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UPPER SEMICONTINUITY PROPERTIES OF VARIABLE SETS IN OPTIMAL CONTROL

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1. INTRODUCTION

In his existence theorems for optimal solutions in control theory, Filippov [1] used the concept of metric upper semicontinuity of subsets of Euclidean spaces E^{r+1} as basic requirements for compact equibounded control spaces. Cesari [2, 3, 4], Lasota and Olech [5], Olech [6,7], and other authors have shown that, when the control space is only closed and not necessarily bounded, then either Kuratovski's concept of upper semicontinuity (property (U)), or Cesari's variant of this concept (property (Q)), are more suitable. In the present paper (§ 2) we introduce a scale of intermediate concepts, or property $Q(\rho)$ for ρ any integer, $0 \leq \rho \leq r + 1$. We prove then (§ 2) that property $Q(\rho)$ for $\rho = 0$ reduces to Kuratovski's property (U), and for $\rho = r + 1$ reduces to Cesari's property (Q). In addition, we prove that property $Q(\rho + 1)$ implies property $Q(\rho)$, $0 \leq \rho \leq r$. In § 3 we prove further statements concerning property $Q(\rho)$. We shall show in subsequent papers [8, 9] that the use of these properties $Q(\rho)$, $0 \leq \rho \leq r + 1$, will allow a considerable reduction of the hypotheses needed in lower closure and existence theorems for optimal solutions in optimization problems with distributed and boundary controls.

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2. PROPERTIES (U), (Q), and Q(ρ) OF SET VALUED FUNCTIONS

Let C be a measurable subset of $t = (t^1, \cdots, t^\nu)$ space E^ν, $\nu \geq 1$, and for every $t \in C$, let A(t) be a non-empty subset of $y = (y^1, \cdots, y^s)$ space E^s, $s \geq 1$. Let A be the set of all $(t, y) \in E^\nu \times E^s$ such that $t \in C$ and $y \in A(t)$. For every point $(t, y) \in A$, let Q(t,y) be a nonempty subset of the z-space E^{r+1}, $z = (z^0, z^1, \cdots, z^r)$, $r \geq 0$. For any point $(t_0, y_0) \in A$, and $\epsilon > 0$ we shall denote by $N_\epsilon(t_0, y_0)$ the set of all $(t, y) \in A$ at a distance $\leq \epsilon$ from (t_0, y_0). Also, given any subset F of a Euclidean space E, we denote by cl F, co F, cl co F the closure of F, the convex of F, and the closure of the convex hull of F, respectively.

We state now the definitions of Kuratovski's property (U) and of Cesari's property (Q). These properties have been studied in Cesari's papers [2, 3, 4].

We say that the sets Q(t, y) have property (Q) at a point $(t_0, y_0) \in A$ provided

$$Q(t_0, y_0) = \bigcap_{\epsilon > 0} \text{cl co } Q(t_0, y_0, \epsilon),$$

where

$$Q(t_0, y_0, \epsilon) = \bigcup_{(t,y) \in N_\epsilon(t_0, y_0)} Q(t,y).$$

We say that the sets Q(t, y) have property (U) at a point $(t_0, y_0) \in A$ provided

$$Q(t_0, y_0) = \bigcap_{\epsilon > 0} \text{cl } Q(t_0, y_0, \epsilon).$$

We say that the sets Q(t, y) have property (U), or (Q), on A provided they have the same property at every point $(t_0, y_0) \in A$. Sets having property (U) are closed, and sets having property (Q) are closed and convex.

We now give the definition of property Q(ρ), $0 \leq \rho \leq r+1$. Property Q(ρ)

is so designed that for $\rho = r+1$ it is equivalent to property (Q) and for $\rho = 0$ it is equivalent to property (U) as we shall prove below.

For any integer ρ, $0 \leq \rho \leq r+1$, we say that the sets $Q(t,y)$ have property $Q(\rho)$ at $(t_o, y_o) \in A$ provided for every $z_o = (z_o^o, \ldots, z_o^r) \in E^{r+1}$,

$$Q(t_o, y_o) \cap \{z \in E^{r+1} \mid z^i = z_o^i,\ i = \rho, \cdots, r\} = \bigcap_{\epsilon>0} \bigcap_{\beta>0} \mathrm{cl\ co}$$

$$[Q(t_o, y_o, \epsilon) \cap \{z \in E^{r+1} \mid |z^i - z_o^i| \leq \beta,\ i = \ , \cdots, r\}].$$

Note that for $\rho = r+1$ we understand here that both sets in braces coincide with E^{r+1}. Also note that, if we denote by P and Q the sets in the first and second members of the equality above then certainly $P \subset Q$. Thus, for property $Q(\rho)$ at the point t_o, y_o we actually require that $P \supset Q$ (for every $z_o \in E^{r+1}$).

We shall use the following notations.

For every $z_o \in E^{r+1}$, ρ a nonnegative integer, $0 \leq \rho \leq r+1$, and $\beta > 0$, let $N_\beta(z_o; \rho)$ be the (cylindrical) set of points in E^{r+1} whose final $(r+1) - \rho$ coordinates are within β of those of z_o; i.e.,

$$N_\beta(z_o; \rho) = \{z \in E^{r+1} \mid |z^i - z_o^i| \leq \beta, \quad i = \rho, \ldots, r\}.$$

Also we denote by $N(z_o; \rho)$ the set

$$N(z_o, \rho) = \{z \in E^{r+1} \mid z^i = z_o^i, \quad i = \rho, \ldots, r\}.$$

Thus, $N_\beta(z_o; 0) = N_\beta(z_o)$, $N(z_o; 0) = \{z_o\}$, and $N_\beta(z_o, r+1) = N(z_o, r+1) = E^{r+1}$. For any subset F or E^{r+1} and number $\eta > 0$ we denote by $(F)_\eta$ the closure of the set of points which are within η of a point of F. We refer to $(F)_\eta$ as the closed η-neighborhood of the set F in E^{r+1}.

(2.i) For every $\epsilon > 0$ and $\beta > 0$ let $Q(\epsilon, \beta)$ be a subset of E^{r+1}. Then

$$\bigcap_{\beta>0} \bigcap_{\epsilon>0} Q(\epsilon,\beta) = \bigcap_{\epsilon>0} \bigcap_{\beta>0} Q(\epsilon,\beta).$$

<u>Proof</u>: Let $z \in \bigcap_{\epsilon>0} \bigcap_{\beta>0} Q(\epsilon, \beta)$. Then $z \in \bigcap_{\beta>0} Q(\epsilon, \beta)$ for every fixed $\epsilon > 0$ and $z \in Q(\epsilon, \beta)$ for every $\beta > 0$ and each fixed $\epsilon > 0$. Now if a property holds for every $\beta > 0$ for each fixed $\epsilon > 0$, then the same property holds for every $\epsilon > 0$ for each fixed $\beta > 0$. Thus $z \in Q(\epsilon, \beta)$ for every $\epsilon > 0$ and each fixed $\beta > 0$. Hence $z \in \bigcap_{\epsilon>0} Q(\epsilon, \beta)$ for each fixed $\beta > 0$ and $z \in \bigcap_{\beta>0} \bigcap_{\epsilon>0} Q(\epsilon, \beta)$. We have proven that $\bigcap_{\epsilon>0} \bigcap_{\beta>0} Q(\epsilon, \beta) \subseteq \bigcap_{\beta>0} \bigcap_{\epsilon>0} Q(\epsilon, \beta)$. The proof of the reverse inequality is similar.

(2.ii) For every $(t, y) \in A$ let $Q(t, y)$ be as in the first paragraph of this section. Let ρ be any integer, $0 \leq \rho \leq r+1$. Then for every $(t_o, y_o) \in E^{\nu+s}$,

$$\bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{cl co}\,(Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho))$$

$$= \bigcap_{\beta>0} \bigcap_{\epsilon>0} \text{cl co}\,(Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho))$$

<u>Proof</u>: This statement follows immediately from (2.i).

We now study the relationships between property $Q(\rho)$ and properties (Q) and (U).

(2.iii) <u>Theorem</u>. For every $(t, y) \in A$ let $Q(t, y)$ be as in the first paragraph of this section. Then the sets $Q(t, y)$ have property $Q(r+1)$ at a point $(t_o, y_o) \in A$ if and only if they have property (Q) at the same point. The sets $Q(t, y)$ have property $Q(o)$ at $(t_o, y_o) \in A$ if and only if they have property (U) at the same point.

Proof: The first statement is apparent from the definitions of properties (Q) and Q(r+1). To prove the second statement, let the sets H and I be defined as follows:

$$H = \bigcap_{\beta>0} \bigcap_{\epsilon>0} \text{cl co } [Q(t_o, y_o, \epsilon) \cap \{z \in E^{r+1} \mid |z - z_o| \leq \beta\}]$$

$$I = \bigcap_{\beta>0} \bigcap_{\epsilon>0} \text{cl } [Q(t_o, y_o, \epsilon) \cap \{z \in E^{r+1} \mid |z - z_o| \leq \beta\}]$$

We need only show that for every $z_o \in E^{r+1}$

$$Q(t_o, y_o) \cap \{z \in E^{r+1} \mid z = z_o\} \supset H$$

if and only if

$$Q(t_o, y_o) \cap \{z \in E^{r+1} \mid z = z_o\} \supset I.$$

The last statement holds if H = I. Clearly, $I \subset H$. We show that $H \subset I$. Suppose that $z_o \in H$. Then, for every $\epsilon > 0$ and every $\beta > 0$, there exists a point $p \in Q(t_o, y_o, \epsilon)$ such that $|p - z_o| \leq \beta$. Hence, for every $\epsilon > 0$ and every $\beta > 0$

$$z_o \in \text{cl}[Q(t_o, y_o, \epsilon) \cap \{z \in E^{r+1} \mid |z - z_o| \leq \beta\}].$$

This statement implies that $z_o \in I$ and that $H \subset I$.

(2.iv) Theorem. Let $Q(t, y) \in E^{r+1}$ be as in the first paragraph of this section, and let ρ be any integer, $0 \leq \rho \leq r$. Then property $Q(\rho + 1)$ implies $Q(\rho)$.

Proof. We need only show that

$$Q(t_o, y_o) \cap \{z \in E^{r+1} \mid z^i = z_o^i, \ i = \rho + 1, \ldots, r\} \supset$$

$$\bigcap_{\epsilon>0}\bigcap_{\beta>0} \text{cl co}\,(Q(t_o, y_o, \epsilon) \cap \{z \in E^{r+1} \mid |z^i - z_o^i| \leq \beta,\ i = \rho + 1, \cdots, r\})$$

implies that

$$Q(t_o, y_o) \cap \{z \in E^{r+1} \mid z^i = z_o^i,\ i = \rho, \cdots, r\} \supset \bigcap_{\epsilon>0}\bigcap_{\beta>0} \text{cl co}$$

$$(Q(t_o, y_o, \epsilon) \cap \{z \in E^{r+1} \mid |z^i - z_o^i| \leq \beta,\ i = \rho, \cdots, r\}).$$

Using property $Q(\rho + 1)$, we have

$$Q(t_o, y_o) \cap \{z \in E^{r+1} \mid z^i = z_o^i,\ i = \rho, \cdots, r\} = \{z \in E^{r+1} \mid z^\rho = z_o^\rho\}$$

$$\cap\ [Q(t_o, y_o) \cap \{z \mid z^i = z_o^i,\ i = \rho + 1, \cdots, r\}]$$

$$\supset \{z \in E^{r+1} \mid z^\rho = z_o^\rho\} \cap \bigcap_{\epsilon>0}\bigcap_{\beta>0} \text{cl co}\,[Q(t_o, y_o, \epsilon)$$

$$\cap \{z \mid z^i - z_o^i| \leq \beta,\ i = \rho + 1, \cdots, r\}]$$

$$= \bigcap_{\beta'>0} \{z \mid |z^\rho - z_o^\rho| \leq \beta'\} \cap \bigcap_{\beta>0}\bigcap_{\epsilon>0} \text{cl co}\,[Q(t_o, y_o, \epsilon)$$

$$\cap \{z \mid |z^i - z_o^i| \leq \beta,\ i = \rho + 1, \cdots, r\}]$$

$$= \bigcap_{\beta>0}\bigcap_{\epsilon>0} (\text{cl co}\,[Q(t_o, y_o\ \epsilon) \cap \{z \mid |z^i - z_o^i| \leq \beta,$$

$$i = \rho + 1, \cdots, r\}] \cap [\{ z \mid |z^\rho - z_o^\rho| \leq \beta\})$$

$$\supset \bigcap_{\beta>0}\bigcap_{\epsilon>0} \text{cl co}\,[Q(t_o, y_o, \epsilon) \cap \{z \mid |z^i - z_o^i| \leq \beta,\ i = \rho, \ldots, r\}]$$

This completes the proof of Theorem (2.iv).

For every $(t, y) \in A$ let $Q(t, y)$ be as in the first paragraph of this section. We say that for a given $(t_o, y_o) \in A$ and $0 \leq \rho \leq r+1$, the set $Q(t_o, y_o)$ is ρ-convex provided

$$z_1 = (z_1^0, \ldots, z_1^{\rho-1}, z_0^\rho, \ldots, z_0^r) \in Q(t_0, y_0),$$

$$z_2 = (z_2^0, \ldots, z_2^{\rho-1}, z_0^\rho, \ldots, z_0^r) \in Q(t_0, y_0),$$

implies

$$\alpha z_1 + (1 - \alpha) z_2 \in Q(t_0, y_0)$$

for all $0 \leq \alpha \leq 1$. In this definition we understand that $(r+1)$-convexity is the usual convexity of $Q(t_0, y_0)$ in E^{r+1}. In other words, $Q(t_0, y_0)$ is ρ-convex provided, for every $z_0 = (z_0^0, \ldots, z_0^r) \in E^{r+1}$, the sets

$$Q(t_0, y_0) \cap \{z \in E^{r+1} \mid z^i = z_0^i, \ i = \rho, \ldots, r\}$$

are convex.

By known properties of convex sets, the set $Q(t_0, y_0)$ is ρ-convex, if and only if for every $\mu \geq 1$, real numbers $\lambda_\alpha \geq 0$, $\lambda_1 + \ldots + \lambda_\mu = 1$, and points

$$z_\gamma = (z_\gamma^0, \ldots, z_\gamma^{\rho-1}, z_0^\rho, \ldots, z_0^\rho) \in Q(t_0, y_0), \ \gamma = 1, \ldots, \mu,$$

we also have

$$\Sigma_\gamma \lambda_\gamma z_\gamma \in Q(t_0, y_0).$$

(2.v) If the sets $Q(t,y)$ have property $Q(\rho)$ at $(t_0, y_0) \in A$ for some ρ, $0 \leq \rho \leq r+1$, then $Q(t_0, y_0)$ is ρ-convex.

Indeed, by the definition of property $Q(\rho)$, the sets

$$Q(t_0, y_0) \cap \{z \in E^{r+1} \mid z^i = z_0^i, \mid i = \rho, \ldots, r\}$$

are closed and convex as intersection of sets which are closed and convex.

We now show that property $Q(\rho + 1)$ is preserved in the sense given below, under addition of a continuous function.

(2.vi) Theorem. For every $t \in C$, $y \in A(t)$, let $Q(t, y) \subset E^{r+1}$ be as in the first paragraph of this section, let ρ be any integer, $0 \leq \rho \leq r$, and $\psi(t)$, $t \in C$, be a real valued continuous function on C. For every $t \in C$, $y \in A(t)$, let $Q_\psi^+(t, y)$ denote the set $Q_\psi^+(t, y) = \{z \in E^{r+1} \mid z = p + (\psi(t), 0,\dots,0)$ for $p \in Q(t, y)\}$. If the sets $Q(t, y)$ have property $Q(\rho + 1)$ on A, then the sets $Q_\psi^+(t, y)$ also have property $Q(\rho + 1)$ on A.

Proof. We designate by $Q_\psi^+(t_o, y_o, \epsilon)$ the set

$$Q_\psi^+(t_o, y_o, \epsilon) = \bigcup_{(t, y) \in A \cap N_\epsilon(t_o, y_o)} Q_\psi^+(t, y)$$

We need only show that for every $(t_o, y_o) \in A$ and $z_o \in E^{r+1}$, $Q_\psi^+(t_o, y_o) \cap N(z_o; \rho+1)$ contains the set

$$\bigcap_{\beta>0} \bigcap_{\epsilon>0} \text{cl co } [Q_\psi^+(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho + 1)].$$

Let us take an arbitrary point p_o in the latter set. Let $\eta > 0$ be an arbitrary positive number and take $\epsilon_o = \epsilon_o(\eta)$ positive so small that

$$\sup_{(t, y) \in A \cap N_{\epsilon_o}(t_o, y_o)} |\psi(t) = \psi(t_o)| \leq \eta/2.$$

Now for every pair of positive numbers (ϵ, β) with $0 \leq \epsilon \leq \epsilon_o$, there exists a point p,

$$p \in \text{co } [Q_\psi^+(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho + 1)] \subset E^{r+1},$$

such that $|p - p_o| \leq \eta/2$. Then p is a convex combination of r+2 points of the set is brackets, or

$$p = \sum_{i=o}^{r+1} \lambda_i p_i + \lambda_i(\psi(t_i), 0,\ldots,0),$$

$$p_i \in Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho + 1) \text{ and } \Sigma\lambda_i = 1,\ \lambda_i \geq 0,\ i = 0,1,\ldots,r+1.$$

Since $|p - p_o| \leq \eta/2$, we have

$$p_o - (\psi(t_o),0,\ldots,0) \in [\text{cl co } (Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho+1))]_\eta \quad (2.1.3)$$

for every $\eta > 0$, $\beta > 0$, and ϵ, $0 < \epsilon \leq \epsilon_o$. Since $Q(t_o, y_o, \epsilon)$ is a subset of $Q(t_o, y_o, \epsilon')$ for every $0 < \epsilon \leq \epsilon'$, equation (2.1.3) holds for arbitrary $\epsilon > 0$, $\beta > 0$, and $\eta > 0$. Since η is arbitrary and the set inside the brackets is closed,

$$p_o - (\psi(t_o),0,\ldots,0) \in \text{cl co } [Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho+1)]$$

for every $\epsilon > 0$ and $\beta > 0$. Hence

$$p_o - (\psi(t_o),0,\ldots,0) \in \bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{cl co } [Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho+1)].$$

Since $Q(t, y)$ has property $Q(\rho+1)$ on C,

$$p_o - (\psi(t_o),0,\ldots,0) \in Q(t_o, y_o) \cap N(z_o; \rho+1).$$

Finally this statement implies

$$p_o \in Q_\psi^+(t_o, y_o) \cap N(z_o; \rho+1)$$

and completes the proof of (2.vi).

We now show that property $Q(\rho)$ is preserved, in the sense given below, upon multiplication by a positive, bounded continuous function.

(2.vi) For every $t \in C$, $y \in A(t)$, let $Q(t, y) \subset E^{r+1}$ be as in the first paragraph of this section, let ρ be any integer, $0 \leq \rho \leq r+1$, and $J(t)$, $t \in C$, be a real valued continuous function with $0 < K^{-1} \leq J(t) \leq K$ for all $t \in C$ and some constant K. For every $t \in C$, $y \in A(t)$ let $Q_J(t, y)$ denote the set $Q_J(t, y) = \{z \in E^{r+1} \mid z = p\,J(t) \text{ for } p \in Q(t, y)\}$. If the sets $Q(t, y)$ have the property $Q(\rho)$ on A, then the sets $Q_J(t, y)$ also have property $Q(\rho)$ on A.

Proof. We only need to show that for every $(t_o, y_o) \in A$ and $z_o \in E^{r+1}$,

$$\bigcap_{\beta>0} \bigcap_{\epsilon>0} \text{cl co}\,(Q_J(t_o, y_o, \epsilon) \cap N_\beta(z_o; \rho)) \subseteq Q_J(t_o, y_o)$$

$$\cap\, N(z_o; \rho).$$

Take a point p_o in the set on the left hand side. Let $\eta > 0$ and $\beta > 0$ be arbitrary numbers. Since

$$|z^i(J(t))^{-1} - z_o^i(J(t_o))^{-1}| \leq |(z^i - z_o^i)(J(t_o))^{-1}| +$$

$$|z^i(J(t_o) - J(t)) \cdot (J(t)J(t_o))^{-1}|,$$

for $|z^i - z_o^i| \leq \beta$ and t sufficiently close to t_o, we may make the difference on the left hand side less than or equal to $2K\beta$. Consequently, we can determine a number $\epsilon_o = \epsilon_o(\eta, \beta) > 0$ small enough so that

(i) $2(|p_o| + 1)K^2 \sup_{(t,y) \in A \cap N_{\epsilon_o}(t_o, y_o)} |J(t) - J(t_o)| \leq \eta/2,$

(ii) if $z \in N_\beta(z_o; \rho)$ then $z(J(t))^{-1} \in N_{2K\beta}(z_o(J(t_o))^{-1}; \rho)$

for all $t \in C \cap \{t \mid |t - t_o| \leq \epsilon_o\}$.

Given ϵ', $0 < \epsilon' \leq \epsilon_o$, there exists a point p such that

$$|p - p_o| \leq \eta/2K, \ |p| \leq 2(|p_o| + 1), \text{ and}$$

$$p \in \text{co } [Q_J(t_o, y_o, \epsilon') \cap N_\beta(z_o; \rho)] \subset E^{r+1}.$$

Then p is the convex combination of r+2 points w_j of the set in brackets, or

$$p = \Sigma_{j=o}^{r+1} \lambda_j w_j; \quad \Sigma_{j=o}^{r+1} \lambda_j = 1, \quad \lambda_j \geq 0,$$

$$w_j \in Q_J(t_o, y_o, \epsilon') \cap N_\beta(z_o; \rho), \ j = 0,1,\ldots,r+1.$$

That is, $w_j = q_j J(t_j)$ with $q_j \in Q(t_j, y_j)$ and

$$(t_j, y_j) \in N_{\epsilon'}(t_o, y_o) \cap A, \ j = 0,1,\ldots,r,r+1.$$

By choice of ϵ' and the statement (ii)

$$q_j \in Q(t_o, y_o, \epsilon') \cap N_{2K\beta}(z_o(J(t_o))^{-1}; \rho).$$

Consider the point $\underline{P}$

$$\underline{P} = (\sum_{j=0}^{r+1} \lambda_j J(t_j))^{-1} (\sum_{j=0}^{r+1} \lambda_j J(t_j)\, q_j)$$

in the set $\mathrm{co}[Q(t_o, y_o, \epsilon) \cap N_{2K\beta}(z_o(J(t_o))^{-1}; \rho)]$.

Since $\underline{P} = (\sum_{j=0}^{r+1} \lambda_j J(t_j))^{-1} \cdot p$, $|\underline{P} - p(J(t_o))^{-1}| \leq |J(t_o) - \sum \lambda_j J(t_j)|$

$$(|J(t_o) \cdot \sum \lambda_j J(t_j)|)^{-1} |p| \leq 2(|p_o| + 1)K^2$$

$$\sup_{(t, y) \in A \cap N_{\epsilon_0}(t_o, y_o)} |J(t) - J(t_o)| \leq \eta/2.$$

Consequently,

$$|\underline{P} - p_o(J(t_o))^{-1}| \leq |\underline{P} - p(J(t_o))^{-1}| + |p(J(t_o))^{-1} - p_o(J(t_o))^{-1}|$$

$$\leq \eta/2 + \eta/2 = \eta.$$

We have

$$p_o(J(t_o))^{-1} \in [\mathrm{cl\ co}\,(Q(t_o, y_o, \epsilon') \cap N_{2K\beta}(z_o(J(t_o))^{-1}; \rho)]_\eta$$

where $\eta > 0$ and $\beta > 0$ are arbitrary and $\epsilon' > 0$ is arbitrary so long as $\epsilon' < \epsilon_o(\eta, \beta)$. Since $Q(t_o, y_o, \epsilon'')$ contains $Q(t_o, y_o, \epsilon')$ for $\epsilon'' > \epsilon'$, the above statement holds for arbitrary positive η, β, and ϵ'. Also, because of the fact that the set inside the brackets is closed and $\eta > 0$ is an arbitrary positive number,

$$p_o(J(t_o))^{-1} \in \mathrm{cl\ co}\,[Q(t_o, y_o, \epsilon') \cap N_{2K\beta}(z_o(J(t_o))^{-1}; \rho)]$$

for arbitrary $\epsilon' > 0$ and $\beta > 0$. Since $Q(t, y)$ has property $Q(\rho)$ on C,

$$p_o(J(t_o))^{-1} \in Q(t_o, y_o) \cap N(z_o(J(t_o))^{-1};\ \rho)$$

and

$$p_o \in Q_J(t_o, y_o) \cap N(z_o; \rho).$$

3. A SUFFICIENT CONDITION FOR PROPERTY $Q(\rho)$

We consider now a situation which occurs often in optimal control theory.

Let A be a closed subset of the (t,y)-space $E^\nu \times E^s$, and for every $(t, y) \in A$ let $U(t, y)$ be a given subset of the u-space E^m, $u = (u^1, \cdots, u^m)$. Let M denote the set of all (t, y, u) with $(t, y) \in A$, $u \in U(t, y)$. Let $f(t, y, u) = (f_o, \ldots, f_r)$ be a given continuous vector function on M, and for every $(t, y) \in A$ let $Q(t, y)$ be the set

$$\begin{aligned} Q(t, y) &= f(f, y, U(t, y)) \\ &= [z \in E^{r+1} \mid z = f(t, y, u)\ u \in U(t, y)] \subset E^{r+1}. \end{aligned}$$

(3.i) Let us assume that A is closed, f continuous on M, the set $Q(t_o, y_o)$ is ρ-convex for some $0 \le \rho \le r+1$, and that the sets $U(t, y)$ are compact, uniformly bounded for (t, y) in a neighborhood $N_\delta(t_o, y_o)$ of (t_o, y_o) in A, and have property (U) at (t_o, y_o). Then the sets $Q(t, y)$ have property $Q(\rho)$ at (t_o, y_o).

<u>Proof</u>. Let B denote a cube of the u-space with $U(t, y) \subset B$ for all $(t, y) \in N_\delta(t_o, y_o)$. Let M_o denote the set of all (t, y, u) with $(t, y) \in N_\delta(t_o, y_o)$, $u \in U(t, y)$. Since $M_o \subset N_\delta(t_o, y_o) \times B$, M_o is bounded. Let $\bar{z} \in E^{r+1}$ be any point

$$\bar{z} \in \bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{cl co } [Q(t_o, y_o, \epsilon) \cap \{z \in E^{r+1} \,|\, |z^i - z_o^i| \leq \beta,$$
$$i = \rho,\ldots,r\}]$$

Then, for every $k = 1, 2,\ldots$, we can select $\epsilon_k > 0$, $\beta_k > 0$, $z_k \in E^{r+1}$, with $\epsilon_k \to 0$, $\beta_k \to 0$, $z_k \to \bar{z}$ as $k \to \infty$, and

$$z_k \in \text{co } [Q(t_o, y_o, \epsilon_k) \cap \{z \in E^{r+1} \,|\, |z^i - z_o^i| \leq \beta_k, i = \rho,\ldots,r\}]$$

Also, for every k, we can select $\lambda_k^\gamma \geq 0$, z_k^γ, $\gamma = 1,\ldots,\mu$, with $\mu = r+2$, such that $0 \leq \lambda_k^\gamma \leq 1$, $\sum_\gamma \lambda_k^\gamma = 1$, and $z_k^\gamma \in [Q(t_o, y_o, \epsilon_k) \cap \{z \in E^{r+1} \,|\, |z^i - z_o^i| \leq \beta_k, i = \rho,\ldots,r]$, $\gamma = 1,\ldots,\mu$, with $z_k^\gamma = (z_k^{o\gamma},\ldots,z_k^{r\gamma}) \in E^{r+1}$, and

$$\bar{z}_k = \sum_\gamma \lambda_k^\gamma z_k^\gamma, \quad k = 1, 2,\ldots.$$

First we see that $|z_k^{i\gamma} - z_o^i| \leq \beta_k$, $i = \rho,\ldots,r$, $r = 1,\ldots,\mu$, $k = 1, 2,\ldots.$ Then we see that for some k_o and $k \geq k_o$ we certainly have $\epsilon_k < \delta$, and hence $z_k^\gamma = f(t_k^\gamma, y_k^\gamma, u_k^\gamma)$ for some $(t_k^\gamma, y_k^\gamma, u_k^\gamma) \in M_o$, namely $(t_k^\gamma, y_k^\gamma) \in N_{\epsilon_k}(t_o, y_o)$, $u_k^\gamma \in U(t_k^\gamma, y_k^\gamma)$, $\gamma = 1,\ldots,\mu$, $k \geq k_o$. Here $(t_k^\gamma, y_k^\gamma) \to (t_o, y_o)$ as $k \to \infty$, $\gamma = 1,\ldots,\mu$. Since M_o is bounded we can select a subsequence, say still [k] for the sake of simplicity, such that as $k \to \infty$, we have $u_k^\gamma \to u^\gamma$, $\gamma = 1,\ldots,\mu$. Then $z_k^\gamma \to z^\gamma$ as $k \to \infty$, with $z^\gamma = f(t_o, y_o, u^\gamma)$ with $|z^{i\gamma} - z_o^i| = 0$, or $z^{i\gamma} = z_o^i$, $i = \rho,\ldots,r$, $r = 1,\ldots,\mu$. We can select the subsequence in such a way that also $\lambda_k^o \to \lambda^\gamma$ as $k \to \infty$, $0 \leq \lambda^\gamma \leq 1$, $\sum_\gamma \lambda^\gamma = 1$.

Note that $(t_k^\gamma, y_k^\gamma, u_k^\gamma) \to (t_o, y_o, u^\gamma)$ as $k \to \infty$, $\gamma = 1,\ldots,\mu$, and property (U) of $U(t, y)$ at (t_o, y_o) yields $u^\gamma \in U(t_o, y_o)$, $\gamma = 1,\ldots,\mu$. Thus $z^\gamma = f(t_o, y_o, u^\gamma) \in Q(t_o, y_o)$, $\gamma = 1,\ldots,\mu$, and also

$$\bar{z} = \sum_\gamma \lambda_\gamma z^\gamma,$$

$$z^\gamma = (z_\gamma^o, \ldots, z_\gamma^{\rho-1}, z_o^\rho, \ldots, z_o^r), \ \gamma = 1, \ldots, \mu.$$

By ρ-convexity of the set $Q(t_o, y_o)$ we conclude that $\bar{z} = Q(t_o, y_o)$. Statement (3.i) is thereby proved.

4. THE UPPER SET PROPERTY

Let C, $A(t)$, A be as in the first paragraph of §2. For every point $(t, y) \in A$, let $Q(t, y)$ be a subset of E^{r+1}, $r \geq 0$. We say that the sets $Q(t, y)$ have <u>the upper set property</u> on A provided, for every $(t, y) \in A$ and for every point $z_o = (z_o^0, z_o^1, \ldots, z_o^r) \in Q(t, y)$, then any other point $\bar{z}_o = (\bar{z}^0, z_o^1, \ldots, z_o^r) \in E^{r+1}$ with $\bar{z}^0 \geq z_o^0$, is also a point of $Q(t, y)$.

The proofs of the following statements are not difficult and will be omitted.

(4.i) If S is a collection of subsets of E^{r+1} each of which has the upper set property, then the union of the sets in S has the upper set property. The closure and the convex hull of a set with the upper set property each have the upper set property.

(4.ii) Let $Q(t, y)$ have the upper set property on A. Let $J(t)$, $t \in C$, be a measurable function on C which satisfies the inequality $0 < K^{-1} \leq J(t) \leq K$ for some constant K and all points $t \in C$. Then the sets $Q_J(t, y)$ defined by

$$Q_J(t, y) = \{z \in E^{r+1} \mid z = q \cdot J(t) \text{ for } q \in Q(t, y)\},$$

have the upper set property on A.

(4.iii) Let $Q(t, y)$ have the upper set property on A. Suppose that $\psi(t)$ is a real valued function defined for $t \in C$. Then the sets $Q_\psi^+(t, y)$ defined by

$$Q_\psi^+(t, y) = \{z \in E^{r+1} \mid z = (\psi(t), 0, \ldots, 0) + q \text{ for } q \in Q(t, y)\}$$

have the upper set property on A.

(4.iv) If the sets $Q(t, y)$ have the upper set property and property (U), then they also have property Q(1).

Proof. Let (t_o, y_o) be any point in A and z_0 any point in E^{r+1}. Let I and D be the sets

$$I = \bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{cl co } (Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; r))$$

$$D = \bigcap_{\epsilon>0} \bigcap_{\beta>0} \text{cl } (Q(t_o, y_o, \epsilon) \cap N_\beta(z_o; r))$$

Property (U) holds provided

$$D \subset Q(t_o, y_o) \cap N_\beta(z_o; r))$$

and property Q(1) holds provided

$$I \subset Q(t_o, y_o) \cap N(z_o; r).$$

Therefore, we need only show that $I \subset D$. Let $\bar{z}$ be any point not in D with $\bar{z} = (\bar{z}^o, z_o^1, \ldots, z_o^r)$. Then

$$\bar{z} \notin \text{cl } Q(t_o, y_o, \epsilon_o) \cap N_\beta(z_o; r)$$

for some ϵ_o and β_o both positive. We denote by π_0 the operation of projection of E^{r+1} onto the z^o-axis of E^{r+1}. We have then

$$\bar{z}^0 \notin \pi_o[\text{cl}(Q(t_o, y_o, \epsilon_o) \cap N_{\beta_o}(z_o; r))].$$

Since $Q(t_o, y_o, \epsilon_o) \cap N_{\beta_o}(z_o; r)$ has the upper set property,

$$\bar{z} \notin \text{cl co } (Q(t_o, y_o, \epsilon_o) \cap N_{\beta_o}(z_o; r)).$$

We have $\bar{z} \notin I$ and may conclude that $I \subset D$.

5. EXAMPLES

Examples of variable sets in E_{r+1} with a property $Q(\rho)$ for some $0 \leq \rho \leq r+1$, but which do not have property (Q), or $Q(r+1)$ arise naturally in control theory.

Example 1. Let us consider the problem of the minimum of the cost functional

$$I[x, u_1, u_2, v] = \iint_G(\zeta^2 + \eta^2 + x^2 + u_1^2 + u_2^2)d\zeta d\eta$$

with differential equations

$$x_\zeta = u_1, \; x_\zeta = u_2, \qquad \text{a.e. in } G,$$

and boundary condition

$$\gamma x = v \qquad \text{s-a.e. on } \Gamma = \partial G,$$

where $G = [(\zeta, \eta) | \xi^2 + \eta^2 \leq 1]$, $\Gamma = \partial G$ is the boundary of G, s is the arc length on Γ, γx the boundary values of x, and the control functions u_1, u_2, v have their values $(u_1, u_2) \in U = E^2$, $v \in V = \{-1\} \cup \{1\}$. Actually, we want to minimize I in the class Ω of all systems (x, u_1, u_2, v) with x any element of the Sobolev space $W_2^1(G)$ satisfying all relations and constraints above and for which I is finite. As we shall see in [8] the following sets are relevant:

$$\tilde{\tilde{Q}}(\zeta, \eta, x) = [(z^0, z^1, z^2) | \; z^0 \geq \zeta^2 + \eta^2 + x^2 + u_1^2 + u_2^2, \; z^1 = u_1,$$
$$z^2 = u_2, \; (u_1, u_2) \in E^2] \subset E^3.$$

$$\tilde{\tilde{R}} = [(z^0, z) | z^0 \geq 0, \; z = v, \; v = \pm 1 \subset E^2.$$

For the sets $\tilde{\tilde{Q}}$ we have $r = 2$, and they have property (Q), or Q(3), in $A = \mathrm{cl}\, G \times E_1$. For the sets $\tilde{\tilde{R}}$ we have $r = 1$, and they have property Q(1) in $B = \Gamma$, have property (U), but they are not convex, and do not have property (Q). We shall see in [9] that the problem above has an absolute minimum in Ω.

Example 2. Let us consider the problem of the minimum of the cost functional

$$I[x, u_1, u_2, v] = \iint_G (x^2 + x_\zeta^2 + x_\eta^2 + u_1^2 + u_2^2(1 - u_2)^2) \, d\zeta d\eta$$
$$+ \int_\Gamma (\gamma x - 1)^2 ds,$$

with differential equations

$$x_{\zeta\zeta} + x_{\eta\eta} = u_1, \qquad x_\zeta + x_\eta = u_2 \qquad \text{a.e. in } G,$$

$$x_\zeta = \cos v, \qquad x_\eta = \sin v, \qquad \text{s-a.e. on } \Gamma = \partial G,$$

where G and Γ are as in example 1, where γx denotes the boundary values of x, and the control functions u_1, u_2, v have their values $(u_1, u_2) \in U = E_2$, $v \in V = E_1$. We want to minimize I in a class Ω of systems (x, u_1, u_2, v) with x any element of the Sobolev space $W_2^2(G)$ satisfying all relations and constraints above, for which I is finite, and satisfying an inequality

$$\|x_{\zeta\zeta}\|_2^2 + \|x_{\zeta\eta}\|_2^2 + \|x_{\eta\eta}\|_2^2 \leq M.$$

Here M is a constant chosen large enough so that Ω is not empty. As we shall see in [8] and [9] the following sets are relevant:

$$\tilde{\tilde{Q}}(y) = [(z^o, z^1, z^2) \mid z^o \geq y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2 (1 - u_2)^2,$$
$$z^1 = u_1, \ z^2 = u_2, \ (u_1, u_2) \in E_2] \subset E_3,$$
$$\tilde{\tilde{R}}(\mathring{y}) = [(z^o, z^1, z^2) \mid z^o \geq (\mathring{y}-1)^2, \ z^1 = \cos v, \ z^2 = \sin v,$$
$$v \in E_1] \subset E_3,$$

where $y = (y_1, y_2, y_3)$ in $\tilde{\tilde{Q}}(y)$, and $\mathring{y}$ in $\tilde{\tilde{R}}(\mathring{y})$ are arbitrary. For both sets we have $r = 2$. The sets $\tilde{\tilde{Q}}$ have property Q(2), but they are not convex and do not have property (Q), or Q(3). The sets $\tilde{\tilde{R}}$ have property Q(1), but they are not convex and do not have property (Q), or Q(3). They all have property Q(o), or (U). We shall see in [9] that this problem has an absolute minimum in Ω.

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