On the Corona of Two Graphs[1])

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Our object in this note is to construct a new and simple operation on two graphs G_1 and G_2, called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G_1 and of G_2.

Consider two permutation groups A and B of order m and n respectively acting on objects sets $X=\{x_1, x_2, \dots, x_d\}$ and $Y=\{y_1, y_2, \dots, y_e\}$. By their *composition* or *wreath product* $A[B]$ we will mean that permutation group of order mn^d acting on $X\times Y$ in which each permutation $\alpha\in A$ and each sequence $(\beta_1, \beta_2, \dots, \beta_d)$ of permutations in B induce the permutation $\gamma=(\alpha; \beta_1, \beta_2, \dots, \beta_d)$ such that $\gamma(x_i, y_j)=(\alpha x_i, \beta_i y_j)$. We write $A\equiv B$ to mean that two permutation groups A and B are not only isomorphic but also permutationally equivalent. More specifically let $h: A\leftrightarrow B$ be an isomorphism. To define $A\equiv B$, we also require another $1-1$ map $f: X\leftrightarrow Y$ between the objects such that for all x in X and α in A, $f(\alpha x)=h(\alpha)f(x)$.

Let graphs[2]) G_1 and G_2 have point sets V_1 and V_2. Then $G_1[G_2]$, their *lexicographic product*, has $V_1\times V_2$ as its set of points, with two of its points $u=(u_1, u_2)$ and $v=(v_1, v_2)$ adjacent whenever u_1 is adjacent to v_1 in G_1, or $u_1=v_1$ and u_2 is adjacent to v_2 in G_2. Let A_1 and A_2 be the groups of graphs G_1 and G_2; then necessary and sufficient conditions for $A_1[A_2]$ to be permutationally equivalent with the group of $G_1[G_2]$ were found by Sabidussi [2]. In order to state this result, we recall from [1] that $\Gamma(G)$ denotes the group of graph G and $\bar{G}$ the *complement* of G, that the *neighborhood* $N(v)$ of a point v of G is the set of points adjacent with v, and the *closed neighborhood* of v is $N(v)\cup\{v\}$.

THEOREM (Sabidussi). *If G_1 is not totally disconnected, then the group of the lexicographic product of two graphs G_1 and G_2 is the wreath product of their groups,*

$$\Gamma(G_1[G_2])\equiv\Gamma(G_1)[\Gamma(G_2)],$$

if and only if the following two conditions hold:

1) *If there are two points in G_1 with the same neighborhood, then G_2 is connected.*

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[2]) The definitions of graphical concepts not included here may be found in the book [1]. In particular, a *graph* G has a finite nonempty set V of points and a set of lines which is a subset of all unordered pairs of points.

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2) *If there are two points in G_1 with the same closed neighborhood, then $\bar{G}_2$ is connected.*

These conditions for the group of the lexicographic products of two graphs to be permutationally equivalent to the composition of their groups are rather complex. This suggests that another operation on graphs be constructed for the purpose of realizing the composition of their groups only up to group isomorphism.

The *corona* $G_1 \circ G_2$ of two graphs G_1 and G_2 (where G_i has p_i points and q_i lines) is defined as the graph G obtained by taking one copy of G_1 and p_1 copies of G_2, and then joining by a line the i'th point of G_1 to every point in the i'th copy of G_2. It follows from the definition of the corona that $G_1 \circ G_2$ has $p_1(1+p_2)$ points and $q_1+p_1q_2+p_1p_2$ lines. This fact allows us to recognize immediately that the associative law never holds: $G_1 \circ (G_2 \circ G_3)$ and $(G_1 \circ G_2) \circ G_3$ are always different graphs. Indeed the first one has $p_1(1+p_2+p_2p_3)$ points while the second has $p_1(1+p_2)(1+p_3)$ points, and these numbers are never equal.

We denote the identity group of degree 1 (and order 1) by E_1 and define the *sum* $A+B$ of two permutation groups A and B as follows. The group $A+B$ acts on the disjoint union $X \cup Y$ and its elements are all the ordered pairs, written $\alpha+\beta$, of permutations $\alpha \in A$, $\beta \in B$, given by

$$(\alpha + \beta)\, z = \begin{cases} \alpha z\,, & z \in X \\ \beta z\,, & z \in Y \end{cases}$$

Note that the trivial graph K_1 consists of one isolated point (of degree zero).

THEOREM 1. *The group of the corona of two graphs G_1 and G_2 can be written explicitly in terms of the following wreath product involving their groups,*

$$\Gamma(G_1 \circ G_2) \equiv \Gamma(G_1)\,[E_1 + \Gamma(G_2)]\,,$$

if and only if at least one of the two graphs G_1 and $\bar{G}_2$ has no isolated points.

Proof. Automorphisms γ of the corona $G_1 \circ G_2$ may be obtained by the action of one automorphism α_1 of G_1 combined with p_1 automorphisms β_i of the graph H_2 obtained from G_2 by adding a new point w to G_2 and joining w by lines with every point of G_2, such that every automorphism β_i leaves w fixed. But we now see that γ acts on the set of points of $G_1 \circ G_2$ precisely in accordance with the definition of the wreath product of the two permutation groups $\Gamma(G_1)$ and $E_1+\Gamma(G_2)$. For α_1 permutes the points of G_1, and hence in the corona graph $G_1 \circ G_2$, α_1 permutes the p_1 copies of H_2 constructed above. Then each of the p_1 automorphisms $\beta_1, \beta_2, \ldots$ permutes one of the copies of H_2 in an adjacency preserving manner, and leaves w fixed. Thus the automorphisms considered so far constitute a permutation group on the $p_1(1+p_2)$ points of $G_1 \circ G_2$ that is permutationally equivalent to the wreath product $\Gamma(G_1)[E_1+\Gamma(G_2)]$.

It remains to be shown that if and only if at least one of the graphs G_1 and $\bar{G}_2$ has no isolated points, no further automorphisms of the corona $G_1 \odot G_2$ can exist. Now to prove the necessity of this condition we will show that such an automorphism really does exist if both G_1 and $\bar{G}_2$ have isolated points.

Indeed let u_i be an isolated point of G_1, and v_j an isolated point of $\bar{G}_2$. Then v_j is joined in G_2 by lines with all the other points of that graph. Now consider in $G_1 \odot G_2$ that copy $G_2^{(i)}$ of G_2 whose points are joined with the point u_i of G_1, and let $v_{j,i}$ be that point of $G_2^{(i)}$ that corresponds to the point v_j of G_2. Since then both u_i and $v_{j,i}$ are joined by lines with all the other points of $G_2^{(i)}$, but with no other points of the corona $G_1 \odot G_2$ (if we except the line joining u_i and $v_{j,i}$), it is obvious that on interchanging u_i and $v_{j,i}$ (but leaving fixed all the other points of $G_1 \odot G_2$), one obtains an automorphism of $G_1 \odot G_2$ that does not belong to the wreath product $\Gamma(G_1)[E_1 + \Gamma(G_2)]$. Hence in order to exclude the existence of such an automorphism it is necessary that at least one of the graphs G_1 or $\bar{G}_2$ have no isolated points.

It remains to prove the sufficiency of these conditions. In other words we have to show that if

(i) G_1 has no isolated points,

or

(ii) $\bar{G}_2$ has no isolated points,

then $G_1 \odot G_2$ does not admit other automorphisms than those of the wreath product $\Gamma(G_1)[E_1 + \Gamma(G_2)]$.

Now it is obvious than an automorphism not belonging to the wreath product $\Gamma(G_1)[E_1 + \Gamma(G_2)]$ would take at least one point of one of the copies of G_2 into a point of G_1. But this is impossible since we will show that if (i) or (ii) holds, in the corona $G_1 \odot G_2$ the degree of any point of G_1 is always higher than that of any other point.

For this purpose let r be the degree in G_1 of any point u_i of G_1; its degree in $G_1 \odot G_2$ will then be $r + p_2$, as seen immediately from the definition of corona. On the other hand, if a point v_j of G_2 has degree s in G_2, the degree of the corresponding point $v_{j,k}$ of any copy $G_2^{(k)}$ $(k = 1, 2, \ldots, p_1)$ will evidently be $s+1$. What we have then to show is that the inequality

$$r + p_2 > s + 1 \tag{1}$$

holds for any pair of points from G_1 and G_2 if (i) or (ii) is satisfied.

Proof of (1) *for the Case* (i). If G_1 has no isolated points, we have $r > 0$ for every point of G_1. Moreover, since G_2 is a graph, it follows that

$$s \leqslant p_2 - 1, \tag{2}$$

and $r + p_2 > p_2 \geqslant s + 1$.

Proof of (1) *for Case* (ii). If $\bar{G}_2$ has no isolated point, and if

$$s = p_2 - 1$$

holds for a point v of G_2, then v would be joined by lines with all the other points of G_2, and hence would be an isolated point of $\bar{G}_2$. Thus

$$s < p_2 - 1 \tag{2'}$$

for all the points of G_2. *A fortiori*, $r + p_2 > s + 1$ (since $r \geqslant 0$).

The main result of this note is an immediate consequence of Theorem 1.

COROLLARY. *The group of the corona $G_1 \circ G_2$ of two graphs is isomorphic to the wreath product $\Gamma(G_1)[\Gamma(G_2)]$ of their groups if and only if at least one of the graphs G_1 and $\bar{G}_2$ has no isolated points.*

Finally it might be observed that the corona operation allows us to obtain another graph whose group is in general isomorphic to the wreath product of the groups of the graphs G_1 and G_2, a sufficient condition being this time only that G_2 have no isolated points. For this purpose, consider the graph $H = G_2 \cup K_1$ obtained from G_2 by adding a new point and no new lines to G_2. This new graph might be described as the complement of $K_1 \circ \bar{G}_2$. For this graph H, the following theorem holds.

THEOREM 2. *If G_2 has no isolated points, then,*

$$\Gamma(G_1 \circ (G_2 \cup K_1)) \equiv \Gamma(G_1)[E_2 + \Gamma(G_2)].$$

Proof. Let $H = G_2 \cup K_1$; then $\bar{H} = K_1 \circ \bar{G}_2$ is obviously a graph without isolated points, and it follows from Theorem 1 that

$$\Gamma(G_1 \circ H) \equiv \Gamma(G_1)[E_1 + \Gamma(H)].$$

Now since the group of a graph and that of its complement are permutationally equivalent, we also have

$$\Gamma(G_1 \circ H) \equiv \Gamma(G_1)[E_1 + \Gamma(\bar{H})],$$

$$\Gamma(G_1 \circ (K_1 \cup G_2)) \equiv \Gamma(G_1)[E_1 + \Gamma(K_1 \circ G_2)].$$

Since the complement of $\bar{G}_2$, namely G_2 itself, has no isolated points, it follows from Theorem 1 that

$$\begin{aligned}\Gamma(\overline{K_1 \circ G_2}) &\equiv \Gamma(K_1)[E_1 + \Gamma(G_2)] \\ &\equiv E_1[E_1 + \Gamma(G_2)], \\ &\equiv E_1 + \Gamma(G_2)\end{aligned}$$

by the definition of wreath product.

REFERENCES

[1] HARARY, F., *Graph Theory* (Addison-Wesley, Reading, Mass. 1969).
[2] SABIDUSSI, G., *The Composition of Graphs*, Duke Math. J. *26*, 693–696 (1959).

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