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LOWER CLOSURE THEOREMS FOR LAGRANGE PROBLEMS OF OPTIMIZATION WITH DISTRIBUTED AND BOUNDARY CONTROLS

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LOWER CLOSURE THEOREMS FOR LAGRANGE PROBLEMS OF OPTIMIZATION WITH DISTRIBUTED AND BOUNDARY CONTROLS*

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I. INTRODUCTION

In this paper we prove lower closure theorems for multidimensional problems of optimization with distributed and boundary controls. The concept of lower closure, introduced by Cesari [1,2,3,4] in connection with his existence theorems for optimal solutions, has the same role for Lagrange problems that Tonelli's lower semicontinuity has for free problems.

The present analysis extends Cesari's theory in [3,4], but differs from it in two respects. First, we use the property $Q(\rho)$, $0 \leq \rho \leq r+1$, of upper semicontinuity of variable sets in E^{r+1}, which we introduced in a previous paper (D. E. Cowles [5]), instead of properties (U) (Kuratowski) and (Q) (Cesari) used in [3,4]. Property $Q(\rho)$ reduces to property (U) for $\rho = 0$, and to property (Q) for $\rho = r+1$, as we proved in [5]. Also, for every $0 \leq \rho \leq r$, property $Q(\rho+1)$ implies property $Q(\rho)$ (see our paper [5]).

As in Cesari's analysis, we first prove a closure theorem (§ 2), which is then used to prove lower closure theorems (§§ 3,4).

2. A CLOSURE THEOREM

Let G be a measurable bounded subset of the t-space E^{ν}, $\nu \geq 1$, t =

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$(t^1,\dots,t^\nu)$. It is not restrictive to assume that G is a subset of the interior of the interval $[-1,1]$, or $-1 \leq t^i \leq 1$, $i = 1,\dots,\nu$. We shall denote by -1 and 1 the points $(-1,\dots,-1)$, $(1,\dots,1)$ respectively. Analogously, for $a = (a^1,\dots,a^\nu)$ and $b = (b^1,\dots,b^\nu)$ with $a^i < b^i$, $i = 1,\dots,\nu$, we shall denote by $[a,b]$ the interval $[t \in E^\nu \mid a^i \leq t \leq b^i,\ i = 1,\dots,\nu]$. We shall denote by $\{t\}_R$ the set of all $t = (t^1,\dots,t^\nu) \in E^\nu$ with $t^1,\dots,t^\nu$ rational.

For every $t \in G$ let $A(t)$ be a closed subset of the y-space E^s, $y = (y^1,\dots,y^s)$. Let A be the set of all points $(t,y) \in E^\nu \times E^s$ with $t \in G$, $y \in A(t)$. For every $(t,y) \in A$ let $U(t,y)$ be a nonempty subset of the u-space E^m, $u = (u^1,\dots,u^m)$. Let M be the set of all $(t,y,u) \in E^\nu \times E^s \times E^m$ with $(t,y) \in A$ and $u \in U(t,y)$. For any subset F of G let A_F, M_F denote the sets

$$A_F = \left\{(t,y) \mid t \in F,\ y \in A(t)\right\} \subset A,$$

$$M_F = \left\{(t,y,u) \mid t \in F,\ y \in A(t),\ u \in U(t,y)\right\} \subset M.$$

Let $f(t,y,u) = (f_o, f_1,\dots,f_2)$ be a continuous r+1 vector function on M, and for any point $(t,y) \in A$ let $Q(t,y) \subset E^{r+1}$ denote the set

$$Q(t,y) = \left\{z \in E^{r+1} \;\middle|\; z = f(t,y,u),\ u \in U(t,y)\right\}.$$

We shall denote below by $\psi(t)$, $J(t)$, $t \in G$, given measurable real valued functions on G, and by $y(t)$, $y_k(t)$, $z(t)$, $z_k(t)$, $u_k(t)$, $t \in G$, $k = 1,2,\dots$, given measurable vector functions on G as follows:

$$y(t) = (y^1,\dots,y^s), \qquad y_k(t) = (y_k^1,\dots,y_k^s),$$

$$z(t) = (z^o,\dots,z^r), \qquad z_k(t) = (z_k^o,\dots,z_k^r),$$

$$u_k(t) = (u_k^1,\dots,u_k^m),$$

for $t \in G$ and $k = 1,2,\dots.$ We shall actually set all these functions equal to zero in $E^\nu - G$, and we take

$$D_k(t) = \int_{[-1,t]} z_k^o(t)\,dt \text{ for } t \in [-1,1],\ k = 1,2,\dots.$$

As in our previous paper [4] we denote by $N_\delta(t_o,y_o)$ the set of all $(t,y) \in A$ at a distance $\leq \delta$ from (t_o,y_o). For any $(t_o,y_o) \in A$ and $\delta > 0$ we denote by $Q(t_o,y_o\ ;\delta)$ the set

$$Q(t_o,y_o\ ;\delta) = \bigcup_{(t,y) \in N_\delta(t_o,y_o)} Q(t,y).$$

Finally, if ρ if any integer, $o \leq \rho \leq r+1$, we say that the subsets $Q(t,y)$ of E^{r+1} have property $Q(\rho)$ at a point $(t_o,y_o) \in A$ provided for every $z_o = (z_o^o,\dots,z_o^r) \in E^{r+1}$,

$$Q(t_o,y_o) \cap \left\{ z \in E^{r+1} \,\middle|\, z^i = z_o^i,\ i = \rho,\dots,r \right\}$$

$$= \bigcap_{\epsilon > 0} \bigcap_{\beta > 0} \text{cl co} \left[Q(t_0, y_0, \epsilon) \cap \left\{ z \in E^{r+1} \,\middle|\, |z^i - z_0^i| \leq \beta,\ i = \rho, \ldots, r \right\} \right]$$

We say that the sets $Q(t,y)$ have property $Q(\rho)$ on A if they have this property at every point $(t_0, y_0) \in A$.

It is suitable to use the notations

$$N_\beta(z_0; \rho) = \left\{ z \in E^{r+1} \,\middle|\, |z^i - z_0^i| \leq \beta,\ i = \rho, \ldots, r \right\}.$$

$$N(z_0; \rho) = \left\{ z \in E^{r+1} \,\middle|\, z^i = z_0^i,\ i = \rho, \ldots, r \right\}.$$

As in [4] we shall say that the sets $Q(t,y)$ have the upper set property provided $(t_0, y_0) \in A$, $z_0 = (z_0^0, z_0^1, \ldots, z_0^r) \in Q(t_0, y_0)$ implies that every other point $\bar{z}_0 = (\bar{z}^0, z_0^1, \ldots, z_0^r) \in E^{r+1}$ with $\bar{z}^0 \geq z_0^s$, is also a point of $Q(t_0, y_0)$.

<u>Lemma 2.1</u> Let G be measurable and assume that for every closed subset F of G the set M_F is closed. Let $f(t,y,u) = (f_0, f_1, \ldots, f_r)$, $(t,y,u) \in M$, be a continuous $(r+1)$-vector function on M, and let $y(t) = (y^1, \ldots, y^s)$, $t \in G$, be a measurable s-vector valued function on G with $y(t) \in A(t)$ a.e. on G. Then, for every measurable $(r+1)$-vector function $E(t) = (E^0, \ldots, E^r)$, $t \in G$, with $E(t) \in Q(t, y(t))$ a.e. on G, there exists a measurable m-vector function $u(t) = (u^1, \ldots, u^m)$, $t \in G$, with $u(t) \in U(t, y(t))$ and $E(t) = f(t, y(t), u(t))$ a.e. in G.

This lemma is a well known consequence of a McShane-Warfield theorem [7] and the proof, therefore, is omitted.

Theorem 2.1 (a closure theorem). Let G be a bounded and measurable subset of E^n, for the sake of simplicity, say $G \subset (-1,1)$. Let us assume that for every closed subset F of G the set M_F is closed. Let $\psi(t) \geq 0$, $t \in G$, be a given L-integrable function in G such that $f(t,y,u) \geq -\psi(t)$ for all $(t,y,u) \in M$. Let $J(t)$, $t \in G$, be a bounded positive measurable function satisfying $0 < K^{-1} \leq J(t) \leq K$, $t \in G$, for some fixed constant K. Let ρ be a given integer, $0 \leq \rho \leq r$, let $f(t,y,u) = (f_0, f_1, \dots f_r)$ be continuous on M, and let us assume that the sets $Q(t,y)$ have the upper set property and property $Q(\rho+1)$ on A. Let us assume that

$$y, y_k \in (L,(G))^s, \quad k = 1,2,\dots, \tag{2.1}$$

$$y_k^i \to y^i \text{ strongly in } L_1(G) \text{ as } k \to \infty, \quad i = 1,\dots,s, \tag{2.2}$$

$$z,\ z_k \in (L_1(G))^{r+1}, \quad k = 1,2,\dots \tag{2.3}$$

$$z_k^j \to z^j \text{ weakly in } L_1(G) \text{ as } k \to \infty, \quad j = 1,\dots,\rho, \tag{2.4}$$

$$z_k^j \to z^j \text{ strongly in } L_1(G) \text{ as } k \to \infty, \quad j = \rho+1,\dots,r, \tag{2.5}$$

$$z_k^j(t) = f_j(t,y_k(t),\ u_k(t)).J(t),\ j = 1,\dots,r,\ k = 1,2,\dots,$$

$$\text{a.e. in } G, \tag{2.6}$$

$$z_k^0(t) = (f_0(t,y_k(t),\ u_k(t)) + \psi(t))J(t),\ k = 1,2,\dots,$$

$$\text{a.e. in } G, \tag{2.7}$$

$$y_k(t) \in A(t),\ u_k(t) \in U(t,y_k(t)),\ k = 1,2,\dots,\ \text{a.e. in } G. \tag{2.8}$$

Let $D_o(t)$, $t \in [-1,1]$ be a monotone nondecreasing (in each variable) function of t with $D(-1) = 0$, and assume that $D_k(t) \to D_o(t)$ pointwise as $k \to \infty$ for every $t \in \{t\}_R \cap [-1,1]$. Let us assume that there is a decomposition $D_o(t) = X(t) + S(t)$ of $D_o(t)$ into two parts $X(t) \geq 0$, $S(t) \geq 0$ both defined on $[-1,1]$ with

$$X(t) = \int_{[-1,t]} z^o(t)dt, \qquad z^o(t) \geq 0 \text{ on } [-1,1], \tag{2.9}$$

$z^o \in L_1(G)$, $z^o(t) = 0$ on $[-1,1] - G$, and $S(t)$ a singular function.

Then $y(t) \in A(t)$ a.e. in G, and there is a measurable function $u(t)$, $t \in G$, such that $u(t) \in U(t,y(t))$ a.e. in G, and

$$z^o(t) = (f_o(t,y(t),\ u(t)) + \psi(t))\ J(t) \tag{2.10}$$

$$z^i(t) = f_i(t,y(t),\ u(t))\ J(t),\ i = 1,\dots,r,\ \text{a.e. in } G. \tag{2.11}$$

Proof. We shall first introduce suitable notations.

For $(t,y)\epsilon A$, we define the following sets:

$$Q_\psi^+(t,y) \equiv \{z \in E^{r+1} | z = \mathbf{p} + (\psi(t),0,\ldots,0) \text{ for } p \in Q(t,y)\}$$

and

$$Q_{\psi,J}(t,y) \equiv \{z | z = pJ(t) \text{ for } p \in Q_\psi(t,y)\}.$$

We will work with subsets C_λ, $\lambda = 1,2,\ldots$, of G. For $t_o \in C_\lambda$ and $(t_o,y_o) \in A$, define

$$Q_{\psi,C_\lambda}^+(t_o,y_o,\epsilon) \equiv \bigcup_{(t,y) \in A\cap N_\epsilon(t_o,y_o) \text{ with } t \in C_\lambda} Q_\psi^+(t,y)$$

and

$$Q_{\psi,J,C_\lambda}(t_o,y_o,\epsilon) \equiv \bigcup_{(t,y) \in A\cap N_\epsilon(t_o,y_o) \text{ with } t \in C_\lambda} Q_{\psi,J}(t,y)$$

For any interval $[a,b] \subseteq E^\nu$ and any function $z(t)$, $t \in E^\nu$, we shall consider the usual differences of order ν relative to the 2^ν vertices of $[a,b]$,

$$\Delta z = \Delta_{[a,b]} z = z(b) - z(a) \text{ if } \nu = 1$$

$$\Delta z = \Delta_{[a,b]} z = z(b^1,b^2) - z(b^1,a^2) - z(b^2,a^1) + z(a^1,a^2)$$

if $\nu = 2$, and so on. Using this notation, we deduce from the pointwise convergence $D_k(t) \to D_o(t)$ for $t \in \{t\}_R$, that for all intervals $I = [a,b] \subset [-1,1]$, having rational coordinates

$$\Delta_I D_k = \int_I z_k^o(t)\, dt \to \Delta_I D_o(t) \text{ as } k \to \infty.$$

Let $t_o = (t_o^1,\dots,t_o^\nu)$ denote any point of G, let $S_o = S_o(t_o)$ denote the distance of t_o from the boundary of $[-1,1]$, and let $q = q_h$ denote any closed hypercube

$$q = \{t = (t^1,\dots,t^\nu) \mid \bar{t}^j \leq t^j \leq \bar{t}^j + h,\ j = 1,2,\dots,\nu\}$$

where $\bar{t}^j$ is rational, $j = 1,2,\dots,\nu$, h is a positive rational with $0 < h < \delta_o/\nu$ and $t_o \in q$. By differentiation of multiple integrals and the definition of a singular function we have

$$\lim_{h\to 0} h^{-\nu} \int_q y^i(t)dt = y^i(t_o), \qquad i = 1,2,\dots,s, \tag{2.12}$$

$$\lim_{h\to 0} h^{-\nu} \int_q z^i(t)dt = z^i(t_o), \qquad i = 1,\dots,r, \tag{2.13}$$

$$\lim_{h\to 0} h^{-\nu} \int_q z^o(t)dt = z^o(t_o), \tag{2.14}$$

$$\lim_{h\to 0} h^{-\nu} \Delta_q S = 0 \tag{2.15}$$

for almost all $t_o \in G$.

For almost all $t \in G$, we have $(t,y_k(t)) \in A$ for all $k = 1,2,\ldots$. The convergences $y_k(t) \to y(t)$ in $(L_1(G))^s$ and $z_k^i(t) \to z^i(t)$ in $L_1(G)$, $i = \rho + 1,\ldots,r$, as $k \to \infty$, imply convergence in measure on G, and hence there is a subsequence $[y_{k_p}(t)]$, $[z_{k_p}^i(t)]$ which converges pointwise almost everywhere in G as $p \to \infty$, $i = \rho + 1,\ldots,r$. For simplicity of notations we denote this subsequence still $[k]$. Let G_o be the set of all $t \in G$ where relations (2.12) through (2.15) hold, where $(t,y_k(t)) \in A$ for all k and where

$$\lim_{k\to\infty} y_k(t) = y(t), \quad \lim_{k\to\infty} z_k^i(t) = z^i(t) \qquad i = \rho + 1,\ldots,r. \tag{2.16}$$

We see that G_o is measurable and $|G_o| = |G|$. Since $A(t)$ is a closed set, $y_k(t) \to y(t)$ and $y_k(t) \in A(t)$ for $t \in G_o$, we have $y(t) \in A(t)$ for $t \in G_o$, that is, $y(t) \in A(t)$ almost everywhere in G.

Because of the pointwise convergences (2.16) on G_o with $|G_o| = |G|$, we know that there are closed sets C_λ, $\lambda = 1,2,\ldots$, with $C_\lambda \subseteq G_o$, $C_\lambda \subseteq C_{\lambda+1}$, $|C_\lambda| > |G_o| - \lambda^{-1}$ such that $y(t)$, $y_k(t)$, $z_k^i(t)$, $z^i(t)$, $J(t)$, and $\psi(t)$ are continuous on C_λ, $i = \rho + 1,\ldots,r$, $k = 1,2,\ldots$, limits (2.16) take place uniformly on C_λ as $k \to \infty$, and this holds for every $\lambda = 1,2,\ldots$. Since G is bounded, each set C_λ is compact and hence $y_k(t), y(t)$, $z_k^i(t)$ and $z^i(t)$, $i = \rho + 1,\ldots,r$, are equicontinous and uniformly continuous on each C_λ.

Let λ be any fixed integer, $\lambda > |G|^{-1}$; hence $|C_\lambda| > 0$. Let $\epsilon > 0$ be an arbitrary positive number. There exists some $\delta_o' = \delta_o'(\epsilon,\lambda) > 0$ such that

$|t-t'| < \delta'_o$ with $t,t' \in C_\lambda$ implies $|y(t) - y(t')| \leq \epsilon$ and $|y_k(t) - y_k(t')| \leq \epsilon$ for every $k = 1,2,\ldots$. Also there exists some $k_o = k_o(\epsilon,\lambda)$ such that $k \geq k_o(\epsilon,\lambda)$, $t \in C_\lambda$, implies $|y(t) - y_k(t)| \leq \epsilon$. Let A_λ, M_λ denote the sets $A_\lambda = A_{C_\lambda} = \{(t,y) | t \in C_\lambda,\ y \in A(t)\} \subset A$, and $M_\lambda = M_{C_\lambda} = \{(t,y,u) | (t,y,u) \in M,\ t \in C_\lambda\}$.

Let $X_\lambda(t)$ and $X^*_\lambda(t)$ be the characteristic functions of the sets C_λ and $[-1,1] - C_\lambda$ so that $X_\lambda(t) + X^*_\lambda(t) = 1$ for $t \in G$. All $X_\lambda(t)$ and $X^*_\lambda(t)z^i(t)$ are in $L_1(G)$ for $i = 1,2,\ldots,\rho$. For every $t_o \in C_\lambda$ we have $X_\lambda(t_o) = 1$ and $X^*_\lambda(t_o)z^i(t_o) = 0$, $i = 1,2,\ldots,\rho$. Then for almost all $t_o \in C_\lambda$ we have

$$\lim_{h\to 0^+} h^{-\nu} \int_q X_\lambda(t)dt = \lim_{h\to 0^+} |q \cap C_\lambda|/|q| = 1 \qquad (2.17)$$

$$\lim_{h\to 0^+} h^{-\nu} \int_q X^*_\lambda(t)\ z^i(t)\ dt = 0 \qquad (2.18)$$

where $i = 1,2,\ldots,\rho$. Let C'_λ be the subset of C_λ where (2.17) and (2.18) occur. Let H and H^* be the sets $H = q \cap C'_\lambda$ and $H^* = q - H$. Then C'_λ is measurable, $C'_\lambda \subseteq C_\lambda \subseteq G_o \subseteq G$ and $|C'_\lambda| = |C_\lambda| > 0$, $|\bigcup_{\lambda=1}^{\infty} C'_\lambda| = |G_o| = |G|$.

Let $\eta > 0$ and $\beta > 0$ be any positive numbers independent of ϵ. Let t_o be any point of C'_λ, and set $y_o = y(t_o)$ and $M_1 = \max\{|z^i(t_o)| + 1\}$, where the maximum is taken over $i = 0,1,\ldots,\rho$.

Let us fix h so small that $0 < h < \epsilon/\nu$, $h < \delta_o/\nu$, $h < \delta'_o$ and also so small that

$$|z^i(t_o) - h^{-\nu} \int_q z^i(t)\, dt| \leqq \min(\eta(r+1)^{-1}, 1), \quad i = 0,1,\dots,\rho, \tag{2.19}$$

$$|1 - (|q|/|H|)| \leq \min\{\eta(r+1)^{-1} M_1^{-1}, 1\}, \tag{2.20}$$

$$|h^{-\nu} \int_q z^i(t)\, X^*_\lambda(t)\, dt| \leqq \eta(r+1)^{-1}, \quad i = 1,2,\dots,\rho. \tag{2.21}$$

$$|h^{-\nu} \Delta_q S| \leqq \eta(r+1)^{-1}, \tag{2.22}$$

$$\sup_{t\in q \cap C_\lambda} |z^i(t) - z^i(t_o)| \leqq \min\{\eta(r+1)^{-1}, \beta\}, \quad i = \rho+1,\dots,r, \tag{2.23}$$

This is possible because of relations (2.12-18).

For any integer $k > 0$, let $z_k(t)$ be the $(r+1)$-vector function $z_k(t) \equiv (z_k^o(t), z_k^1(t), \dots, z_k^r(t))$, $t \in G$.

For $t \in H$ and $k \geq k_o(\epsilon,\lambda)$ we have

$$|t-t_o| \leqq \nu h \leqq \min\{\epsilon, \delta_o, \delta_o'\}, \qquad |y_k(t) - y(t_o)| \leqq |y_k(t) - y_k(t_o)|$$

$$+ |y_k(t_o) - y(t_o)| \leqq \epsilon + \epsilon = 2\epsilon,$$

and hence $(t, y_k(t)) \in N_{3\epsilon}(t_o, y_o)$ for $t \in H$ and $k \geq k_o(\epsilon,\lambda)$.

For all $t \in H$ we have, therefore,

$$z_k(t) \in Q_{\psi, J, C_\lambda}(t_o, y_o, 3\epsilon), \qquad k \geq k_o(\epsilon,\lambda). \tag{2.24}$$

The hypothesis of weak convergence of $z_k^i(t)$ to $z^i(t)$, as $k \to \infty$ implies that

$$\lim_{k\to\infty} \int_H z_k^i(t)dt = \int_H z^i(t)dt, \qquad i = 1,2,\ldots,\rho.$$

We can determine an integer $k' = k'(t_o,\epsilon,\lambda,\eta,\beta)$, $k' \geqq k_o(\epsilon,\lambda)$, such that for $k \geq k'$ we have

$$\sup_{t\in H} |z_k^i(t) - z^i(t)| \leqq \min\{\eta(r+1)^{-1},\beta\}, \quad i = \rho+1,\ldots,r, \tag{2.25}$$

$$|\int_H z_k^i(t)\,dt - \int_H z^i(t)\,dt| \leqq \eta(r+1)^{-1}|H|, \quad i = 1,\ldots,\rho. \tag{2.26}$$

Now for $k \geqq k'(t_o,\epsilon,\lambda,\eta,\beta)$, and $i = 1,2,\ldots,\rho$, we have

$$|z^i(t_o) - |H|^{-1} \int_H z_k^i(t)\,dt| \leqq$$

$$|z^i(t_o) - |q|^{-1} \int_q z^i(t)\,dt| + |(|q|^{-1} - |H|^{-1})\int_q z^i(t)\,dt|$$

$$|\,|H|^{-1} \int_H (z^i(t) - z_k^i(t))\,dt| + |\,|H|^{-1} \int_{H^*} z^i(t)\,dt\,|$$

$$\leqq d_1 + d_2 + d_3 + d_4. \tag{2.27}$$

By (2.19) we have $d_1 \leqq \eta(r+1)^{-1}$, by (2.26) we have $d_3 \leqq \eta(r+1)^{-1}$, and by (2.21) we have $d_4 \leqq \eta(r+1)^{-1}$. Also $d_2 = |\,1 - |q|(|H|)^{-1}|\cdot||q|^{-1}\int_q z^i(t)\,dt|$ $\leqq (r+1)^{-1}M_1^{-1}M_1\eta$ by the definition of M_1, (2.19) and (2.20). Thus (2.27) yields for $i = 1,2,\ldots,\rho$,

$$|z^i(t_o) - |H|^{-1} \int_H z_k^i(t)\, dt| \leqq 4(r+1)^{-1}\eta. \tag{2.28}$$

For $i = \rho + 1,\ldots,r$, we have

$$|z^i(t_o) - |H|^{-1} \int_H z_k^i(t)\, dt| \leqq ||H|^{-1} \int_H (z^i(t_o) - z_k^i(t))\, dt|$$

$$\leqq \sup_{t\in H} |z_k^i(t) - z^i(t_o)|. \tag{2.29}$$

Using (2.25) and (2.23) we have for $i = \rho+1,\ldots,r$,

$$|z^i(t_o) - |H|^{-1} \int_H z_k^i(t)\, dt| \leqq 2(r+1)^{-1}\eta \tag{2.30}$$

For $i = 0$, because of assumption (2.7),

$$h^{-\nu}\int_{H^*} z_k^0(t)\, dt \geqq 0 \text{ for all } k. \tag{2.31}$$

We also have

$$h^{-\nu}\int_q z_k^0(t)\, dt = h^{-\nu}\Delta_q D_k \tag{2.32}$$

and

$$h^{-\nu} \int_q z^0(t)\, dt = h^{-\nu}\Delta_q D_0.$$

On the other hand, $D_0 = X + S$ and by (2.22)

$$h^{-\nu}\Delta_q D_0 = h^{-\nu}\Delta_q X + h^{-\nu}\Delta_q S \tag{2.33}$$

with $|h^{-\nu}\Delta_q S| \leqq (r+1)^{-1}\eta$. Also, since $\Delta_q D_k$ approaches $\Delta_q D_0$ as k approaches infinity, $q \subseteq G$ having vertices with rational coordinates, and we can determine $k'(t_o,\epsilon,\lambda,\eta,\beta)$ above so that for $k \geq k'$ we have also

$$|h^{-\nu}\Delta_q D_k - h^{-\nu}\Delta_q D_0| \leqq \eta(r+1)^{-1}. \tag{2.34}$$

Finally, (2.32), (2.33) and (2.34) yield

$$|h^{-\nu}\int_q z_k^0(t)\,dt - h^{-\nu}\int_q z^0(t)\,dt| = |h^{-\nu}\Delta_q D_k - h^{-\nu}\Delta_q X| \leqq$$

$$|h^{-\nu}\Delta_q D_k - h^{-\nu}\Delta_q D_0| + |h^{-\nu}\Delta_q S| \leqq 2(r+1)^{-1}\eta. \tag{2.35}$$

We have

$$z^0(t_o) - |H|^{-1}\int_H z_k^0(t)\,dt \geqq$$

$$- |z^0(t_o) - |q|^{-1}\int_q z^0(t)\,dt|$$

$$-|(|q|^{-1} - |H|^{-1})\int_q z^0(t)\,dt|$$

$$-|\,|H|^{-1}\int_q (z^0(t) - z_k^0(t))\,dt|$$

$$+ |H|^{-1}\int_{H*} z_k^0(t)\,dt = d_5 + d_6 + d_7 + d_8 \tag{2.36}$$

By (2.19) we have $d_5 \geqq -\eta\,(r+1)^{-1}$, by (2.20) and (2.35) we have $d_7 \geqq -4(r+1)^{-1}\eta$, and by (2.31) we have $d_8 \geqq 0$. Also,

$$d_6 = -|(1-|q|(|H|)^{-1})|\cdot||q|^{-1}\int_q z^0(t)\ dt| \geqq -(r+1)^{-1}\eta,$$

where we have used (2.20), (2.19), and the definition of M_1. Hence, (2.36) yields

$$z^0(t_o) - |H|^{-1}\int z_k^0(t)\ dt \geqq -6\eta(r+1)^{-1}. \tag{2.37}$$

We have also

$$|z^i(t_o) - |H|^{-1}\int_H z_k^i(t)\ dt| \leqq 4\eta(r+1)^{-1}, \tag{2.28}$$

$i = 1,2,\ldots,\rho$, and

$$|z^i(t_o) - |H|^{-1}\int_H z^i(t)\ dt| \leqq 2\eta(r+1)^{-1}, \tag{2.30}$$

$i = \rho + 1,\ldots,r.$

Equations (2.23) and (2.25) imply that

$$z_k(t) \in N_{2\beta}(z(t_o);\ \rho + 1) \text{ for } t \in H. \tag{2.38}$$

Because of (2.24) and (2.38)

$$|H|^{-1}\int_H z_k(t)\ dt \in \text{clco}\,(Q_{\psi,J,C\lambda}(t_o,y_o,3\epsilon) \cap N_{2\beta}(z(t_o);\ \rho + 1)).$$

The latter set has the upper set property by statements (3.i), (3.ii) and (3.iii) of [4]. From (2.37), (2.28), and (2.30),

$$z(t_o) \in (\text{clco}(Q_{\psi,J,C_\lambda}(t_o,y_o,3\epsilon) \cap N_{2\beta}(z(t_o) ; \rho + 1))_{12\eta}$$

for every $\eta > 0$, $\beta > 0$, and $\epsilon > 0$. Since η is an arbitrary positive number and the set inside the parenthesis is closed,

$$z(t_o) \in \text{clco}\,(Q_{\psi,J,C_\lambda}(t_o,y_o,3\epsilon) \cap N_{2\beta}(z(t_o) ; \rho + 1))$$

for every $\epsilon > 0$ and every $\beta > C$. By statements (2.v) and (2.vi) of [4], the set $Q_{\psi,J}(t,y)$ has property $Q(\rho+1)$ on A_λ. Therefore, $z(t_o) \in Q_{\psi,J}(t_o,y_o)$ and

$$z(t_o)(J(t_o))^{-1} - (\psi(t_o),0,\dots,0) \in Q(t_o,y_o)$$

for $t_o \in C$. But $|\bigcup_{\lambda=1}^{\infty} C_\lambda| = |G|$. Therefore,

$$z(t)(J(t))^{-1} - (\psi(t),0,\dots,0) \in Q(t,y(t))$$

for almost all $t \in G$. The conclusion of the theorem now follows from lemma (2.1).

3. A PRELIMINARY LOWER CLOSURE THEOREM

We shall use the same notations as in § 2, in particular let $f(t,y,u)$ denote a vector function $f(t,y,u) = (f_o,f_1,\dots,f_r)$ defined on M. Here we shall consider the functional

$$I\,[y,u] = \int_G f_o(t,y(t), u(t))\, J(t)\, dt.$$

Instead of the sets $Q(t,y)$ of § 2, we shall consider here the sets

$$\tilde{\tilde{Q}}(t,y) = \{z = (z^o, z^1, \ldots, z^{r+1}),\ z^o \geq f_o(t,y,u),$$

$$z^i = f_i(t,y,u),\ i = 1,\ldots,r,\ u \in U(t,y)\} \subset E^{r+1}$$

defined for every $(t,y) \in A$.

Theorem 3.1 (a lower closure theorem). Let G, $A(t)$, A, $U(t,y)$, M, and M_F be defined as in § 2, G measurable, A closed, M_F closed for every closed subset F of G. Let $f(t,y,u) = (f_o, f_1, \ldots, f_r)$ be continuous on M, let ρ be any integer, $0 \leq \rho \leq r$, and let us assume that the sets $\tilde{\tilde{Q}}(t,y)$ have property $Q(\rho+1)$ on A. Let $\psi(t)$, $J(t)$, $t \in G$, be measurable functions real valued on G with $0 < K^{-1} \leq J(t) \leq K$ for all $t \in G$ and some constant K. Let $y(t) = (y^1, \ldots, y^s)$, $y_k(t) = (y_k^1, \ldots, y_k^s)$, $z(t) = (z^1, \ldots, z^r)$, $z_k(t) = (z_k^1, \ldots, z_k^r)$, $u_k(t) = (u_k^1, \ldots, u_k^m)$, $t \in G$, be as in Theorem (2.1), $k = 1,2,\ldots$, satisfying (2.1-6) and (2.8-9), and let us assume that $\lim_{k\to\infty} I[y_k, u_k] = a_o < +\infty$. Then $y(t) \in A(t)$ ae. on G, and there exists a measurable function $u(t) = (u^1, \ldots, u^m)$, $t \in G$, such that $u(t) \in U(t,y(t))$, $z^i(t) = f_i(t,y(t), u(t))\, J(t)$, $i = 1,\ldots,m$, ae. on G, and $I[y,u] \leq a_o$.

Proof First set $D_k(t)$, $t \in [-1,1]$, equal to

$$D_k(t) = \int_{[-1,t]} (f_0(t,y_k(t),u_k(t)) + \psi(t))J(t)\, dt,$$

and let e be the real number, $e = \int_{[-1,1]} J(t)\psi(t)\, dt$. We may assume without loss of generality that $0 \leqq D_k(t) \leqq a_0 + e + 1$. Then using a diagonal process, we may extract a subsequence of the original sequence, say still $k = 1,2,\ldots$, so that $D_k(t)$ converges pointwise to a number $D_0(t)$ for each point $t \in [-1,1]$ having all rational coordinates. We have defined $D_0(t)$ on the rationals. For a point $t = (t^1,\ldots,t^\nu) \in [-1,1]$ having at least one irrational coordinate we define $D_0(t) = \sup D_0(\bar{t})$, where sup is taken over all $\bar{t} = (\bar{t}^1,\ldots,\bar{t}^\nu)$, with $\bar{t}^i$ rational, $\bar{t}^i \leq t^i$, $i = 1,\ldots,\nu$.

Hence, $D_0(t)$ is defined on $[-1,1]$. Also, since the functions $D_k(t)$, $k = 1,2,\ldots$, are nonnegative, monotone nondecreasing (**in** each variable) and equibounded, $D_0(t)$ is nonnegative, monotone nondecreasing and bounded. Since $D_0(t)$ is of bounded variation, we may, using the Lebesgue decomposition theorem, decompose $D_0(t)$ as $D_0(t) = X(t) + S(t)$, where

$$X(t) = \int_{[-1,t]} z^0(\tau)\, d\tau, \qquad t \in [-1,1]$$

$S(t) \geq 0$ is singular and monotone decreasing, and $z^0(t) \geq 0$ is L-integrable in $[-1,1]$ and zero on $[-1,1] - G$.

We now set up an auxiliary problem to which we will apply theorem 2.1. In this situation, $A(t)$ and A are defined as above. We define the set $\tilde{U}(t,y)$ as

$$\tilde{U}(t,y) \equiv \{(u^0,u) \in E^{m+1} \mid u^0 \geqq f_0(t,y,u),\ u \in U(t,y)\}$$

for $(t,y) \in A$, and we define the set $\tilde{M}$ as

$$\tilde{M} \equiv \{(t,y,\tilde{u}) \mid (t,y) \in A \text{ and } \tilde{u} \in \tilde{U}(t,y)\}$$

$$= \{(t,y,u^0,u) \mid (t,y) \in A \text{ and } u^0 \geqq f_0(t,y,u)$$

for $u \in U(t,y)\}$.

For any subset F of G, the set $\tilde{M}_F$ is defined as the set of $(t,y,\tilde{u}) \in \tilde{M}$ for which $t \in F$. For each closed subset F of G, since f_0 is continuous and M_F is assumed to be closed, $\tilde{M}_F$ is closed.

Let $\tilde{f}(t,y,\tilde{u}) = (u^0, f_1(t,y,u), \ldots, f_r(t,y,u))$.

The set $Q(t,y) = \{z \in E^{r+1} \mid z = \tilde{f}(t,y,u),\ u \in U(t,y)\}$ has the upper set property, and we have assumed that it also has property $Q(\rho+1)$ on A.

Let $y(t)$, $y_k(t)$, $z(t)$, $z_k(t)$, $J(t)$ and $\psi(t)$ be as in the statement and proof of this theorem, with

$$z_k(t) = (z_k^0(t),\ z_k^1(t), \ldots, z_k^r(t)),$$

$$z_k^0(t) = (u_k^0(t) + \psi(t)) \cdot J(t) \qquad k = 1,2,\ldots,$$

Let $\tilde{u}_k(t)$ be the control vector for the auxiliary problem defined by

$$\tilde{u}_k(t) = (f_0(t,y_k(t),u_k(t)),\ u_k(t)),\ k = 1,2,\ldots.$$

We have $D_k(t)$, $D_0(t)$, $X(t)$ and $S(t)$ defined as above.

Relations (2.1-6), (2.8-9) of Theorem (2.1) hold by hypothesis, and we have just verified (2.7), as well as the pointwise convergence $D_k(t) \to D_o(t)$ as $k \to \infty$ for every $t = (t^1, \ldots, t^\nu)$, $t \in [-1,1]$, with rational coordinates. Thus, theorem (2.1) holds in the present situation. Hence, $y(t) \in A(t)$ ae. on G, and there is a measurable function $\tilde{u}(t) = (u^o,u)$

$= (u^o, u^1, \ldots, u^m)$, $t \in G$, such that

$$u^o \geq f_o(t,y)t),u(t)), \quad u(t) \in U(t,y(t)),$$

$$z^i(t) = f_i(t,y(t), u(t))\, J(t), \quad i = 1,\ldots,r,$$

and

$$z^o(t) = [u^o(t) + \psi(t)]\, J(t) \text{ ae. on } G.$$

We set $u^o(t) = 0$ on $[-1,1] - G$, and then, because of $S(1) \geq 0$, we have

$$\int_{[-1,1]} u^0(t)J(t)dt = \int_{[-1,1]} (z^0(t) - \psi(t)J(t))dt$$

$$= X(1) - e = (\, X(1) + S(1)\,) - (e + S(1)\,)$$

$$= a_0 + e - (\, e + S(1)\,) = a_0 - S(1) \leqq a_0.$$

Because $f_0(t,y(t),u(t))$ is a continuous function of three measurable functions, it is measurable. Also, $\psi(t)$ and $u^0(t)$ are integrable and we have $-\psi(t) \leq f_o(t,y(t),u(t)) \leq u^o(t)$, and $K^{-1} \leq J(t) \leq K$ ae on G. Thus, $f_o(t,y(t), u(t))\, J(t)$ is integrable on G, and

$$I[y,u] = \int_G f_o(t,y(t), u(t))\, J(t)\, dt =$$

$$= \int_{[-1,1]} f_o(t,y(t), u(t)\, J(t)\, dt \leq \int_{[-1,1]} u^O(t)\, J(t)\, dt$$

$$\leq a_o.$$

Theorem 3.1 is thereby proved.

4. A LOWER CLOSURE THEOREM FOR OPTIMIZATION PROBLEMS WITH DISTRIBUTED AND AND BOUNDARY CONTROLS

In this section we state and prove a lower closure theorem for optimization problems with controls, state variables, constraints, and state equations on both the domain and its boundary.

We shall first introduce a few definitions.

We begin with C. B. Morrey's definition of regular transformations of class K from his paper [8].

Let T and S be subsets of Euclidean spaces. A transformation $x = x(y)$ of T onto S is said to be of class K provided it is one to one and continuous, and the functions $x = x(y)$ and $y = y(x)$ satisfy a uniform Lipschitz condition on each compact subset of T and S respectively. The transformation is said to be regular if, in addition, the functions $x(y)$ and $y(x)$ satisfy a uniform Lipschitz condition on the whole of T and S respectively.

Let G be a bounded measurable subset of E^{ν}, $\nu \geq 1$, whose boundary will be denoted by ∂G.

Let Γ_j, $j = 1,2,\dots,N$, be subsets of ∂G, each of which is the image under a regular transformation t_j of class K of a bounded interval R_j' of $E^{\nu-1}$. Let Γ be a closed subset of $\bigcup_{j=1}^{N}\Gamma_j$, and let μ be a measure defined on $\bigcup_{j=1}^{N}\Gamma_j$. For each $j = 1,\dots,N$, we assume that if e is a subset of Γ_j, measurable with respect to μ, then $E = t_j^{-1}(e)$ is measurable with respect to Lebesgue $(\nu\text{-}1)$-dimensional measure $|\ |$ on R'_j. Also, we assume the converse, so that measurable sets on Γ_j and R'_j correspond under t_j, $j = 1,2,\ldots,N$. We assume that there is a constant $K > 1$ such that if $e = t_j(E)$ is μ-measurable, then

$$K^{-1}|E| \leq \mu(e) \leq K|E| \tag{4.1}$$

independent of $j = 1,\ldots,N$. Since μ induces a measure on each set R'_j via the transformation t_j, $j = 1,2,\ldots,N$, we may define $J_j(\bar{t})$, $\bar{t} \in R'_j$, as the functions in $L_1(R'_j)$ which satisfy the equations

$$\mu(t_j(E)) = \int_E J_j(\bar{t})\, d\bar{t} \tag{4.2}$$

for every measurable subset E of R'_j, $j = 1,2,\ldots,N$.

For every $t \in cl(G)$, let $A(t)$ be a nonempty closed subset of y-space E^s. Let A be the set of all points (t,y) with $t \in cl(G)$ and $y \in A(t)$. For every $(t,y) \in A$, let $U(t,y)$ be a nonempty subset of u-space E^m. Let M be the set of all $(t,y,u) \in E^\nu \times E^s \times E^m$ such that $(t,y) \in A$ and $u \in U(t,y)$.

For every $t \in \Gamma$, let $B(t)$ be a nonempty closed subset of $\mathring{y}$-space $E^{s'}$. Let B be the set of all $(t,\mathring{y})$ with $t \in \Gamma$ and $\mathring{y} \in B(t)$. For every $(t,\mathring{y}) \in B$, let $V(t,\mathring{y})$ be a nonempty closed subset of v-space $E^{m'}$. Let $\mathring{M}$ be the set of all $(t,\mathring{y},v) \in E^\nu \times E^{s'} \times E^{m'}$ with $(t,\mathring{y}) \in B$ and $v \in V(t,\mathring{y})$.

Let $\tilde{f}(t,y,u) = (f_0, f_1,\ldots,f_r)$ be a continuous $(r+1)$-vector function on M, and let us consider the sets

$$\tilde{\tilde{Q}}(t,y) = \{z = (z^0,\dots,z^r) \in E^{r+1} \mid z^0 \geqq f_0(t,y,u),$$

$$(z^1,\dots,z^r) = (f_1,\dots,f_r)(t,y,u),\ u \in U(t,y)\} \quad (4.3)$$

Let $\tilde{g}(t,\overset{\circ}{y},v) = (g_0,\dots,g_{r'})$ be a continuous $(r'+1)$-vector function on $\overset{\circ}{M}$, and let us consider the sets

$$\tilde{\tilde{R}}(t,\overset{\circ}{y}) = \{z = (z^0,\dots,z^{r'}) \in E^{r'+1} \mid z^0 \geqq g_0(t,\overset{\circ}{y},v)$$

$$(z^1,\dots,z^{r'}) = (g_1,\dots,g_{r'})(t,\overset{\circ}{y},v),\ v \in V(t,y)\} \quad (4.4)$$

We assume that there are two functions $\psi(t)$, $t \in G$ and $\overset{\circ}{\psi}(t)$, $t \in \Gamma$, such that $f_0(t,y,u) \geqq -\psi(t)$ for all (t,y,u) in M, $\psi(t) \geqq 0$, $\psi(t) \in L_1(G)$, and $g_0(t,\overset{\circ}{y},v) \geqq -\overset{\circ}{\psi}(t)$ for all $(t,\overset{\circ}{y},v)$ in $\overset{\circ}{M}$, $\overset{\circ}{\psi}(t) \geqq 0$, $\overset{\circ}{\psi}(t) \in L_1(\Gamma)$.

We consider here the functional

$$I[y,\overset{\circ}{y},u,v] = \int_G f_0(t,y(t),u(t))\,dt + \int_\Gamma g_0(t,\overset{\circ}{y}(t),\ v(t))\,d\mu.$$

In the lower closure theorem below we shall deal with sequences of functions all defined on G and Γ:

$$z(t) = (z^1,\dots,z^r), \qquad z_k(t) = (z_k^1,\dots,z_k^r),$$

$$y(t) = (y^1,\dots,y^s), \qquad y_k(t) = (y_k^1,\dots,y_k^s),$$

$$u_k(t) = (u_k^1,\dots,u_k^m), \qquad t \in G,\ k = 1,2,\dots$$

$$\overset{\circ}{z}(t) = (\overset{\circ}{z}{}^1,\dots,\overset{\circ}{z}{}^{r'}), \qquad \overset{\circ}{z}_k(t) = (\overset{\circ}{z}{}_k^1,\dots,\overset{\circ}{z}{}_k^1,\dots,\overset{\circ}{z}{}_k^{r'}),$$

$$\overset{\circ}{y}(t) = (\overset{\circ}{y}{}^1, \dots, \overset{\circ}{y}{}^{s'}), \qquad \overset{\circ}{y}_k(t) = (\overset{\circ}{y}{}^1_k, \dots, \overset{\circ}{y}{}^{s'}_k),$$

$$v_k(t) = (v^1_k, \dots, v^{m'}_k), \qquad t \in \Gamma,\ k = 1,2,\dots.$$

<u>Theorem</u> 4.1 (a lower closure theorem). Let G be bounded and measurable, A, B, M, $\overset{\circ}{M}$ closed, $f(t,y,u)$ continuous on M, $g(t,\overset{\circ}{y},v)$ continuous on $\overset{\circ}{M}$, and assume that for some integers ρ, ρ', $0 \le \rho \le r$, $o \le \rho' \le r'$, the sets $\tilde{\tilde{Q}}(t,y)$ have property $Q(\rho+1)$ on A, and the sets $\tilde{\tilde{R}}(t,\overset{\circ}{y})$ have property $Q(\rho'+1)$ on B.

Let us assume that there are functions $\psi(t) \ge 0$, $t \in G$, $\psi \in L_1(G)$ and $\overset{\circ}{\psi}(t) \ge 0$, $t \in \Gamma$, $\overset{\circ}{\psi} \in L_1(\Gamma)$, such that $f_o(t,y,u) \ge -\psi(t)$ for all $(t,y,u) \in M$, and $g_o(t,\overset{\circ}{y},v) \ge -\overset{\circ}{\psi}(t)$ for all $(t,\overset{\circ}{y},v) \in \overset{\circ}{M}$.

Let us assume that the functions $z^i(t)$, $z^i_k(t)$, $y^j(t)$, $y^j_k(t)$, $i = 1,\dots r$, $j = 1,\dots,s$, are in $L_1(G)$, that the functions $u^j_k(t)$ are measurable on G, $j = 1,\dots,m$, that $f_o(t,y_k(t), u_k(t)) \in L_1(G)$, and that

$$y_k(t) \in A(t),\ u_k(t) \in U(t,y_k(t)),\ z^i_k(t) = f_i(t,y_k(t), u_k(t))$$

$$\text{a.e. on } G,\ k = 1,2,\dots \tag{4.5}$$

Let us assume that the functions $\overset{\circ}{z}{}^i(t)$, $\overset{\circ}{z}{}^i_k(t)$, $\overset{\circ}{y}{}^j(t)$, $\overset{\circ}{y}{}^j_k(t)$, $i = 1,\dots,r'$, $j = 1,\dots,s'$, are in $L_1(\Gamma)$, that the functions $v^j_k(t)$ are measurable in Γ, $j = 1,\dots,m'$, that $g_o(t,\overset{\circ}{y}_k(t), v_k(t)) \in L_1(\Gamma)$, and that

$$\overset{\circ}{y}_k(t) \in B(t),\ v_k(t) \in V(t,\overset{\circ}{y}_k(t)),\ \overset{\circ}{z}{}^i_k(t) = g_i(t,\overset{\circ}{y}_k(t),v_k(t))$$

μ.-a.e. on Γ, $k = 1,2,\ldots.$ (4.6)

Finally, let us assume that as $k \to \infty$ we have

$$z_k^i(t) \to z^i(t) \text{ weakly in } L_1(G),\ i = 1,\ldots,\rho, \tag{4.7}$$

$$z_k^i(t) \to z^i(t) \text{ strongly in } L_1(G),\ i = \rho+1,\ldots,r, \tag{4.8}$$

$$y_k^j(t) \to y^j(t) \text{ strongly in } L_1(G),\ j = 1,\ldots,s, \tag{4.9}$$

$$\mathring{z}_k^i(t) \to \mathring{z}^i(t) \text{ weakly in } L_1(\Gamma),\ i = 1,\ldots,\rho', \tag{4.10}$$

$$\mathring{z}_k^i(t) \to \mathring{z}^i(t) \text{ strongly in } L_1(\Gamma),\ i = \rho'+1,\ldots,r', \tag{4.11}$$

$$\mathring{y}_k^j(t) \to \mathring{y}^j(t) \text{ strongly in } L_1(\Gamma),\ j = 1,\ldots,s',$$

$$\underline{\lim}_{k\to\infty}\ I\,[y_k,\mathring{y}_k,\ u_k,v_k] \le a_o < +\infty. \tag{4.12}$$

Then, $y(t) \in A(t)$ a.e. on G, $\mathring{y}(t) \in B(t)$ μ-a.e. on Γ, and there are measurable functions $u(t) = (u^1,\ldots,u^m)$, $t \in G$, and μ- measurable functions $v(t) = (v^1,\ldots,v^{m'})$, $t \in \Gamma$, such that $f_o(t,y(t),\ u(t)) \in L,(G)$, $g_o(t,\mathring{y}(t),$ $v(t)) \in L_1(\Gamma)$, and such that

$u(t) \in U(t,y(t))$, $z^i(t) = f_i(t,y(t),\ u(t))$, $i = 1,\ldots,r$, a.e. on G,

$v(t) \in V(t,\mathring{y}(t))$, $\mathring{z}^i(t) = g_i(t,\mathring{y}(t),\ v(t))$, $i = 1,\ldots,r'$, μ-a e

on Γ, $I\,[y,\mathring{y},u,v] \le a_o$.

<u>Proof</u>: We may write

$$I(y_k,\mathring{y}_k,u_k,v_k) = \int_G f_o(t,y_k(t),u_k(t))\,dt + \sum_{j=1}^{N} \int_{\Delta_j} g_o(t_j(\bar{t}),\ \mathring{y}_k(t_j(\bar{t})),\ v_k(t_j(\bar{t})))\ J_j(\bar{t})\ d\bar{t},$$

where

$$\Delta_j = \mathrm{cl}\ t_j^{-1}\ [\Gamma \cap (\Gamma_j - \bigcup_{i=1}^{j-1} \Gamma_i)], \quad k = 1,2,\ldots.$$

We may assume that $I[y_k,\mathring{y}_k,u_k,v_k] \leqq a_o + 1$. Since $f_o(t,y,u) \geqq -\psi(t)$ for all (t,y,u) in M, the integrals of f_o on G are uniformly bounded below by the number $\int -\mathring{\psi}(t)\,dt$. Also, because of the fact that $g_o(t,\mathring{y},v) \geqq -\mathring{\psi}(t)$ and $J(t) \leqq K$, the integrals of g_o on Δ_j are uniformly bounded below by the numbers

$$\int_{\Delta_j} -\mathring{\psi}(t_j(\bar{t}))\ K\ d\bar{t}, \qquad j = 1,2,\ldots,N.$$

Hence, each of the integrals in $I(y_k,\mathring{y}_k,u_k,v_k)$ on G and Δ_j, $j = 1,2,\ldots,N$, is uniformly bounded above and below. We may, assume, therefore without loss of generality, that

$$\int_{\Delta_j} g_o(t_j(\bar{t}),\ \mathring{y}_k(t_j(\bar{t})),\ v_k(t_j(\bar{t})))\cdot\ J_j(\bar{t})\ d\bar{t}$$

approaches a finite limit a_j, $j = 1,2,\ldots,N$, as k approaches infinity.

We then have

$$\lim_{k\to\infty} \int_G f_o(t,y_k(t),\ u_k(t))\ dt = a_o - \sum_{j=1}^{N} a_j.$$

We shall apply lower closure theorem 3.1 on each set $\Delta_j \subseteq R'_j$, $j = 1,2,\ldots,N$. Here we have $B(\bar{t}) \equiv B(t_j(\bar{t}))$, $\bar{t} \in \Delta_j$, which is a nonempty closed subset of $\mathring{y}$-space $E^{s'}$. Let B_j, $j = 1,2,\ldots,N$, be the set of all $(\bar{t},\mathring{y})$ with $t \in \Delta_j$ and $\mathring{y} \in B(\bar{t})$. For every $(\bar{t},\mathring{y}) \in B_j$, let $V(\bar{t},\mathring{y})$ be defined as the set $V(\bar{t},\mathring{y}) \equiv V(t_j(\bar{t}),\mathring{y})$. Let $\mathring{M}_j$, $j = 1,2,\ldots,N$, be the set

$$\mathring{M}_j = \{(\bar{t},\mathring{y},v) \mid (\bar{t},\mathring{y}) \in B_j \text{ and } v \in V(\bar{t},\mathring{y})\}.$$

Since $\mathring{M}$ is assumed to be closed and Δ_j is closed, $\mathring{M}_j$ is a closed subset of $E^{\nu} \times E^{s'} \times E^{m'}$.

The function $\tilde{g}(t_j(\bar{t}),\ \mathring{y},\ v) = (g_o \ldots, g_{r'})(t_j(\bar{t}),\mathring{y},v)$ is a continuous $(r'+1)$-vector function on $\mathring{M}_j$. Let $\tilde{\tilde{R}}_j(\bar{t},\mathring{y})$ be the set

$$\{z \in E^{r'+1} \mid z^o \geq g_o(t_j(\bar{t}),\mathring{y},v),\ (z,\ldots,z^{r'}) = (g_1,\ldots,g_{r'})$$

$$(t_j(\bar{t}),\mathring{y},v), \text{ for } v \in V(\bar{t},\mathring{y})\} \text{ for } (\bar{t},\mathring{y}) \in B_j\}.$$

Then, we have $\tilde{\tilde{R}}_j(\bar{t},\mathring{y}) = \tilde{\tilde{R}}(t_j(\bar{t}),\mathring{y})$ for $(\bar{t},\mathring{y}) \in B_j$, $j = 1,2,\ldots,N$. Now if $(\bar{t}',\mathring{y}) \in N_\epsilon(\bar{t},\mathring{y}) \cap B_j$, then $(t_j(\bar{t}'),\mathring{y})$ is in $N_{\bar{K}\epsilon+\epsilon}(t,\mathring{y})$ B, where $\bar{K}$ is the Lipschitz constant of the transformation t_j and $t = t_j(\bar{t})$. Hence, for each $j = 1,2,\ldots,N$, and $(\bar{t},\mathring{y}) \in B_j$,

$$\tilde{\tilde{R}}_j(\bar{t},\mathring{y},\epsilon) \subseteq \tilde{\tilde{R}}(t_j(\bar{t}),\mathring{y},\bar{K}\,\epsilon + \epsilon), \text{ and}$$

$$\bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{clco}\,(\tilde{\tilde{R}}_j(\bar{t},\mathring{y},\epsilon) \cap N_\beta(z_o;\, \rho' + 1))$$

is a subset of

$$\bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{clco}(\tilde{\tilde{R}}(t_j(\bar{t}),\, \mathring{y},\, \bar{K}\,\epsilon + \epsilon) \cap N_\beta(z_o;\, \rho' + 1)).$$

We see that, since $\tilde{\tilde{R}}(t,\mathring{y})$ has property $\tilde{\tilde{Q}}(\rho'+1)$ on B, the set $\tilde{\tilde{R}}(\bar{t},\mathring{y})$ has property $Q(\rho'+1)$ on B_j, for each $j = 1,2,\ldots,N$.

We have $(\bar{t},\mathring{y}_k(t_j(\bar{t}))) \in B_j$ and $v_k(t_j(\bar{t}))$ in $V(t_j(\bar{t}),\mathring{y}_k(t_j(\bar{t})))$, a.e. on Δ_j, $j = 1,2,\ldots,N$, and $k = 1,2,\ldots$. For $j = 1,\ldots,N$, $i = 1,\ldots,r'$, and $k = 1,2,\ldots$, let us take

$$\mathring{z}^i(j;\,\bar{t}) = \mathring{z}^i(t_j(\bar{t}))\cdot J_j(\bar{t}),\quad \mathring{z}^i_k(j;\,\bar{t}) = \mathring{z}^i_k(t_j(\bar{t}))\cdot J_j(\bar{t}).$$

By virtue of the convergence relations (4.10), (4.11), and the relation $K^{-1}|E| \leqq \mu(e) \leqq K|E|$, $e = t_j(E)$, we have

$$0 < K^{-1} \leqq J_j(\bar{t}) \leqq K, \quad \text{a.e. on } \Delta_j$$

$$\mathring{z}^i_k(j;\bar{t}) \to \mathring{z}^i(j;\bar{t}) \text{ weakly in } L_1(\Delta_j),\ i = 1,2,\ldots,\rho',$$

$$\mathring{z}^i_k(j;\bar{t}) \to \mathring{z}^i(j;\bar{t}) \text{ strongly in } L_1(\Delta_j),\ i = \rho'+1,\ldots,r',$$

$j = 1,2,\ldots,N$, as k approaches ∞. Finally, $\mathring{\psi}(t_j(\bar{t}))$ is in $L_1(\Delta_j)$,

$\mathring{\psi}(t_j(\bar{t})) \geqq 0$, and

$$g_o(t_j(\bar{t}),\mathring{y},v) \geqq -\mathring{\psi}(t_j(\bar{t})) \text{ for all } (\bar{t},\mathring{y},v)$$

in $\mathring{M}_j$, $j = 1,2,\ldots,N$.

Applying lower closure theorem, 3.1 we see that $\mathring{y}(t_j(\bar{t})) \in B(t_j(\bar{t}))$ and there are measurable controls $v_j(\bar{t})$, $\bar{t} \in \Delta_j$, such that $v_j(\bar{t}) \in V(\bar{t},\mathring{y}(t_j(\bar{t})))$,

$$\mathring{z}^i(t_j(\bar{t})) = g_i(t_j^{-1}(\bar{t})),\ v_j(\bar{t})), \text{ and } \int_{\Delta j} g_o(t,\ \mathring{y}(t),\ v_j(t))d\mu \leq a_j\ ,$$

$j = 1,2,\ldots,N$. Setting $v(t) = v_j(t_j^{-1}(t))$ on $t_j(\Delta_j)$, we see that there is a measurable control $v(t)$, $t \in \Gamma$, such that $\mathring{y}(t) \in B(t)$, $v(t) \in V(t,\mathring{y}(t))$,

$$z^{\circ i}(t) = g_i(t,\mathring{y}(t),v(t)) \qquad \mu \text{ a.e. on } \Gamma,$$

and

$$\int_\Gamma g_o(t,\mathring{y}(t),v(t))\ d\mu \leqq \sum_{j=1}^N a_j.$$

On G itself, we have exactly the situation of the lower closure theorem with $J(t) = 1$. Therefore, $y(t) \in A(t)$, a.e. on G, and there exists a measurable control $u(t)$, $t \in G$, with

$$u(t) \in U(t,y(t)),\ z^i(t) = f_i(t,y(t),u(t)), \text{ a.e. on } G,\ i=1,\ldots,r,$$

$$\int_G f_o(t,y(t),u(t)\ dt \leqq a_o - \sum_{j=1}^N a_j.$$

The conclusion of theorem 4.1 follows from the conclusions of this and the preceding paragraphs.

Remark 1. Suppose that all of the hypotheses of theorem 4.1 hold except that $I(y,\overset{\circ}{y},u,v)$ is written as

$$I(y,\overset{\circ}{y},u,v) = \int_G f_o(t,y(t),u(t))\,dt + \int_\Gamma g_o(t,\overset{\circ}{y}(t),v(t))d\mu + T(y(t),\overset{\circ}{y}(t))$$

We see that we could have proven the same lower closure theorem for $I(y,\overset{\circ}{y},u,v)$ provided that $T(y,\overset{\circ}{y}) \leq \underline{\lim}_{k\to\infty} T(y_k,\overset{\circ}{y}_k)$ and $\int_G f_o(t,y_k(t),u_k(t))\,dt + \int_\Gamma g_o(t,\overset{\circ}{y}_k(t),v_k(t))\,d\mu$ approaches a finite limit as $k \to \infty$.

Remark 2. We mention a variant of theorem 4.1. We may assume that G and Γ are made up of a finite number of components $G_1,\ldots,G_d$ and $\Gamma_1,\ldots,\Gamma_d$, and that, in each of these, there is a different system of control equations similar to the ones on G and Γ in theorem 4.1. Also, we mention that the sets Γ_j throughout this paper are thought of as subsets of the boundary ∂G of G because this will be the main application we have in mind, but actually the sets Γ_j could be subsets of G instead, or even abstract sets in no way connected with G.

Examples

The following two examples illustrate the use of the intermediate properties $Q(\rho)$, $0 \leq \rho \leq r$, used in connection with lower closure theorems in the present paper. Both examples have been mentioned already in [5].

Example 1. Let us consider the problem of the minimum of the cost functional $I[x,u_1,u_2,v] = \iint_G (\zeta^2+\eta^2+x^2+u_1^2+u_2^2)d\zeta dy$ with differential equations

$$x_\zeta = u_1, \qquad x_\eta = u_2, \qquad \text{a.e. in } G,$$

and boundary conditions

$$\gamma x = v \qquad s \text{ a.e. on } \Gamma = \partial G,$$

where $G = [(\zeta,\eta)|\zeta^2 + \eta^2 \leq 1]$, Γ is the boundary of G, s is the arc length on Γ, γx the boundary values of x, and the control functions u_1, u_2, v have their values $(u_1,u_2) \in U = E^2$, $v \in V = \{-1\} \cup \{1\}$. Actually, we want to minimize I in the class Ω of all systems (x,u_1,u_2,v) with u_1,u_2 measurable in G, v measurable on Γ, and x any element of the Sobolev space $W_2^1(G)$. We shall consider here the sets

$$\tilde{\tilde{Q}}(\zeta,\eta,x) = [(z^o,z^1,z^2)|z^o \geq \zeta^2+\eta^2+x^2+u_1^2+u_2^2,\ z^1 = u_1,\ z^2 = u_2,$$

$$(u_1,u_2) \in E^2] \subset E^3, \quad \tilde{\tilde{R}} = [(z^o,z)|z^o \geq 0,\ z = v,\ v = \pm 1] \subset E^2.$$

We have here $r = 2$, the sets $\tilde{\tilde{Q}}$ have property (Q), or Q(3), in $A = d\Gamma \times E^1$ We have also $r' = 1$, the sets $\tilde{\tilde{R}}$ have property Q(1) in $B = \Gamma$, have property (U), but they are not convex and do not have property (Q).

In the search of the minimum of I in Ω we can limit ourselves to those elements $(x_o, u_1, u_2, v) \in \Omega$ with $I \leq M$ for some constant M. Here $f_o = \zeta^2 + \eta^2 + x^2 + u_1^2 + u_2^2$, $g_o = 0$, hence $\psi = 0$, $\mathring{\psi} = 0$. We take $z(t) = (x_\zeta, x_\eta)$, $y(t) = x$, $\mathring{z}(t) = \gamma x$, $\mathring{y}(t) = 0$. If $[x_k]$ is a minimizing sequence, hence $\|x_k\|_{W_2^1} \leq N$ for some constant N, there is a subsequence, say still [k] for the sake of simplicity, such that $x_k \to x$ weakly in $W_2^1(G)$, $z_k \to z$ weakly in $(L_2(G))^2$, $y_k \to y$ strongly in $L_2(G)$, $\mathring{z}_k \to \mathring{z}$ strongly in $L_2(\Gamma)$, $\mathring{y}_k \to \mathring{y}$ strongly in $L_2(\Gamma)$. Lower closure theorem (4.1) may be applied witj $\rho = 2$, $\rho' = 0$.

Example 2. Let us consider the problem of the minimum of the cost functional

$$I[x,u_1,u_2,v] = \iint_G (x^2+x_\zeta^2+x_\eta^2+u_1^2+u_2^2(1-u_2)^2)\, d\zeta\, d\eta + \int_\Gamma (\gamma x-1)^2 ds$$

with differential equations

$$x_{\zeta\zeta} + x_{\eta\eta} = u_1, \qquad u_\zeta + u_\eta = u_2 \qquad \text{a.e. in } G,$$

$$\gamma x_\zeta = \cos v, \quad \gamma x_\eta = \sin v, \qquad s\text{- a.e. on } \Gamma = \partial G,$$

where G and Γ are as in example 1, where γx denotes the boundary values of x, and the control functions u_1, u_2, v have their values $(u_1, u_2) \in U = E^2$, $v \in V = E^1$. We want to minimize I in a class Ω of systems (x, u_1, u_2, v) with u_1, u_2 measurable in G, v measurable on Γ, x any element of the Sobolev space $W_2^2(G)$ satisfying an inequality $\|x_{\zeta\zeta}\|_2^2+\|x_{\zeta\eta}\|_2^2+\|x_{\eta\eta}\|_2^2 \leq M$ (M a constant large enough so that Ω is not empty). We shall consider here the sets

$$\tilde{\tilde{Q}}(y) = [(z^o, z^1, z^2) \,|\, z^o \geq y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2(1-u_2)^2,\ z^1 = u_1,$$

$$z^2 = u_2,\ (u_1, u_2) \in E^2] \subset E^3,$$

$$\tilde{\tilde{R}}(\mathring{y}) = [(z^o, z^1, z^2) \,|\, z^o \geq (\mathring{y}-1)^2,\ z^1 = \cos v,\ z^2 = \sin v,\ v \in E^1]$$

$$\subset E^3,$$

where $y = (y_1, y_2, y_3)$ in $\tilde{\tilde{Q}}(y)$, and $\mathring{y}$ in $\tilde{\tilde{R}}(\mathring{y})$ are arbitrary. Here we have $r = 2$, $r' = 2$. The sets $\tilde{\tilde{Q}}$ have property Q(2), but they are not convex, and do not have property (Q), or Q(3). The sets $\tilde{\tilde{R}}$ have property Q(1), but they are not convex, and do not have property (Q), or Q(3). They all have property Q(0), or (U). Here $f_o = y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2(1 - u_2)^2$, $g_o = (y-1)^2$, and we can take $\psi = 0$, $\mathring{\psi} = 0$. We have here $z(t) = (x_{\zeta\zeta} + x_{\zeta\eta}, x_\zeta + x_\eta)$, $y(t) = (x, x_\zeta, x_\eta)$, $\mathring{z}(t) = (\gamma x_\zeta, \gamma x_\eta)$, $\mathring{y}(t) = \gamma x$. If $[x_k]$ is any sequence $x_k \in \{x\}_\Omega$, then there is a subsequence, say still [k], such that $x_k \to x$ weakly in $W_2^2(G)$, $(z_k)^1 \to (z)^1$ weakly in $L_2(G)$, $(z_k)^2 \to (z)^2$ strongly in $L_2(G)$, $y_k \to y$ strongly in $(L_2(G))^3$, $\mathring{z}_k \to \mathring{z}$ strongly in $L_2(\Gamma)$, and $\mathring{y}_k \to \mathring{y}$ strongly in $L_2(\Gamma)$. Lower closure theorem (4.1) applies with $\rho = 1$ and $\rho' = 0$.

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