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LOWER CLOSURE THEOREMS FOR LAGRANGE PROBLEMS
OF OPTIMIZATION WITH DISTRIBUTED AND BOUNDARY CONTROLS

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I. INTRODUCTION

In this paper we prove lower closure theorems for multidimensional problems of optimization with distributed and boundary controls. The concept of lower closure, introduced by Cesari [1,2,3,4] in connection with his existence theorems for optimal solutions, has the same role for Lagrange problems that Tonelli's lower semicontinuity has for free problems.

The present analysis extends Cesari's theory in [3,4], but differs from it in two respects. First, we use the property \( Q(\rho), 0 \leq \rho \leq r+1, \) of upper semicontinuity of variable sets in \( E^{r+1} \), which we introduced in a previous paper (D. E. Cowles [5]), instead of properties (U) (Kuratowski) and (Q) (Cesari) used in [3,4]. Property \( Q(\rho) \) reduces to property (U) for \( \rho = 0 \), and to property (Q) for \( \rho = r+1 \), as we proved in [5]. Also, for every \( 0 \leq \rho \leq r \), property \( Q(\rho+1) \) implies property \( Q(\rho) \) (see our paper [5]).

As in Cesari's analysis, we first prove a closure theorem (§ 2), which is then used to prove lower closure theorems (§§ 3,4).

2. A CLOSURE THEOREM

Let \( G \) be a measurable bounded subset of the \( t \)-space \( E^v, \ v \geq 1, \ t = \)...
It is not restrictive to assume that $G$ is a subset of the interior of the interval $[-1,1]$, or $-1 \leq t^i \leq 1$, $i = 1,\ldots,v$. We shall denote by $-1$ and $1$ the points $(-1,\ldots,-1)$, $(1,\ldots,1)$ respectively. Analogously, for $a = (a^1,\ldots,a^v)$ and $b = (b^1,\ldots,b^v)$ with $a^i < b^i$, $i = 1,\ldots,v$, we shall denote by $[a,b]$ the interval $\{t \in E^v \mid a^i \leq t^i \leq b^i, i = 1,\ldots,v\}$. We shall denote by $\{t\} \subset \mathbb{R}$ the set of all $t = (t^1,\ldots,t^v) \in E^v$ with $t^1,\ldots,t^v$ rational.

For every $t \in G$ let $A(t)$ be a closed subset of the $y$-space $E^S$, $y = (y^1,\ldots,y^S)$. Let $A$ be the set of all points $(t,y) \in E^v \times E^S$ with $t \in G$, $y \in A(t)$. For every $(t,y) \in A$ let $U(t,y)$ be a nonempty subset of the $u$-space $E^m$, $u = (u^1,\ldots,u^m)$. Let $M$ be the set of all $(t,y,u) \in E^v \times E^S \times E^m$ with $(t,y) \in A$ and $u \in U(t,y)$. For any subset $F$ of $G$ let $A_F$, $M_F$ denote the sets

$$A_F = \{(t,y) \mid t \in F, y \in A(t)\} \subset A,$$

$$M_F = \{(t,y,u) \mid t \in F, y \in A(t), u \in U(t,y)\} \subset M.$$

Let $f(t,y,u) = (f^1(t,y,u),\ldots,f^r(t,y,u))$ be a continuous $r+1$ vector function on $M$, and for any point $(t,y) \in A$ let $Q(t,y) \subset E^{r+1}$ denote the set

$$Q(t,y) = \left\{ z \in E^{r+1} \mid z = f(t,y,u), u \in U(t,y) \right\}.$$

We shall denote below by $\psi(t)$, $J(t)$, $t \in G$, given measurable real valued functions on $G$, and by $y(t)$, $y_k(t)$, $z(t)$, $z_k(t)$, $u_k(t)$, $t \in G$, $k = 1,2,\ldots$, given measurable vector functions on $G$ as follows:
\[ \begin{align*}
    y(t) &= (y^1, \ldots, y^s), \\
    y_k(t) &= (y^1_k, \ldots, y^s_k), \\
    z(t) &= (z^0, \ldots, z^r), \\
    z_k(t) &= (z^0_k, \ldots, z^r_k), \\
    u_k(t) &= (u^1_k, \ldots, u^m_k),
\end{align*} \]

for \( t \in G \) and \( k = 1, 2, \ldots \). We shall actually set all these functions equal to zero in \( \mathbb{E}^y - G \), and we take

\[ D_k(t) = \int_{[-1,t]} z^0_k(t) \, dt \quad \text{for} \quad t \in [-1,1], \quad k = 1, 2, \ldots. \]

As in our previous paper [4] we denote by \( N^{(t_o, y_o)}(\delta) \) the set of all \((t, y) \in A\) at a distance \( \leq \delta \) from \((t_o, y_o)\). For any \((t_o, y_o) \in A\) and \( \delta > 0 \) we denote by \( Q(t_o, y_o ; \delta) \) the set

\[ Q(t_o, y_o ; \delta) = \bigcup_{(t, y) \in N^{(t_o, y_o)}(\delta)} Q(t, y). \]

Finally, if \( p \) is any integer, \( 0 \leq p \leq r+1 \), we say that the subsets \( Q(t, y) \) of \( \mathbb{E}^{r+1} \) have property \( Q(p) \) at a point \((t_o, y_o) \in A\) provided for every \( z_o = (z^0_o, \ldots, z^r_o) \in \mathbb{E}^{r+1} \),

\[ Q(t_o, y_o) \cap \left\{ z \in \mathbb{E}^{r+1} \mid z^i = z^i_o, \; i = p, \ldots, r \right\} \]
\[ = \cap_{\varepsilon > 0} \cap_{\rho > 0} \text{cl} \text{co} \left[ Q(t_o, y_o, \varepsilon) \cap \left\{ z \in E^{r+1} \mid |z^i - z^i_0| \leq \beta, i = \rho, \ldots, r \right\} \right] \]

We say that the sets \( Q(t, y) \) have property \( Q(\rho) \) on \( A \) if they have this property at every point \( (t_o, y_o) \in A \).

It is suitable to use the notations

\[ N_\beta(z_o; \rho) = \left\{ z \in E^{r+1} \mid |z^i - z^i_0| \leq \beta, i = \rho, \ldots, r \right\}. \]

\[ N(z_o; \rho) = \left\{ z \in E^{r+1} \mid z^i - z^i_0 = z^i_0, i = \rho, \ldots, r \right\}. \]

As in [4] we shall say that the sets \( Q(t, y) \) have the upper set property provided \( (t_o, y_o) \in A, z_o = (z^0_o, z^1_o, \ldots, z^r_o) \in Q(t_o, y_o) \) implies that every other point \( z_o' = (z^0_o', z^1_o, \ldots, z^r_o) \in E^{r+1} \) with \( z^0_o' \geq z^0_o \) is also a point of \( Q(t_o, y_o) \).

**Lemma 2.1** Let \( G \) be measurable and assume that for every closed subset \( F \) of \( G \) the set \( M_F \) is closed. Let \( f(t, y, u) = (f^1_o, f^1_1, \ldots, f^1_r), (t, y, u) \in M \), be a continuous \((r+1)\)-vector function on \( M \), and let \( y(t) = (y^1, \ldots, y^s), t \in G \), be a measurable \( s \)-vector valued function on \( G \) with \( y(t) \in A(t) \text{ a.e. on } G \).

Then, for every measurable \((r+1)\)-vector function \( E(t) = (E^0, \ldots, E^r), t \in G \), with \( E(t) \in Q(t, y(t)) \text{ a.e. on } G \), there exists a measurable \( m \)-vector function \( u(t) = (u^1, \ldots, u^m), t \in G \), with \( u(t) \in U(t, y(t)) \) and \( E(t) = f(t, y(t), u(t)) \text{ a.e. in } G \).
This lemma is a well known consequence of a McShane-Warfield theorem [7] and the proof, therefore, is omitted.

**Theorem 2.1** (a closure theorem). Let \( G \) be a bounded and measurable subset of \( \mathbb{E}^n \), for the sake of simplicity, say \( G \subset (-1,1) \). Let us assume that for every closed subset \( F \) of \( G \) the set \( M_F \) is closed. Let \( \psi(t) \geq 0, \ t \in G \), be a given \( L \)-integrable function in \( G \) such that \( f(t,y,u) \geq -\psi(t) \) for all \( (t,y,u) \in M \). Let \( \mathcal{J}(t), \ t \in G \), be a bounded positive measurable function satisfying \( 0 < K^{-1} \leq \mathcal{J}(t) \leq K, \ t \in G \), for some fixed constant \( K \). Let \( \rho \) be a given integer, \( 0 \leq \rho \leq r \), let \( f(t,y,u) = (f_0, f_1, \ldots, f_r) \) be continuous on \( M \), and let us assume that the sets \( Q(t,y) \) have the upper set property and property \( Q(\rho+1) \) on \( A \). Let us assume that

\[
y, y_k \in (L_1(G))^s, \ k = 1,2,\ldots, \tag{2.1}
\]

\[
y^i_k \to y^i \text{ strongly in } L_1(G) \text{ as } k \to \infty, \ i = 1, \ldots, s, \tag{2.2}
\]

\[
z, z_k \in (L_1(G))^{r+1}, \ k = 1,2,\ldots \tag{2.3}
\]

\[
z^j_k \to z^j \text{ weakly in } L_1(G) \text{ as } k \to \infty, \ j = 1, \ldots, \rho, \tag{2.4}
\]

\[
z^j_k \to z^j \text{ strongly in } L_1(G) \text{ as } k \to \infty, \ j = \rho+1, \ldots, r, \tag{2.5}
\]
\[
\begin{align*}
\frac{z_j^k(t)}{k} = f_j(t, y_k(t), u_k(t)) J(t), \quad j = 1, \ldots, r, \quad k = 1, 2, \ldots, \\
a.e. \text{ in } G, \\
z_k^0(t) = (f_0(t, y_k(t), u_k(t)) + \psi(t)) J(t), \quad k = 1, 2, \ldots, \\
a.e. \text{ in } G, \\
y_k(t) \in A(t), \quad u_k(t) \in U(t, y_k(t)), \quad k = 1, 2, \ldots, \quad a.e. \text{ in } G. (2.8)
\end{align*}
\]

Let \( D_0(t), \ t \in [-1, 1] \) be a monotone nondecreasing (in each variable) function of \( t \) with \( D(-1) = 0 \), and assume that \( D_k(t) + D_0(t) \) pointwise as \( k \to \infty \) for every \( t \in [t]_R \cap [-1, 1] \). Let us assume that there is a decomposition \( D_0(t) = X(t) + S(t) \) of \( D_0(t) \) into two parts \( X(t) \geq 0, \ S(t) \geq 0 \) both defined on \([-1, 1]\) with

\[
X(t) = \int_{[-1, t]} z_0^o(t) dt, \quad z_0^o(t) \geq 0 \text{ on } [-1, 1], \quad (2.9)
\]

\( z_0^o \in L_1(G), \ z_0^o(t) = 0 \text{ on } [-1, 1] - G, \) and \( S(t) \) a singular function.

Then \( y(t) \in A(t) \) a.e. in \( G \), and there is a measurable function \( u(t), \ t \in G \), such that \( u(t) \in U(t, y(t)) \) a.e. in \( G \), and

\[
\begin{align*}
\frac{z_0^i(t)}{i} &= (f_0(t, y(t), u(t)) + \psi(t)) J(t) \\
(2.10)
\end{align*}
\]

\[
\begin{align*}
\frac{z_i(t)}{i} &= f_i(t, y(t), u(t)) J(t), \quad i = 1, \ldots, r, \quad a.e. \text{ in } G. (2.11)
\end{align*}
\]
Proof. We shall first introduce suitable notations.

For \((t,y)\in A\), we define the following sets:

\[ Q^+_{\psi}(t,y) \equiv \{ z \in E^{r+1} \mid z = p + (\psi(t),O,\ldots,O) \text{ for } p \in Q(t,y) \} \]

and

\[ Q_{\psi,J}(t,y) \equiv \{ z \mid z = pJ(t) \text{ for } p \in Q_{\psi}(t,y) \}. \]

We will work with subsets \(C^\lambda\), \(\lambda = 1,2,\ldots\), of \(G\). For \(t_o \in C^\lambda\) and \((t_o,y_o) \in A\), define

\[ Q^+_{\psi,C}(t_o,y_o,\epsilon) \equiv \bigcup_{(t,y) \in \text{ANN}_{\epsilon}(t_o,y_o)} Q^+_{\psi}(t,y) \]

with \(t \in C^\lambda\)

and

\[ Q_{\psi,J,C}(t_o,y_o,\epsilon) \equiv \bigcup_{(t,y) \in \text{ANN}_{\epsilon}(t_o,y_o)} Q_{\psi,J}(t,y) \]

with \(t \in C^\lambda\).

For any interval \([a,b] \subseteq E^v\) and any function \(z(t)\), \(t \in E^v\), we shall consider the usual differences of order \(v\) relative to the \(2^v\) vertices of \([a,b]\),

\[ \Delta z = \Delta_{[a,b]} z = z(b) - z(a) \text{ if } v = 1 \]

\[ \Delta z = \Delta_{[a,b]} z = z(b^1, b^2) - z(b^1, a^2) - z(b^2, a^1) + z(a^1, a^2) \]
if \( v = 2 \), and so on. Using this notation, we deduce from the pointwise convergence \( D_k(t) \to D_0(t) \) for \( t \in \{ t \}_R \), that for all intervals \( I = [a, b] \subset [-1, 1] \), having rational coordinates

\[
\Delta^D_k = \int_I z^D_k(t) \, dt - \Delta^D_0(t) \text{ as } k \to \infty.
\]

Let \( t = (t^1_0, \ldots, t^v_0) \) denote any point of \( G \), let \( S = S(t_0) \) denote the distance of \( t_0 \) from the boundary of \([-1, 1] \), and let \( q = q_h \) denote any closed hypercube

\[
q = \{ t = (t^1, \ldots, t^v) \mid \bar{t}^j \leq t^j \leq \bar{t}^j + h, \ j = 1, 2, \ldots, v \}
\]

where \( \bar{t}^j \) is rational, \( j = 1, 2, \ldots, v \), \( h \) is a positive rational with \( 0 < h < \frac{\varepsilon}{S} \) and \( t_0 \in q \). By differentiation of multiple integrals and the definition of a singular function we have

\[
\lim_{h \to 0} h^{-v} \int_y^i(t) \, dt = y^i(t_0), \quad i = 1, 2, \ldots, s, \quad (2.12)
\]

\[
\lim_{h \to 0} h^{-v} \int z^i(t) \, dt = z^i(t_0), \quad i = 1, \ldots, r, \quad (2.13)
\]

\[
\lim_{h \to 0} h^{-v} \int z^0(t) \, dt = z^0(t_0), \quad (2.14)
\]

\[
\lim_{h \to 0} h^{-v} \Delta S = 0 \quad (2.15)
\]

for almost all \( t_0 \in G \).
For almost all \( t \in G \), we have \((t, y_k(t)) \in A\) for all \( k = 1, 2, \ldots\). The convergences \( y_k(t) \to y(t) \) in \( L_1(G)^s \) and \( z_k^i(t) \to z^i(t) \) in \( L_1(G) \),
i = \rho + 1, \ldots, r,\text{ as } k \to \infty,\text{ imply convergence in measure on } G,\text{ and hence}
there is a subsequence \([y_k(t)]_p, [z_k^i(t)]_p\) which converges pointwise almost everywhere in \( G \) as \( p \to \infty, \ i = \rho + 1, \ldots, r \). For simplicity of notations we denote this subsequence still \([k]\). Let \( G_0 \) be the set of all \( t \in G \) where relations (2.12) through (2.15) hold, where \((t, y_k(t)) \in A\) for all \( k \) and where

\[
\lim_{k \to \infty} y_k(t) = y(t), \quad \lim_{k \to \infty} z_k^i(t) = z^i(t) \quad i = \rho + 1, \ldots, r.
\]

(2.16)

We see that \( G_0 \) is measurable and \(|G_0| = |G|\). Since \( A(t) \) is a closed set, \( y_k(t) \to y(t) \) and \( y_k(t) \in A(t) \) for \( t \in G_0 \), we have \( y(t) \in A(t) \) for \( t \in G_0 \),
that is, \( y(t) \in A(t) \) almost everywhere in \( G \).

Because of the pointwise convergences (2.16) on \( G_0 \) with \(|G_0| = |G|\),
we know that there are closed sets \( C_{\lambda}, \lambda = 1, 2, \ldots, \) with \( C_{\lambda} \subseteq G_0 \),
\( C_{\lambda} \subseteq C_{\lambda+1}, |C_\lambda| > |G_0| - \lambda^{-1} \) such that \( y(t), y_k(t), z_k^i(t), z^i(t), J(t), \) and
\( \psi(t) \) are continuous on \( C_{\lambda}, i = \rho + 1, \ldots, r, k = 1, 2, \ldots, \) limits (2.16)
take place uniformly on \( C_{\lambda} \) as \( k \to \infty \), and this holds for every \( \lambda = 1, 2, \ldots \).
Since \( G \) is bounded, each set \( C_{\lambda} \) is compact and hence \( y_k(t), y(t), z_k^i(t) \)
and \( z^i(t), i = \rho + 1, \ldots, r, \) are equicontinuous and uniformly continuous on each \( C_{\lambda} \).

Let \( \lambda \) be any fixed integer, \( \lambda > |G|^{-1} \); hence \(|C_\lambda| > 0\). Let \( \epsilon > 0 \) be an arbitrary positive number. There exists some \( \delta_0' = \delta_0'(\epsilon, \lambda) > 0 \) such that
\(|t-t'| < \varepsilon_o \) with \( t, t' \in C_\lambda \) implies \(|y(t) - y(t')| \leq \varepsilon \) and \(|y_k(t) - y_k(t')| \leq \varepsilon \) for every \( k = 1, 2, \ldots \). Also there exists some \( k_o = k_o(\varepsilon, \lambda) \) such that \( k \geq k_o(\varepsilon, \lambda), t \in C_\lambda \), implies \(|y(t) - y_k(t)| \leq \varepsilon \). Let \( A_{C_\lambda}, M_{C_\lambda} \) denote the sets \( A_{C_\lambda} = \{ (t, y) | t \in C_\lambda, y \in A(t) \} \subseteq A \), and \( M_{C_\lambda} = \{ (t, y, u) | (t, y, u) \in M, t \in C_\lambda \} \).

Let \( X_{\lambda}(t) \) and \( X^*_\lambda(t) \) be the characteristic functions of the sets \( C_\lambda \) and \([-1,1] - C_\lambda \) so that \( X_{\lambda}(t) + X^*_\lambda(t) = 1 \) for \( t \in G \). All \( X_{\lambda}(t) \) and \( X^*_\lambda(t)z^i(t) \) are in \( L_1(G) \) for \( i = 1, 2, \ldots, \rho \). For every \( t_o \in C_\lambda \) we have \( X_{\lambda}(t_o) = 1 \) and \( X^*_\lambda(t_o)z^i(t_o) = 0, i = 1, 2, \ldots, \rho \). Then for almost all \( t_o \in C_\lambda \) we have

\[
\lim_{h \to 0} + \int_{q} X_{\lambda}(t) dt = \lim_{h \to 0} + \frac{|q \cap C_\lambda|}{|q|} = 1 \quad (2.17)
\]

\[
\lim_{h \to 0} + \int_{q} X^*_\lambda(t) z^i(t) dt = 0 \quad (2.18)
\]

where \( i = 1, 2, \ldots, \rho \). Let \( C'_\lambda \) be the subset of \( C_\lambda \) where (2.17) and (2.18) occur. Let \( H \) and \( H^* \) be the sets \( H = q \cap C'_\lambda \) and \( H^* = q - H \). Then \( C'_\lambda \) is measurable, \( C'_\lambda \subseteq C_\lambda \subseteq G_o \subseteq G \) and \( |C'_\lambda| = |C_\lambda| > 0, \bigcup_{\lambda=1}^{\infty} C'_\lambda = |G_o| = |G| \).

Let \( \eta > 0 \) and \( \beta > 0 \) be any positive numbers independent of \( \varepsilon \). Let \( t_o \) be any point of \( C'_\lambda \), and set \( y_o = y(t_o) \) and \( M_1 = \max\{|z^i(t_o)| + 1\} \), where the maximum is taken over \( i = 0, 1, \ldots, \rho \).

Let us fix \( h \) so small that \( 0 < h < \varepsilon/v, h < \delta_o/v, h < \delta'_o \) and also so small that
\[ |z_i(t_o) - h^{-\nu} \int_t^{t_o} z_i(t) \, dt| \leq \min \{ \eta^{-1}(r+1) \, l \}, \quad i = 0,1,\ldots,\rho, \]  
(2.19)

\[ |1 - (|q|/|H|)| \leq \min \{ \eta^{-1}(r+1) \, M_1^{-1} \, l \}, \]  
(2.20)

\[ |h^{-\nu} \int_t^{t_o} z_i(t) X_{\lambda}(t) \, dt| \leq \eta^{-1}(r+1), \quad i = 1,2,\ldots,\rho. \]  
(2.21)

\[ |h^{-\nu} A S| \leq \eta^{-1}(r+1), \]  
(2.22)

\[ \sup_{t \in q} \left| z_i(t) - z_i(t_o) \right| \leq \min \{ \eta^{-1}(r+1), \beta \}, \quad i = \rho + 1,\ldots,r, \]  
(2.23)

This is possible because of relations (2.12-18).

For any integer \( k > 0 \), let \( z_k(t) \) be the \((r+1)\)-vector function \( z_k(t) \equiv (z_k^o(t),z_k^1(t),\ldots,z_k^r(t)) \), \( t \in G \).

For \( t \in H \) and \( k \geq k_o(\epsilon,\lambda) \) we have

\[ |t-t_o| \leq \nu h \leq \min \{ \epsilon,\delta^o,\delta^o' \}, \quad |y_k(t) - y(t_o)| \leq |y_k(t) - y_k(t_o)| \]

\[ + |y_k(t_o) - y(t_o)| \leq \epsilon + \epsilon = 2 \epsilon, \]

and hence \( (t,y_k(t)) \in N_{\epsilon}(t_o,y_o) \) for \( t \in H \) and \( k \geq k_o(\epsilon,\lambda) \).

For all \( t \in H \) we have, therefore,

\[ z_k(t) \in Q_{\Psi,J,C_\lambda}(t_o,y_o,3\epsilon), \quad k \geq k_o(\epsilon,\lambda). \]  
(2.24)

The hypothesis of weak convergence of \( z_k(t) \) to \( z(t) \), as \( k \to \infty \) implies
that

$$\lim_{k \to \infty} \int_{H} z^i(t) \, dt = \int_{H} z^i(t) \, dt, \quad i = 1, 2, \ldots, \rho.$$  

We can determine an integer \( k' = k'(t_0, \epsilon, \lambda, \eta, \beta) \), \( k' \geq k(\epsilon, \lambda) \), such that for \( k \geq k' \) we have

$$\sup_{t \in H} |z^i_k(t) - z^i(t)| \leq \min \{ \eta(r+1)^{-1} \beta \}, \quad i = \rho + 1, \ldots, r, \quad (2.25)$$

$$|\int_{H} z^i_k(t) \, dt - \int_{H} z^i(t) \, dt| \leq \eta(r+1)^{-1} |H|, \quad i = 1, \ldots, \rho. \quad (2.26)$$

Now for \( k \leq k'(t_0, \epsilon, \lambda, \eta, \beta) \), and \( i = 1, 2, \ldots, \rho \), we have

$$|z^i(t_0) - |H|^{-1} \int_{H} z^i_k(t) \, dt| \leq$$

$$|z^i(t_0) - |q|^{-1} \int_{q} z^i(t) \, dt| + |(|q|^{-1} - |H|^{-1}) \int_{q} z^i(t) \, dt|$$

$$+ |H|^{-1} \int_{H} (z^i_k(t) - z^i_k(t)) \, dt| + |H|^{-1} \int_{H^*} z^i(t) \, dt|$$

$$\leq d_1 + d_2 + d_3 + d_4. \quad (2.27)$$

By (2.19) we have \( d_1 \leq \eta(r+1)^{-1} \), by (2.26) we have \( d_2 \leq \eta(r+1)^{-1} \), and by (2.21) we have \( d_3 \leq \eta(r+1)^{-1} \). Also \( d_2 = |1 - |q||(H)|^{-1}||q|^{-1} \int_{q} z^i(t) \, dt| \leq (r+1)^{-1} M_1^{1} \eta \) by the definition of \( M_1 \), (2.19) and (2.20). Thus (2.27) yields for \( i = 1, 2, \ldots, \rho \),
\[ |z'(t_o) - |H|^{-1} \int_H z_k'(t) \, dt| \leq \eta (r+1)^{-1}. \]  
(2.28)

For \( i = \rho + 1, \ldots, r \), we have

\[ |z'(t_o) - |H|^{-1} \int_H z_k'(t) \, dt| \leq ||H|^{-1} \int_H (z(t_o) - z_k(t)) \, dt| \]

\[ \leq \sup_{t \in H} |z_k'(t) - z_{i^*}'(t)\rangle. \]  
(2.29)

Using (2.25) and (2.23) we have for \( i = \rho + 1, \ldots, r \),

\[ |z'(t_o) - |H|^{-1} \int_H z_k'(t) \, dt| \leq 2(r+1)^{-1} \eta \]  
(2.30)

For \( i = 0 \), because of assumption (2.7),

\[ h^{-\nu} \int_H z_0'(t) \, dt = 0 \text{ for all } k. \]  
(2.31)

We also have

\[ h^{-\nu} \int q z_0'(t) \, dt = h^{-\nu} \Delta q_k \]  
(2.32)

and

\[ h^{-\nu} \int q z_0'(t) \, dt = h^{-\nu} \Delta q_0'. \]

On the other hand, \( D_0 = X + S \) and by (2.22)

\[ h^{-\nu} \Delta q D_0 = h^{-\nu} \Delta X + h^{-\nu} \Delta S \]  
(2.33)
with \( |h^{-\nu} \Delta q_k S| \leq (r+1)^{-1}\eta \). Also, since \( \Delta D_k \) approaches \( \Delta D_0 \) as \( k \) approaches infinity, \( G \) having vertices with rational coordinates, and we can determine \( k'(t_o, \epsilon, \lambda, \eta, \beta) \) above so that for \( k \geq k' \) we have also

\[
|h^{-\nu} \Delta q_k D_k - h^{-\nu} \Delta q_0 D_0| \leq \eta (r+1)^{-1}.
\] (2.34)

Finally, (2.32), (2.33) and (2.34) yield

\[
|\int_{z^0(t)}^{D_k} - \int_{z^0(t)}^{D_0} dt| = |h^{-\nu} \Delta q_k D_k - h^{-\nu} \Delta q_0 X| \leq 2(r+1)^{-1}\eta.
\] (2.35)

We have

\[
z^0(t_o) = |H|^{-1} \int_{H^k} z^0(t) dt \geq \\
-|z^0(t_o) - |q|^{-1} \int z^0(t) dt| \\
-|(|q|^{-1} - |H|^{-1}) \int z^0(t) dt| \\
-|H|^{-1} \int (z^0(t) - z^0_k(t)) dt| \\
+ |H|^{-1} \int_{H^k} z^0(t) dt = d_2 + d_6 + d_7 + d_8
\] (2.36)

By (2.19) we have \( d_2 \geq -\eta (r+1)^{-1} \), by (2.20) and (2.35) we have \( d_7 \geq -4(r+1)^{-1}\eta \), and by (2.31) we have \( d_8 \geq 0 \). Also,
\[ d^6 = -\left| (1-|q|(|H|^{-1}) \right| \cdot \frac{1}{2} \left| \int z^0(t) \, dt \right| \geq -(r+1)^{-1} \eta, \]

where we have used (2.20), (2.19), and the definition of \( M_1 \). Hence, (2.36) yields

\[ z^0(t_o) - |H|^{-1} \int z^0_k(t) \, dt \geq -6n(r+1)^{-1}. \]

(2.37)

We have also

\[ |z^i(t_o) - |H|^{-1} \int z^i_H(t) \, dt| \leq 4\eta(r+1)^{-1}, \]

(2.28)

\[ i = 1, 2, \ldots, \rho, \text{ and} \]

\[ |z^i(t_o) - |H|^{-1} \int z^i_H(t) \, dt| \leq 2\eta(r+1)^{-1}, \]

(2.30)

\[ i = \rho + 1, \ldots, r. \]

Equations (2.23) and (2.25) imply that

\[ z_k(t) \in \mathbb{N}_{\rho}(z(t_o); \rho + 1) \text{ for } t \in H. \]

(2.38)

Because of (2.24) and (2.38)

\[ |H|^{-1} \int z_k(t) \, dt \in \text{clco} \left( Q(t_o, \psi, \delta, 3\varepsilon) \cap \mathbb{N}(z(t_o); \rho + 1) \right). \]

The latter set has the upper set property by statements (3.1), (3.11) and (3.1ii) of [4]. From (2.37), (2.28), and (2.30),

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\[ z(t_o) \in \text{clco}(Q, J, C_{\lambda}^{(t_o, y_o, 3\epsilon)} \cap N_{\beta}(z(t_o); \rho + 1))_{12\eta} \]

for every \( \eta > 0, \beta > 0, \) and \( \epsilon > 0. \) Since \( \eta \) is an arbitrary positive number and the set inside the parenthesis is closed,

\[ z(t_o) \in \text{clco}(Q, J, C_{\lambda}^{(t_o, y_o, 3\epsilon)} \cap N_{\beta}(z(t_o); \rho + 1)) \]

for every \( \epsilon > 0 \) and every \( \beta > C. \) By statements (2.v) and (2.vi) of [4], the set \( Q, J(t,y) \) has property \( Q(\rho + 1) \) on \( A_\lambda. \) Therefore, \( z(t_o) \in Q, J(t_o,y_o) \)

and

\[ z(t_o)(J(t_o))^{-1} - (\psi(t_o), 0, \ldots, 0) \in Q(t_o, y_o) \]

for \( t_o \in C. \) But \( \bigcup_{\lambda=1}^{\infty} C_\lambda = |G|. \) Therefore,

\[ z(t)(J(t))^{-1} - (\psi(t), 0, \ldots, 0) \in Q(t, y(t)) \]

for almost all \( t \in G. \) The conclusion of the theorem now follows from lemma (2.1).

3. A PRELIMINARY LOWER CLOSURE THEOREM

We shall use the same notations as in §2, in particular let \( f(t,y,u) \) denote a vector function \( f(t,y,u) = (f_0, f_1, \ldots, f_r) \) defined on \( M. \) Here we shall consider the functional

\[ I[y,u] = \int_G f_0(t,y(t), u(t)) J(t) \, dt. \]
Instead of the sets $Q(t,y)$ of $S^2$, we shall consider here the sets

$$\tilde{Q}(t,y) = \{ z = (z^0, z^1, \ldots, z^{r+1}), z^0 \geq f_o(t,y,u),$$

$$z^i = f_i(t,y,u), i = 1,\ldots,r, u \in U(t,y) \} \subset E^{r+1}$$

defined for every $(t,y) \in A$.

**Theorem 3.1** (a lower closure theorem). Let $G, A(t), A, U(t,y), M,$ and $M_F$ be defined as in $S^2$, $G$ measurable, $A$ closed, $M_F$ closed for every closed subset $F$ of $G$. Let $f(t,y,u) = (f^o, f^1, \ldots, f^r)$ be continuous on $M$, let $\rho$ be any integer, $0 \leq \rho \leq r$, and let us assume that the sets $\tilde{Q}(t,y)$ have property $Q(\rho+1)$ on $A$. Let $\psi(t), J(t), t \in G$, be measurable functions real valued on $G$ with $0 < K^{-1} \leq J(t) \leq K$ for all $t \in G$ and some constant $K$. Let $y(t) = (y^1, \ldots, y^s), y_k(t) = (y^1_k, \ldots, y^s_k), z(t)$

$= (z^1, \ldots, z^r), z_k(t) = (z^1_k, \ldots, z^r_k), u_k(t) = (u^1_k, \ldots, u^m_k), t \in G$, be as in Theorem (2.1), $k = 1,2,\ldots$, satisfying (2.1-6) and (2.8-9), and let us assume that $\lim_{k \to \infty} I [y_k, u_k] = a_o < +\infty$. Then $y(t) \in A(t)$ ae. on $G$, and there exists a measurable function $u(t) = (u^1, \ldots, u^m), t \in G$, such that $u(t) \in U(t,y(t)), z^i(t) = f_i(t,y(t), u(t)) J(t), i = 1,\ldots,m$, ae. on $G$, and $I [y,u] \leq a_o$.

**Proof** First set $D_k(t), t \in [-1,1], \text{equal to}$

$$D_k(t) = \int_{[-1,t]} (f_0(t,y_k(t), u_k(t)) + \psi(t)) J(t) \, dt,$$
and let \( e \) be the real number, \( e = \int_{[-1,1]} J(t) \psi(t) \, dt \). We may assume without loss of generality that \( 0 \leq D_k(t) \leq a_0 + e + 1 \). Then using a diagonal process, we may extract a subsequence of the original sequence, say still \( k = 1, 2, \ldots, \) so that \( D_k(t) \) converges pointwise to a number \( D_0(t) \) for each point \( t \in [-1,1] \) having all rational coordinates. We have defined \( D_0(t) \) on the rationals. For a point \( t = (t^1, \ldots, t^\nu) \in [-1,1] \) having at least one irrational coordinate we define \( D_0(t) = \sup D_0(\tilde{t}) \), where \( \sup \) is taken over all \( \tilde{t} = (\tilde{t}^1, \ldots, \tilde{t}^\nu) \), with \( \tilde{t}^i \) rational, \( \tilde{t}^i \leq t^i, i = 1, \ldots, \nu \).

Hence, \( D_0(t) \) is defined on \([-1,1]\). Also, since the functions \( D_k(t), k = 1, 2, \ldots, \) are nonnegative, monotone nondecreasing (in each variable) and equibounded, \( D_0(t) \) is nonnegative, monotone nondecreasing and bounded. Since \( D_0(t) \) is of bounded variation, we may, using the Lebesgue decomposition theorem, decompose \( D_0(t) \) as \( D_0(t) = X(t) + S(t) \), where

\[
X(t) = \int_{[-1,t]} z_0(\tau) \, d\tau, \quad t \in [-1,1]
\]

\( S(t) \geq 0 \) is singular and monotone decreasing, and \( z_0(t) \geq 0 \) is \( L \)-integrable in \([-1,1]\) and zero on \([-1,1] - G \).

We now set up an auxiliary problem to which we will apply theorem 2.1. In this situation, \( A(t) \) and \( A \) are defined as above. We define the set \( \tilde{U}(t, y) \) as

\[
\tilde{U}(t, y) = \{(u, u^0) \in \mathbb{E}^{m+1} | u^0 \geq f_0(t, y, u), u \in U(t, y)\}
\]

for \((t, y) \in A\), and we define the set \( \tilde{M} \) as

\[
\tilde{M} = \{(t, y, \tilde{u}) \mid (t, y) \in A \text{ and } \tilde{u} \in \tilde{U}(t, y)\}
\]

\[
= \{(t, y, u^0, u) \mid (t, y) \in A \text{ and } u^0 \geq f_0(t, y, u)\}
\]
for $u \in U(t,y)$.}

For any subset $F$ of $G$, the set $\tilde{M}_F$ is defined as the set of $(t,y,u) \in \tilde{M}$ for which $t \in F$. For each closed subset $F$ of $G$, since $f_0$ is continuous and $M_F$ is assumed to be closed, $\tilde{M}_F$ is closed.

Let $\tilde{f}(t,y,u) = (u^0, f_1(t,y,u), \ldots, f_r(t,y,u))$.

The set $Q(t,y) = \{z \in E^{r+1} | z = \tilde{f}(t,y,u), u \in U(t,y)\}$ has the upper set property, and we have assumed that it also has property $Q(r+1)$ on $A$.

Let $y(t)$, $y_k(t)$, $z(t)$, $z_k(t)$, $J(t)$ and $\psi(t)$ be as in the statement and proof of this theorem, with

$$z_k(t) = (z_0^1(t), z_1^1(t), \ldots, z_r^1(t)),$$

$$z_k^0(t) = (u_k^0(t) + \psi(t) \cdot J(t)) \quad k = 1, 2, \ldots,$$

Let $\tilde{u}_k^0(t)$ be the control vector for the auxiliary problem defined by

$$\tilde{u}_k^0(t) = (f_0(t,y_k(t), u_k(t)), \ u_k(t)), \ k = 1, 2, \ldots.$$

We have $D_k(t)$, $D_0(t)$, $X(t)$ and $S(t)$ defined as above.

Relations (2.1-6), (2.8-9) of Theorem (2.1) hold by hypothesis, and we have just verified (2.7), as well as the pointwise convergence $D_k(t) \to D_0(t)$ as $k \to \infty$ for every $t = (t^1, \ldots, t^r)$, $t \in [-1,1]$, with rational coordinates. Thus, theorem (2.1) holds in the present situation. Hence, $y(t) \in A(t)$ ae. on $G$, and there is a measurable function $\tilde{u}(t) = (u^0, u)$
\( = (u^0, u^1, \ldots, u^m), \ t \in G, \ \text{such that} \)

\[
\quad u^0 \geq f^0_o(t, y(t), u(t)), \ u(t) \in U(t, y(t)), \\
\quad z^i(t) = f^i_1(t, y(t), u(t)) \ J(t), \ i = 1, \ldots, r,
\]

and

\[
\quad z^0(t) = [u^0(t) + \psi(t)] \ J(t) \ \text{ae. on } G.
\]

We set \( u^0(t) = 0 \) on \([-1,1] - G \), and then, because of \( S(l) \geq 0 \), we have

\[
\quad \int_{[-1,1]} u^0(t)J(t) dt = \int_{[-1,1]} (z^0(t) - \psi(t)J(t)) dt \\
\quad = X(1) - e = (X(1) + S(1)) - (e + S(1)) \\
\quad = a_0^* + e - (e + S(1)) = a_0^* - S(1) \neq a_0^*.
\]

Because \( f^0_o(t, y(t), u(t)) \) is a continuous function of three measurable functions, it is measurable. Also, \( \psi(t) \) and \( u^0(t) \) are integrable and we have \( -\psi(t) \leq f^0_o(t, y(t), u(t)) \leq u^0(t) \), and \( K^{-1} \leq J(t) \leq K \ \text{ae on } G \). Thus, \( f^0_o(t, y(t), u(t)) \ J(t) \) is integrable on \( G \), and

\[
I[y, u] = \int_G f^0_o(t, y(t), u(t)) \ J(t) \ dt = \\
\quad = \int f^0_o(t, y(t), u(t)) \ J(t) \ dt \leq \int u^0(t) \ J(t) \ dt \leq \int [-1,1] u^0(t) \ J(t) \ dt \\
\quad \leq a_0^*.
\]
Theorem 3.1 is thereby proved.

4. A LOWER CLOSURE THEOREM FOR OPTIMIZATION PROBLEMS WITH DISTRIBUTED AND  
AND BOUNDARY CONTROLS

In this section we state and prove a lower closure theorem for optimization problems with controls, state variables, constraints, and state  
equations on both the domain and its boundary.

We shall first introduce a few definitions.

We begin with C. B. Morrey's definition of regular transformations of  
class K from his paper [8].

Let $T$ and $S$ be subsets of Euclidean spaces. A transformation  
x = $x(y)$ of $T$ onto $S$ is said to be of class $K$ provided it is one to one  
and continuous, and the functions $x = x(y)$ and $y = y(x)$ satisfy a uniform  
Lipschitz condition on each compact subset of $T$ and $S$ respectively. The  
transformation is said to be regular if, in addition, the functions $x(y)$  
and $y(x)$ satisfy a uniform Lipschitz condition on the whole of $T$ and $S$  
respectively.

Let $G$ be a bounded measurable subset of $E^\nu$, $\nu \geq 1$, whose boundary  
will be denoted by $\partial G$.

Let $\Gamma_j$, $j = 1, 2, \ldots, N$, be subsets of $\partial G$, each of which is the image  
under a regular transformation $t_j$ of class $K$ of a bounded interval $R_j$ of  
$E^{\nu-1}$. Let $\Gamma$ be a closed subset of $\bigcup_{j=1}^N \Gamma_j$, and let $\mu$ be a measure defined  
on $\bigcup_{j=1}^N \Gamma_j$. For each $j = 1, \ldots, N$, we assume that if $e$ is a subset of $\Gamma_j$,  
measurable with respect to $\mu$, then $E = t_j^{-1}(e)$ is measurable with respect  

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to Lebesgue \((v-1)\)-dimensional measure \(| | \) on \(\mathbb{R}^n_j\). Also, we assume the converse, so that measurable sets on \(\Gamma_j\) and \(\mathbb{R}^n_j\) correspond under \(t_j\), \(j = 1, 2, \ldots, N\). We assume that there is a constant \(K > 1\) such that if \(e = t_j(E)\) is \(\mu\)-measurable, then

\[
K^{-1}|E| \leq \mu(e) \leq K|E|
\]

(4.1)

independent of \(j = 1, \ldots, N\). Since \(\mu\) induces a measure on each set \(\mathbb{R}^n_j\) via the transformation \(t_j\), \(j = 1, 2, \ldots, N\), we may define \(J_j(\tilde{t})\), \(\tilde{t} \in \mathbb{R}^n_j\), as the functions in \(L_1(\mathbb{R}^n_j)\) which satisfy the equations

\[
\mu(t_j(E)) = \int_{E} J_j(\tilde{t}) \, d\tilde{t}
\]

(4.2)

for every measurable subset \(E\) of \(\mathbb{R}^n_j\), \(j = 1, 2, \ldots, N\).

For every \(t \in \text{cl}(G)\), let \(A(t)\) be a nonempty closed subset of \(\gamma\)-space \(E^s\). Let \(A\) be the set of all points \((t, y)\) with \(t \in \text{cl}(G)\) and \(y \in A(t)\).

For every \((t, y) \in A\), let \(U(t, y)\) be a nonempty subset of \(u\)-space \(E^m\). Let \(M\) be the set of all \((t, y, u) \in E^v \times E^s \times E^m\) such that \((t, y) \in A\) and \(u \in U(t, y)\).

For every \(t \in \Gamma\), let \(B(t)\) be a nonempty closed subset of \(\gamma\)-space \(E^{s'}\). Let \(B\) be the set of all \((t, \bar{y})\) with \(t \in \Gamma\) and \(\bar{y} \in B(t)\). For every \((t, \bar{y}) \in B\), let \(V(t, \bar{y})\) be a nonempty closed subset of \(v\)-space \(E^{m'}\). Let \(M\) be the set of all \((t, \bar{y}, v) \in E^v \times E^s \times E^{m'}\) with \((t, \bar{y}) \in B\) and \(v \in V(t, \bar{y})\).

Let \(\tilde{f}(t, y, u) = (f_0, f_1, \ldots, f_r)\) be a continuous \((r+1)\)-vector function on \(M\), and let us consider the sets
\[ \tilde{\mathbb{O}}(t,y) = \{ z = (z^0, \ldots, z^r) \in \mathbb{E}^{r+1} \mid z^0 \geq f^0(t,y,u), \] 
\[(z^1, \ldots, z^r) = (f^1, \ldots, f^r)(t,y,u), \ u \in U(t,y) \} \ (4.3) \]

Let \( \tilde{g}(t,y,v) = (g^0, \ldots, g^r_v) \) be a continuous \((r'+1)\)-vector function on \( \tilde{\mathbb{R}} \), and let us consider the sets

\[ \tilde{\mathbb{R}}(t,y) = \{ z = (z^0, \ldots, z^{r'}) \in \mathbb{E}^{r'+1} \mid z^0 \geq g^0(t,y,v) \] 
\[(z^1, \ldots, z^{r'}) = (g^1, \ldots, g^r_v)(t,y,v), \ v \in V(t,y) \} \ (4.4) \]

We assume that there are two functions \( \psi(t), t \in G \) and \( \tilde{\psi}(t), t \in \Gamma \), such that \( f^0(t,y,u) \geq -\psi(t) \) for all \((t,y,u)\) in \( M \), \( \psi(t) \geq 0 \), \( \psi(t) \in L_1(G) \), and \( g^0(t,y,v) \geq -\tilde{\psi}(t) \) for all \((t,y,v)\) in \( \tilde{R} \), \( \tilde{\psi}(t) \geq 0 \), \( \tilde{\psi}(t) \in L_1(\Gamma) \).

We consider here the functional

\[ I[y,y_v,u,v] = \int_G f^0(t,y(t),u(t)) \, dt + \int_{\Gamma} g^0(t,y(t),v(t)) \, dv. \]

In the lower closure theorem below we shall deal with sequences of functions all defined on \( G \) and \( \Gamma \):

\[ z(t) = (z^1, \ldots, z^r), \quad z_k(t) = (z^1_k, \ldots, z^r_k), \]
\[ y(t) = (y^1, \ldots, y^s), \quad y_k(t) = (y^1_k, \ldots, y^s_k), \]
\[ u_k(t) = (u^1_k, \ldots, u^m_k), \quad t \in G, \ k = 1, 2, \ldots \]
\[ \tilde{z}(t) = (\tilde{z}^1, \ldots, \tilde{z}^r), \quad \tilde{z}_k(t) = (\tilde{z}^1_k, \ldots, \tilde{z}^r_k), \]

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\[ \dot{y}(t) = (\dot{y}, \ldots, \dot{y}'), \quad \ddot{y}_k(t) = (\ddot{y}_k, \ldots, \ddot{y}_k'), \]

\[ v_k(t) = (v_k, \ldots, v_k'), \quad t \in \Gamma, \ k = 1, 2, \ldots. \]

**Theorem 4.1** (a lower closure theorem). Let \( G \) be bounded and measurable, \( A, B, M, M \) closed, \( f(t, y, u) \) continuous on \( M \), \( g(t, \dot{y}, v) \) continuous on \( \dot{G} \), and assume that for some integers \( \rho, \rho', 0 \leq \rho \leq r, 0 \leq \rho' \leq r' \), the sets \( Q(t, y) \) have property \( Q(\rho + 1) \) on \( A \), and the sets \( \tilde{Q}(t, y) \) have property \( Q(\rho' + 1) \) on \( B \).

Let us assume that there are functions \( \psi(t) \geq 0, t \in G, \psi \in L_1(G) \) and \( \dot{\psi}(t) \geq 0, t \in \Gamma, \dot{\psi} \in L_1(\Gamma) \), such that \( f_\psi(t, y, u) \geq -\psi(t) \) for all \( (t, y, u) \in M \), and \( g_\psi(t, \dot{y}, v) \geq -\dot{\psi}(t) \) for all \( (t, \dot{y}, v) \in \dot{M} \).

Let us assume that the functions \( z^i(t), z^i_k(t), y^j(t), y^j_k(t), i = 1, \ldots, r, j = 1, \ldots, s, \) are in \( L_1(G) \), that the functions \( u^j_k(t) \) are measurable on \( G, j = 1, \ldots, m, \) that \( f_\psi(t, y_k(t), u_k(t)) \in L_1(G) \), and that

\[ y_k(t) \in A(t), u_k(t) \in U(t, y_k(t)), z^i_k(t) = f^i(t, y_k(t), u_k(t)) \]

a.e. on \( G, k = 1, 2, \ldots. \) \( (4.5) \)

Let us assume that the functions \( z^i(t), z^i_k(t), y^j(t), y^j_k(t), i = 1, \ldots, r', j = 1, \ldots, s', \) are in \( L_1(\Gamma) \), that the functions \( y^j_k(t) \) are measurable in \( \Gamma, j = 1, \ldots, m', \) that \( g_\psi(t, y_k(t), v_k(t)) \in L_1(\Gamma) \), and that

\[ y_k(t) \in B(t), v_k(t) \in V(t, y_k(t)), z^i_k(t) = g^i(t, y_k(t), v_k(t)) \]
\[ \mu\text{-a.e. on } \Gamma, \; k = 1,2,\ldots \] (4.6)

Finally, let us assume that as \( k \to \infty \) we have

\[ z_i^k(t) \to z_i(t) \text{ weakly in } L_1(G), \; i = 1,\ldots,\rho, \] (4.7)

\[ z_i^k(t) \to z_i(t) \text{ strongly in } L_1(G), \; i = \rho + 1,\ldots,\nu, \] (4.8)

\[ y_j^k(t) \to y_j(t) \text{ strongly in } L_1(G), \; j = 1,\ldots,\omega, \] (4.9)

\[ z_i^\circ_k(t) \to z_i^\circ(t) \text{ weakly in } L_1(\Gamma), \; i = 1,\ldots,\rho', \] (4.10)

\[ z_i^\circ_k(t) \to z_i^\circ(t) \text{ strongly in } L_1(\Gamma), \; i = \rho' + 1,\ldots,\nu', \] (4.11)

\[ y_j^\circ_k(t) \to y_j^\circ(t) \text{ strongly in } L_1(\Gamma), \; j = 1,\ldots,\omega', \]

\[ \lim_{k \to \infty} I[y_k^\circ, y_k^\circ, u_k^\circ, v_k^\circ] \leq a_0 < +\infty. \] (4.12)

Then, \( y(t) \in A(t) \) a.e. on \( G, \; y(t) \in B(t) \) \( \mu \)-a.e. on \( \Gamma \), and there are measurable functions \( u(t) = (u_1^1,\ldots,u_m^m), \; t \in G \), and \( \mu \)-measurable functions \( v(t) = (v_1^1,\ldots,v_m^m), \; t \in \Gamma \), such that \( f^\circ_0(t,y(t),u(t)) \in L_1(G), \; g^\circ_0(t,y(t),v(t)) \in L_1(\Gamma) \), and such that

\[ u(t) \in U(t,y(t)), \; z_i^1(t) = f_i^1(t,y(t),u(t)), \; i = 1,\ldots,\nu, \text{ a.e. on } G, \]

\[ v(t) \in V(t,y(t)), \; z_i^\circ(t) = g_i^\circ(t,y(t),v(t)), \; i = 1,\ldots,\nu', \text{ \( \mu \)-a.e. on } \Gamma, \]

\[ I[y,y,u,v] \leq a_0. \]
Proof: We may write

$$I(y_k, \dot{y}_k, u_k, v_k) = \int_G f_o(t, y_k(t), u_k(t)) \, dt +$$

$$\sum_{j=1}^{N} \int_{\Delta_j} g_0(t_j(\tilde{t}), \dot{y}_k(t_j(\tilde{t})), v_k(t_j(\tilde{t}))) \, J_j(\tilde{t}) \, d\tilde{t},$$

where

$$\Delta_j = \text{cl} \, t_j^{-1} [\Gamma \cap (\Gamma_j - \sum_{i=1}^{j-1} \Gamma_i)], \quad k = 1, 2, \ldots.$$ 

We may assume that $$I(y_k, \dot{y}_k, u_k, v_k) \leq a_o + 1.$$ Since $$f_o(t, y, u)$$

$$\equiv -\psi(t)$$

for all $$(t, y, u)$$ in $$M$$, the integrals of $$f_o$$ on $$G$$ are uniformly bounded below by the number $$\int -\psi(t) \, dt.$$ Also, because of the fact that $$g_0(t, \dot{y}, v) \equiv -\psi(t)$$ and $$J(t) \leq K$$, the integrals of $$g_0$$ on $$\Delta_j$$ are uniformly bounded below by the numbers

$$\int_{\Delta_j} -\psi(t_j(\tilde{t})) \, K \, d\tilde{t}, \quad j = 1, 2, \ldots, N.$$ 

Hence, each of the integrals in $$I(y_k, \dot{y}_k, u_k, v_k)$$ on $$G$$ and $$\Delta_j$$, $$j = 1, 2, \ldots, N$$, is uniformly bounded above and below. We may, assume, therefore without loss of generality, that

$$\int_{\Delta_j} g_0(t_j(\tilde{t}), \dot{y}_k(t_j(\tilde{t})), v_k(t_j(\tilde{t}))) \, J_j(\tilde{t}) \, d\tilde{t}$$

approaches a finite limit $$a_j$$, $$j = 1, 2, \ldots, N$$, as $$k$$ approaches infinity.
We then have
\[
\lim_{k \to \infty} \int_{G \cap \mathcal{C}} f\left( t, y(\mathbf{z}_k), u_0(t) \right) dt = a_0 - \sum_{j=1}^{N} a_j.
\]

We shall apply lower closure theorem 3.1 on each set \( \Lambda_j \subseteq \mathbb{R}^r_+ \), \( j = 1, 2, \ldots, N \). Here we have \( \mathcal{B}(\tilde{t}) = \mathcal{B}(t_j(\tilde{t})) \), \( \tilde{t} \in \Lambda_j \), which is a nonempty closed subset of \( \mathbb{R}^s \). Let \( \mathcal{B}_j, j = 1, 2, \ldots, N \), be the set of all \( (\bar{t}, \bar{y}) \) with \( \bar{t} \in \Lambda_j \) and \( \bar{y} \in \mathcal{B}(\tilde{t}) \). For every \( (\bar{t}, \bar{y}) \in \mathcal{B}_j \), let \( \mathcal{V}(\bar{t}, \bar{y}) \) be defined as the set \( \mathcal{V}(\bar{t}, \bar{y}) = \mathcal{V}(t_j(\tilde{t}), \bar{y}) \). Let \( \mathcal{M}_j, j = 1, 2, \ldots, N \), be the set
\[
\mathcal{M}_j = \{ (\bar{t}, \bar{y}, v) \mid (\bar{t}, \bar{y}) \in \mathcal{B}_j \text{ and } v \in \mathcal{V}(\bar{t}, \bar{y}) \}.
\]

Since \( \mathcal{M}_j \) is assumed to be closed and \( \Lambda_j \) is closed, \( \mathcal{M}_j \) is a closed subset of \( \mathbb{E}^r \times \mathbb{E}^s \times \mathbb{E}^m \).

The function \( \tilde{g}(t_j(\tilde{t}), \bar{y}, v) = (g_0, \ldots, g_r)(t_j(\tilde{t}), \bar{y}, v) \) is a continuous \((r'+1)\)-vector function on \( \mathcal{M}_j \). Let \( \tilde{\mathcal{M}}_j(\bar{t}, \bar{y}) \) be the set
\[
\{ z \in \mathbb{E}^{r'+1} \mid z^0 = \tilde{g}(t_j(\tilde{t}), \bar{y}, v), (z, \ldots, z^{r'}) = (g_1, \ldots, g_r), (t_j(\tilde{t}), \bar{y}, v), \text{ for } v \in \mathcal{V}(\bar{t}, \bar{y}) \text{ for } (\bar{t}, \bar{y}) \in \mathcal{B}_j \}.
\]

Then, we have \( \tilde{\mathcal{M}}_j(\bar{t}, \bar{y}) = \tilde{\mathcal{M}}(t_j(\tilde{t}), \bar{y}) \) for \( (\bar{t}, \bar{y}) \in \mathcal{B}_j \), \( j = 1, 2, \ldots, N \). Now if \( (\bar{t}', \bar{y}) \in \mathcal{M}_j(\bar{t}, \bar{y}) \cap \mathcal{B}_j \), then \( (t_j(\tilde{t}'), \bar{y}) \) is in \( \mathcal{N}_{K\epsilon + \epsilon}(t_j(\tilde{t}'), \bar{y}) \) \( \mathcal{B} \), where \( K \) is the Lipschitz constant of the transformation \( t_j \) and \( t = t_j(\tilde{t}) \). Hence, for each \( j = 1, 2, \ldots, N \), and \( (\bar{t}, \bar{y}) \in \mathcal{B}_j \),
\[ \tilde{\mathcal{R}}_j(t, \dot{y}, \varepsilon) \subseteq \tilde{\mathcal{R}}(t_j(\ddot{t}), \dot{y}, \bar{K} \varepsilon + \varepsilon), \text{ and} \]

\[ \cap_{\varepsilon > 0} \cap_{\bar{K} > 0} \text{clco} \left( \tilde{\mathcal{R}}_j(t, \dot{y}, \varepsilon) \cap N_{\bar{K}}(z_0; \rho' + 1) \right) \]

is a subset of

\[ \cap_{\varepsilon > 0} \cap_{\bar{K} > 0} \text{clco} \left( \tilde{\mathcal{R}}(t_j(\ddot{t}), \dot{y}, \bar{K} \varepsilon + \varepsilon) \cap N_{\bar{K}}(z_0; \rho' + 1) \right). \]

We see that, since \( \tilde{\mathcal{R}}(t, \dot{y}) \) has property \( \tilde{\mathcal{Q}}(\rho' + 1) \) on \( B_j \), the set \( \tilde{\mathcal{R}}(t, \dot{y}) \) has property \( \mathcal{Q}(\rho' + 1) \) on \( B_j \), for each \( j = 1, 2, \ldots, N \).

We have \((\ddot{t}, \dot{y}, k(t_j(\ddot{t}))) \in B_j \) and \( v_k(t_j(\ddot{t})) \) in \( V(t_j(\ddot{t}), k(t_j(\ddot{t}))) \), a.e. on \( \Delta_j \), \( j = 1, 2, \ldots, N \), and \( k = 1, 2, \ldots \). For \( j = 1, \ldots, N \), \( i = 1, \ldots, r' \), and \( k = 1, 2, \ldots \), let us take

\[ z^i(j; \ddot{t}) = z^i(t_j(\ddot{t})), J_j(\ddot{t}), z^i_k(j; \ddot{t}) = z^i_k(t_j(\ddot{t})). \]

By virtue of the convergence relations (4.10), (4.11), and the relation \( K^{-1}|E| \equiv \mu(e) \equiv K|E| \), \( e = t_j(E) \), we have

\[ 0 < K^{-1} \equiv J_j(\ddot{t}) \equiv K, \text{ a.e. on } \Delta_j \]

\[ z^i_k(j; \ddot{t}) + z^i(j; \ddot{t}) \text{ weakly in } L_1(\Delta_j), \ i = 1, 2, \ldots, \rho', \]

\[ z^i_k(j; \ddot{t}) + z^i(j; \ddot{t}) \text{ strongly in } L_1(\Delta_j), \ i = \rho' + 1, \ldots, r', \]

\( j = 1, 2, \ldots, N \), as \( k \) approaches \( \infty \). Finally, \( \tilde{\mathcal{Q}}(t_j(\ddot{t})) \) is in \( L_1(\Delta_j) \),

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\[ \dot{\psi}(t_j(\bar{t})) \geq 0, \quad \text{and} \]
\[ g_0(t_j(\bar{t}), \dot{y}, v) \geq -\dot{\psi}(t_j(\bar{t})) \quad \text{for all } (\bar{t}, \dot{y}, v) \]
in \( \hat{M}_j \), \( j = 1, 2, \ldots, N \).

Applying lower closure theorem, 3.1 we see that \( \dot{y}(t_j(\bar{t})) \in B(t_j(\bar{t})) \) and there are measurable controls \( v_j(\bar{t}), \bar{t} \in \Delta_j \), such that \( v_j(\bar{t}) \in V(\bar{t}, \dot{y}(t_j(\bar{t}))) \),
\[ z_{\dot{i}}^i(t_j(\bar{t})) = g_1(t_j^{-1}(\bar{t})), v_j(\bar{t})) \text{, and } \int_{\Delta_j} g_0(t, \dot{y}(t), v_j(t)) \, du \leq a_j, \]
\( j = 1, 2, \ldots, N \). Setting \( v(t) = v_j(t_j^{-1}(t)) \) on \( t_j(\Delta_j) \), we see that there is a measurable control \( v(t), t \in \Gamma \), such that \( \dot{y}(t) \in B(t), v(t) \in V(t, \dot{y}(t)) \),
\[ z^i(t) = g_1(t_0 \dot{y}(t), v(t)) \quad \mu \text{ a.e. on } \Gamma, \]
and
\[ \int_{\Gamma} g_0(t, \dot{y}(t), v(t)) \, du \leq \sum_{j=1}^{N} a_j. \]

On \( G \) itself, we have exactly the situation of the lower closure theorem with \( J(t) = 1 \). Therefore, \( y(t) \in A(t) \), a.e. on \( G \), and there exists a measurable control \( u(t), t \in G \), with
\[ u(t) \in U(t, y(t)), z_1^i(t) = f_1(t, y(t), u(t)), \quad \text{a.e. on } G, \ i=1, \ldots, r, \]
\[ \int_{G} f_0(t, y(t), u(t)) \, dt \leq a_0 - \sum_{j=1}^{N} a_j. \]

The conclusion of theorem 4.1 follows from the conclusions of this and
the preceding paragraphs.

**Remark 1.** Suppose that all of the hypotheses of theorem 4.1 hold except that $I(y_0, y, u, v)$ is written as

$$I(y_0, y, u, v) = \int_G f_0(t, y(t), u(t)) \, dt + \int_{\Gamma_0} g_0(t, y(t), v(t)) \, d\mu + T(y(t), y_0(t))$$

We see that we could have proven the same lower closure theorem for $I(y_0, y, u, v)$ provided that $T(y, y_0) \leq \lim_{k \to \infty} T(y_k, y_k)$ and $\int_G f_0(t, y_k(t), u_k(t)) \, dt + \int_{\Gamma_0} g_0(t, y_k(t), v_k(t)) \, d\mu$ approaches a finite limit as $k \to \infty$.

**Remark 2.** We mention a variant of theorem 4.1. We may assume that $G$ and $\Gamma$ are made up of a finite number of components $G_1, \ldots, G_d$ and $\Gamma_1, \ldots, \Gamma_d$, and that, in each of these, there is a different system of control equations similar to the ones on $G$ and $\Gamma$ in theorem 4.1.

Also, we mention that the sets $\Gamma_j$ throughout this paper are thought of as subsets of the boundary $\partial G$ of $G$ because this will be the main application we have in mind, but actually the sets $\Gamma_j$ could be subsets of $G$ instead, or even abstract sets in no way connected with $G$.

**Examples**

The following two examples illustrate the use of the intermediate properties $Q(\rho), 0 \leq \rho \leq r$, used in connection with lower closure theorems in the present paper. Both examples have been mentioned already in [5].
Example 1. Let us consider the problem of the minimum of the cost functional
\[ I[x, u_1, u_2, v] = \int \int (\xi^2 + \eta^2 + x^2 + u_1^2 + u_2^2) d\xi d\eta \]
with differential equations
\[ x_\xi = u_1, \quad x_\eta = u_2, \quad \text{a.e. in } G, \]
and boundary conditions
\[ \gamma x = v \quad \text{s a.e. on } \Gamma = \partial G, \]
where \( G = \{(\xi, \eta) | \xi^2 + \eta^2 \leq 1\}, \) \( \Gamma \) is the boundary of \( G, \) \( s \) is the arc length on \( \Gamma, \) \( \gamma x \) the boundary values of \( x, \) and the control functions \( u_1, u_2, v \) have their values \( (u_1, u_2) \in U = E^2, \) \( v \in V = (-1) \cup (1). \) Actually, we want to minimize \( I \) in the class \( \Omega \) of all systems \( (x, u_1, u_2, v) \) with \( u_1, u_2 \) measurable in \( G, \) \( v \) measurable on \( \Gamma, \) and \( x \) any element of the Sobolev space \( W^1_2(G). \) We shall consider here the sets
\[ q(\xi, \eta, x) = [(z^2, \xi^2, \eta^2)|z^2 + \xi^2 + \eta^2 + x^2 + u_1^2 + u_2^2, \quad z_1 = u_1, \quad z_2 = u_2, \]
\[ (u_1, u_2) \in E^2 \subseteq E^3, \quad \tilde{R} = [(z^2, z)|z^2 \geq 0, \quad z = v, \quad v = \pm 1 \subseteq E^2. \]
We have here \( r = 2, \) the sets \( \tilde{Q} \) have property (Q), or Q(3), in \( A = d\Gamma \times E^1. \) We have also \( r' = 1, \) the sets \( \tilde{R} \) have property Q(1) in \( B = \Gamma, \) have property (U), but they are not convex and do not have property (Q).

In the search of the minimum of \( I \) in \( \Omega \) we can limit ourselves to those
elements \((x_0, u_1, u_2, v) \in \Omega\) with \(I \leq M\) for some constant \(M\). Here \(f = \zeta^2 + \eta^2 + x^2 + u_1^2 + u_2^2, g = 0\), hence \(\psi = 0, \phi = 0\). We take \(z(t) = (x_0, x), y(t) = x, \xi(t) = \gamma x, \eta(t) = 0\). If \([x_k]\) is a minimizing sequence, hence \(\|x_k\|_{W^1_2} \leq N\) for some constant \(N\), there is a subsequence, say still \([k]\) for the sake of simplicity, such that \(x_k \rightharpoonup x\) weakly in \(W^1_2(G)\), \(z_k \rightharpoonup z\) weakly in \((L^2(G))^2\), \(y_k \rightarrow y\) strongly in \(L^2(G)\), \(\xi_k \rightarrow \xi\) strongly in \(L^2(\Gamma)\), \(\eta_k \rightarrow \eta\) strongly in \(L^2(\Gamma)\). Lower closure theorem (4.11) may be applied with \(\rho = 2, \rho' = 0\).

Example 2. Let us consider the problem of the minimum of the cost functional

\[
I[x, u_1, u_2, v] = \int_G (x^2 + x\eta u_1^2 + u_2^2 + (1 - u_2)^2) \, d\zeta \, d\eta + \int_\Gamma (\gamma x - 1)^2 \, ds
\]

with differential equations

\[
\begin{align*}
\frac{d}{dt} x_0 + x_\zeta = u_1, \\
\frac{d}{dt} u_1 + u_\zeta = u_2
\end{align*}
\]

a.e. in \(G\),

\[
\gamma x_\zeta = \cos v, \quad \gamma x_\eta = \sin v
\]

s.a.e. on \(\Gamma = \partial G\),

where \(G\) and \(\Gamma\) are as in example 1, where \(\gamma x\) denotes the boundary values of \(x\), and the control functions \(u_1, u_2, v\) have their values \((u_1, u_2) \in U = E^2, v \in V = E\). We want to minimize \(I\) in a class \(\Omega\) of systems \((x, u_1, u_2, v)\) with \(u_1, u_2\) measurable in \(G\), \(v\) measurable on \(\Gamma\), \(x\) any element of the Sobolev space \(W^2_2(G)\) satisfying an inequality \(\|x_\zeta\|_2^2 + \|x_\eta\|_2^2 + \|x\|_2^2 \leq M\) (\(M\) a constant large enough so that \(\Omega\) is not empty). We shall consider here the sets
\( \tilde{Q}(y) = \{(z^0, z^1, z^2) | z^0 \geq y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2 (1 - u_2)^2, z^1 = u_1, z^2 = u_2, (u_1, u_2) \in E^2\} \subset E^3, \)

\( \tilde{R}(\hat{y}) = \{(z^0, z^1, z^2) | z^0 \geq (\hat{y} - 1)^2, z^1 = \cos v, z^2 = \sin v, v \in E^1\} \subset E^3, \)

where \( y = (y_1, y_2, y_3) \) in \( \tilde{Q}(y) \), and \( \hat{y} \) in \( \tilde{R}(\hat{y}) \) are arbitrary. Here we have \( r = 2, r' = 2 \). The sets \( \tilde{Q} \) have property \( Q(2) \), but they are not convex, and do not have property \((Q)\), or \( Q(3) \). The sets \( \tilde{R} \) have property \( Q(1) \), but they are not convex, and do not have property \((Q)\), or \( Q(3) \). They all have property \( Q(0) \), or \((U)\). Here \( f_0 = y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2 (1 - u_2)^2, g_0 = (y - 1)^2 \), and we can take \( \psi = 0, \tilde{\psi} = 0 \). We have here \( z(t) = (x_\xi + x_\eta, x_\xi + x_\eta), y(t) = (x, x_\xi, x_\eta), \tilde{z}(t) = (\gamma x_\xi, \gamma x_\eta), \tilde{y}(t) = \gamma x \). If \([x_k] \) is any sequence \( x_k \in [x]_H \), then there is a subsequence, say still \([k] \), such that \( x_k \rightharpoonup x \) weakly in \( W^2_2(G) \), \( (z_k)^1 \rightharpoonup (z)^1 \) weakly in \( L^2(G) \), \( (z_k)^2 \rightharpoonup (z)^2 \) strongly in \( L^2(G) \), \( y_k \rightharpoonup y \) strongly in \( (L^2(G))^3 \), \( \tilde{z}_k \rightharpoonup \tilde{z} \) strongly in \( L^2(\Gamma) \), and \( \tilde{y}_k \rightharpoonup \tilde{y} \) strongly in \( L^2(\Gamma) \). Lower closure theorem \((4.1)\) applies with \( \rho = 1 \) and \( \rho' = 0 \).
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