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Characteristic functional equations of polynomials and the Morera–Carleman theorem

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Abstract. Several characteristic functional equations satisfied by classes of polynomials of bounded degree are examined in connection with certain generalizations of the Morera–Carleman Theorem. Certain functional equations which have nonanalytic polynomial solutions are also considered.

1. Introduction

It was shown by H. Haruki [2] that a certain functional relation characterizes polynomials of bounded degree. More specifically let

$$\omega_l = e^{2\pi i/(l+1)}, \qquad l = 0, 1, \ldots . \tag{1}$$

Then the relation

$$\sum_{k=0}^{n+1} \omega_{n+1}^k f(z + \omega_{n+1}^k \zeta) = 0 \tag{2}$$

characterizes the family of all polynomials of degree not exceeding n in the sense that (2) is satisfied by this family and they are the only solutions of (2) among all continuous functions in the complex plane.

Functional equations characterizing polynomials have been studied recently by Shigeru Haruki [3] and M. A. McKiernan [5]. More general functional equations of this type have been studied by Halina Świątak [6], [7], J. H. B. Kemperman [4] and others (see also references in [6] and [7]). The above mentioned studies deal with functional equations mainly in real Euclidean spaces or finite dimensional vector spaces. The methods used are basically real variable methods in [4], [5], [7] and [9] and distributional methods in [6].

In this note functional equations of the general form (2) are studied but with

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emphasis on the complex analytic properties of the solutions with certain connections to classical results in complex variable theory pointed out. These considerations lead to certain generalizations of the Morera–Carleman theorem which is of independent interest. The method applied is similar to the one used by L. Zalcman [9] coupled with certain distributional considerations.

A possible application of some results to a numerical approach is also mentioned.

2. Functional relations

LEMMA 1. *Let* $F(z)=\sum_{k=0}^{n} a_k z^k$. *Then*

$$\sum_{k=0}^{l} F(z+\omega_l^k\zeta)=(l+1)\sum_{k=0}^{[n/(l+1)]} \frac{\zeta^{k(l+1)}}{[k(l+1)]!}F^{(k(l+1))}(z) \tag{3}$$

$l=0,1,\ldots,$ *and* ω_l *is defined by* (1).

Proof. Denote the left-hand side of (3) by S. Then

$$\begin{aligned} S&=\sum_{k=0}^{l}\sum_{m=0}^{n} a_m(z+\omega_l^k\zeta)^m\\ &=\sum_{m=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\omega_l^{kj}a_m\binom{m}{j}z^{m-j}\zeta^j\\ &=\sum_{j=0}^{n}\sum_{m=j}^{n}\sum_{k=0}^{l}\omega_l^{kj}a_m\binom{m}{j}z^{m-j}\zeta^j. \end{aligned} \tag{4}$$

The sum $\sum_{k=0}^{l}\omega_l^{kj}$ equals $(l+1)$ if $j=p(l+1)$ for some nonnegative integer p and vanishes otherwise. The expression

$$\sum_{m=j}^{n} a_m\binom{m}{j}z^{m-j}\zeta^j=\zeta^j\sum_{i=0}^{n-j}a_{i+j}\binom{i+j}{j}z^i=\frac{\zeta^j}{j!}F^{(j)}(z).$$

Substituting the last two sums into (4) we deduce (3). Consider a few particular cases.

(a) $l=0$

$$F(z+\zeta)=\sum_{k=0}^{n}\frac{\zeta^k}{k!}F^{(k)}(z)$$

which is the Taylor expansion for polynomials.

(b) $l \geq n$

$$\frac{1}{(l+1)}\sum_{k=0}^{l} F(z+\omega_l^k\zeta)=F(z) \tag{5}$$

which is a well-known averaging property of polynomials. Relation (2) follows from it by differentiation with respect to ζ and replacing n, F' by $(n+1)$, f, respectively.

(c) $[n/(l+1)]=1$

$$\frac{1}{(l+1)}\sum_{k=0}^{l} F(z+\omega_l^k\zeta)=F(z)+\frac{\zeta^{l+1}}{(l+1)!}F^{(l+1)}(z). \tag{6}$$

Differentiating with respect to ζ and replacing F' by G we have

$$\frac{1}{(l+1)}\sum_{k=0}^{l} \omega_l^k G(z+\omega_l^k\zeta)=\frac{\zeta^l}{l!}G^{(l)}(z) \tag{7}$$

for a polynomial $G(z)$ of degree p such that $l \leq p \leq 2l$.

(d) We may add another geometric interpretation of (5). For a fixed $n \leq l$ let Q denote the point z, and let P_k be the points $P_k=Q+\omega_l^k\zeta$, $k=0, 1, \ldots, l+1$. The points P_k are located equidistantly on the circle with center Q and radius $|\zeta|$ and $P_{l+1}=P_0$. Differentiating (5) with respect to ζ we have

$$\sum_{k=0}^{l} (P_k-Q)F(P_k)=0.$$

Applying the last equation to $F(((w-z)/\omega_l)+z)$ instead of $F(w)$ we have

$$\sum_{k=0}^{l} (P_k-Q)F(P_{k-1})=0$$

where $P_{-1}=P_l$. Therefore

$$\sum_{k=0}^{l} (P_{k+1}-P_k)F(P_k)=0.$$

We notice that this is a Riemann sum for the integral of F on the circle of radius $|\zeta|$ and center z. We summarize the last result as

COROLLARY 1. *If $F(z)$ is a polynomial of degree n then there exist vanishing Riemann sums corresponding to arbitrary degree of refinement on every circle in the complex plane.*

In this form Corollary 1 leads to a possibility of establishing a method to the basic complex variable theory by defining analytic functions in terms of convergent power series. Cauchy's theorem for analytic functions for circles follows. Then one can construct piecewise continuously differentiable homeomorphisms to more general regions, such as any piecewise continuously differentiable Jordan curve by radial projections. The advantage of such an approach will be mainly pedagogical since most of the proofs will be based on known properties of integrals and series studied in calculus and will thus provide and opportunity of review.

3. Connection with the Morera–Carleman theorem

The functional relation (3) satisfied by polynomials naturally leads to the question of the extent to which (3) characterizes polynomials. We shall pose this question in a somewhat more general form, the answer to which will depend on two extensions of the Morera–Carleman theorem [9]

THEOREM 1. *Let ρ_i $i=1,2,\ldots,m$, $\rho_1=1$ be fixed positive numbers. Let $C(w)=\bigcup_{i=1}^{m} C(w, r_i)$, where $C(w, r_i)$ denotes the circle of radius r_i centered at w. If f is a continuous function on a domain D such that*

$$\int_{C(w)} f\,dz=0$$

whenever $C(w)$ and its interior lie in D and $r_i/r_1=\rho_i$, $i=1,2,\ldots,m$, then f is analytic in D.

Proof. The proof follows the method applied in [9]. First assume f is continuously differentiable. By Green's theorem

$$\int_{C(w,r_i)} f\,dz=2i\iint_{D(w,r_i)} \frac{\partial f}{\partial \bar{z}}\,dx\,dy$$

where $z = x + iy$, $D(w, r_i)$ is the interior of $C(w, r_i)$ and $\partial f/\partial \bar{z} = \frac{1}{2}(\partial f/\partial x + i\, \partial f/\partial y)$. Let z_0 be a fixed point of D. For sufficiently small r_k we have

$$\int_{C(z_0, r_k)} f\, dz = 2i \iint_{D(z_0, r_k)} \frac{\partial f}{\partial \bar{z}}\, dx\, dy.$$

Hence

$$\sum_{k=1}^{m} \frac{\rho_k^2}{\pi r_k^2} \iint_{D(z_0, r_k)} \frac{\partial f}{\partial \bar{z}}\, dx\, dy = 0.$$

Let $r_1 \to 0$. It follows that

$$\frac{\partial f}{\partial \bar{z}}(z_0) \sum_{k=1}^{m} \rho_k^2 = 0.$$

This establishes the theorem for C^1-functions. In the general case we smoothen f by a convolution with mollifiers (see [8], [9]). These are real valued functions $\phi(z)$ on R^2 satisfying $\phi \in C^\infty$, $\phi \geq 0$, $\iint \phi\, dx\, dy = 1$ and $\phi = 0$ for $|z| \geq 1$. For example $\phi(z) = c \exp(1/(|z|^2 - 1))$ if $|z| < 1$ and vanishing otherwise. The constant c is adjusted so that $\iint \phi\, dx\, dy = 1$. If $\phi_\varepsilon(x) = \varepsilon^{-2}\phi(z/\varepsilon)$ for $\varepsilon > 0$ then the function

$$f_\varepsilon(z) = \iint f(z - \zeta)\phi_\varepsilon(\zeta)\, d\xi\, d\eta$$

where $\zeta = \xi + i\eta$ and the domain of integration if R^2, is a C^∞ function defined for all z whose distance from the boundary of D is at least ε.

Moreover it is well-known that on every compact subset of D f_ε converges uniformly to f as $\varepsilon \to 0$. If $f \in C^1$ then also $Df_\varepsilon \to Df$ uniformly for any first order derivative Df of f. Some of the properties of f_ε which are not used here will be applied in the proof of theorem 3 in the sequel.

To complete the proof one notices that for sufficiently small r_k and ε

$$\begin{aligned}\int_{C(z_0)} f_\varepsilon\, dz &= \int_{C(z_0)} dz \iint f(z - \zeta)\phi_\varepsilon(\zeta)\, d\xi\, d\eta \\ &= \iint \phi_\varepsilon(\zeta)\, d\xi\, d\eta \int_{C(z_0 - \zeta)} f\, dz = 0.\end{aligned}$$

By the first part of the proof, f_ε is analytic at z_0 and therefore f is analytic at z_0. Since z_0 is an arbitrary point of D the theorem is established.

We remark that reasoning as in Theorem 1 we can also prove a somewhat stronger

THEOREM 1′. *Theorem 1 holds under the more general conditions where $C(w)$ is replaced formally by $C_1(w)=\bigcup_{k=1}^m a_k C(w, r_k)$ $\Big($in the sense that*

$$\int_{C_1(w)} f\,dz=\sum_{k=1}^m a_k \int_{C(w,r_k)} f\,dz\Big),$$

provided $\sum_{k=1}^m a_k\rho_k^2\neq 0$.

The proof is similar.

The connection between Theorem 1 and the functional relations discussed in Section 2 becomes apparent in the following theorem. For the sake of simplicity we shall henceforth assume that all functions are defined on the entire complex plane.

THEOREM 2. *Let f be a continuous function defined on R^2 which satisfies the functional equation*

$$\sum_{k=1}^m a_k f(z+a_k\zeta)=g(z,\zeta) \tag{8}$$

where a_k are fixed complex numbers, not all zero, and $g(z,\zeta)$ is analytic in ζ for fixed z. Then f is an entire function.

Proof. Integrate both sides of (8) with respect to ζ on $|\zeta|=r$. We have

$$\int_{\bigcup_{k=1} C(z,r|a_k|)} f(w)\,dw=\sum_{k=1}^m \int_{C(z,r|a_k|)} f(w)\,dw=0.$$

One applies now Theorem 1 with $\rho_i=|a_i|/|a_1|$, $r=r_1/|a_1|$ assuming $a_1\neq 0$.

One notices that equations (2) and (7) are particular cases of the functional equation (8). Similarly applying Theorem 1′ one has

THEOREM 2′. *Let f be a continuous function on R^2 which satisfies the functional equation*

$$\sum_{k=1}^m f(z+b_k\zeta)=g(z,\zeta) \tag{9}$$

where $g(z, \zeta)$ *is analytic in* ζ *for fixed* z *and the constants* b_k *are such that* $\sum_{k=1}^m b_k \neq 0$. *Then* f *is an entire function.*

Proof. We may assume $b_k \neq 0$ for all $k = 1, 2, \ldots, m$ since any summand on the left-hand side of (9) for which $b_k = 0$ can be absorbed in the function g.

We apply Theorem 1′ with $a_k = 1/b_k$, $r_k = |b_k|$, $\rho_k = |b_k|/|b_1|$. Then

$$\sum_{k=1}^{m} a_k \rho_k^2 = \frac{1}{|b_1|^2} \sum_{k=1}^{m} \bar{b}_k \neq 0.$$

It is important to observe that the condition $\sum_{k=1}^m b_k \neq 0$ is necessary. For instance consider the example $m = 2$, $f(z+\zeta)+f(z-\zeta) = g(z, \zeta)$ where $f(z) = c\bar{z} + a(z)$ for any entire function $a(z)$ and any constant $c \neq 0$.

The relations (3), (5) and (6) are particular cases of Theorem 2′. However these will be dealt separately and more generally in

THEOREM 3. *Let* f *be a continuously differentiable function on* R^2 *which satisfies the functional equation*

$$\sum_{k=1}^{m} f(z + b_k \zeta) = g(z, \zeta) \tag{10}$$

where $g(z, \zeta)$ *is analytic in* ζ *for fixed* z *and the* b_k *are constants which satisfy*

$$\sum_{k=1}^{m} b_k^j = 0 \quad \text{for} \quad j = 1, \ldots, p$$

but

$$\sum_{k=1}^{m} b_k^{p+1} \neq 0$$

$p = 1, 2, \ldots$. *Then, and only then,* f *has the form*

$$f(z) = P_p(\bar{z}) + A(z) \tag{11}$$

where $P_p(z)$ *is a polynomial of degree not exceeding* p *and* $A(z)$ *is an arbitrary entire function.*

Proof. We may assume, as before, without loss of generality that all $b_k \neq 0$. For fixed $z = z_0$, w and r $(r > 0)$ we integrate (10) with respect to ζ on the circle

$|\zeta-w|=r$ and in the kth summand make the substitution $\eta=z_0+b_k\zeta$. Then applying Green's theorem we have

$$0=\sum_{k=1}^{m}\frac{1}{\pi b_k r^2}\int_{C(z_0+b_kw,\,|b_k|\,r)} f(\eta)\,d\eta=2i\sum_{k=1}^{m}\frac{\bar{b}_k}{\pi\,|b_k|^2\,r^2}\iint_{D(z_0+b_kw,\,|b_k|\,r)}\frac{\partial f}{\partial \bar{z}}\,dx\,dy.$$

Since w is arbitrary, letting $r\to 0$ we deduce

$$\sum_{k=1}^{m}\bar{b}_k h(z_0+b_kz)=0, \tag{12}$$

where $h=\partial f/\partial \bar{z}$. Assume now that f is sufficiently often differentiable. Differentiate (12) with respect to z at $z=0$. Then

$$\frac{\partial h}{\partial z}(z_0)=0.$$

Since z_0 is arbitrary $\partial h/\partial z=0$ so that $h=h(\bar{z})$. Now differentiate (12) p times with respect to $\bar{z}$ at $\bar{z}=0$. Then

$$\frac{\partial^p h}{\partial \bar{z}^p}(z_0)=0,$$

so that $h(\bar{z})=P_{p-1}(\bar{z})$, where P_{p-1} is a polynomial of degree at most $(p-1)$. By hypothesis however h is only continuous. We apply therefore the mollifiers introduced in the proof of Theorem 1. Since $f\in C^1$ we have [3]

$$\begin{aligned}&\lim_{\varepsilon\to 0} f_\varepsilon=f\\&\lim_{\varepsilon\to 0}\frac{\partial f_\varepsilon}{\partial \bar{z}}=h\end{aligned} \tag{13}$$

the limit being uniform on compact subsets of the plane. Let $h_\varepsilon=\partial f_\varepsilon/\partial \bar{z}$. By the previous argument applied to the function f_ε we have

$$h_\varepsilon=P_{p-1,\varepsilon}(\bar{z}) \tag{14}$$

where $P_{p-1,\varepsilon}$ is a polynomial of degree at most $(p-1)$ whose coefficients depend on ε. By (13) and (14) the polynomials $P_{p-1,\varepsilon}$ have a limit as $\varepsilon\to 0$ uniformily on compact subsets. Integrating (14) we deduce

$$f_\varepsilon = \int_0^{\bar z} P_{p-1,\varepsilon}(\zeta)\, d\zeta + A_\varepsilon \tag{15}$$

where A_ε is an entire function depending on ε. Since $f_\varepsilon \to f$ as $\varepsilon \to 0$ and $P_{p-1,\varepsilon}$ has a uniform limit as $\varepsilon \to 0$ it follows that $A_\varepsilon \to A$ where A is an entire function as $\varepsilon \to 0$. (11) follows from (15).

Remark. It is possible that theorem 3 remains valid if f is assumed to be merely continuous but we were not able to prove this by the above method. We have

COROLLARY 2. (a) *The most general continuously differentiable solution having the averaging property* (5) *is a sum of two polynomials of degree not exceeding l in the variables z and $\bar z$ correspondingly.*

(b) *The most general continuously differentiable solution F of a relation of the form*

$$\sum_{k=0}^{l} F(z + \omega_l^k \zeta) = \sum_{k=0}^{[n/(l+1)]} a_k(z) \zeta^{k(l+1)}$$

is a polynomial of degree at most n.

The above results have also another aspect which is somewhat auxiliary to the considerations made here but may be worthwhile to notice. The characteristic relations which have been pointed out for classes of polynomials in z and $\bar z$ of bounded degree could possibly serve as a means of determining the degree of interpolatory polynomials in the complex plane, particularly in light of the fact that some of the converging methods interpolate functions on equidistant points on circles [1].

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