{8}$$

The vector representation of this current is

$$J = \delta(\overline{r} - \overline{r}')\hat{x}. \tag{9}$$

Equation (9) represents a dipole directed in the x-direction and parallel to the interface between regions II and III. It has been shown by Sommerfeld that the magnetic vector potential of this structure needs to have two components so that the appropriate boundary conditions are satisfied. This dictates that $\overline{A}$ must have one component parallel to the current source and another parallel to the interface;

$$\overline{A}^{i\,2} = A^i_x\hat{x} + A^i_z\hat{z}. \tag{10}$$

The integral equation which relates the magnetic vector potential to the current of interest is written as

$$\overline{A}^i = \mu \int\int\int_V \overline{\overline{G}}^i \cdot \overline{J}dV \tag{11}$$

where the dyadic Green's function $\overline{\overline{G}}^i$ is uniquely defined by the structure under investigation and takes on the general form

$$\overline{\overline{G}}^i = \begin{bmatrix} G^i_{xx}\hat{x}\hat{x} & G^i_{xy}\hat{x}\hat{y} & G^i_{xz}\hat{x}\hat{z} \\ G^i_{yx}\hat{y}\hat{x} & G^i_{yy}\hat{y}\hat{y} & G^i_{yz}\hat{y}\hat{z} \\ G^i_{zx}\hat{z}\hat{x} & G^i_{zy}\hat{z}\hat{y} & G^i_{zz}\hat{z}\hat{z} \end{bmatrix}.$$

[2] i denotes for which region (I, II, or III) the equation applies as defined in Fig.2

In the case of a single directional current in the x-direction, the dyadic equation has only two components,

$$\overline{\overline{G}}^i = G^i_{xx}\hat{x}\hat{x} + G^i_{zx}\hat{z}\hat{x}. \tag{12}$$

From equations (1)-(5) our electric field is related to the magnetic vector potential by

$$\overline{E} = \frac{\nabla \times \overline{H}}{j\omega\epsilon} = \frac{1}{j\omega\epsilon\mu}\nabla \times (\nabla \times \overline{A}) = \frac{1}{j\omega\epsilon\mu}(k^2\overline{A} + \nabla\nabla \cdot \overline{A}) \tag{13}$$

therefore

$$E_x = -j\omega\left[A_x + \frac{1}{(k^i)^2}\frac{\delta}{\delta x}\left(\frac{\delta A_x}{\delta x} + \frac{\delta A_z}{\delta z}\right)\right] \tag{14}$$

$$E_y = -j\omega\left[\frac{1}{(k^i)^2}\frac{\delta}{\delta y}\left(\frac{A_x}{\delta x} + \frac{\delta A_z}{\delta z}\right)\right] \tag{15}$$

$$E_z = -j\omega\left[A_z + \frac{1}{(k_i)^2}\frac{\delta}{\delta z}\left(\frac{\delta A_x}{\delta x} + \frac{\delta A_z}{\delta z}\right)\right] \tag{16}$$

and through Maxwell's equations,

$$H_x = \frac{1}{\mu}\frac{\delta A_z}{\delta y} \tag{17}$$

$$H_y = \frac{1}{\mu}\left(\frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x}\right) \tag{18}$$

$$H_z = -\frac{1}{\mu}\left(\frac{\delta A_x}{\delta y}\right). \tag{19}$$

## 2.2 Derivation of Green's function

For our structure pictured in Fig.1 (an air bridge on an enclosed thin microstrip), we require only the derivation of the Green's function for an x- and z-directed current. For the variation of current in the y-direction, we have assumed a Maxwellian distribution. Our first step will be to derive the Green's function considering the x and z components of the current separately.

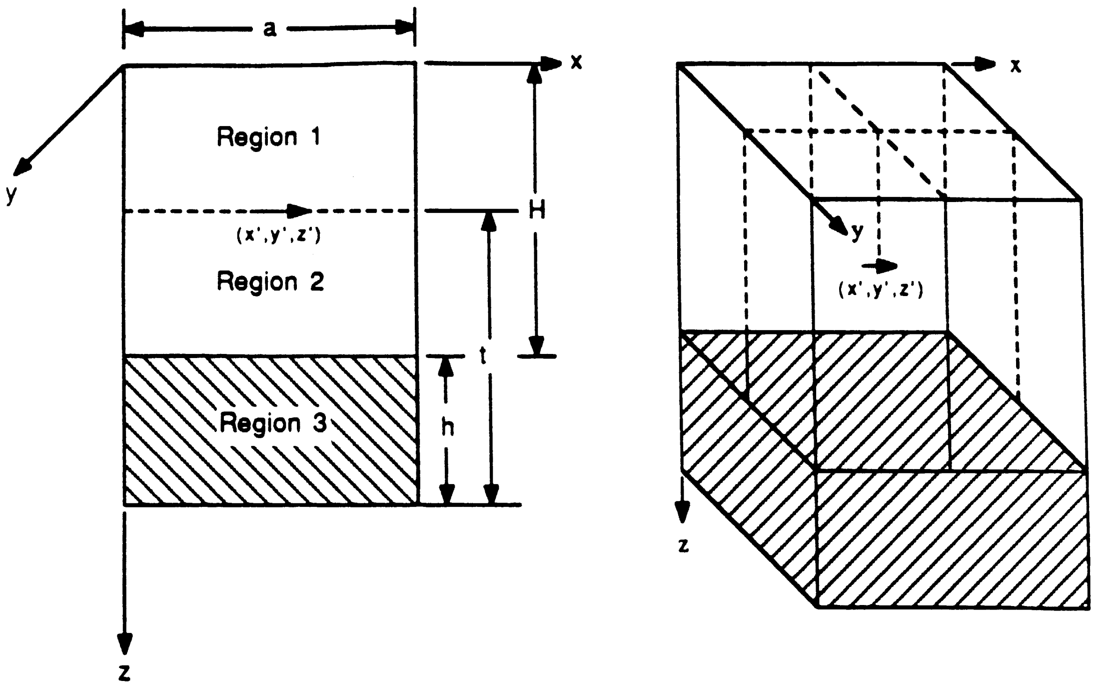



Figure 2: x-directed current above dielectric in enclosed microstrip structure.

### 2.2.1 Derivation of the Green's function for a x-directed current

With equations (14)-(19) the Green's function can be related to the electric and magnetic fields which have conceivably defined values dictated by the structures electrical characteristics. We begin by stating that the tangential electric fields are zero everywhere on the surface of the walls of the structure;

$$E^{i}_{y,z} = 0 \; at \; x = 0, a \tag{20}$$

$$E^{i}_{x,z} = 0 \; at \; y = 0, b \tag{21}$$

$$E^{I}_{x,y} = 0 \; at \; z = 0 \tag{22}$$

$$E^{III}_{x,y} = 0 \; at \; z = c. \tag{23}$$

Employing separation of variables to the expressions and the established boundary conditions (20)-(23), the following general forms of the green's functions can be derived for each region of interest

$$G^{I}_{xx} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A^{I} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k^{I}_{z} z) \quad (24)$$

$$G^{I}_{zx} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B^{I} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k^{I}_{z} z) \quad (25)$$

$$G^{II}_{xx} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) (A^{II} sin(k^{II}_{z} z) + C^{II} cos(k^{II}_{z} z)) \quad (26)$$

$$G^{II}_{zx} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) (B^{II} sin(k^{II}_{z} z) + D^{II} cos(k^{II}_{z} z)) \quad (27)$$

$$G^{III}_{xx} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A^{III} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k^{III}_{z}(z-c)) \quad (28)$$

$$G^{III}_{zx} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B^{III} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k^{III}_{z}(z-c)). \quad (29)$$

In equations (24)-(29) the eigenvalues $k_x$, $k_y$, and $k_z$ satisfy the following relations:

$$(k^{i})^2 = (k^{i}_{x})^2 + (k^{i}_{y})^2 + (k^{i}_{z})^2$$
$$\text{where } k_x = \left(\frac{m\pi}{a}\right)$$
$$\text{and } k_y = \left(\frac{m\pi}{b}\right)$$
$$k^2 = \omega^2 \mu \epsilon. \quad (30)$$

To determine the unknown coefficients $A^{I,II,III}$ and $B^{I,II,III}$, one should apply boundary conditions at the two interfaces between Regions I, II, and III. Since at the boundary between Regions I and II there exists no magnetic charge, the normal magnetic field must be continuous across the boundary. Therefore $H_z^I = H_z^{II}$, so

$$A^I sin(k_z^I z') = A^{II} sin(k_z^{II} z') + C^{II} sin(k_z^{II} z') \tag{31}$$

Since there is no electric current in the y direction, the tangential magnetic field is continuous. By substituting this relation into (18), one obtains $H_x^I = H_x^{II}$, therefore

$$B^I cos(k_z^I z') = B^{II} sin(k_z^{II} z') + D^{II} cos(k_z^{II} z') \tag{32}$$

Also, since there is no magnetic current in the x direction, the tangential electric field is continuous. Therefore $E_y^I = E_y^{II}$;

$$A^I sin(k_z^I z') - B^I sin(k_z^I z') =$$
$$B^{II} cos(k_z^{II} z') - D^{II} sin(k_z^{II} z') + A^{II} sin(k_z^{II} z') + C^{II} sin(k_z^{II} z') \tag{33}$$

By applying similar boundary conditions on the interface between Region II and III, the following equations result;

$$A^{II} sin(k_z^{II} H) + C^{II} cos(k_z^{II} H) = A^{III} sin(k_z^{III}(H - c)) \tag{34}$$

$$B^{II} sin(k_z^{II} H) + D^{II} cos(k_z^{II} H) = B^{III} cos(k_z^{III}(H - c)) \tag{35}$$

$$k_z^{II}(A^{II} cos(k_z^{II} H) - C^{II} sin(k_z^{II} H)) = A^{III} k_z^{III} cos(k_z^{III}(H - c)) \tag{36}$$

$$\begin{aligned} & A^{II} sin(k_z^{II} H) + C^{II} cos(k_z^{II} H) \\ & + \frac{k_z^{II}}{-(\frac{n\pi}{a})} \left(B^{II} cos(k_z^{II} H) - D^{II} sin(k_z^{II} H)\right) \\ & = \frac{1}{\epsilon_{III}} \left(A^{III} sin(k_z^{III}(H - c)) + \frac{B^{III} k_z^{III}}{(\frac{n\pi}{a})} sin(k_z^{III}(H - c))\right). \end{aligned} \tag{37}$$

Since there does exist an electric current between Regions I and II, the magnetic field between these two boundaries is discontinuous. This discontinunity can be considered by integrating the inhomogenous helmholtz expression over the boundary and using orthogonality. This results in

$$\frac{ab}{\rho}\left(k_z^I A^I cos(k_z^I z') - k_z^{II} A^{II} cos(k_z^{II} z') + k_z^{II} C^{II} sin(k_z^{II} z')\right)$$
$$= -cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \qquad (38)$$

$$\text{where } \rho = \begin{cases} 4 & \text{when } n \neq 0 \\ 2 & \text{when } n = 0 \end{cases}$$

Using these eight relations derived from the boundary conditions, the eight unknown constants, $A^I$, $B^I$, $A^{II}$, $B^{II}$, $C^{III}$, $D^{II}$, $A^{III}$, $B^{III}$ are found. The resulting Green's functions for a x-directed current above the dielectric upon simplification are

$$G_{xx}^I = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho\left[\frac{k_z^{III} - k_z tan(k_z^{III}(H-c))tan(k_z H)}{k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))} - cot(k_z H)\right]}{abk_z[cot(k_z z')cos(k_z z') - sin(k_z z')]}$$
$$cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z^I z) \qquad (39)$$

$$G_{zx}^I = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho(1 - tan^2(k_z H))\left(\frac{n\pi}{a}\right)(\epsilon_r^{III} - 1)tan(k_z^{III}(H-c))}{ab[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))][cot(k_z z')cos(k_z z') - sin(k_z z')]}$$
$$cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \frac{1}{k_z^{III} tan(k_z^{III}(H-c)) - \epsilon_r^{III} k_z tan(k_z H)}$$

$$sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^I z) \tag{40}$$

$$\begin{aligned} G_{xx}^{II} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho\left[k_z^{III} - k_z tan(k_z^{III}(H-c))tan(k_z H)\right]}{abk_z[cot(k_z z')cos(k_z z') - sin(k_z z')][k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))]} \\ & cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z^{II} z) cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) + \\ & \frac{-\rho}{abk_z[cot(k_z z')cos(k_z z') - sin(k_z z')]} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^{II} z) \\ & cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \end{aligned} \tag{41}$$

$$\begin{aligned} G_{zx}^{II} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho\,(1 - tan^2(k_z H))\left(\frac{n\pi}{a}\right)(\epsilon_r^{III} - 1) tan(k_z^{III}(H-c))}{ab[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))][cot(k_z z')cos(k_z z') - sin(k_z z')]} \\ & \frac{1}{[k_z^{III} tan(k_z^{III}(H-c)) - \epsilon_r^{III} tan(k_z H)]} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^{II} z) \\ & cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \end{aligned} \tag{42}$$

$$\begin{aligned} G_{xx}^{III} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho\,(1 - tan^2(k_z H)) cos(k_z H)}{ab[cot(k_z z')cos(k_z z') - sin(k_z z')]cos(k_z^{III}(H-c))} \\ & \frac{1}{[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))]} \\ & cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z^{III}(z-c)) cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \end{aligned} \tag{43}$$

$$\begin{aligned} G_{zx}^{III} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho(1 - tan^2(k_z H))\left(\frac{n\pi}{a}\right)(\epsilon_r^{III} - 1) tan(k_z^{III}(H-c))}{ab[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))][cot(k_z z')cos(k_z z') - sin(k_z z')]} \\ & \frac{sec(k_z^{III}(H-c))}{[k_z^{III} tan(k_z(H-c)) - \epsilon_r^{III} tan(k_z^{III} H)]} \\ & sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^{III}(z-c)) cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \end{aligned} \tag{44}$$

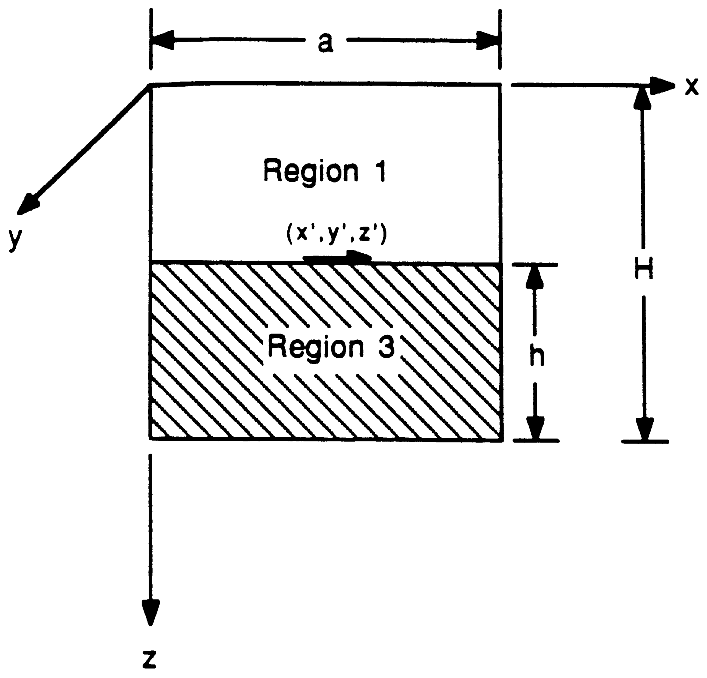



Figure 3: Structure when conducting strip is lowered on to the dielectric.

If the conducting strip is lowered on to the dielectric, we will have only two regions; air representing Region I and dielectric representing Region III.

To formulate the following Green's fuctions for the strip on the dielectric , one simply lets $z' = H$ in the previous expressions.

$$G^I_{xx} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho\, tan(k_z^{III}(H-c))}{ab(cos(k_z H))[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))]} cos\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z^I z) \quad (45)$$

$$G^I_{zx} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\rho\, tan(k_z H)\left(\frac{n\pi}{a}\right)(\epsilon_r^{III}-1) tan(k_z^{III}(H-c))}{ab(cos(k_z H))[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))]}$$

$$\frac{1}{[k_z^{III}tan(k_z^{III}(H-c))-\epsilon_r^{III}k_z tan(k_z H)]}cos\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right)$$
$$sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)cos(k_z^I z) \quad (46)$$

$$G_{xx}^{III} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\rho\, tan(k_z H)}{abcos(k_z^{III}(H-c))[k_z^{III}tan(k_z H)-k_z tan(k_z^{III}(H-c))]}$$
$$cos\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)sin(k_z^{III}(z-c))cos\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right) \quad (47)$$

$$G_{zx}^{III} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\rho(\epsilon_r^{III}-1)tan(k_z^{III}(H-c)tan(k_z H)\left(\frac{n\pi}{a}\right)}{abcos(k_z(H-c))[k_z^{III}tan(k_z H)-k_z tan(k_z^{III}(H-c))]}$$
$$\frac{1}{[k_z^{III}tan(k_z^{III}(H-c))-\epsilon_r^{III}k_z tan(k_z H)]}sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)cos(k_z^{III}(z-c))$$
$$cos\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right) \quad (48)$$

### 2.2.2 Derivation of the Green's function for a z-directed current.

The structure used to consider the z-directed current is pictured in Fig.4. We will simplify our problem separating it into two parts; a primary field problem and a secondary field problem as shown in Fig.5. For a z-directed current only one component of the magnetic vector potential is needed to satisfy the boundary conditions;

$$\overline{A} = \hat{z}Az \quad (49)$$

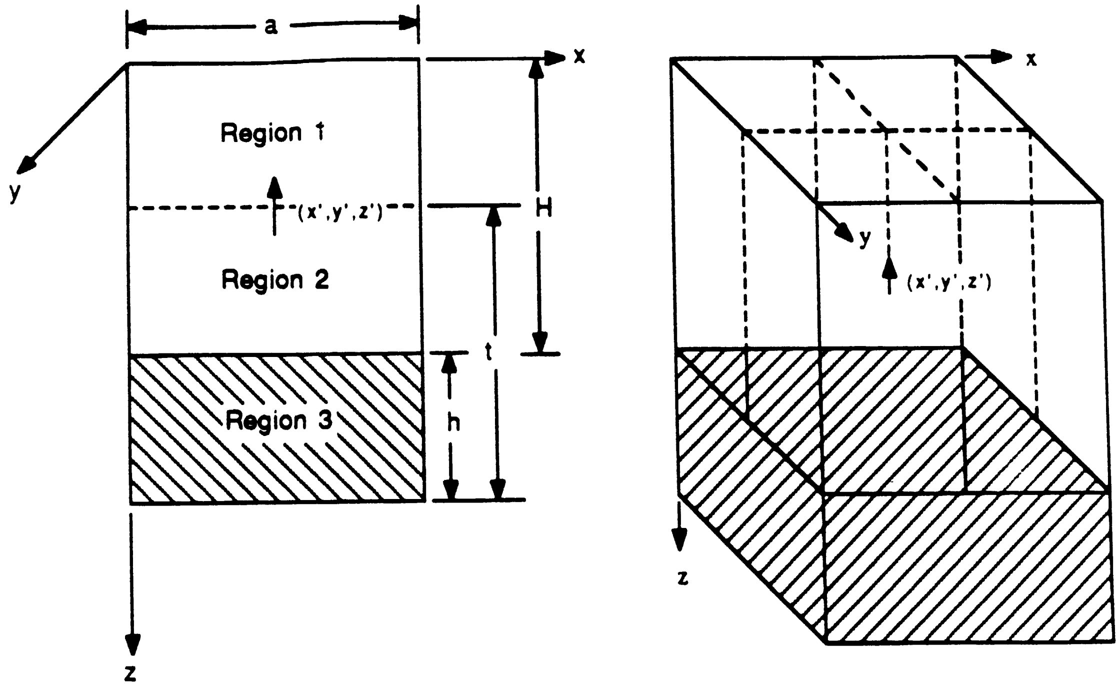



Figure 4: A z-directed current above a dielectric substrate.

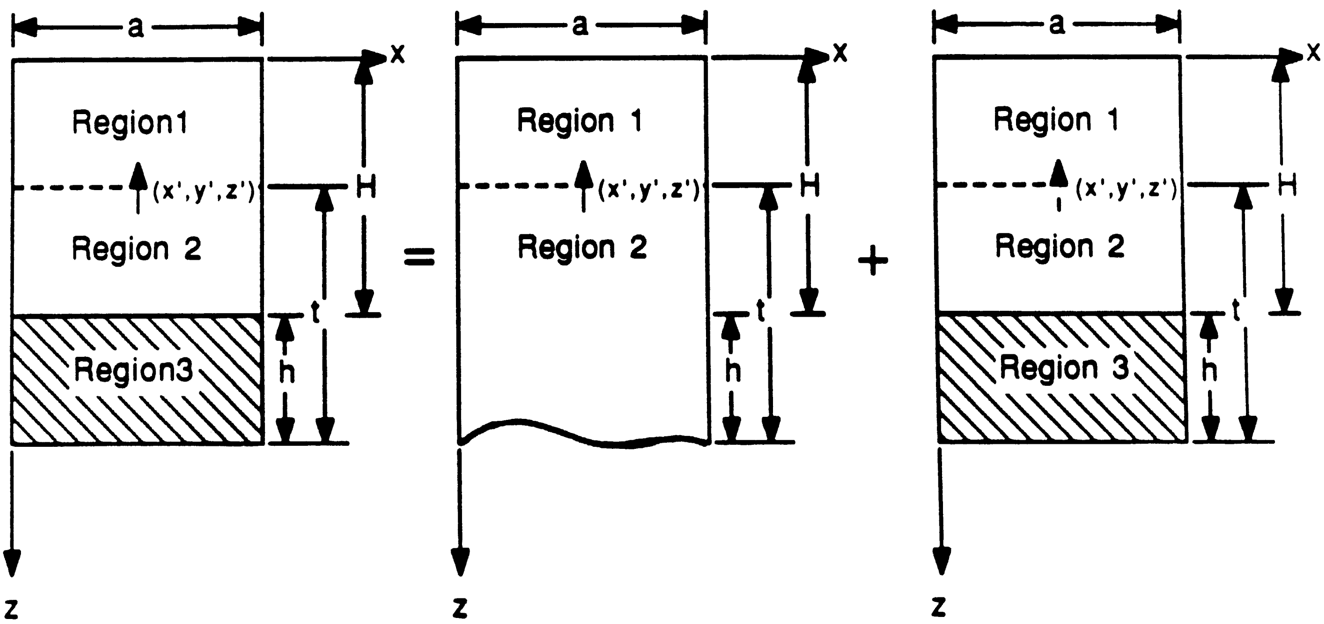



Figure 5: The problem is divided into two parts, a primary and secondary problem.

where

$$A_z^I = A_{zp}^I + A_{zs}^I \tag{50}$$

$$A_z^{II} = A_{zp}^{II} + A_{zs}^{II} \tag{51}$$

$$A_z^{III} = A_{zs}^{III}. \tag{52}$$

From our definition of Magnetic vector potentials,

$$\overline{H} = \frac{\overline{\nabla} \times \overline{A}}{\mu} \tag{53}$$

and from Maxwell's equations,

$$\overline{E} = \frac{\nabla \times \overline{H}}{j\omega\epsilon} = \frac{1}{j\omega\epsilon\mu} \nabla \times (\nabla \times \overline{A}) = \frac{1}{j\omega\epsilon\mu}(k^2\overline{A} + \overline{\nabla\nabla} \cdot \overline{A}) \tag{54}$$

therefore, for the z-directed current,

$$E_z = \frac{1}{j\omega\mu\epsilon}\left(k^2 + \frac{\delta^2}{\delta z^2}\right) A_z \tag{55}$$

$$E_y = \frac{1}{j\omega\mu\epsilon}\left(\frac{\delta^2 A_z}{\delta y \delta z}\right) \tag{56}$$

$$E_x = \frac{1}{j\omega\mu\epsilon}\left(\frac{\delta^2 A_z}{\delta x \delta z}\right) \tag{57}$$

$$H_z = 0 \tag{58}$$

$$H_y = \frac{1}{\mu}\left(\frac{-\delta A_z}{\delta x}\right) \tag{59}$$

$$H_x = \frac{1}{\mu}\left(\frac{\delta A_z}{\delta y}\right). \tag{60}$$

Now we must apply the boundary conditions of the structure to obtain the general (primary) solution. The first readily known boundary conditions are those on the walls where the tangential electric fields become zero. Therefore

$$E_x^I = 0 \; at \; z = 0 \; y = 0, b \tag{61}$$

$$E_x^{II} = 0 \ at \ y = 0, b \tag{62}$$

$$E_y^I = 0 \ at \ z = 0; \ x = 0, a \tag{63}$$

$$E_y^{II} = 0 \ at \ x = 0, a \tag{64}$$

$$E_z^I = 0 \ at \ x = 0, a \ y = 0, b \tag{65}$$

$$E_z^{II} = 0 \ at \ x = 0, a \ y = 0, b \tag{66}$$

In the primary field problem, the inhomogenous differential equation takes on the form

$$\nabla^2 A_z + k_o^2 A_z = -\mu \overline{J}. \tag{67}$$

The solution to the above equation when the listed boundary conditions are satisfied is of the form

$$A_{zp}^I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A^I sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^I z) \tag{68}$$

$$A_{zp}^{II} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A^{II} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) e^{-jk_z^{II}(z-z')} \tag{69}$$

In the secondary field problem, we will derive a solution that satisfies the homogenous differential equation. This is due to the fact that we do not have a current source in the secondary field problem. In both field problems, the electric fields must satisfy the same boundary conditions on the conducting walls. As a result, the secondary fields are of the same dependence with respect to the x and y coordinates, therefore

$$A_{zs}^{I} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f^{I}\left(j\frac{4\mu}{abk_z}\right) sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right)$$
$$sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z z) \quad (70)$$

$$A_{zs}^{II} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (f_s^{II} sin(k_z z) + f_c^{II} cos(k_z z))\left(j\frac{4\mu}{abk_z}e^{jk_z z'}\right)$$
$$sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) cos(k_z z')$$
$$sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) \quad (71)$$

For Region III we have the same standing wave solution as in Region 1 except for the fact that the conducting wall has been moved by (H+h) along the z-axis as reflected in our z dependence below

$$A_{zs}^{III} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} j\frac{4\mu}{abk_z^{III}}e^{jk_z z'} f^{III} sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right)$$
$$sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^{III}(z-(H+h))). \quad (72)$$

In the case of a delta function source, $A_{zp}$ will give a $G_{zzp}$ component in the dyadic Green's function and similarly, a $A_{zs}$ will give a $G_{zzs}$. These two components are related to $G_{zz}$ by the relation

$$G_{zz} = G_{zzp} + G_{zzs} \quad (73)$$

Since we have a current source between Regions I and II, our magnetic field is discontinuous so we must integrate the inhomogenous helmholtz equation over the interface which results in

$$\lim_{\alpha\to 0}\int_{z'-\alpha}^{z'+\alpha}(\nabla^2+k_o^2)A_z dz = -\mu\lim_{\alpha\to 0}\int_{z'-\alpha}^{z'+\alpha}\delta(x-x')\delta(y-y')\delta(z-z')dz, \quad (74)$$

and upon simplification, one obtains

$$\lim_{\alpha\to 0}\left(\frac{\delta}{\delta z}A_z \mid_{z'-\alpha}^{z'+\alpha}\right) = -\mu\delta(x-x')\delta(y-y') \quad (75)$$

$$\frac{\delta A_z^{II}}{\delta z}\mid_{z'} - \frac{\delta A_z^{I}}{\delta z}\mid_{z'} = \mu\delta(x-x')\delta(y-y') \quad (76)$$

from (25), one obtains

$$\frac{ab}{4}\left(k_z^I sin(k_z^I z')A_z^I - jk_z^{II}A_z^{II}\right) = sin\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right). \quad (77)$$

One more equation is needed to solve for the two unknowns $A_p^I$, and $A_p^{II}$. Utilizing the fact that the E field in the z- direction must be continuous at the boundary between regions I and II; one obtains

$$A^I cos(k_z z') = A^{II}. \quad (78)$$

These equations are solved for the unknown coefficients resulting in

$$A_p^I = j\frac{4\mu}{abk_z}e^{jk_z z'} sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \quad (79)$$

$$A_p^{II} = j\frac{4\mu}{abk_z}e^{jk_z z'} sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) cos(k_z z'). \quad (80)$$

By substituting these expressions into our general forms (68) and (69), the zz-component of the Green's function takes the form;

$$\begin{aligned} G_{zzp}^I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} j\frac{4\mu}{abk_z}e^{jk_z z'} sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) \\ & \quad sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z z) \end{aligned} \quad (81)$$

$$G_{zzp}^{II} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} j\frac{4\mu}{abk_z}e^{jk_z z'} sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) cos(k_z z')$$
$$sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) e^{-jk_z^{II}(z-z')} \quad (82)$$

In order to determine the four unknowns of the secondary field problem, boundary conditions on the air-dielectric must be applied to the structure of Fig.5c.

First, the z-component of the electric field is continuous across the boundary between regions I and II; $E_z^I = E_z^{II}$. Therefore

$$f^I cos(k_z z') - f_s^{II} sin(k_z z') - f_c^{II} cos(k_z z') = 0 \quad (83)$$

Integrating the homogenous helmholtz equation (there is no electric source in the secondary problem) across the boundary and using orthogonality, one obtains

$$f_s^{II} cos(k_z z') - f_c^{II} sin(k_z z') + f^I sin(k_z z') = 0. \quad (84)$$

From (83) and (84), we conclude that

$$f_s^{II} = 0 \quad (85)$$

and

$$f_I = f_c^{II}. \quad (86)$$

From the boundary conditions at the dielectric interface, $E_x^{II} = E_x^{III}$ one obtains

$$je^{-jk_z H} cos(k_z z') + f_c^{II} sin(k_z H) = \frac{-f^{III} sin(k_z^{III} h)}{\epsilon_r^{III}\mu_r^{III}} \quad (87)$$

and from $H_x^{II} = H_x^{III}$,

$$\frac{e^{-jk_z H} cos(k_z z') + f_c^{II} cos(k_z H)}{k_z} = \frac{1}{u_r^{III}} f^{III} \frac{cos(k_z^{III} h)}{k_z^{III}}. \quad (88)$$

The resultant Green's function for this problem will have a zz component only, which is given by

$$G^I_{zz} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(e^{jk_z z'} + f^I)j\frac{4\mu}{abk_z}sin\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right)$$
$$sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)cos(k_z z) \quad (89)$$

$$G^{II}_{zz} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} j\frac{4\mu}{abk_z}e^{jk_z z'}(e^{-jk_z z}cos(k_z z) + f^{II}_s sin(k_z z) + f^{II}_c cos(k_z z))$$
$$sin\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right)$$
$$sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right) \quad (90)$$

$$G^{III}_{zz} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} j\frac{4\mu}{abk^{III}_z}f^{III}sin\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right)$$
$$sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)cos(k^{III}_z(z-(H+h))). \quad (91)$$

After applying all the necessary boundary conditions, the Green's function takes on the following form

$$G^I_{zz} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\epsilon^{III}_r k_z cos(k_z(z'-H))cos(k^{III}_z h) + k^{III}_z sin(k_z(z'-H))sin(k^{III}_z h)}{k^2_z\epsilon^{III}_r sin(k_z H)cos(k^{III}_z h) + k^{III}_z k_z sin(k^{III}_z h)cos(k_z H)}$$
$$\frac{4\mu}{ab}sin\left(\frac{n\pi x'}{a}\right)sin\left(\frac{m\pi y'}{b}\right)$$
$$sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)cos(k_z z) \quad (92)$$

$$G^{II}_{zz} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\epsilon_r^{III} k_z cos(k_z(z-H))cos(k_z^{III}h) + k_z^{III} sin(k_z(z-H))sin(k_z^{III}h)}{k_z^2 \epsilon_r^{III} sin(k_z H)cos(k_z^{III}h) + k_z^{III} k_z sin(k_z^{III}h)cos(k_z H)}$$
$$\frac{4\mu}{ab} sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) cos(k_z z')$$
$$sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) \quad (93)$$

$$G^{III}_{zz} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\epsilon_r^{III} k_z \mu_r^{III} cos(k_z z')}{k_z^2 \epsilon_r^{III} sin(k_z H)cos(k_z^{III}h) + k_z^{III} sin(k_z^{III}h)cos(k_z H)}$$
$$\frac{4\mu}{ab} sin\left(\frac{n\pi x'}{a}\right) sin\left(\frac{m\pi y'}{b}\right) cos(k_z z')$$
$$sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^{III}(z-(h+H))) \quad (94)$$

#### 2.2.3 Summary of Green's function determination

In this chapter we have determined the unique Green's function for the air-bridge structure (Fig. 1). This was accomplished by working with Maxwell's equations to establish a tractable equation and by representation of our source as dirac delta functions. We then applied boundary conditions to solve for the unknown coefficients and formulated a solution. We now have expressions that will give the resulting field produced by a point source directed in the x or z direction as required to analyze the air-bridge structure.

### 2.3 Application of Method of Moments

The method of moments is a numerical technique used for solving functional equations which cannot be solved in closed form. By reducing the functional relation to a matrix equation, known techniques can be used to solve the resulting matrix equation. This method is computationally intensive but with the advent of faster computers, the method has become feasible.

To apply the method of moments in the specific case of an air-bridge, one should follow the steps outlined below:

1. Use the integral equation (11) derived in section 1 along with the relations (13) and (5) so one obtains an integral equation that relates the current to the electric and magnetic fields respectively. A general form of this is

$$L_{op}(\overline{J}_s) = \overline{g} \begin{pmatrix} \overline{E} \\ \overline{H} \end{pmatrix} \tag{95}$$

where $L_{op}$ is an integral operator operating along with the derived Green's function and $\overline{g}$ is a vector function of either the electric field $\overline{E}$ or magnetic field $\overline{H}$.

2. Represent the current on the conducting strip as a sum of coefficients multiplied by a pre-determined basis function,

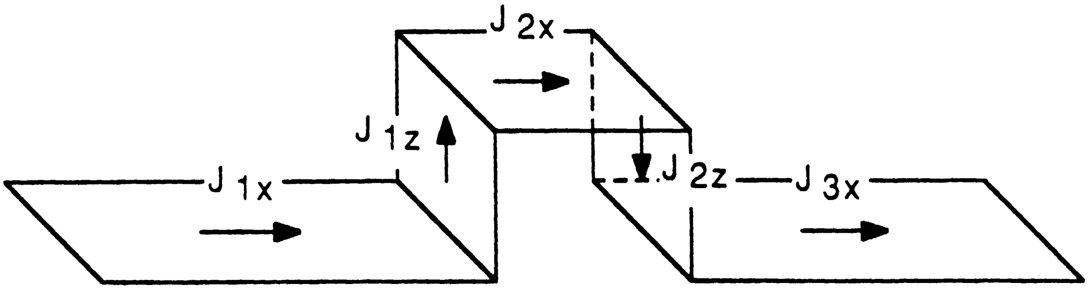
$$\overline{J}_s = \sum_{q=1}^{Nia} I_q \overline{J}_q \tag{96}$$

In equation (96) where $I_q$ represents the complex coefficients, Nia represents the number of sections the strip is to be divided into and $\overline{J}_q$ represents the chosen basis functions which represent the current distribution.

3. Discretize the integral equation by minimizing the resuting error function $\delta E$ on the surface of the conducting strips.

In applying the above steps to the problem of an air-bridge we have set up the integral equation for the electric field of the form

$$\overline{E}^i = -j\omega\mu \int\int\int_V \left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right) \cdot \overline{\overline{G}}^i \cdot \overline{J} dV. \tag{97}$$




a. One dimensional view highlighting current directions

Figure 6: Currents are assigned variables by direction.

In proceeding to the second step, we seperate the problem into five different sections as pictured below (this figure is taken in part from fig.1).

For $J_{1x}$ and $J_{3x}$ we will model the current as a sum of an incident current A, a reflected current B, and the sum of incremental currents $I_q$. For $J_{2x}$, $J_{1z}$, and $J_{2z}$ only a sum of incremental currents is required. Implementing this convention results in

$$J_{1x} = A_{1x}e^{jkx} + B_{1x}e^{-jkx} + \sum_{q=1}^{N1x} I_{q1x}\overline{J}_{1xq} \tag{98}$$

$$J_{2x} = \sum_{q=1}^{N2x} I_{q2x}\overline{J}_{2xq} \tag{99}$$

$$J_{3x} = A_{3x}e^{jkx} + B_{3x}e^{-jkx} + \sum_{q=1}^{N3x} I_{q3x}\overline{J}_{3xq} \tag{100}$$

$$J_{1z} = \sum_{q=1}^{N1z} I_{q1z}\overline{J}_{1zq} \tag{101}$$

$$J_{2z} = \sum_{q=1}^{N2z} I_{q2z}\overline{J}_{2zq}. \tag{102}$$

The basis functions the x-directed currents in this case are the same and are defined as being

$$J_{ixq}(x', y') = \begin{cases} \frac{sin(k(x'-x_{q-1}))}{sin(kl_x)} & x_{q-1} \leq x' \leq x_q \\ \frac{sin(k(x_{q+1}-x'))}{sin(kl_x)} & x_q \leq x' \leq x_{q+1} \\ \left(\frac{2}{\pi w}\right)\frac{1}{\left(1-\left(\frac{2y'}{w}\right)^2\right)^{\frac{1}{2}}} & -\frac{w}{2} \leq y' \leq \frac{w}{2} \end{cases} \tag{103}$$

where for the y-direction we have used a maxwellian distribution function.

The first step in discretizing the integral equation (97) is to evaluate the error that our mathematical representation of the electric field will produce. This is done by evaluating the electric field produced by one section of current on the conductor on another section of the conductor. Since the electric field on any part of the strip must be zero, the value of our integral evaluates the error. Then, by the concept of least square estimation, when one takes the inner product of the basis function along with the error and sets the result to zero, the error is minimized.

Proceeding to do this for our problem here we first account for the different components of the fields produced by a current of given orientation.

$$E_x = E_{1xx} + E_{2xx} + E_{3xx} + E_{1xz} + E_{2xz} \tag{104}$$

and

$$E_z = E_{1zx} + E_{2zx} + E_{3zx} + E_{1zz} + E_{2zz}. \qquad (105)$$

First, we shall consider the x-directed field produced by a x-directed current on the dielectric

$$E_{1xx} = A_{1xx} \int_{-\frac{a}{2}}^{\frac{a}{2}} cos\left(\frac{n\pi}{a}x'\right) e^{-jkx'}dx' \int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\left[\frac{\rho\, tan(k_z^{III}(H-c))}{ab(cos(k_zH))[k_z^{III}tan(k_zH) - k_z tan(k_z^{III}(H-c))]}\right.$$

$$\left(k^2 - \left(\frac{n\pi}{a}\right)^2\right) cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi x}{b}\right) sin(k_z z) +$$

$$\frac{\rho\, tan(k_zH)\left(\frac{n\pi}{a}\right)(\epsilon_r^{III}-1)tan(k_z^{III}(H-c))}{abcos(k_zH)[k_z^{III}tan(k_zH) - k_z tan(k_z^{III}(H-c))]}$$

$$\left.\frac{1}{[k_z^{III}tan(k_z^{III}(H-c)) - \epsilon_r^{III}k_z tan(k_zH)]} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi x}{b}\right) cos(k_z z)\right] +$$

$$R_{1xx} \int_{-\frac{a}{2}}^{\frac{a}{2}} cos\left(\frac{n\pi}{a}x'\right) e^{jkx'}dx' \int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\left[\frac{\rho\, tan(k_z^{III}(H-c))}{ab(cos(k_zH))[k_z^{III}tan(k_zH) - k_z tan(k_z^{III}(H-c))]}\right.$$

$$\left(k^2 - \left(\frac{n\pi}{a}\right)^2\right) cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi x}{b}\right) sin(k_z z) +$$

$$\frac{\rho\, tan(k_z H) \left(\frac{n\pi}{a}\right) (\epsilon_r^{III} - 1) tan(k_z^{III}(H - c))}{abcos(k_z H)[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))]}$$

$$\left.\frac{1}{[k_z^{III} tan(k_z^{III}(H - c)) - \epsilon_r^{III} k_z tan(k_z H)]} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi x}{b}\right) cos(k_z z)\right] +$$

$$\sum_{q=1}^{N_{1x}} I_{q1x} \left( \int_{-\frac{x}{2}}^{0} \frac{sin(k(x' - x_{q-1}))}{sin(kl_x)} cos\left(\frac{n\pi}{a} x'\right) dx' + \right.$$

$$\left. \int_{0}^{\frac{x}{2}} \frac{sin(k(x_{q+1} - x'))}{sin(kl_x)} cos\left(\frac{n\pi}{a} x'\right) dx' \right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1 - \left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b} y'\right) dy'$$

$$\left[ \left( k^2 - \left(\frac{n\pi}{a}\right)^2 \right) \frac{\rho\, tan(k_z^{III}(H - c))}{ab(cos(k_z H))[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))]} \right.$$

$$cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi x}{b}\right) sin(k_z z) +$$

$$\frac{\rho\, tan(k_z H) \left(\frac{n\pi}{a}\right) (\epsilon_r^{III} - 1) tan(k_z^{III}(H - c))}{abcos(k_z H)[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))]}$$

$$\frac{1}{[k_z^{III} tan(k_z^{III}(H - c)) - \epsilon_r^{III} k_z tan(k_z H)]}$$

$$\left. \left(\frac{n\pi}{a}\right) k_z cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z z) \right] \quad (106)$$

~~Integrating this expression over the surface of the conducting microstrip where~~

the x-component of the electric field can exist, one obtains

For a x-directed current, a z-directed field is also produced. Proceeding with the same procedure as above one obtains

$$E_{1zx} = A_{1zx} \int_{-\frac{a}{2}}^{\frac{a}{2}} cos\left(\frac{n\pi}{a}x'\right) e^{-jkx'} dx' \int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\left[\frac{\rho\, tan(k_z^{III}(H-c))}{ab(cos(k_zH))[k_z^{III}tan(k_zH) - k_z tan(k_z^{III}(H-c))]}\right.$$

$$\left(\frac{-n\pi}{a}\right) sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) k_z cos(k_z z) +$$

$$\frac{\rho\, tan(k_zH)\left(\frac{n\pi}{a}\right)(\epsilon_r^{III}-1)tan(k_z^{III}(H-c))}{abcos(k_zH)[k_z^{III}tan(k_zH) - k_z tan(k_z^{III}(H-c))]}$$

$$\frac{1}{[k_z^{III}tan(k_z^{III}(H-c)) - \epsilon_r^{III}k_z tan(k_zH)]}$$

$$\left.(k^2-k_z^2)sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z z)\right] +$$

$$R_{1zx} \int_{-\frac{a}{2}}^{\frac{a}{2}} cos\left(\frac{n\pi}{a}x'\right) e^{jkx'} dx' \int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\left[\frac{\rho\, tan(k_z^{III}(H-c))}{ab(cos(k_zH))[k_z^{III}tan(k_zH) - k_z tan(k_z^{III}(H-c))]}\right.$$

$$\left(\frac{-n\pi}{a}\right) sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) k_z cos(k_z z) +$$

$$\frac{\rho \, tan(k_z H) \left(\frac{n\pi}{a}\right) (\epsilon_r^{III} - 1) tan(k_z^{III}(H - c))}{abcos(k_z H)[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))]}$$

$$\frac{1}{[k_z^{III} tan(k_z^{III}(H - c)) - \epsilon_r^{III} k_z tan(k_z H)]}$$

$$(k^2 - k_z^2) sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z z)\Big] +$$

$$\sum_{q=1}^{N_{1x}} I_{q1x} \left( \int_{-\frac{x}{2}}^{0} \frac{sin(k(x' - x_{q-1}))}{sin(kl_x)} cos\left(\frac{n\pi}{a} x'\right) dx' + \right.$$

$$\left. \int_0^{\frac{x}{2}} \frac{sin(k(x_{q+1} - x'))}{sin(kl_x)} cos\left(\frac{n\pi}{a} x'\right) dx' \right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1 - \left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b} y'\right) dy'$$

$$\left[ \frac{\rho \, tan(k_z^{III}(H - c))}{ab(cos(k_z H))[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))]} \right.$$

$$\left(\frac{-n\pi}{a}\right) sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) k_z cos(k_z z) +$$

$$\frac{\rho \, tan(k_z H) \left(\frac{n\pi}{a}\right) (\epsilon_r^{III} - 1) tan(k_z^{III}(H - c))}{abcos(k_z H)[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))]}$$

$$\frac{1}{[k_z^{III} tan(k_z^{III}(H - c)) - \epsilon_r^{III} k_z tan(k_z H)]}$$

$$(k^2 - k_z^2) sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z z)\Big] \tag{107}$$

For a z-directed current, one must consider the fields above and below the source seprately as they have different forms in those two respective regions. Also, as before, both x- and z-components of the electric field are produced.

First we consider the x-directed field produced by a z-directed current above the source where

$$E_{1axz} = \sum_{q=1}^{N_{1z}} I_{q1z} \left( \int_{\frac{z}{2}}^{0} \frac{sin(k(z' - z_{q-1}))}{sin(kl_z)} (\epsilon_r^{III} k_z cos(k_z(z' - H)) cos(k_z^{III} h) + \right.$$

$$k_z^{III} sin(k_z(z' - H)) sin(k_z^{III} h) dz') +$$

$$\int_0^{-\frac{z}{2}} \frac{sin(k(z_{q+1} - z'))}{sin(kl_z)} (\epsilon_r^{III} k_z cos(k_z(z' - H)) cos(k_z^{III} h) +$$

$$\left. k_z^{III} sin(k_z(z' - H)) sin(k_z^{III} h) dz') \right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1 - \left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b} y'\right) dy'$$

$$\frac{4\mu}{ab} \frac{1}{\epsilon_r^{III} k_z^2 sin(k_z H) cos(k_z^{III} h) + k_z^{III} k_z cos(k_z H) sin(k_z^{III} h)} sin\left(\frac{n\pi}{a} x'\right)$$

$$\left(\frac{n\pi}{a}\right) cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) (-k_z) sin(k_z z) \tag{108}$$

For the x-directed field below the source produced by a z-directed current, one obtains

$$E_{1bxz} = \sum_{q=1}^{N_{1z}} I_{q1z} \left( \int_{\frac{l}{2}}^{0} \frac{sin(k(z' - z_{q-1}))}{sin(kl_z)} cos(k_z z')dz' + \int_{0}^{-\frac{l}{2}} \frac{sin(k(z_{q+1} - z'))}{sin(kl_z)} cos(k_z z')dz' \right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1 - \left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\frac{4\mu}{ab} \frac{\epsilon_r^{III} k_z sin(k_z(z-H))cos(k_z^{III}h) + k_z^{III} cos(k_z(z-H))sin(k_z^{III}h)}{k_z \epsilon_r^{III} sin(k_z H)cos(k_z^{III}h) + k_z^{III} sin(k_z^{III}h)cos(k_z H)} sin\left(\frac{n\pi}{a}x'\right)$$

$$\left(\frac{n\pi}{a}\right) cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) \tag{109}$$

For the z-directed field above the source produced by a z-directed current, one obtains

$$E_{1azz} = \sum_{q=1}^{N_{1z}} I_{q1z} \left( \int_{\frac{l}{2}}^{0} \frac{sin(k(z' - z_{q-1}))}{sin(kl_z)} (\epsilon_r^{III} k_z cos(k_z(z' - H))cos(k_z^{III}h) + \right.$$

$$k_z^{III} sin(k_z(z' - H))sin(k_z^{III}h)dz') +$$

$$\int_{0}^{-\frac{l}{2}} \frac{sin(k(z_{q+1} - z'))}{sin(kl_z)} (\epsilon_r^{III} k_z cos(k_z(z' - H))cos(k_z^{III}h) +$$

$$\left. k_z^{III} sin(k_z(z' - H))sin(k_z^{III}h)dz') \right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1 - \left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\frac{4\mu}{ab}\frac{1}{\epsilon_r^{III}k_z^2 sin(k_zH)cos(k_z^{III}h)+k_z^{III}k_z cos(k_zH)sin(k_z^{III}h)}sin\left(\frac{n\pi}{a}x'\right)$$

$$(k^2-k_z^2)sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)cos(k_zz) \tag{110}$$

For the z-directed field below the source produced by a z-directed current, one obtains

$$E_{1bzz} = \sum_{q=1}^{N_{1z}} I_{q1z}\left(\int_{\frac{z}{2}}^{0}\frac{sin(k(z'-z_{q-1}))}{sin(kl_z)}cos(k_zz')dz' + \int_0^{-\frac{z}{2}}\frac{sin(k(z_{q+1}-z'))}{sin(kl_z)}cos(k_zz')dz'\right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}}\left(\frac{2}{\pi w}\right)\frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}}sin\left(\frac{m\pi}{b}y'\right)dy'$$

$$\frac{4\mu}{ab}sin\left(\frac{n\pi}{a}x'\right)$$

$$\frac{\epsilon_r^{III}k_z sin(k_z(z-H))cos(k_z^{III}h)+k_z^{III}cos(k_z(z-H))sin(k_z^{III}h)}{k_z\epsilon_r^{III}sin(k_zH)cos(k_z^{III}h)+k_z^{III}sin(k_z^{III}h)cos(k_zH)}$$

$$(k^2-k_z^2)sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right) \tag{111}$$

For a x-directed field in the region above a x-directed current which is above the dielectric, one obtains

$$E_{1axxa} = \sum_{q=1}^{N_{2x}} I_{q2x}\left(\int_{-\frac{x}{2}}^{0}\frac{sin(k(x'-x_{q-1}))}{sin(kl_x)}cos\left(\frac{n\pi}{a}x'\right)dx' +\right.$$

$$\left.\int_0^{\frac{x}{2}}\frac{sin(k(x_{q+1}-x'))}{sin(kl_x)}cos\left(\frac{n\pi}{a}x'\right)dx'\right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\left[\left(k^2-\left(\frac{n\pi}{a}\right)^2\right) \frac{\rho\left[\frac{k_z^{III}-k_z tan(k_z^{III}(H-c))tan(k_z H)}{k_z^{III} tan(k_z H)-k_z tan(k_z^{III}(H-c))} - cot(k_z H)\right]}{abk_z[cot(k_z z')cos(k_z z') - sin(k_z z')]}\right.$$

$$cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi x}{b}\right) sin(k_z z) +$$

$$\frac{\rho(1-tan^2(k_z H))\left(\frac{n\pi}{a}\right)(\epsilon_r^{III}-1)tan(k_z^{III}(H-c))}{ab[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))][cot(k_z z')cos(k_z z') - sin(k_z z')]}$$

$$\frac{1}{k_z^{III} tan(k_z^{III}(H-c)) - \epsilon_r^{III} k_z tan(k_z H)}$$

$$\left.\left(\frac{n\pi}{a}\right) k_z cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right)(-k_z sin(k_z z)\right] \tag{112}$$

For the x-directed field located below the x-directed current source which is above the dielectric, we obtain

$$E_{1bxxa} = \sum_{q=1}^{N_{2x}} I_{q2x}\left(\int_{-\frac{a}{2}}^{0} \frac{sin(k(x'-x_{q-1}))}{sin(kl_x)} cos\left(\frac{n\pi}{a}x'\right) dx' +\right.$$

$$\left.\int_0^{\frac{a}{2}} \frac{sin(k(x_{q+1}-x'))}{sin(kl_x)} cos\left(\frac{n\pi}{a}x'\right) dx'\right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b}y'\right) dy'$$

$$\left[\left(k^2-\left(\frac{n\pi}{a}\right)^2\right) \frac{\rho\left[k_z^{III} - k_z tan(k_z^{III}(H-c))tan(k_z H)\right]}{abk_z[cot(k_z z')cos(k_z z') - sin(k_z z')][k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H -}\right.$$

$$cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) sin(k_z^{II} z) +$$

$$\frac{-\rho}{abk_z[cot(k_z z')cos(k_z z') - sin(k_z z')]} cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z^{II} z) +$$

$$\frac{\rho(1 - tan^2(k_z H))\left(\frac{n\pi}{a}\right)(\epsilon_r^{III} - 1)tan(k_z^{III}(H - c))}{ab[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))][cot(k_z z')cos(k_z z') - sin(k_z z')]}$$

$$\frac{1}{k_z^{III} tan(k_z^{III}(H - c)) - \epsilon_r^{III} k_z tan(k_z H)}$$

$$\left.\left(\frac{n\pi}{a}\right) k_z cos\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right)(-k_z) sin(k_z z)\right]$$

For a z-directed field above the current produced from a x-directed current above the dielectric, one obtains the expression

$$E_{1azxa} = \sum_{q=1}^{N_{2x}} I_{q2x} \left( \int_{-\frac{a}{2}}^{0} \frac{sin(k(x' - x_{q-1}))}{sin(kl_x)} cos\left(\frac{n\pi}{a} x'\right) dx' + \right.$$

$$\left. \int_{0}^{\frac{a}{2}} \frac{sin(k(x_{q+1} - x'))}{sin(kl_x)} cos\left(\frac{n\pi}{a} x'\right) dx' \right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{2}{\pi w}\right) \frac{1}{\left[1 - \left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}} sin\left(\frac{m\pi}{b} y'\right) dy'$$

$$\left(-\frac{n\pi}{a}\right) sin\left(\frac{n\pi}{a} x\right) \frac{\rho \left[\frac{k_z^{III} - k_z tan(k_z^{III}(H-c)) tan(k_z H)}{k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H-c))} - cot(k_z H)\right]}{abk_z[cot(k_z z')cos(k_z z') - sin(k_z z')]}$$

$$sin\left(\frac{m\pi y}{b}\right)(k_z)cos(k_z z) +$$

$$\frac{\rho(1-tan^2(k_z H))\left(\frac{n\pi}{a}\right)(\epsilon_r^{III}-1)tan(k_z^{III}(H-c))}{ab[k_z^{III}tan(k_z H)-k_z tan(k_z^{III}(H-c))][cot(k_z z')cos(k_z z')-sin(k_z z')]}$$

$$\frac{1}{k_z^{III}tan(k_z^{III}(H-c))-\epsilon_r^{III}k_z tan(k_z H)}$$

$$\left.(k^2-k_z^2)sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)cos(k_z z)\right] \tag{114}$$

For a z-directed field below the current produced from a x-directed current above the dielectric, one obtains the expression

$$E_{1bzxa} = \sum_{q=1}^{N_{2x}} I_{q2x}\left(\int_{-\frac{x}{2}}^{0}\frac{sin(k(x'-x_{q-1}))}{sin(kl_x)}cos\left(\frac{n\pi}{a}x'\right)dx'+\right.$$

$$\left.\int_0^{\frac{x}{2}}\frac{sin(k(x_{q+1}-x'))}{sin(kl_x)}cos\left(\frac{n\pi}{a}x'\right)dx'\right)$$

$$\int_{-\frac{w}{2}}^{\frac{w}{2}}\left(\frac{2}{\pi w}\right)\frac{1}{\left[1-\left(\frac{2y'}{w}\right)^2\right]^{\frac{1}{2}}}sin\left(\frac{m\pi}{b}y'\right)dy'$$

$$\left[\frac{\rho\left[k_z^{III}-k_z tan(k_z^{III}(H-c))tan(k_z H)\right]}{abk_z[cot(k_z z')cos(k_z z')-sin(k_z z')][k_z^{III}tan(k_z H)-k_z tan(k_z^{III}(H-c))]}\right.$$

$$\left(-\frac{n\pi}{a}\right)sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)k_z cos(k_z^{II}z) +$$

$$\frac{-\rho}{abk_z[cot(k_z z')cos(k_z z')-sin(k_z z')]}\left(-\frac{n\pi}{a}\right)sin\left(\frac{n\pi x}{a}\right)sin\left(\frac{m\pi y}{b}\right)(-k_z)sin(k_z^{II}z) +$$

$$\frac{\rho(1 - tan^2(k_z H)) \left(\frac{n\pi}{a}\right) (\epsilon_r^{III} - 1) tan(k_z^{III}(H - c))}{ab[k_z^{III} tan(k_z H) - k_z tan(k_z^{III}(H - c))][cot(k_z z')cos(k_z z') - sin(k_z z')]}$$

$$\frac{1}{k_z^{III} tan(k_z^{III}(H - c)) - \epsilon_r^{III} k_z tan(k_z H)}$$

$$(k^2 - k_z^2) sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{b}\right) cos(k_z z)\Big] \tag{115}$$

### 2.4 Matrix Equation

The resulting matrix equation is formed and upon inversion the unknowns can be obtained for a given exciation.

## 3 Scattering Parameters

Using the derived current distribution on the conductors, one can apply an ideal transmission line model to determine the scattering parameters.

## 4 Summary

From this general analysis of the Green's function and the ultimate determination of the scattering parameters of an air-bridge, the formulation used here could be applied to a variety of structures whose geometry requires a three-dimentional analysis.

Other planned work includes the application of this work to other structures and the use of air-bridges as a curcuit element in the construction of other passive microwave circuits.

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