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Random Demand Satisfaction in Unreliable Production–Inventory–Customer Systems *

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Abstract. A method for calculating the probability of customer demand satisfaction in production–inventory–customer systems with Markovian machines, finite finished goods buffers, and random demand is developed. Using this method, the degradation of this probability as a function of demand variability is quantified. In addition, it is shown by examples that the probability of customer demand satisfaction depends primarily on the coefficient of variation, rather than on the complete distribution, of the demand.

Keywords: due-time performance, finished goods buffer, random demand, Markovian reliability

1. Introduction

Reliable satisfaction of customer demand is an important issue of manufacturing systems performance. Although a number of measures for customer demand satisfaction may be considered, in the automotive industry it is often quantified by the probability to ship to the customer a required number of parts during a fixed period of time. We refer to this performance measure as the Due-Time Performance (DTP). For the case of *unreliable* production and *reliable* (constant) customer demand, a method for calculating DTP was developed by Jacobs and Meerkov (1995), Tan (1998, 1999), Li and Meerkov (2000a, 2000b, 2001, 2002). In reality, however, the demand is almost always variable (even if purchasing agreements state, otherwise). Therefore, analysis of systems with *variable demand* is of importance. The current paper is devoted to this topic.

Specifically, we consider a production–inventory–customer (PIC) system, which consists of a Production Subsystem (PS), Inventory Subsystem (IS), and Customer Subsystem (CS) (see figure 1; the parameters of the subsystems, indicated in this figure, are defined in section 2). It is assumed that the PS has its up- and downtimes distributed exponentially, the IS has a finite capacity, and the CS generates the demand modeled as a sequence of independent, identically distributed random variables with fixed expected values but otherwise distributed arbitrarily. Under these conditions, we provide an analytical method for DTP calculation. Using this method, we quantify DTP degradation as

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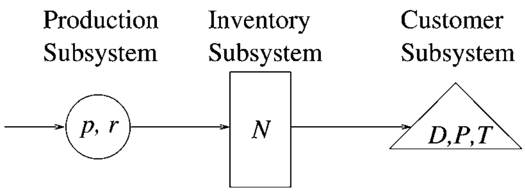

Figure 1. Production–inventory–customer system.

a function of demand variability. In addition, we show by examples that DTP depends, in fact, only on the coefficient of variation rather than on the complete distribution of the demand. This leads to a hypothesis that DTP for all demand distributions with identical coefficients of variation is practically the same.

Production/inventory systems have been considered in numerous publications for over two decades (see monographs by Buzacott and Shanthikumar (1993), Hopp and Spearman (1996), Altiok (1997) and Zipkin (2001), and articles by Gavish and Graves (1981), Federgruen and Zipkin (1986), Altiok (1989), Srinivasan and Lee (1991), Ciarallo, Akella, and Morton (1994), Zijm and Vanhoutum (1994), Glasserman and Tayur (1996), Rubio and Wein (1996), Wang and Gerchak (1996), Moinzadeh and Aggarwal (1997), Gullu, Onol, and Erkip (1997, 1999), Gullu (1998), Liberopoulos and Dallery (2002)). The demand is almost always assumed to be random. The production is either deterministic or random and, in many models, instantaneous. The finished goods buffer is always assumed (perhaps, tacitly) to be infinite. The problems considered typically center on *optimal replenishment policies* and often use queueing theory methods. Although this literature offers many important results and insights, the issue of Due-Time Performance for random demand has not been addressed.

The main contribution of this paper is in providing a method for calculating DTP in the framework of a PIC model, which captures realistic features of automotive manufacturing systems, i.e., random production, random demand, and finite finished goods buffers (FGB).

The outline of the paper is as follows. In section 2, the model of the production–inventory–customer system at hand is introduced. Section 3 provides a method for DTP calculation. Effects of the demand randomness are investigated in sections 4 and 5. Finally, in section 6, the conclusions are formulated. The proofs are given in the appendix.

2. Model and problem formulation

The PIC system considered in this work is defined by the following assumptions.

Production subsystem

(i) The production subsystem has two states: up and down. When up, it is capable of producing one part per unit of time; when the PS is down, no production takes place.

(ii) The uptime and the downtime of PS are random variables distributed exponentially with parameters p and r, respectively. Thus, PS is characterized by the pair (p, r).

Inventory subsystem

(iii) The inventory subsystem is characterized by its capacity, $0 \leqslant N < \infty$.

Interaction of PS and IS

(iv) PS is blocked at time t if FGB is full at time t. PS is never starved.

Customer subsystem

(v) From the point of view of the customer, the time axis is divided into "epochs", each consisting of T units of time. At the end of epoch, unfinished part is scrapped.

(vi) At the end of each epoch i, $i = 1, 2, \ldots$, the customer requires $D(i)$ parts to be available for shipment. $D(i)$ is assumed to be a random variable taking values D_1, ..., D_J with probabilities $P_1, \ldots, P_J$, respectively. Thus, the customer is characterized by the triplet (D, P, T) where $D = [D_1, \ldots, D_J]$, $P = [P_1, \ldots, P_J]$ and T is the shipping period.

(vii) The expected value of $D(i)$ is $\overline{D}$ for all i. To avoid triviality, it is assumed that

$$\overline{D} \leqslant T \frac{r}{p+r}, \tag{1}$$

i.e., that the average demand is not larger than the average production capacity during the shipping period T.

(viii) $D(1), D(2), \ldots$, is a sequence of independent identically distributed (IID) random variables.

Interaction of IS and CS

(ix) At the beginning of epoch i, parts are removed from the FGB in the amount of $\min(H(i-1), D(i))$, where $H(i-1)$ is the number of parts in the FGB at the end of $(i-1)$th epoch. If $H(i-1) \geqslant D(i)$, the shipment is complete; if $H(i-1) < D(i)$, the balance of the shipment, i.e., $D(i) - H(i-1)$ parts, is to be produced by the PS during the shipping period T. Parts produced are immediately removed from the FGB and prepared for shipment, until the shipment is complete, i.e., $D(i)$ parts are available. If the shipment is complete before the end of the epoch, the system continues operating, but with the parts being accumulated in the FGB, either until the end of the epoch or until the PS is blocked, whichever occurs first. If the shipment is not complete by the end of the epoch, an incomplete shipment is sent to the customer. No backlog is allowed.

Remark 1. The assumptions on exponential up- and downtime (see (ii)), scrapping (see (v)), IID sequencing (see (viii)), and no backloging (see (ix)) are introduced to simplify calculations. In future work, we plan to remove some of them. In particular, production systems with non-exponential distributions will be considered as well as non-IID demand.

Assumptions (i)–(ix) define the production–inventory–customer system under consideration. In the time scale of the epoch and in an appropriately defined state space, it represents an irreducible Markov chain with a finite number of states. Therefore, it has a unique stationary probability distribution (Hoel, Port, and Stone, 1972), i.e., steady state. We refer to this steady state as the "normal system operation".

Let $\hat{t}(i)$ be the number of parts produced by the PS in epoch i during the normal system operation. Then DTP can by expressed as

$$\text{DTP} = \Pr\big(H(i-1) + \hat{t}(i) \geqslant D(i)\big). \tag{2}$$

The problem addressed in this paper is: *Given production–inventory–customer system* (i)–(ix)*, develop a method for calculating DTP and quantify effects of the demand variability on customer demand satisfaction.*

3. DTP calculation

Let $t(i)$ denote the number of parts produced during epoch i if no blockage occurs. Introduce the following quantities:

$$\begin{aligned}
\mathcal{P}(x) &= \Pr\big(t(i) \geqslant x\big), && x \in \{0, 1, \ldots, T\},\\
r_{k,l,j} &= \Pr\big(t(i) = D(i) + k - l\big), && k = 1, \ldots, N-1,\ l = 0, 1, \ldots, N,\\
& && D(i) = D_j,\ j = 1, \ldots, J,\\
\hat{r}_{N,l,j} &= \Pr\big(t(i) \geqslant D(i) + N - l\big), && l = 0, 1, \ldots, N,\ D(i) = D_j,\ j = 1, \ldots, J.
\end{aligned}$$

These quantities can be calculated as follows. As it has been shown by Jacobs and Meerkov (1995),

$$\begin{aligned}
\mathcal{P}(x) = {} & \frac{r\mathrm{e}^{-px}}{p+r}\left[1 + \sum_{j=2}^{\infty} \frac{(px)^{j-1}}{(j-1)!}\left(1 - \mathrm{e}^{-r(T-x)} \sum_{k=0}^{j-2} \frac{[r(T-x)]^k}{k!}\right)\right]\\
& + \frac{p\mathrm{e}^{-px}}{p+r} \sum_{j=1}^{\infty} \frac{(px)^{j-1}}{(j-1)!}\left[1 - \mathrm{e}^{-r(T-x)} \sum_{k=0}^{j-1} \frac{[r(T-x)]^k}{k!}\right].
\end{aligned} \tag{3}$$

To calculate $r_{k,l,j}$, the following expression can be used:

$$r_{k,l,j} = \mathcal{P}(D_j + k - l) - \mathcal{P}(D_j + k - l + 1). \tag{4}$$

The $\hat{r}_{N,l,j}$ can be calculated as

$$\hat{r}_{N,l,j} = \mathcal{P}(D_j + N - l). \tag{5}$$

Introduce matrix $\mathcal{R}$ and vector Z_0 defined by

$$\mathcal{R} = \begin{pmatrix} \sum_{j=1}^{J}(r_{1,1,j}-r_{1,0,j})P_j-1 & \sum_{j=1}^{J}(r_{1,2,j}-r_{1,0,j})P_j & \ldots & \sum_{j=1}^{J}(r_{1,N,j}-r_{1,0,j})P_j \\ \sum_{j=1}^{J}(r_{2,1,j}-r_{2,0,j})P_j & \sum_{j=1}^{J}(r_{2,2,j}-r_{2,0,j})P_j-1 & \ldots & \sum_{j=1}^{J}(r_{2,N,j}-r_{2,0,j})P_j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{J}(\hat{r}_{N,1,j}-\hat{r}_{N,0,j})P_j & \sum_{j=1}^{J}(\hat{r}_{N,2,j}-\hat{r}_{N,0,j})P_j & \ldots & \sum_{j=1}^{J}(\hat{r}_{N,N,j}-\hat{r}_{N,0,j})P_j-1 \end{pmatrix}, \tag{6}$$

$$Z_0 = \begin{pmatrix} \sum_{j=1}^{J} r_{1,0,j}P_j \\ \sum_{j=1}^{J} r_{2,0,j}P_j \\ \vdots \\ \sum_{j=1}^{J} \hat{r}_{N,0,j}P_j \end{pmatrix}. \tag{7}$$

Matrix $\mathcal{R}$ is nonsingular due to the uniqueness of the stationary probability distribution defined by system (i)–(ix).

Theorem 1. Under assumptions (i)–(ix),

$$\text{DTP} = \sum_{k=0}^{N}\sum_{j=1}^{J} \mathcal{P}(D_j - k)P_j z_k, \tag{8}$$

where $z_k = \Pr(H(i-1) = k)$, $k = 0, 1, \ldots, N$, and vector $Z = [z_1, z_2, \ldots, z_N]^T$ is calculated according to

$$Z = -\mathcal{R}^{-1}Z_0. \tag{9}$$

Proof. See appendix. □

Remark 2. When the demand is deterministic, i.e., $D(i) = \overline{D}$, $\forall i$, expression (8) simplifies to

$$\text{DTP} = \sum_{k=0}^{N} \mathcal{P}(\overline{D} - k)z_k, \tag{10}$$

where $\mathcal{R}$ and Z_0 are given by

$$\mathcal{R} = \begin{pmatrix} r_{1,1} - r_{1,0} - 1 & r_{1,2} - r_{1,0} & \dots & r_{1,N} - r_{1,0} \\ r_{2,1} - r_{2,0} & r_{2,2} - r_{2,0} - 1 & \dots & r_{2,N} - r_{2,0} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{r}_{N,1} - \hat{r}_{N,0} & \hat{r}_{N,2} - \hat{r}_{N,0} & \dots & \hat{r}_{N,N} - \hat{r}_{N,0} - 1 \end{pmatrix}, \tag{11}$$

$$Z_0 = \begin{pmatrix} r_{1,0} \\ r_{2,0} \\ \vdots \\ \hat{r}_{N,0} \end{pmatrix}. \tag{12}$$

Here the third subscript j in $r_{k,l,j}$ and $\hat{r}_{N,l,j}$ becomes superfluous and, therefore, omitted. These expressions coincide with those derived by Li and Meerkov (2000a, 2002).

Remark 3. Theorem 1 can be useful for DTP evaluation in production–inventory–customer systems with more than one exponential machine. Specifically, similar to Li and Meerkov (2000a), it is possible to show that the following numerical fact holds:

$$\mathrm{DTP}_M \geqslant \mathrm{DTP}_1. \tag{13}$$

Here DTP_M is the due-time performance of M-machine PIC system (with in-process buffering) and DTP_1 is the due-time performance in one-machine case (i)–(ix), where the machine is the aggregation of the machines of the M-machine system with the aggregation carried out using the method of Chiang, Kuo, and Meerkov (2000). Thus, the results obtained in this paper are applicable to M-machine PIC systems as well (in the sense of the lower bound (13)).

Remark 4. As it follows from theorem 1, the calculation of DTP involves evaluations of infinite sums in (3) and matrix inversion in (9). In our calculations, the infinite sums have been evaluated by truncating higher order terms so that the first neglected term is less than 10^{-10}. (The terms are decreasing due to the factorials in the denominators of (3).) The inversion of matrix $\mathcal{R}$ turned out to be not time-consuming since the total time of DTP calculation was within a few seconds for all systems analyzed (see remark 5 below).

Remark 5. Although the DTP of system (i)–(ix) can be estimated using simulations, it would require, however, much more time than the analytical calculations. For instance, for systems analyzed in this paper, simulations would require on the average from 40 minutes ($N = 45$) to two hours ($N = 200$) using a 600 MHz PC whereas analytical calculations are carried out within 2 and 6 seconds, respectively.

Table 1
Uniform PMFs considered.

CV	D_i	P_j
0.1	12, 13, ..., 17, 18	1/7
0.25	9, 10, ..., 20, 21	1/13
0.4	5, 6, ..., 24, 25	1/21
0.6	4, 15, 26	1/3
0.8	3, 27	1/2

4. DTP degradation as a function of demand variability

It is well known that the demand variability leads to a decrease in customer demand satisfaction (Buzacott and Shanthikumar, 1993; Hopp and Spearman, 1996; Altiok, 1997; Zipkin, 2001). It is obvious, that DTP, being a measure of customer demand satisfaction, also suffers from uncertainties in the demand. However, the level of DTP degradation for various values of FGB has not been quantified. Using the method developed above, we provide such a quantification below.

To analyze the behavior of DTP as a function of the demand variability, we consider five uniform demand distributions shown in table 1. All of them have $\overline{D} = 15$ and the coefficients of variation (CV) ranging from 0.1 to 0.8.

To analyze DTP degradation for various FGB capacities, we consider IS with N taking values 1, 5, 15, and 45.

To investigate the properties of DTP for various levels of customer demand, we use the notion of the load factor (LF). This notion has been introduced by Jacobs and Meerkov (1995) as the ratio of the average demand to the average production volume during the shipping period, i.e.,

$$L = \frac{\overline{D}}{Te}, \tag{14}$$

where e is the PS efficiency, i.e., $e = r/(p + r)$. Below, we consider three levels of LF: low ($L = 0.9159$), medium ($L = 0.9511$), and high ($L = 0.9892$).

The parameters of the three production subsystems and the shipping periods that result in the above mentioned load factors are shown in table 2. To ensure fairness in the comparison, the efficiencies of each PS are chosen to be the same, $e = 0.6066$, as are the relative shipping periods defined by

$$\tau = \frac{T}{1/r + 1/p} \tag{15}$$

and chosen to be $\tau = 1.0918$.

Table 2
Systems analyzed.

	Load factor	p	r	T
System 1	Low			
	$L = 0.9159$	0.0667	0.1028	27
System 2	Medium			
	$L = 0.9511$	0.0692	0.1067	26
System 3	High			
	$L = 0.9892$	0.0720	0.1110	25

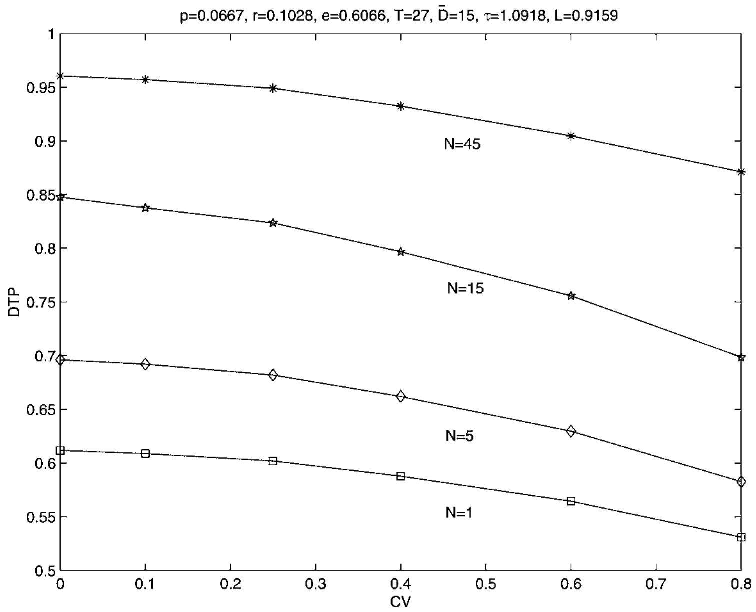

Figure 2. DTP degradation as a function of demand randomness: system 1, low LF case.

The behavior of DTP as a function of CV of the demand is shown in figures 2–4. These figures include also the case of deterministic demand (CV = 0). Examining these data, we conclude the following.

1. As expected, for all values of LF and N, larger CVs lead to smaller DTPs. Similarly, larger LFs lead to smaller DTPs. Finally, for any CV and LF, larger N results in larger DTP.
2. Unexpectedly, the curves representing DTP as function of CV for all LF with identical N are practically collinear. This implies that *the percent of DTP degradation as a function of CV is almost independent of LF, as long as N remains the same*. This conclusion is quantified in table 3 using the ratio of DTPs for CV = 0.8 and CV = 0.

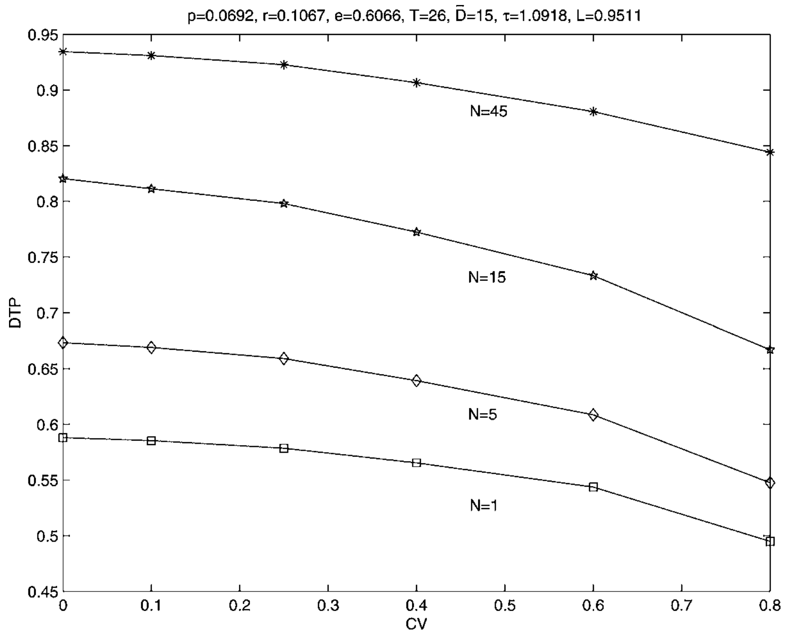

Figure 3. DTP degradation as a function of demand randomness: system 2, medium LF case.

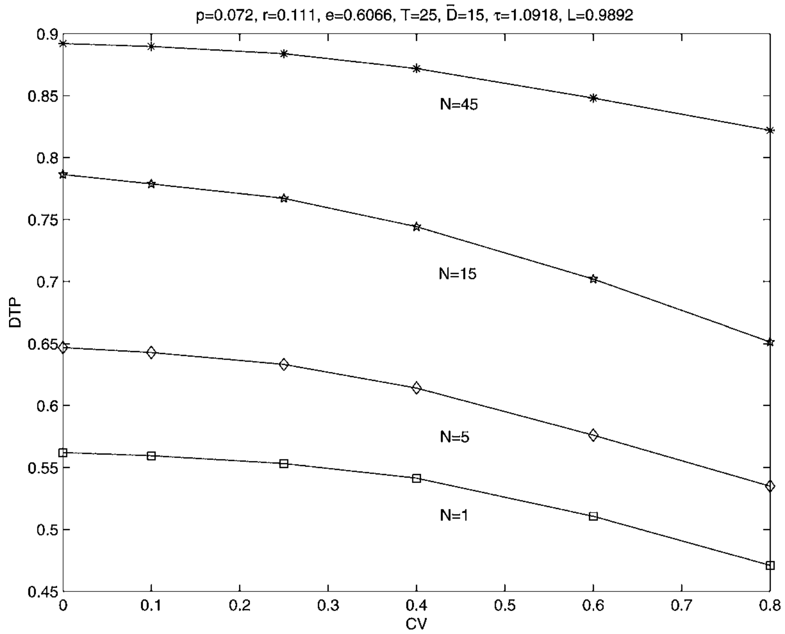

Figure 4. DTP degradation as a function of demand randomness: system 3, high LF case.

Table 3
Effect of demand randomness on DTP degradation (DTP(CV = 0.8)/DTP(CV = 0)).

	High LF	Medium LF	Low LF
$N = 1$	0.8379	0.8415	0.8678
$N = 5$	0.8270	0.8131	0.8368
$N = 15$	0.8284	0.8130	0.8241
$N = 45$	0.9214	0.9035	0.9071

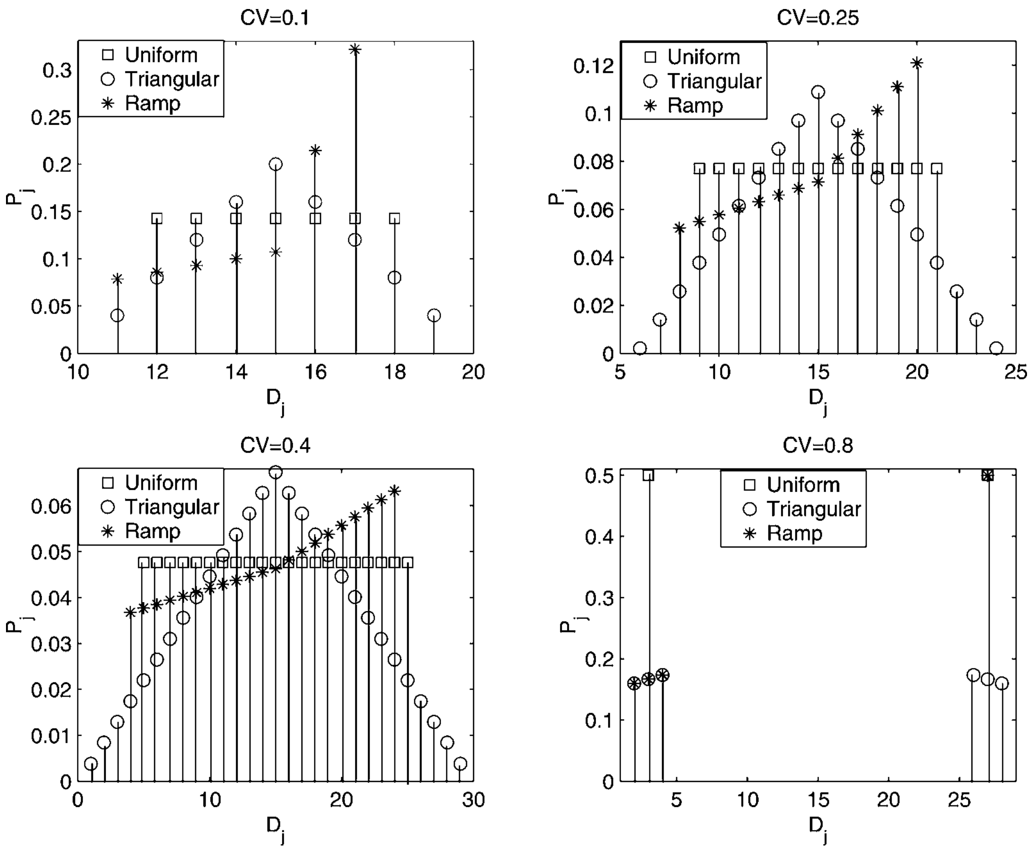

Figure 5. Probability mass functions considered.

5. DTP as a function of demand CV

The method of section 3 implies that DTP is a *functional* of the probability mass function (PMF) of the demand. Below, using examples, we show that DTP for various types of PMFs remains the same as long as their CVs are identical. This implies that DTP can be viewed as a *function* of CV, no matter what the shape of PMF may be.

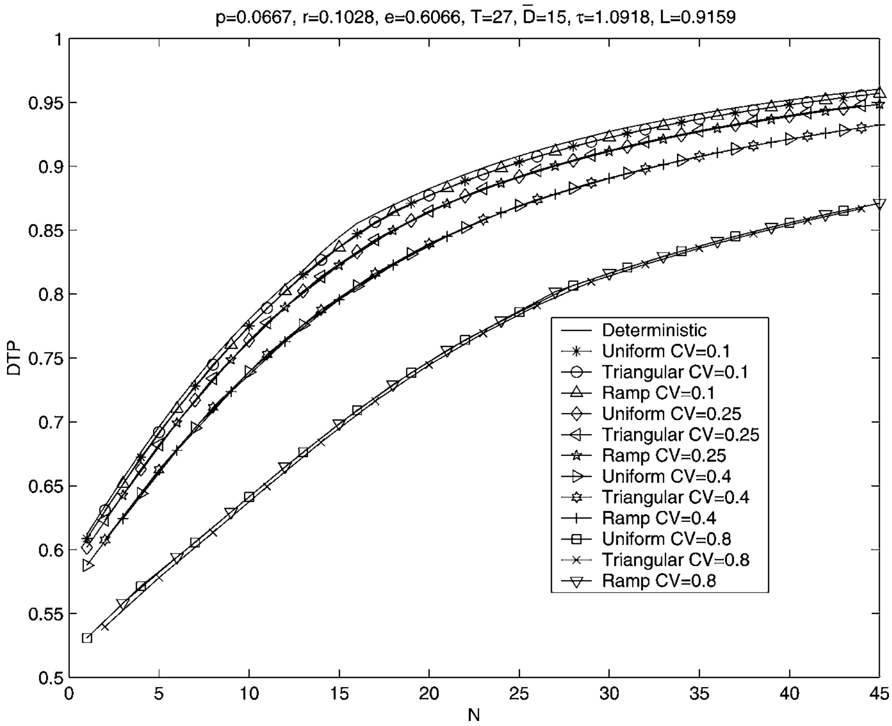

Figure 6. DTP as function of demand CV: system 1, low LF case.

The PMFs that we consider in this work are uniform, triangular, and "ramp" (see figure 5). We consider three groups of these distributions, with CVs equal to 0.1, 0.25, 0.4, and 0.8. The expected values of all distributions are 15.

Using these PMFs and theorem 1, we calculate DTP as a function of N for the three systems defined in table 2. The results are shown in figures 6–8. From these data we conclude:

1. For every value of N, DTP for all PMFs considered are almost the same, as long as CVs are identical.

2. The largest differences occur for the smallest N and the largest CV. Table 4 quantifies these differences, which take place due to the effects of higher moments of corresponding distribution.

3. Based on these results, the following hypothesis can be formulated. *In the production–inventory–customer system defined by assumptions* (i)–(ix)*, DTP for all demand distributions with identical coefficients of variation remains practically the same.*

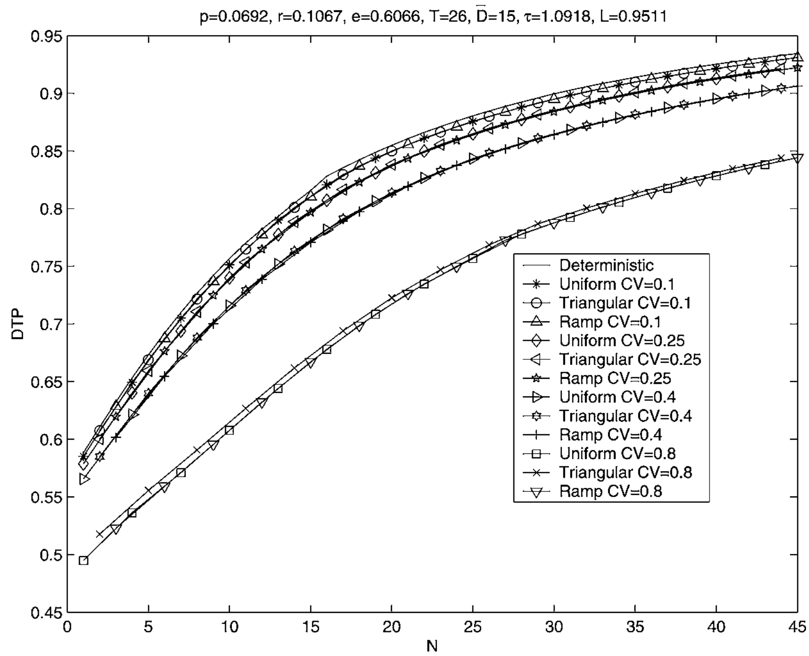

Figure 7. DTP as function of demand CV: system 2, medium LF case.

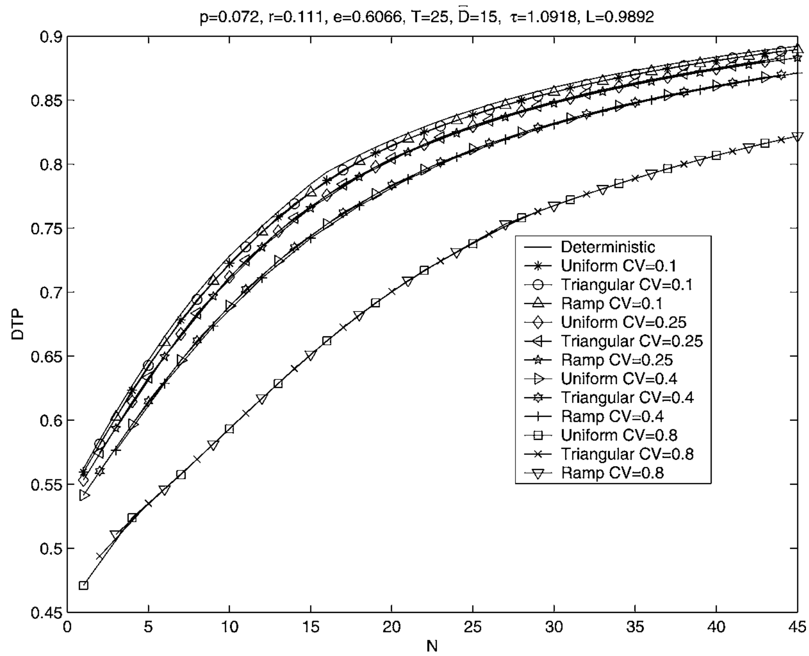

Figure 8. DTP as function of demand CV: system 3, high LF case.

Table 4
DTP for different distributions of the demand (CV $= 0.8$, $N = 1$).

	Uniform	Triangular	"Ramp"
High LF	0.4709	0.4782	0.4708
Medium LF	0.4949	0.4992	0.4948
Low LF	0.5308	0.5256	0.5307

6. Conclusions

This paper provides an analytical method for evaluating due-time performance (DTP) in production–inventory–customer systems with Markovian machines, finite inventory, and random demand. Using this method, the degradation of DTP as a function of demand variability is quantified and it is shown, by examples, that DTP is practically independent of a particular type of the demand distribution, as long as its coefficient of variation (CV) remains fixed. This property is important in applications since CVs of the demand can be relatively easily evaluated in practice, whereas determining the complete probability mass function is a formidable task.

Acknowledgments

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Appendix. Proof of theorem 1

The logic of the proof is as follows:

Step 1. Derive the characterization of the probability mass function, $\Pr(H(i) = k)$, for $k = 0, 1, \ldots, N$, where $H(i)$ is the number of parts in FGB at the end of the ith epoch.

Step 2. Combine the results of step 1 into matrix–vector form and solve for the probability mass function.

Step 3. From the above calculation, express the DTP in terms of $\Pr(H(i) = k)$ and obtain the claim of the theorem.

Below these three steps are carried out.

Step 1. For $k = 1, 2, \ldots, N - 1$,

$$\begin{aligned} z_k &\triangleq \Pr\big(H(i) = k\big) \\ &= \Pr\big(k \leqslant \hat{t}(i) + H(i-1) - D(i) < k + 1\big) \end{aligned}$$

$$= \sum_{l=0}^{N} \Pr\big(k \leqslant \hat{t}(i) + H(i-1) - D(i) < k+1 \big| H(i-1) = l\big) \Pr\big(H(i-1) = l\big)$$
$$= \sum_{l=0}^{N} \Pr\big(D(i) + k - l \leqslant \hat{t}(i) < D(i) + k + 1 - l\big) \Pr\big(H(i-1) = l\big)$$
$$= \sum_{l=0}^{N} \big[\Pr\big(\hat{t}(i) \geqslant D(i) + k - l\big) - \Pr\big(\hat{t}(i) \geqslant D(i) + k - l + 1\big)\big] \Pr\big(H(i-1) = l\big)$$
$$= \sum_{l=0}^{N} \big[\Pr\big(t(i) \geqslant D(i) + k - l\big) - \Pr\big(t(i) \geqslant D(i) + k - l + 1\big)\big] \Pr\big(H(i-1) = l\big).$$

Due to stationary, we obtain

$$\begin{aligned} z_k &= \sum_{l=0}^{N} \big[\Pr\big(t(i) \geqslant D(i) + k - l\big) - \Pr\big(t(i) \geqslant D(i) + k - l + 1\big)\big] \Pr\big(H(i) = l\big), \\ &= \sum_{l=0}^{N} \Bigg[\sum_{j=1}^{J} \Pr\big(t(i) \geqslant D(i) + k - l \big| D(i) = D_j\big) \Pr\big(D(i) = D_j\big) \\ &\qquad - \sum_{j=1}^{J} \Pr\big(t(i) \geqslant D(i) + k - l + 1 \big| D(i) = D_j\big) \Pr\big(D(i) = D_j\big) \Bigg] \Pr\big(H(i) = l\big) \\ &= \sum_{l=0}^{N} \sum_{j=1}^{J} \big[\Pr\big(t(i) \geqslant D_j + k - l\big) - \Pr\big(t(i) \geqslant D_j + k - l + 1\big)\big] P_j z_l \\ &= \sum_{l=0}^{N} \sum_{j=1}^{J} \big[\mathcal{P}(D_j + k - l) - \mathcal{P}(D_j + k - l + 1)\big] P_j z_l \\ &= \sum_{l=0}^{N} \sum_{j=1}^{J} r_{k,l,j} P_j z_l, \end{aligned} \tag{A.1}$$

where $\mathcal{P}(D_j + k - l)$ and $\mathcal{P}(D_j + k - l + 1)$ are calculated according to Jacobs and Meerkov (1995):

$$\begin{aligned} \mathcal{P}(x) = \frac{r\mathrm{e}^{-px}}{p+r} &\left[1 + \sum_{j=2}^{\infty} \frac{(px)^{j-1}}{(j-1)!} \left(1 - \mathrm{e}^{-r(T-x)} \sum_{k=0}^{j-2} \frac{[r(T-x)]^k}{k!} \right) \right] \\ &+ \frac{p\mathrm{e}^{-px}}{p+r} \sum_{j=1}^{\infty} \frac{(px)^{j-1}}{(j-1)!} \left[1 - \mathrm{e}^{-r(T-x)} \sum_{k=0}^{j-1} \frac{[r(T-x)]^k}{k!} \right]. \end{aligned} \tag{A.2}$$

For $k = N$,

$$\begin{aligned}
z_N &= \Pr\big(\hat{t}(i) + H(i-1) - D(i) = N\big) \\
&= \Pr\big(t(i) + H(i-1) - D(i) \geqslant N\big) \\
&= \sum_{l=0}^{N} \Pr\big(t(i) + H(i-1) - D(i) \geqslant N \big| H(i-1) = l\big)\Pr\big(H(i-1) = l\big) \\
&= \sum_{l=0}^{N} \Pr\big(t(i) \geqslant D(i) + N - l\big)\Pr\big(H(i) = l\big) \\
&= \sum_{l=0}^{N}\sum_{j=1}^{J} \Pr\big(t(i) \geqslant D(i) + N - l \big| D(i) = D_j\big)\Pr\big(D(i) = D_j\big)\Pr\big(H(i) = l\big) \\
&= \sum_{l=0}^{N}\sum_{j=1}^{J} \Pr\big(t(i) \geqslant D_j + N - l\big) P_j z_l \\
&= \sum_{l=0}^{N}\sum_{j=1}^{J} \mathcal{P}(D_j + N - l) P_j z_l \\
&= \sum_{l=0}^{N}\sum_{j=1}^{J} \hat{r}_{N,l,j} z_l. \qquad \text{(A.3)}
\end{aligned}$$

For $k = 0$, we write:

$$\Pr\big(H(i) = 0\big) = 1 - \sum_{k=1}^{N} \Pr\big(H(i) = k\big), \qquad \text{(A.4)}$$

i.e.,

$$z_0 = 1 - \sum_{k=1}^{N} z_k. \qquad \text{(A.5)}$$

Step 2. Substituting z_0 from (A.5) into (A.1) and (A.3), we obtain:

$$\begin{aligned}
&\left[\sum_{j=1}^{J}(r_{1,1,j} - r_{1,0,j})P_j - 1\right] z_1 + \sum_{j=1}^{J}(r_{1,2,j} - r_{1,0,j})P_j z_2 + \cdots \\
&\quad + \sum_{j=1}^{J}(r_{1,N,j} - r_{1,0,j})P_j z_N = -\sum_{j=1}^{J} r_{1,0,j} P_j, \\
&\sum_{j=1}^{J}(r_{2,1,j} - r_{2,0,j})P_j z_1 + \left[\sum_{j=1}^{J}(r_{2,2,j} - r_{2,0,j})P_j - 1\right] z_2 + \cdots \\
&\quad + \sum_{j=1}^{J}(r_{2,N,j} - r_{2,0,j})P_j z_N = -\sum_{j=1}^{J} r_{2,0,j} P_j,
\end{aligned}$$

$$\vdots$$

$$\sum_{j=1}^{J}\big(\hat{r}_{N,1,j} - \hat{r}_{N,0,j}\big)P_j z_1 + \sum_{j=1}^{J}\big(\hat{r}_{N,2,j} - \hat{r}_{N,0,j}\big)P_j z_2 + \cdots$$
$$+ \left[\sum_{j=1}^{J}\big(\hat{r}_{N,N,j} - \hat{r}_{N,0,j}\big)P_j - 1\right] z_N = -\sum_{j=1}^{J}\hat{r}_{N,0,j}P_j,$$

or, in matrix–vector form,

$$\mathcal{R}Z = -Z_0, \tag{A.6}$$

where $\mathcal{R}$, and Z_0 are defined in (6) and (7), respectively, and $\mathcal{R}$ is nonsingular. Thus,

$$Z = -\mathcal{R}^{-1}Z_0. \tag{A.7}$$

Step 3. From the definition of DTP, using the total probability formula, we have:

$$\begin{aligned}
\mathrm{DTP} &= \Pr\big(\hat{t}(i) + H(i-1) \geqslant D(i)\big) \\
&= \sum_{k=0}^{N} \Pr\big(\hat{t}(i) + H(i-1) \geqslant D(i) \big| H(i-1) = k\big)\Pr\big(H(i-1) = k\big) \\
&= \sum_{k=0}^{N} \Pr\big(\hat{t}(i) \geqslant D(i) - k\big)\Pr\big(H(i-1) = k\big) \\
&= \sum_{k=0}^{N} \Pr\big(t(i) \geqslant D(i) - k\big)\Pr\big(H(i-1) = k\big) \\
&= \sum_{k=0}^{N} \Pr\big(t(i) \geqslant D(i) - k\big)\Pr\big(H(i) = k\big) \\
&= \sum_{k=0}^{N}\sum_{j=1}^{J} \Pr\big(t(i) \geqslant D(i) - k\big)\Pr\big(D(i) = D_j\big)\Pr\big(H(i) = k\big) \\
&= \sum_{k=0}^{N}\sum_{j=1}^{J} \mathcal{P}(D_j - k)P_j z_k.
\end{aligned} \tag{A.8}$$

Theorem 1 is proved. □

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