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The Path Integral Approach to Financial Modeling and Options Pricing*

VADIM LINETSKY
Financial Engineering Program, Department of Industrial and Operations Engineering, University of Michigan, 272 IOE Building, 1205 Beal Avenue, Ann Arbor, MI 48109-2117, U.S.A.; e-mail: linetsky@engin.umich.edu

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Abstract. In this paper we review some applications of the path integral methodology of quantum mechanics to financial modeling and options pricing. A path integral is defined as a limit of the sequence of finite-dimensional integrals, in a much the same way as the Riemannian integral is defined as a limit of the sequence of finite sums. The risk-neutral valuation formula for path-dependent options contingent upon multiple underlying assets admits an elegant representation in terms of path integrals (Feynman–Kac formula). The path integral representation of transition probability density (Green's function) explicitly satisfies the diffusion PDE. Gaussian path integrals admit a closed-form solution given by the Van Vleck formula. Analytical approximations are obtained by means of the semiclassical (moments) expansion. Difficult path integrals are computed by numerical procedures, such as Monte Carlo simulation or deterministic discretization schemes. Several examples of path-dependent options are treated to illustrate the theory (weighted Asian options, floating barrier options, and barrier options with ladder-like barriers).

Key words: options pricing, financial derivatives, path integrals, stochastic models

1. Introduction

In this paper we consider some applications of the path integral formalism of quantum mechanics to financial modeling. Path integrals constitute one of the basic tool of modern quantum physics. They were introduced in physics by Richard Feynman in 1942 in his Ph.D. thesis on path integral formulation of quantum mechanics (Feynman, 1942, 1948; Feynman and Hibbs, 1965; Kac, 1949, 1951, 1980; Fradkin, 1965; Simon, 1979; Schulman, 1981; Glimm and Jaffe, 1981; Freidlin, 1985; Dittrich and Reuter, 1994). In classical deterministic physics, time evolution of dynamical systems is governed by the *Least Action Principle*. Classical equations of motion, such as Newton's equations, can be viewed as the Euler–Lagrange equations for a minimum of a certain *action functional*, a time integral of the Lagrangian function defining the dynamical system. Their deterministic solutions, trajectories of the classical dynamical system, minimize the action functional (the least action principle). In quantum, i.e. probabilistic, physics, one talks about probabilities of different paths a quantum (stochastic) dynamical system can take. One defines a

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measure on the set of all possible paths from the initial state x_i to the final state x_f of the quantum (stochastic) dynamical system, and expectation values (averages) of various quantities dependent on paths are given by *path integrals* over all possible paths from x_i to x_f (path integrals are also called *sums over histories*, as well as *functional integrals*, as the integration is performed over a set of continuous functions of time (paths)). The classical action functional is evaluated to a real number on each path, and the exponential of the negative of this number gives a weight of the path in the path integral. According to Feynman, *a path integral is defined as a limit of the sequence of finite-dimensional multiple integrals*, in a much the same way as the Riemannian integral is defined as a limit of the sequence of finite sums. The path integral representation of averages can also be obtained directly as the *Feynman–Kac solution to the partial differential equation* describing the time evolution of the quantum (stochastic) dynamical system (Schrodinger equation in quantum mechanics or diffusion (Kolmogorov) equation in the theory of stochastic processes).

In finance, the fundamental principle is the absence of arbitrage (Ross, 1976; Cox and Ross, 1976; Harrison and Kreps, 1979; Harrison and Pliska, 1981; Merton, 1990; Duffie, 1996). In finance it plays a role similar to the least action principle and the energy conservation law in natural sciences. Accordingly, similar to physical dynamical systems, one can introduce Lagrangian functions and action functionals for financial models. Since financial models are stochastic, expectations of various quantities contingent upon price paths (*financial derivatives*) are given by path integrals, where the action functional for the underlying risk-neutral price process defines a risk-neutral measure on the set of all paths. Averages satisfy the Black–Scholes partial differential equation, which is a finance counterpart of the Schrodinger equation of quantum mechanics, and the risk-neutral valuation formula is interpreted as the Feynman–Kac representation of the PDE solution. Thus, the path-integral formalism provides a natural bridge between the risk-neutral martingale pricing and the arbitrage-free PDE-based pricing.

To the best of our knowledge, applications of path integrals and related techniques from quantum physics to finance were first systematically developed in the eighties by Jan Dash (see Dash, 1988, 1989 and 1993). His work influenced the author of the present paper as well. See also Esmailzadeh (1995) for applications to path-dependent options and Eydeland (1994) for applications to fixed-income derivatives and interesting numerical algorithms to compute path integrals. This approach is also very close to the semigroup pricing developed by Garman (1985), as path integrals provide a natural representation for *pricing semigroup kernels*, as well as to the Green's functions approach to the term structure modeling of Beaglehole and Tenney (1991) and Jamshidian (1991). See also Chapters 5–7 and 11 in the monograph Duffie (1996) for the Feynman–Kac approach in finance and references therein.

It is the purpose of this paper to give an introductory overview of the path integral approach to financial modeling and options pricing and demonstrate that path integrals and Green's functions constitute both a natural theoretical concept and a practical computational tool in finance, especially for *path-dependent derivatives*.

The rest of this paper is organized as follows. In Section 2, we give an overview of the general framework of *path-integral options pricing*. We start by considering a single-asset Black–Scholes model as an example. Then, we develop the path integral formalism for a multi-asset economy with asset- and time-dependent volatilities and correlations. The central result here is a general *path integral representation (Feynman–Kac formula) for a path-dependent option* contingent upon a finite number of underlying asset prices. The path integration measure is given by an exponential of the negative of the action functional for the risk-neutral price process. This formula constitutes a basis for practical calculations of path-dependent options. In Section 3, we give a brief overview of the main techniques to evaluate path integrals. Gaussian path integrals are calculated analytically by means of the Van Vleck formula. Certain initially non-Gaussian integrals may be reduced to Gaussians by changes of variables, time re-parametrizations and projections. Finally, essentially non-Gaussian path integrals must be evaluated either numerically by Monte Carlo simulation or a deterministic discretization scheme, such as binomial or trinomial trees, or by analytical approximations such as the semi-classical or WKB approximation. In Section 4, three examples of path-dependent options are given to illustrate the theory (weighted Asian options, floating barrier options and barrier options with ladder-like barriers).

2. Risk-Neutral Valuation and Wiener–Feynman Path Integrals

2.1. Black–Scholes Example

We begin by reviewing the Black–Scholes model (Black and Scholes (1973) and Merton (1973); see also Hull (1996) and Duffie (1996)). A *path-independent* option is defined by its payoff at expiration at time T

$$\mathcal{O}_F(S_T, T) = F(S_T), \tag{2.1}$$

where F is a given function of the terminal asset price S_T. We assume we live in the Black–Scholes world with continuously compounded risk-free interest rate r and a single risky asset following a standard geometric Brownian motion

$$\frac{\mathrm{d}S}{S} = m\ \mathrm{d}t + \sigma\ \mathrm{d}z \tag{2.2}$$

with constant drift rate m and volatility σ (for simplicity we assume no dividends). Then the standard absence of arbitrage argument leads us to constructing a replicating portfolio consisting of the underlying asset and the risk-free bond and to the Black–Scholes PDE for the present value of the option at time t preceeding expiration

$$\frac{\sigma^2}{2} S^2 \frac{\partial^2 \mathcal{O}_F}{\partial S^2} + rS\frac{\partial \mathcal{O}_F}{\partial S} - r\mathcal{O}_F = -\frac{\partial \mathcal{O}_F}{\partial t} \tag{2.3}$$

with initial condition (2.1) (more precisely, terminal condition since we solve backwards in time). This is the backward Kolmogorov equation for the risk-neutral diffusion process (2.2) with drift rate equal to the risk-free rate r. Introducing a new variable $x = \ln S$ which follows a standard arithmetic Brownian motion

$$\mathrm{d}x = \left(m - \frac{\sigma^2}{2}\right)\ \mathrm{d}t + \sigma\ \mathrm{d}z, \tag{2.4}$$

Equations (2.1), (2.3) reduce to

$$\frac{\sigma^2}{2}\frac{\partial^2 \mathcal{O}_F}{\partial x^2} + \mu\frac{\partial \mathcal{O}_F}{\partial x} - r\mathcal{O}_F = -\frac{\partial \mathcal{O}_F}{\partial t}, \tag{2.5a}$$

$$\mu = r - \frac{\sigma^2}{2}, \tag{2.5b}$$

$$\mathcal{O}_F(\mathrm{e}^{x_T}, T) = F(\mathrm{e}^{x_T}). \tag{2.5c}$$

A unique solution to the Cauchy problem (2.5) is given by the Feynman–Kac formula (see, e.g., Duffie, 1996; see also Ito and McKean, 1974; Durrett, 1984; Freidlin, 1985; Karatzas and Shreve, 1992)

$$\mathcal{O}_F(S,t) = \mathrm{e}^{-r\tau} E_{(t,S)}\left[F(S_T)\right], \quad \tau = T - t, \tag{2.6}$$

where $E_{(t,S)}[.]$ denotes averaging over the risk-neutral measure conditional on the initial price S at time t. This average can be represented as an integral over the set of all paths originating from (t, S), *path integral*. It is defined as a *limit of the sequence of finite-dimensional multiple integrals*, in a much the same way as the standard Riemannian integral is defined as a limit of the sequence of finite sums (Feynman, 1942 and 1948; Feynman and Hibbs, 1965). We will first present the final result and then give its derivation. In Feynman's notation, the average in (2.6) is represented as follows ($x = \ln S,\ x_T = \ln S_T$):

$$\begin{aligned}\mathcal{O}_F(S,t) &= \mathrm{e}^{-r\tau} E_{(t,S)}\left[F(\mathrm{e}^{x_T})\right]\\ &= \mathrm{e}^{-r\tau}\int_{-\infty}^{\infty}\left(\int_{x(t)=x}^{x(T)=x_T} F\left(\mathrm{e}^{x_T}\right)\mathrm{e}^{-A_{BS}[x(t')]}\mathcal{D}x(t')\right)\ \mathrm{d}x_T.\end{aligned} \tag{2.7}$$

A key object appearing in this formula is the Black–Scholes action functional $A_{BS}[x(t')]$ defined on paths $\{x(t'), t \leq t' \leq T\}$ as a time integral of the Black–Scholes Lagrangian function

$$\begin{aligned}&A_{BS}[x(t')] = \int_t^T \mathcal{L}_{BS}\ \mathrm{d}t', \qquad \mathcal{L}_{BS} = \frac{1}{2\sigma^2}(\dot{x}(t') - \mu)^2,\\ &\dot{x}(t') := \frac{\mathrm{d}x}{\mathrm{d}t'}.\end{aligned} \tag{2.8}$$

This action functional defines the path integration measure.

The path integral in (2.7) is defined as follows. First, paths are discretized. Time to expiration τ is discretized into N equal time steps Δt bounded by $N+1$ equally spaced time points $t_i = t + i\Delta t$, $i = 0, 1, \ldots, N$, $\Delta t = (T-t)/N$. Discrete prices at these time points are denoted by $S_i = S(t_i)$ ($x_i = x(t_i)$ for the logarithms). The discretized action functional becomes a function of $N+1$ variables x_i ($x_0 \equiv x$, $x_N \equiv x_T$)

$$A_{BS}(x_i) = \frac{\mu^2\tau}{2\sigma^2} - \frac{\mu}{\sigma^2}(x_T - x) + \frac{1}{2\sigma^2\Delta t}\sum_{i=0}^{N-1}(x_{i+1} - x_i)^2. \tag{2.9}$$

This is obtained directly from the Definition (2.8) by first noting that

$$\mathcal{L} = \frac{1}{2\sigma^2}\dot{x}^2 - \frac{\mu}{\sigma^2}\dot{x} + \frac{\mu^2}{2\sigma^2}, \tag{2.10a}$$

$$A_{BS}[x(t')] = \frac{\mu^2\tau}{2\sigma^2} - \frac{\mu}{\sigma^2}(x_T - x) + A_0[x(t')], \tag{2.10b}$$

$$A_0[x(t')] = \int_t^T \mathcal{L}_0 \, \mathrm{d}t', \tag{2.10c}$$

where $\mathcal{L}_0$ is the Lagrangian for a zero-drift process $\mathrm{d}x = \sigma\,\mathrm{d}z$ (martingale)

$$\mathcal{L}_0 = \frac{1}{2\sigma^2}\dot{x}^2, \tag{2.11}$$

and then substituting

$$\int_t^T \cdots \mathrm{d}t' \to \sum_{i=0}^{N-1} \ldots \Delta t, \qquad \dot{x} \to \frac{x_{i+1} - x_i}{\Delta t}.$$

Now, the path integral over all paths from the initial state $x(t)$ to the final state x_T is defined as a limit of the sequence of finite-dimensional multiple integrals:

$$\int_{x(t)=x}^{x(T)=x_T} F(\mathrm{e}^{x_T})\ \mathrm{e}^{-A_{BS}[x(t')]}\mathcal{D}x(t')$$

$$:= \lim_{N\to\infty} \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{N-1} F(\mathrm{e}^{x_T})\ \mathrm{e}^{-A_{BS}(x_i)}\frac{\mathrm{d}x_1}{\sqrt{2\pi\sigma^2\Delta t}}\cdots\frac{\mathrm{d}x_{N-1}}{\sqrt{2\pi\sigma^2\Delta t}}. \tag{2.12}$$

This definition of path integrals is used in physics to describe quantum (probabilistic) phenomena. It can be shown (see, e.g., Kac, 1951; Kac, 1980; Glimm and Jaffe, 1981; Simon, 1979; Freidlin, 1985) that this definition is completely rigorous and the limit does converge. In this paper, however, we will follow a heuristic approach to path integrals leaving out the technical details.

Since the payoff in Equation (2.7) depends only on the terminal state x_T, the payoff function F can be moved outside of the path integral, and it can be re-written as follows:

$$\mathcal{O}_F(S,t) = \mathrm{e}^{-r\tau}\int_{-\infty}^{\infty} F(\mathrm{e}^{x_T})\ \mathrm{e}^{(\mu/\sigma^2)(x_T-x)-(\mu^2\tau/2\sigma^2)}\mathcal{K}(x_T,T|x,t)\ \mathrm{d}x_T, \tag{2.13}$$

where $\mathcal{K}(x_T,T|x,t)$ is the transition probability density for zero-drift Brownian motion $\mathrm{d}x=\sigma\ \mathrm{d}z$ (probability density for the terminal state x_T at time T conditional on the initial state x at time t), or Green's function (also called *propagator* in quantum physics) (see, e.g., Schulman (1981)):

$$\begin{aligned}\mathcal{K}(x_T,T|x,t) &= \int_{x(t)=x}^{x(T)=x_T} \mathrm{e}^{-A_0[x(t')]}\mathcal{D}x(t') \\ &:= \lim_{N\to\infty}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{N-1}\exp\left(-\frac{1}{2\sigma^2\Delta t}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2\right) \\ &\quad\times\frac{\mathrm{d}x_1}{\sqrt{2\pi\sigma^2\Delta t}}\cdots\frac{\mathrm{d}x_{N-1}}{\sqrt{2\pi\sigma^2\Delta t}}.\end{aligned} \tag{2.14}$$

The multiple integral here is Gaussian and is calculated using the following identity

$$\int_{-\infty}^{\infty} \mathrm{e}^{-a(x-z)^2-b(z-y)^2}\ \mathrm{d}z = \sqrt{\frac{\pi}{a+b}}\exp\left[-\frac{ab}{a+b}(x-y)^2\right]. \tag{2.15}$$

This is proved by completing the squares in the exponential. Using (2.15) consider the integral on x_1 in (2.14)

$$\frac{1}{2\pi\sigma^2\Delta t}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma^2\Delta t}\left((x_2-x_1)^2+(x_1-x_0)^2\right)\right]\ \mathrm{d}x_1 \tag{2.16a}$$

which equals

$$\frac{1}{\sqrt{2\pi\sigma^2(2\Delta t)}}\exp\left[-\frac{(x_2-x_0)^2}{2\sigma^2(2\Delta t)}\right]. \tag{2.16b}$$

Thus the effect of the x_1 integration is to change Δt to $2\Delta t$ (both in the square root and in the exponential) and to replace $(x_2-x_1)^2+(x_1-x_0)^2$ by $(x_2-x_0)^2$.

The integral over x_2 changes $2\Delta t$ to $3\Delta t$ (both in the square root and in the exponential) and yields the term $(x_3 - x_0)^2$. This procedure is continued for all $N - 1$ integrals. Finally, Δt becomes $N\Delta t$, which is just τ, and $(x_T - x_0)^2$ appears in the exponential. Since there is no longer any dependence on N, the limit operation is trivial and we finally obtain the result

$$\mathcal{K}(x_T, T \mid x, t) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(x_T - x)^2}{2\sigma^2\tau}\right), \tag{2.17}$$

which is, as expected, the normal density. This is the fundamental solution of the zero-drift diffusion equation

$$\frac{\sigma^2}{2}\frac{\partial^2\mathcal{K}}{\partial x^2} = -\frac{\partial\mathcal{K}}{\partial t} \tag{2.18a}$$

with initial condition at $t = T$

$$\mathcal{K}(x_T, T \mid x, T) = \delta(x_T - x), \tag{2.18b}$$

where $\delta(x)$ is the Dirac delta function. Certainly, in this simple case one can also solve the diffusion equation directly. First, a formal solution to the Cauchy problem (2.18) can be written as

$$\mathcal{K}(x_T, T \mid x, t) = \exp\left(\tau\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\right)\delta(x_T - x). \tag{2.19}$$

If we now represent the delta function as a Fourier integral, we obtain

$$\begin{aligned}\mathcal{K}(x_T, T \mid x, t) &= \exp\left(\tau\frac{\sigma^2}{2}\ \frac{\partial^2}{\partial x^2}\right)\int_{-\infty}^{\infty} \mathrm{e}^{ip(x_T - x)}\frac{\mathrm{d}p}{2\pi} \\ &= \int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}\tau\sigma^2p^2 + ip(x_T - x)\right)\frac{\mathrm{d}p}{2\pi} \\ &= \frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(-\frac{(x_T - x)^2}{2\sigma^2\tau}\right),\end{aligned} \tag{2.20}$$

where we have used the standard Gaussian integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{a}{2}y^2 + by\right)\ \mathrm{d}y = \frac{1}{\sqrt{2\pi a}}\exp\left(\frac{b^2}{2a}\right). \tag{2.21}$$

This proves that the path integral (2.14) indeed represents the fundamental solution of diffusion Equation (2.18).

It is useful to note that the Green's function for diffusion with constant drift rate μ is obtained by multiplying the zero-drift Green's function by the drift-dependent factor (see (2.13)):

$$\mathcal{K}^{\mu}(x_T,T\mid x,t) = \mathrm{e}^{(\mu/\sigma^2)(x_T-x)-(\mu^2\tau/2\sigma^2)}\mathcal{K}(x_T,T\mid x,t)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(-\frac{(x_T-x-\mu\tau)^2}{2\sigma^2\tau}\right). \tag{2.22a}$$

It is easy to check directly that $\mathcal{K}^{\mu}$ is the fundamental solution of diffusion equation with drift

$$\frac{\sigma^2}{2}\frac{\partial^2\mathcal{K}^{\mu}}{\partial x^2}+\mu\frac{\partial\mathcal{K}^{\mu}}{\partial x} = -\frac{\partial\mathcal{K}^{\mu}}{\partial t}. \tag{2.22b}$$

The transition probability density satisfies the fundamental *Chapman–Kolmogorov semigroup property* (continuous-time Markov property) (see Garman, 1985, for semigroups in finance)

$$\mathcal{K}(x_3,t_3\mid x_1,t_1) = \int_{-\infty}^{\infty}\mathcal{K}(x_3,t_3\mid x_2,t_2)\mathcal{K}(x_2,t_2\mid x_1,t_1)\ \mathrm{d}x_2. \tag{2.23}$$

Now one can see that the definition of the path integral (2.14) can be obtained by repeated use of the Chapman–Kolmogorov equation:

$$\mathcal{K}(x_T,T\mid x,t) = \lim_{N\to\infty}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{N-1}\mathcal{K}(x_T,T\mid x_{N-1},t_{N-1})$$
$$\cdots\mathcal{K}(x_1,t_1\mid x,t)\ \mathrm{d}x_1\cdots\mathrm{d}x_{N-1}. \tag{2.24}$$

Finally, substituting $\mathcal{K}$ into Equation (2.13) one obtains the Black–Scholes formula for path-independent options

$$\mathcal{O}_F(S,t) = \mathrm{e}^{-r\tau}\int_{-\infty}^{\infty}F(\mathrm{e}^{x_T})\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(-\frac{(x_T-x-\mu\tau)^2}{2\sigma^2\tau}\right)\ \mathrm{d}x_T. \tag{2.25}$$

For a call option with the payoff $\mathrm{Max}(\mathrm{e}^{x_T}-K,0)$ one obtains after performing the integration

$$C(S,t) = SN(d_2)-\mathrm{e}^{-r\tau}KN(d_1), \tag{2.26}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right)+\mu\tau}{\sigma\sqrt{\tau}},\qquad d_2 = d_1+\sigma\sqrt{\tau}. \tag{2.27}$$

In the Black–Scholes example of diffusion with constant coefficients and path-independent payoffs there exists a closed-form solution for the transition probability density as a normal density, and one certainly does not need the path-integrals machinery in this simple case. However, the path integral point of view becomes very useful for more complex models, especially for path-dependent options, general volatilities and drifts and derivatives contingent upon several underlying assets. American options are valued in this framework by the procedure of Geske and Johnson (1984) (see Dash (1988)).

2.2. The Feynman–Kac Approach to Pricing Path-Dependent Options

Consider now a *path-dependent* option defined by its payoff at expiration

$$\mathcal{O}_F(T) = F[S(t')], \tag{2.28}$$

where $F[S(t')]$ is a given *functional* on price paths $\{S(t'), t \le t' \le T\}$, rather than a function dependent just on the terminal asset price. We assume the risk-neutral price process

$$\begin{aligned} \frac{\mathrm{d}S}{S} &= r\ \mathrm{d}t + \sigma\ \mathrm{d}z, \qquad x = \ln S, \\ \mathrm{d}x &= \mu\ \mathrm{d}t + \sigma\ \mathrm{d}z, \qquad \mu = r - \frac{\sigma^2}{2}. \end{aligned} \tag{2.29}$$

Then the present value of this path-dependent option at the inception of the contract t is given by the Feynman–Kac formula

$$\begin{aligned} \mathcal{O}_F(S,t) &= \mathrm{e}^{-r\tau} E_{(t,S)}\left[F[S(t')]\right] \\ &= \mathrm{e}^{-r\tau} \int_{-\infty}^{\infty} \left(\int_{x(t)=x}^{x(T)=x_T} F\left[\mathrm{e}^{x(t')}\right]\ \mathrm{e}^{-A_{BS}[x(t')]} \mathcal{D}x(t') \right)\ \mathrm{d}x_T, \end{aligned} \tag{2.30}$$

where the average is over the risk-neutral process. Since now $F[\mathrm{e}^{x(t')}]$ depends on the *entire path*, it cannot be simply moved outside of the path integral as we did in the previous section in the Black–Scholes case.

Let us first consider a special case. Suppose the payoff functional F can be represented in the form

$$F = f(S_T)\ \mathrm{e}^{-I[S(t')]}, \tag{2.31}$$

where $f(S_T)$ depends only on the terminal asset price S_T, and I is a functional on price paths from (t, S) to (T, S_T) that can be represented as a time integral

$$I[S(t')] = \int_t^T V(x(t'), t')\ \mathrm{d}t', \tag{2.32}$$

of some *potential* $V(x,t')(x = \ln S)$. Then the Feynman–Kac formula (2.30) reduces to

$$\mathcal{O}_F(S,t)$$
$$= \mathrm{e}^{-r\tau} \int_{-\infty}^{\infty} f(\mathrm{e}^{x_T})\ \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2\tau/2\sigma^2)}\, \mathcal{K}_V(x_T, T \mid x, t)\ \mathrm{d}x_T, \tag{2.33}$$

where $\mathcal{K}_V$ is the Green's function (transition probability density) for zero-drift *Brownian motion with killing at rate* $V(x,t')$ (see, e.g., Ito and McKean, 1974; Karlin and Taylor, 1981; Durrett, 1984):

$$\mathcal{K}_V(x_T, T \mid x, t) = \int_{x(t)=x}^{x(T)=x_T} \exp\left(-\int_t^T (\mathcal{L}_0 + V)\ \mathrm{d}t'\right) \mathcal{D}x(t'). \tag{2.34}$$

This is the Feynman–Kac representation of the fundamental solution of zero-drift diffusion PDE with potential V

$$\frac{\sigma^2}{2}\frac{\partial^2 \mathcal{K}_V}{\partial x^2} - V(x,t)\mathcal{K}_V = -\frac{\partial \mathcal{K}_V}{\partial t} \tag{2.35a}$$

and initial condition

$$\mathcal{K}_V(x_T, T \mid x, T) = \delta(x_T - x). \tag{2.35b}$$

It is easy to see that the option price (2.33) satisfies the Black–Scholes PDE with potential

$$\frac{\sigma^2}{2}\frac{\partial^2 \mathcal{O}_F}{\partial x^2} + \mu\frac{\partial \mathcal{O}_F}{\partial x} - (r + V(x,t))\mathcal{O}_F = -\frac{\partial \mathcal{O}_F}{\partial t} \tag{2.36}$$

and the terminal condition $\mathcal{O}_F(S_T, T) = f(S_T)$. It can be interpreted as the Black–Scholes equation with an effective risk-free rate $r + V(x,t)$ and continuous dividend yield $V(x,t)$. Now consider a more general path-dependent payoff that can be represented as a function of the terminal asset price as well as a set of n in some sense 'elementary' functionals $I^i[S(t')]$ on price paths

$$F[\mathrm{e}^{x(t')}] = F(\mathrm{e}^{x_T}, I^i), \qquad I^i = I^i[\mathrm{e}^{x(t')}]. \tag{2.37}$$

Some examples of such functionals and corresponding path-dependent options are: weighted average price (weighted Asian options), maximum or minimum prices (lookback and barrier options) and occupation times (range notes, step options and more general occupation time derivatives, see Linetsky, 1996). We can employ the following trick to move the function F outside of the path integral in Equation

(2.30) (Dash, 1993). First, introduce auxiliary variables λ^i by inserting the Dirac delta function as follows

$$\begin{aligned} F(\mathrm{e}^{x_T}, I^i) &= \int_{\mathbf{R}^n} \delta^n(\lambda^i - I^i) F(\mathrm{e}^{x_T}, \lambda^i) d^n\lambda, \\ \delta^n(\lambda^i - I^i) &\equiv \delta(\lambda^1 - I^1) \dots \delta(\lambda^n - I^n). \end{aligned} \tag{2.38}$$

Next, the delta function is represented as a Fourier integral

$$\begin{aligned} F(\mathrm{e}^{x_T}, I^i) \;&=\; \int_{\mathbf{R}^n} F(\mathrm{e}^{x_T}, \lambda^i) \mathcal{F}_\lambda^{-1} \left[\exp\left(-i \sum_{i=1}^n p_i I^i \right) \right] d^n\lambda \\ &=\; \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \exp\left(i \sum_{i=1}^n p_i (\lambda^i - I^i) \right) F(\mathrm{e}^{x_T}, \lambda^i) d^n p \, d^n\lambda. \end{aligned} \tag{2.39}$$

Finally, substituting this back into Equation (2.30) we arrive at the pricing formula for path-dependent options

$$\begin{aligned} \mathcal{O}_F(S,t) \;=\; \mathrm{e}^{-r\tau} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} F(\mathrm{e}^{x_T}, \lambda^i) \; \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2\tau/2\sigma^2)} \\ \times \mathcal{P}(x_T, \lambda^i, T \mid x, t) \, d^n\lambda \; \mathrm{d}x_T, \end{aligned} \tag{2.40}$$

where $\mathcal{P}$ is the joint probability density for the terminal state x_T and terminal values λ^i of the Brownian functionals I^i at expiration T conditional on the initial state x at inception t. It is given by the inverse Fourier transform

$$\begin{aligned} \mathcal{P}(x_T, \lambda^i, T \mid x, t) \;&=\; \mathcal{F}_\lambda^{-1} \left[\mathcal{K}_{I,p}(x_T, T \mid x, t) \right] \\ &=\; \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \exp\left(i \sum_{i=1}^n p_i \lambda^i \right) \mathcal{K}_{I,p}(x_T, T \mid x, t) d^n p, \end{aligned} \tag{2.41}$$

of the Green's function

$$\mathcal{K}_{I,p}(x_T, T \mid x, t) = \int_{x(t)=x}^{x(T)=x_T} \exp\left(-A_0 - i \sum_{i=1}^n p_i I^i \right) \mathcal{D}x(t') \tag{2.42}$$

with respect to the parameters p_i. If the elementary functionals can be represented as time integrals

$$I^i = \int_t^T v^i(x(t'), t') \; \mathrm{d}t' \tag{2.43}$$

of some potentials $v^i(x, t')$, then the path integral (2.42) takes the standard form (2.34) with potential

$$V(x, t, p_i) = i \sum_{i=1}^n p_i \, v^i(x, t). \tag{2.44}$$

If the potentials v^i are non-negative functions, then Laplace transform can be used in place of the Fourier transform. Consider a path-dependent payoff $F(\mathrm{e}^{x_T}, I)$, where $I = \int_t^T v(x(t'), t')\ dt'$ and $v(x, t')$ is non-negative. Then

$$\begin{aligned} F(\mathrm{e}^{x_T}, I) &= \int_0^\infty \delta(\lambda - I) F(\mathrm{e}^{x_T}, \lambda)\ \mathrm{d}\lambda = \int_0^\infty F(\mathrm{e}^{x_T}, \lambda) \mathcal{L}_\lambda^{-1} \left[\mathrm{e}^{-sI}\right]\ \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_0^\infty \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathrm{e}^{s(\lambda - I)} F(\mathrm{e}^{x_T}, \lambda)\ \mathrm{d}s\ \mathrm{d}\lambda. \end{aligned} \tag{2.45}$$

Here the auxiliary variable λ takes only non-negative values, and we represent the Dirac delta function as an inverse Laplace transform. Then the path-dependent pricing formula takes the form:

$$\begin{aligned} \mathcal{O}_F(S, t) &= \mathrm{e}^{-r\tau} \int_{-\infty}^\infty \int_0^\infty F(\mathrm{e}^{x_T}, \lambda)\ \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2 \tau / 2\sigma^2)} \\ &\quad \times \mathcal{P}(x_T, \lambda, T \mid x, t)\ \mathrm{d}\lambda\ \mathrm{d}x_T, \end{aligned} \tag{2.46}$$

where $\mathcal{P}$ is the joint probability density for the final state x_T and the terminal value λ of the Brownian functional I conditional on the initial state x at time t. It is given by the inverse Laplace transform

$$\mathcal{P}(x_T, \lambda, T \mid x, t) = \mathcal{L}_\lambda^{-1} \left[\mathcal{K}_V(x_T, T \mid x, t)\right] \tag{2.47}$$

of the Green's function for zero-drift Brownian motion with killing at rate $V(x, t) = s\, v(x, t)$ given by the path integral (2.34). It is the Feynman–Kac representation of the fundamental solution of zero-drift diffusion PDE with potential $V(x, t)$ (2.35). It is easy to see that the density (2.47) satisfies a three-dimensional PDE:

$$\frac{\sigma^2}{2} \frac{\partial^2 \mathcal{P}}{\partial x^2} - v(x, t) \frac{\partial \mathcal{P}}{\partial \lambda} = -\frac{\partial \mathcal{P}}{\partial t}. \tag{2.48}$$

In summary, to price a path-dependent claim with the payoff contingent both on the terminal asset price and the terminal value λ of some functional I on price paths that can be represented as a time integral of non-negative potential $v(x, t)$:

(1) find the Green's function of Brownian motion with killing at rate $V = s\, v(x, t)$ by solving the PDE (2.35) or calculating the path integral (2.34);
(2) invert the Laplace transform with respect to s to find the joint density for x_T and λ (2.47); and
(3) calculate the discounted expectation (2.46).

Equations (2.46, 2.47) together with (2.34, 2.35) constitute the traditional form of the Feynman–Kac approach. It allows one to compute joint probability densities of Brownian functionals and the terminal state given the initial state.

So far we considered pricing *newly-written* path-dependent options at the inception of the contract t. Now consider a *seasoned* option at some time t^* during the life of the contract, $t \leq t^* \leq T$, with the terminal payoff $F(S, I)$, where $I = \int_t^T v(x(t'), t')\ \mathrm{d}t'$, and $v(x, t')$ is non-negative. The functional I is additive and can be represented as a sum $I = I_f + I_u$, where I_f is the value of the functional on already fixed price observations on the time interval from the contract inception t to date t^*, $I_f = \int_t^{t^*} v(x(t'), t')\ \mathrm{d}t$, and I_u is the functional on yet unknown segment of the price path from time t^* to expiration T, $I_u = \int_{t^*}^T v(x(t'), t')\ \mathrm{d}t'$. Then the seasoned option price at time t^* is a function of the current asset price $S^* = S(t^*)$, the value I_f of the functional I accumulated to date t^*, and current time t^*:

$$\mathcal{O}_F(S^*, I_f, t^*) = \mathrm{e}^{-r\tau^*} \int_{-\infty}^{\infty} \int_0^{\infty} F(\mathrm{e}^{x_T}, I_f + \lambda)\ \mathrm{e}^{(\mu/\sigma^2)(x_T - x^*) - (\mu^2\tau^*/2\sigma^2)} \times \mathcal{P}(x_T, \lambda, T \mid x^*, t^*)\ \mathrm{d}\lambda\ \mathrm{d}x_T, \tag{2.49}$$

where $\tau^* = T - t^*$ and $x^* = \ln S^*$. It is easy to see that, under suitable technical conditions, from (2.48) it follows that the seasoned option price (2.49) satisfies the following three-dimensional PDE in variables x^*, I_f and t^* (Wilmott, Dewyne and Howison, 1993):

$$\frac{\sigma^2}{2}\frac{\partial^2 \mathcal{O}_F}{\partial x^{*2}} + \mu \frac{\partial \mathcal{O}_F}{\partial x^*} - r\mathcal{O}_F + v(x^*, t^*)\frac{\partial \mathcal{O}_F}{\partial I_f} = -\frac{\partial \mathcal{O}_F}{\partial t^*}. \tag{2.50}$$

2.3. Valuation of Multi-Asset Derivatives with General Parameters

Consider a general D-dimensional diffusion process x^μ, $\mu = 1, 2, \ldots, D$,

$$\mathrm{d}x^\mu = a^\mu\ \mathrm{d}t + \sum_{a=1}^{D} \sigma_a^\mu\ \mathrm{d}z^a, \tag{2.51}$$

$$a^\mu = a^\mu(\mathbf{x}, t), \qquad \sigma_a^\mu = \sigma_a^\mu(\mathbf{x}, t),$$

where $\mathrm{d}z^a$, $a = 1, 2, \ldots, D$, are standard uncorrelated Wiener processes

$$E\left[\mathrm{d}z^a\ \mathrm{d}z^b\right] = \delta^{ab}\ \mathrm{d}t \tag{2.52}$$

(δ^{ab} is the Kroeneker symbol, $\delta^{ab} = 1$ if $a = b$ and zero otherwise). Suppose the risk-free rate is $r(\mathbf{x}, t)$ and Equation (2.51) describes a D-dimensional risk-neutral price process with the risk-neutral drift (boldface letters $\mathbf{x}$ denote D-dimensional vectors – prices of D traded assets in our economy)

$$a^\mu(\mathbf{x}, t) = r(\mathbf{x}, t)x^\mu - D^\mu(\mathbf{x}, t) \tag{2.53}$$

(D^μ are dividends). Consider a path-dependent option with the payoff at expiration

$$\mathcal{O}_F(T) = F[\mathbf{x}(t')]. \tag{2.54}$$

Then the present value $\mathcal{O}_F(\mathbf{x}, t)$ at the inception of the contract t is given by the Feynman–Kac formula

$$\mathcal{O}_F(\mathbf{x}, t) = \int_{\mathbf{R}^D} \left(\int_{\mathbf{x}(t)=\mathbf{x}}^{\mathbf{x}(T)=\mathbf{x}_T} F[\mathbf{x}(t')] \; \mathrm{e}^{-A[\mathbf{x}(t')]} \mathcal{D}\mathbf{x}(t') \right) d^D x_T. \tag{2.55}$$

Here A is the action functional

$$A = \int_t^T \mathcal{L} \; \mathrm{d}t' \tag{2.56}$$

with the Lagrangian function $\mathcal{L}$ for the process (2.51) given by (see, e.g., Langouche, Roekaerts and Tirapegui, 1980 and 1982; Freidlin, 1985)

$$\mathcal{L} = \frac{1}{2} \sum_{\mu,\nu=1}^{D} g_{\mu\nu}(\mathbf{x}, t')(\dot{x}^\mu(t') - a^\mu(\mathbf{x}, t'))(\dot{x}^\nu(t') - a^\nu(\mathbf{x}, t')) + r(\mathbf{x}, t'), \tag{2.57}$$

where $g_{\mu\nu} = g_{\nu\mu}$ is an inverse of the variance-covariance matrix $g^{\mu\nu} = g^{\nu\mu}$

$$\sum_{\rho=1}^{D} g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu, \qquad g^{\mu\nu} = \sum_{a=1}^{D} \sigma_a^\mu \sigma_a^\nu. \tag{2.58}$$

Readers familiar with the Riemannian geometry will recognize $g_{\mu\nu}$ as the Riemannian metric and σ_μ^a as components of the local frame (vielbein).

The general multi-asset path-integral (2.55) is defined as a limit of the sequence of finite-dimensional multiple integrals similar to the one-dimensional example. A discretized action functional is given by

$$\begin{aligned} A(\mathbf{x}_i) \;=\; & \frac{1}{2} \sum_{i=0}^{N-1} \sum_{\mu,\nu=1}^{D} g_{\mu\nu}(\mathbf{x}_i, t_i) \left(\frac{\Delta x_i^\mu}{\Delta t} - a^\mu(\mathbf{x}_i, t_i) \right) \left(\frac{\Delta x_i^\nu}{\Delta t} - a^\nu(\mathbf{x}_i, t_i) \right) \Delta t \\ & + \sum_{i=0}^{N-1} r(\mathbf{x}_i, t_i) \Delta t, \qquad \Delta x_i^\mu = x_{i+1}^\mu - x_i^\mu, \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} & \int_{\mathbf{x}(t)=\mathbf{x}}^{\mathbf{x}(T)=\mathbf{x}_T} F[\mathbf{x}(t')] \; \mathrm{e}^{-A[\mathbf{x}(t')]} \mathcal{D}\mathbf{x}(t') \\ & \quad := \lim_{N\to\infty} \underbrace{\int_{\mathbf{R}^D} \cdots \int_{\mathbf{R}^D}}_{N-1} F(\mathbf{x}_i) \exp(-A(\mathbf{x}_i)) \\ & \qquad \times \prod_{i=1}^{N-1} \frac{d^D \mathbf{x}_i}{\sqrt{(2\pi)^D \; \det(g^{\mu\nu}(\mathbf{x}_i, t_i)) \Delta t}}. \end{aligned} \tag{2.60}$$

The determinant $\det(g^{\mu\nu}(\mathbf{x}_i, t_i))$ of the variance-covariance matrix $g^{\mu\nu}(\mathbf{x}_i, t_i)$ appearing in the square roots defines the integration measure over intermediate points $\mathbf{x}_i$. This discretization scheme is called *pre-point discretization* and is consistent with the Ito's calculus. One could choose a different discretization, such as mid-point or symmetric discretization. The mid-point discretization is consistent with the Stratanovich calculus rather than Ito's. Theoretically, different discretization schemes are equivalent (see Langouche, Roekaerts and Tirapegui, 1980 and 1982, for detailed discussions). However, in practice different discretizations have different numerical convergence properties and it may be advantageous to use one scheme over the other for a particular calculation (see also Karlin and Taylor, 1981, for a discussion of Ito's vs. Stratanovich calculus).

For path-independent options, when the payoff depends only on terminal states $\mathbf{x}_T, \mathcal{O}_F(\mathbf{x}_T, T) = F(\mathbf{x}_T)$, the option value satisfies the backward PDE

$$\mathcal{H}\,\mathcal{O}_F = -\frac{\partial \mathcal{O}_F}{\partial t}, \tag{2.61}$$

where $\mathcal{H}$ is a second order differential operator (generator of the diffusion process (2.51) with the killing term $r(\mathbf{x}, t)$)

$$\mathcal{H} = \frac{1}{2}\sum_{\mu,\nu=1}^{D} g^{\mu\nu}(\mathbf{x}, t)\frac{\partial^2}{\partial x^\mu \partial x^\nu} + \sum_{\mu=1}^{D} a^\mu(\mathbf{x}, t)\frac{\partial}{\partial x^\mu} - r(\mathbf{x}, t). \tag{2.62}$$

The proof that the path integral (2.55) for path-independent options indeed solves the PDE (2.61) is as follows. Consider the fundamental solution $\mathcal{K}(\mathbf{x}_T, T \mid \mathbf{x}, t)$ of the PDE (2.61) with initial condition

$$\mathcal{K}(\mathbf{x}_T, T \mid \mathbf{x}, T) = \delta^D(\mathbf{x}_T - \mathbf{x}). \tag{2.63}$$

For a short time interval $\Delta t = t_2 - t_1$, it can be represented up to the second order $O(\Delta t^2)$ similar to the Black–Scholes case (2.20) (in contrast to the case with constant parameters, it is only valid up to the second order in Δt in the general case):

$$\begin{aligned}\mathcal{K}(\mathbf{x}_2, t_2 \mid \mathbf{x}_1, t_1) &= (1 + \Delta t\,\mathcal{H} + O(\Delta t^2))\delta^D(\mathbf{x}_2 - \mathbf{x}_1) \\ &\approx \exp(\Delta t\,\mathcal{H})\,\delta^D(\mathbf{x}_2 - \mathbf{x}_1).\end{aligned} \tag{2.64}$$

Again introducing the Fourier integral representation of the delta function, we have

$$\begin{aligned}\mathcal{K}(\mathbf{x}_2, t_2 \mid \mathbf{x}_1, t_1) &\approx \exp(\Delta t\,\mathcal{H})\int_{\mathbf{R}^D} \exp\left\{ i\sum_{\mu=1}^{D} p_\mu (x_2^\mu - x_1^\mu)\right\}\frac{d^D p}{(2\pi)^D} \\ &= \int_{\mathbf{R}^D} \exp\left\{ -\frac{\Delta t}{2}\sum_{\mu,\nu=1}^{D} g^{\mu\nu}(\mathbf{x}_1, t_1) p_\mu p_\nu \right.\end{aligned}$$

$$+\, i \sum_{\mu=1}^{D} \left(x_2^\mu - x_1^\mu - a^\mu(\mathbf{x}_1, t_1)\Delta t\right) p_\mu$$

$$\left. -r(\mathbf{x}_1, t_1)\Delta t \right\} \frac{d^D p}{(2\pi)^D}$$

$$= \frac{1}{\sqrt{(2\pi\Delta t)^D \ \det(g^{\mu\nu}(\mathbf{x}_1, t_1))}}$$

$$\times \exp\left\{ -\frac{1}{2} \sum_{\mu,\nu=1}^{D} g_{\mu\nu}(\mathbf{x}_1, t_1) \left(\frac{x_2^\mu - x_1^\mu}{\Delta t} - a^\mu(\mathbf{x}_1, t_1) \right) \right.$$

$$\left. \times \left(\frac{x_2^\nu - x_1^\nu}{\Delta t} - a^\nu(\mathbf{x}_1, t_1) \right) \Delta t - r(\mathbf{x}_1, t_1)\Delta t \right\}. \quad (2.65)$$

To obtain this result we have used the following standard multi-dimensional Gaussian integral

$$\frac{1}{(2\pi)^D} \int_{\mathbf{R}^D} \exp\left(-\frac{1}{2} \sum_{\mu,\nu=1}^{D} A^{\mu\nu} y_\mu y_\nu + \sum_{\mu=1}^{D} B^\mu y_\mu \right) d^D y$$

$$= \frac{1}{\sqrt{(2\pi)^D \ \det(A^{\mu\nu})}} \exp\left(-\frac{1}{2} \sum_{\mu,\nu=1}^{D} (A^{-1})_{\mu\nu} B^\mu B^\nu \right). \quad (2.66)$$

Having at our disposal the short-time transition probability density, we can obtain the density for a finite time interval $\tau = T - t$ in the continuous time limit by successively applying the Chapman–Kolmogorov semigroup property

$$\mathcal{K}(\mathbf{x}_T, T \mid \mathbf{x}, t) \ = \ \lim_{N\to\infty} \underbrace{\int_{\mathbf{R}^D} \cdots \int_{\mathbf{R}^D}}_{N-1} \mathcal{K}(\mathbf{x}_T, T \mid \mathbf{x}_{N-1}, t_{N-1})$$

$$\cdots \mathcal{K}(\mathbf{x}_1, t_1 \mid \mathbf{x}, t) d^D\mathbf{x}_1 \cdots d^D\mathbf{x}_{N-1}. \quad (2.67)$$

Substituting the expression (2.65) for the short-time densities and recognizing that the individual exponentials of short-time densities combine to form the expression (2.59) for the discretized action, we finally obtain

$$\mathcal{K}(\mathbf{x}_T, T \mid \mathbf{x}, t) \ = \ \lim_{N\to\infty} \underbrace{\int_{\mathbf{R}^D} \cdots \int_{\mathbf{R}^D}}_{N-1} \exp(-A(\mathbf{x}_i))$$

$$\times \prod_{i=1}^{N-1} \frac{d^D \mathbf{x}_i}{\sqrt{(2\pi)^D \ \det(g^{\mu\nu}(\mathbf{x}_i, t_i))\Delta t}}$$

$$:= \int_{\mathbf{x}(t)=\mathbf{x}}^{\mathbf{x}(T)=\mathbf{x}_T} \mathrm{e}^{-A[\mathbf{X}(t')]} \mathcal{D}\,\mathbf{x}(t'). \tag{2.68}$$

Thus, we have proved that the path integral (2.68) indeed represents the fundamental solution of diffusion PDE (2.61). Then Equation (2.55) for path-independent options is simply

$$\mathcal{O}_F(\mathbf{x}, t) = \int_{\mathbf{R}^D} F(\mathbf{x}_T)\mathcal{K}(\mathbf{x}_T, T \mid \mathbf{x}, t) d^D \mathbf{x}_T. \tag{2.69}$$

This concludes the proof that (2.55) indeed solves the Cauchy problem (2.61).

For path-dependent payoffs one must employ the procedure outlined in the previous section to move the functional F outside of the path integral. This will result in the appearance of a non-trivial potential $V(x,t)$ in the exponential in path integral (2.68) and in the PDE (2.61) for transition probability density:

$$\mathcal{H}\,\mathcal{K}_V - V(x,t)\mathcal{K}_V = -\frac{\partial \mathcal{K}_V}{\partial t}. \tag{2.70}$$

3. Evaluation of Path Integrals

The Feynman–Kac formula (2.55) is a powerful and versatile tool for obtaining both closed-form and approximate solutions to financial derivatives valuation problems. A number of techniques are available to evaluate path integrals (2.60), (2.68). They fall into three broad categories: exact analytical solutions, analytical approximations, and numerical approximations.

Analytical solutions are available for Gaussian path integrals and those that can be reduced to Gaussians by changes of variables, re-parametrizations of time and projections. Suppose the Lagrangian function $\mathcal{L}$ (2.57) is at most quadratic in $\mathbf{x}$ and $\dot{\mathbf{x}}$. Then the closed-form solution for the Gaussian path integral (2.68) is given by the *Van Vleck formula*:

$$\int_{\mathbf{x}(t_1)=\mathbf{x}_1}^{\mathbf{x}(t_2)=\mathbf{x}_2} \mathrm{e}^{-A[\mathbf{x}(t)]} \mathcal{D}\,\mathbf{x}(t)$$

$$= \sqrt{\det\left(-\frac{1}{2\pi}\frac{\partial^2 A_{Cl}(\mathbf{x}_2, \mathbf{x}_1)}{\partial x_2^\mu \ \partial x_1^\nu}\right)} \exp\left\{-A_{Cl}(\mathbf{x}_2, \mathbf{x}_1)\right\}. \tag{3.1}$$

Here $A_{Cl}(\mathbf{x}_2, \mathbf{x}_1)$ is the action functional (2.56) evaluated along a classical solution of the Euler–Lagrange equations

$$\frac{\delta A}{\delta x^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0 \tag{3.2}$$

with the boundary conditions

$$x^{\mu}_{Cl}(t_1) = x^{\mu}_1, \qquad x^{\mu}_{Cl}(t_2) = x^{\mu}_2. \tag{3.3}$$

The determinant appearing in (3.1) is called *Van Vleck determinant*. Note that, in general, the explicit evaluation of $A_{Cl}(\mathbf{x}_2, \mathbf{x}_1)$ may be quite complex due to complicated classical solutions $\mathbf{x}_{Cl}(t)$ of the Euler–Lagrange Equations (3.2).

Models admitting closed-form solutions due to the Van Vleck formula include Gaussian models and models that can be reduced to Gaussians by changes of variables, re-parametrizations of time and projections. Examples of the former category include the Black–Scholes model and mean reversion models (Ornstein–Uhlenbeck, or harmonic oscillator, processes). The later category includes the Cox–Ingersoll–Ross model (Bessel process which is the radial part of the multi-dimensional Brownian motion (projection)).

To illustrate the use of the Van Vleck formula, let us again consider the Black–Scholes example. The Euler–Lagrange Equation (3.2) for the Black–Scholes action

$$A_0 = \frac{1}{2\sigma^2}\int_t^T \dot{x}^2 \, \mathrm{d}t' \tag{3.4}$$

simply states that acceleration vanishes in the absence of external forces (Newton's law)

$$\ddot{x} = 0. \tag{3.5}$$

The solution with boundary conditions (3.3) is a classical trajectory from x_1 to x_2 – a straight line connecting the two points:

$$x_{Cl}(t') = \frac{x_1(T - t') + x_2(t' - t)}{T - t}. \tag{3.6}$$

The action functional evaluates on this trajectory to

$$A_{Cl} = \frac{1}{2\sigma^2}\int_t^T \frac{(x_2 - x_1)^2}{\tau^2} \, \mathrm{d}t' = \frac{(x_2 - x_1)^2}{2\sigma^2\tau}. \tag{3.7}$$

Substituting this result into the Van Vleck formula (3.1) we again obtain the normal density

$$\frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(x_2 - x_1)^2}{2\sigma^2\tau}\right). \tag{3.8}$$

Furthermore, the Van Vleck formula serves as a starting point for *semiclassical (general moments) expansion.* The first term in the semiclassical expansion is called

WKB (or semiclassical) approximation. It approximates the non-Gaussian model by a suitable Gaussian.

Finally, more complex path integrals can only be evaluated numerically. Monte Carlo simulation has long been one of the favorite techniques for computing path integrals numerically (see, e.g., Metropolis et al. (1953), Creutz et al. (1983)). Monte Carlo simulation simply approximates the path integral by a sum over a finite number of sample paths. Deterministic low-discrepancy algorithms (quasi Monte Carlo) may be especially appropriate for simulations in finance, as they sample paths more efficiently than unstructured pseudo Monte Carlo (Birge, 1995; Joy, Boyle and Tan, 1995; Paskov and Traub, 1995). Finally, different finite-difference techniques for solving the backward PDE can also be alternatively viewed as discretization schemes for path integrals. For interesting numerical algorithms for computing path integrals in finance see Eydeland (1994).

4. Examples

4.1. Weighted Asian Options

Asian options are options with the payoff dependent on the average price of the underlying asset over a specified period of time. The average price over a time period preceeding expiration, rather than just a terminal price, has two main advantages. First, it smooths the option's payoff and prevents it from being determined by the underlying price at a single instant in time. A given terminal asset price may be unnaturally biased or manipulated. The later has been a concern in certain commodity markets dominated by large institutions whose actions might temporarily distort prices.

Another need of using the average price often arises in corporate hedging situations. For example, many corporations exchange foreign currency for domestic currency at regular intervals over a period of time. Asian-style derivatives provide a cheaper alternative to hedging each individual transaction. They hedge only the average exchange rate over a period of time, thus significantly reducing the hedge costs. Moreover, if individual transaction dates are unknown in advance, it is impossible to hedge each individual transfer precisely, but it is still possible to hedge the average exchange rate over time. See, e.g., Kemna and Vorst, 1990; Levy and Turnbull, 1992; Turnbull and Wakeman, 1991; Chance and Rich, 1995 and references therein for details on usage and pricing of Asian options.

To accommodate hedging of cash flows that may not be equal in amount, but rather follow a specific schedule, *weighted or flexible Asian options* (WAOs) have been recently introduced (Dash, 1993; Zhang, 1994 and 1995a). Specific weighted averaging schemes are used in these options. WAOs became quite popular in foreign exchange and energy markets in particular.

In case when the weighted averaging is geometric, since the geometric average of a lognormal variate is itself lognormally distributed, a closed-form pricing formula can be easily obtained. Consider a WAO with the payoff at expiration

$$\mathcal{O}_F(T) = F(S_T, I), \tag{4.1.1}$$

where S_T is the terminal asset price, I is a weighted average of the logarithm of the asset price, $x = \ln S$, over a specified time period $t_0 \leq t' \leq T$ preceeding expiration

$$I = \int_{t_0}^{T} w(t')x(t')\ \mathrm{d}t', \tag{4.1.2}$$

$w(t')$ is a given weight function specified in the contract and normalized so that

$$\int_{t_0}^{T} w(t') = 1, \tag{4.1.3}$$

and F is a given function of S_T and I. The weighted geometric average is given by the exponential of I. Some examples of the possible choices for the weight function are:

Standard option with payoff dependent on S_T only

$$w(t') = \delta(T - t'); \tag{4.1.4a}$$

Standard (equally weighted) Asian option, continuous averaging

$$w(t') = \frac{1}{T - t_0}; \tag{4.1.4b}$$

Standard (equally weighted) Asian option, discrete averaging

$$w(t') = \frac{1}{N+1}\sum_{i=0}^{N} \delta(t' - t_i); \tag{4.1.4c}$$

Discrete weighted averaging

$$w(t') = \sum_{i=0}^{N} w_i \delta(t' - t_i), \qquad \sum_{i=0}^{N} w_i = 1, \tag{4.1.4d}$$

where w_i are specified weights and $t_i = t + ih,\ h = (T - t_0)/N,\ i = 0, \ldots, N$. Here $N + 1$ is the total number of price observations to construct the weighted average price specified in the contract and h is the time interval between two observations. By using Dirac's delta functions both continuous and discrete sampling can be treated uniformly.

Some examples of the payoff function F are:

Weighted average price call

$$F(I) = \text{Max}(e^I - K, 0); \tag{4.1.5a}$$

Weighted average strike call

$$F(S_T, I) = \text{Max}(S_T - e^I, 0); \tag{4.1.5b}$$

Digital weighted average price call

$$F(I) = D\,\theta(e^I - K), \tag{4.1.5c}$$

where θ is the Heavyside step function ($\theta(x) = 1\,(0)$ for $x \geq 0\,(x < 0)$) and D is a fixed payoff amount if the average price is above strike K at expiration. Asian puts are defined similarly.

In discrete case (4.1.4d), a geometric weighted average price e^I is given by

$$e^I = \prod_{i=0}^{N} S(t_i)^{w_i}. \tag{4.1.6}$$

If the current time t when we price the option is inside the averaging interval, $t_0 < t < T$ (seasoned Asian option), then

$$I = I_f + I_u, \tag{4.1.7a}$$

where I_f is the weighted average of already fixed price observations ($x_f = \ln S_f$)

$$I_f = \int_{t_0}^{t} w(t')x_f(t')\ \mathrm{d}t', \tag{4.1.7b}$$

and I_u is the average of yet uncertain prices

$$I_u = \int_{t}^{T} w(t')x(t')\ \mathrm{d}t'. \tag{4.1.7c}$$

If $t < t_0$ (forward-starting Asian option), then it is convenient to extend the definition of the weight function to the entire interval $t \leq t' \leq T$ by setting

$$w(t') \equiv 0 \quad \text{for} \quad t \leq t' < t_0. \tag{4.1.8}$$

Then Equation (4.1.7a) is always true ($I_f \equiv 0$ if $t < t_0$).

The present value at time t of a weighted Asian option with the payoff $F(S_T, I)$ is given by the average:

$$\mathcal{O}_F(S,t) = \mathrm{e}^{-r\tau} E_{(t,S)} \left[F(S_T, I_f + I_u) \right] . \tag{4.1.9}$$

According to the methodology developed in Section 2.2, this average reduces to (note that since $x \in \mathbf{R}$, the linear potential $v(x,t') = \omega(t')x$ is unbounded, and we use the Fourier transform rather than the Laplace, and integrate from $-\infty$ to ∞):

$$\begin{aligned} &\mathcal{O}_F(S,t) \\ &\quad = \mathrm{e}^{-r\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mathrm{e}^{x_T}, \lambda + I_f)\, \mathcal{P}^{\mu}(x_T, \lambda, T \mid x, t)\, \mathrm{d}\lambda\, \mathrm{d}x_T, \end{aligned} \tag{4.1.10}$$

where $\mathcal{P}^{\mu}$ is the joint probability density for the logarithm of the terminal state x_T and the weighted average of the logarithm of the asset prices λ at expiration T conditional on the initial state x at time t. It is given by the inverse Fourier transform

$$\begin{aligned} &\mathcal{P}^{\mu}(x_T, \lambda, T \mid x, t) \\ &\quad = \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2\tau/2\sigma^2)}\, \mathcal{F}_{\lambda}^{-1} \left[\mathcal{K}_V(x_T, T \mid x, t) \right] \\ &\quad = \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2\tau/2\sigma^2)} \int_{-\infty}^{\infty} \mathrm{e}^{ip\lambda} \mathcal{K}_V(x_T, T \mid x, t) \frac{dp}{2\pi}. \end{aligned} \tag{4.1.11}$$

Here $\mathcal{K}_V$ is the Green's function for zero-drift Brownian motion with potential

$$V(x,t') = ip\, \omega(t')\, x \tag{4.1.2}$$

given by the path integral (2.34). Since the potential is linear in x, the path integral is Gaussian and thus can be evaluated in closed form:

$$\begin{aligned} &\mathcal{K}_V(x_T, T \mid x, t) \\ &\quad = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\left\{ -\frac{(x_T - x)^2}{2\sigma^2\tau} - ip(\tau_1 x_T + \tau_2 x) - \tau\chi\sigma^2 p^2 \right\}, \end{aligned} \tag{4.1.13}$$

where

$$\tau_1 = \frac{\bar{\tau}}{\tau}, \qquad \tau_2 = 1 - \tau_1, \qquad \bar{\tau} = \int_t^T w(t')(t' - t)\, \mathrm{d}t', \tag{4.1.14a}$$

$$\chi = \frac{1}{\tau^2} \int_t^T \int_t^{t'} w(t')w(t'')(T - t')(t'' - t)\, \mathrm{d}t''\, \mathrm{d}t'. \tag{4.1.14b}$$

It is a classic result (see, e.g., Feynman and Hibbs, 1965; Schulman, 1981). In physics, this Green's function describes a quantum particle in an external time-dependent electric field $pw(t')$.

Now the integral over p in (4.1.11) is Gaussian and the inverse Fourier transform can be evaluated in closed form yielding the result for the density:

$$\mathcal{P}^{\mu}(x_T, \lambda, T \mid x, t)$$

$$= \frac{1}{2\pi\sigma^2\tau\sqrt{2\chi}} \exp\left\{-\frac{(x_T - x - \mu\tau)^2}{2\sigma^2\tau} - \frac{(\lambda - \tau_1 x_T - \tau_2 x)^2}{4\chi\sigma^2\tau}\right\}. \tag{4.1.15}$$

This is a bivariate normal density for two random variables x_T and λ (weighted average of the logarithm of the asset price) at time T conditional on the state x at time t. It can be re-written in the standard form

$$\begin{aligned}\mathcal{P}^{\mu}(x_T, \lambda, T \mid x, t) \;=\; & \frac{1}{2\pi\sigma_{x_T}\sigma_\lambda\sqrt{1-\rho^2}} \\ & \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_T - \mu_{x_T})^2}{\sigma_{x_T}^2} + \frac{(\lambda - \mu_\lambda)^2}{\sigma_\lambda^2}\right.\right. \\ & \left.\left. - \; \frac{2\rho(x_T - \mu_{x_T})(\lambda - \mu_\lambda)}{\sigma_{x_T}\sigma_\lambda}\right)\right\},\end{aligned} \tag{4.1.16}$$

with the means

$$\mu_{x_T} = x + \mu\tau, \qquad \mu_\lambda = x + \mu\bar{\tau}, \tag{4.1.17a}$$

standard deviations

$$\sigma_{x_T} = \sigma\sqrt{\tau}, \qquad \sigma_\lambda = \sigma\sqrt{\psi\tau}, \tag{4.1.17b}$$

and the correlation coefficient

$$\rho = \frac{\tau_1}{\sqrt{\psi}}, \tag{4.1.17c}$$

$$\psi = 2\chi + \tau_1^2. \tag{4.1.17d}$$

Here $\sigma\sqrt{\psi}$ is the volatility of the weighted average and ρ is the correlation coefficient between the weighted average and x_T.

Formulas (4.1.10), (4.1.16) allow one to price any geometric weighted Asian options with payoffs dependent both on the terminal asset price and the geometric average.

Let us consider a particular case when the payoff function is independent of x_T. Then we can perform the Gaussian integration over x_T in (4.1.10), (4.1.15) and arrive at

$$\mathcal{O}_F(S,t)$$

$$= \mathrm{e}^{-r\tau} \int_{-\infty}^{\infty} F(\lambda + I_f) \frac{1}{\sqrt{2\pi\psi\sigma^2\tau}} \exp\left\{-\frac{(\lambda - x - \mu\bar{\tau})^2}{2\psi\sigma^2\tau}\right\} \,\mathrm{d}\lambda. \tag{4.1.18}$$

This coincides with the Black–Scholes formula (2.25) with *re-scaled volatility* $\sigma \to \sigma\sqrt{\psi}$ and *drift rate* $\mu \to \mu\tau_1$. All the information about the weight function w is encoded in the volatility and drift rate multipliers ψ and τ_1.

In particular, for payoffs (4.1.5a) and (4.1.5c) we have:

Digital weighted average price call

$$\begin{aligned} C_D(S,t) &= D\ \mathrm{e}^{-r\tau} \int_{\ln K - I_f}^{\infty} \frac{1}{\sqrt{2\pi\psi\sigma^2\tau}} \exp\left\{-\frac{(\lambda - x - \mu\bar{\tau})^2}{2\psi\sigma^2\tau}\right\} \,\mathrm{d}\lambda \\ &= D\ \mathrm{e}^{-r\tau} N(d_1), \end{aligned} \tag{4.1.19a}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + I_f + \mu\bar{\tau}}{\sigma\sqrt{\psi\tau}}; \tag{4.1.19b}$$

Weighted average price call

$$C(S,t)$$

$$\begin{aligned} &= \mathrm{e}^{-r\tau} \int_{\ln K - I_f}^{\infty} \left(\mathrm{e}^{\lambda + I_f} - K\right) \frac{1}{\sqrt{2\pi\psi\sigma^2\tau}} \exp\left\{-\frac{(\lambda - x - \mu\bar{\tau})^2}{2\psi\sigma^2\tau}\right\} \,\mathrm{d}\lambda \\ &= \mathrm{e}^{-q\tau + I_f} S N(d_2) - \mathrm{e}^{-r\tau} K N(d_1), \end{aligned} \tag{4.1.20a}$$

where d_1 is given above and

$$d_2 = d_1 + \sigma\sqrt{\psi\tau}, \qquad q = r\tau_2 + \frac{\sigma^2}{2}(\tau_1 - \psi) = r - \mu\tau_1 - \frac{\psi\sigma^2}{2}. \tag{4.1.20b}$$

At the start of the averaging period ($I_f = 0$), this formula for the weighted geometric average price call coincides with the Black–Scholes formula with re-scaled volatility $\sigma\sqrt{\psi}$ and continuous dividend yield q.

The average strike options are a particular case of options to exchange one asset for another; the terminal price and the weighted geometric average with volatility and correlation given by Equations (4.1.17) are the two underlying variables in this case.

Now let us consider different choices for the weight function. For (4.1.4a) we have

$$\bar{\tau} = \tau, \qquad \chi = 0, \qquad \psi = 1, \tag{4.1.21}$$

and Equation (4.1.20) becomes the standard Black–Scholes formula.

Consider the case of standard (equally weighted) continuously averaged geometric Asian options (4.1.4b). If $t = t_0$, i.e., the pricing time coincides with the start of the averaging period, we have

$$\bar{\tau} = \frac{\tau}{2}, \qquad \chi = \tfrac{1}{24}, \qquad \psi = \tfrac{1}{3}. \tag{4.1.22}$$

Thus, volatility of the equally weighted geometric average is $\sigma/\sqrt{3}$ (the well-known $\sqrt{3}$-rule).

Now consider the case of discrete weighted averaging (4.1.4d) and set $t = t_0$. The coefficients τ_1 and χ (4.1.14) reduce to

$$\tau_1 = \frac{1}{N}\sum_{k=1}^{N} k w_k, \tag{4.1.23}$$

$$\chi = \frac{1}{2N^2}\sum_{k=1}^{N} k(N-k)w_k^2 + \frac{1}{N^2}\sum_{k=2}^{N}\sum_{l=1}^{k-1} l(N-k)w_k w_l. \tag{4.1.24}$$

Substituting this into Equation (4.1.17d), we arrive at the discretized expression for the volatility multiplier

$$\psi = \frac{1}{N}\sum_{k=1}^{N} k w_k^2 + \frac{2}{N}\sum_{k=2}^{N}\sum_{l=1}^{k-1} l\, w_k w_l. \tag{4.1.25}$$

In the case of arithmetic averaging, the second state variable is a weighted average price

$$I = \int_t^T w(t')\ \mathrm{e}^{x(t')}\ \mathrm{d}t'. \tag{4.1.26}$$

The potential $v(x,t') = \omega(t')e^x$ is non-negative and the joint density for x_T and I at time T conditional on the state x at time t is given by the inverse Laplace transform

$$\mathcal{P}^{\mu}(x_T, \lambda, T \mid x, t) = \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2\tau/2\sigma^2)}\, \mathcal{L}_{\lambda}^{-1}\left[\mathcal{K}_V(x_T, T \mid x, t)\right] \tag{4.1.27}$$

of the Green's function for Brownian motion with killing at rate

$$V(x,t') = sw(t')\ \mathrm{e}^{x} \tag{4.1.28}$$

satisfying the PDE (2.35a) with potential V and initial condition (2.35b) (the inverse Laplace transform is taken with respect to the variable s). This PDE cannot be solved in closed form for arbitrary $w(t')$, and one must resort to one of the approximation procedures. However, in the special case of equally weighted continuous averaging, an analytical solution does exist. When $w(t')$ is independent of time t', potential (4.1.28) defines the so-called *Liouville model* known in quantum physics. The corresponding closed-form expression for arithmetic Asian options involves Bessel functions (see Geman and Yor (1993) and Geman and Eydeland (1995)).

4.2. Floating Barrier Options

Our second example are floating barrier options. This is an interesting example of a two-asset path-dependent option. Barrier options have increasingly gained popularity over the recent years. A wide variety of barrier options are currently traded over the counter. Closed-form pricing formulas for barrier options can be readily derived by employing the method of images. In addition to the original eight types of barrier options priced by Rubinstein and Reiner, 1991, double-barrier (Kunitomo and Ikeda, 1992), partial barrier (Heynen and Kat, 1994a; Zhang, 1995a) and outside barrier options (Heynen and Kat, 1994b; Rich, 1996; Zhang, 1995b) were studied recently.

A key observation is that due to the reflection principle the zero-drift transition probability density for down-and-out options with barrier B (Brownian motion with absorbing barrier at the level B) is given by the difference of two normal densities (Merton, 1973; Rubinstein and Reiner, 1991) ($b := \ln B$):

$$\begin{aligned}&\mathcal{K}_B(x_T, T \mid x,t)\\ &\quad = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\left(\exp\left\{-\frac{(x-y)^2}{2\sigma^2\tau}\right\} - \exp\left\{-\frac{(x+y-2b)^2}{2\sigma^2\tau}\right\}\right).\end{aligned} \tag{4.2.1}$$

Floating barrier options (FBOs) are options on the underlying *payoff asset* with the barrier proportional to the price of the second *barrier asset*. To illustrate, consider a call on the underlying S_1. In a standard down-and-out call, the barrier is set at some constant pre-specified price level B (out-strike) at the contract inception. The option is extinguished (knocked out) as soon as the barrier is hit. For a floating down-and-out call, the barrier B is set to be *proportional to the price of the second asset* S_2, $B = \Lambda S_2$, where Λ is a specified constant. Thus, a floating knock-out contract is in effect as long as the price of the underlying payoff asset stays above the price of the barrier asset times Λ, $S_1 > \Lambda S_2$, and is extinguished as soon as S_1 hits the floating barrier ΛS_2.

Just as standard barrier options, FBOs can be used to reduce premium payments and as building blocks in more complex transactions. For example, suppose a US-based investor wishes to purchase a call on a foreign currency S_1, say Swiss Franc. Furthermore, suppose he holds a view that another foreign currency S_2, which is correlated with S_1 with the correlation coefficient ρ, say Deutsche Mark, is going to stay below S_1/Λ during the option's lifetime, where Λ is a given fixed threshold cross-currency exchange rate. Then he may elect to add a floating knock-out provision, $S_1 > \Lambda S_2$, to the call to reduce his premium payment. Similarly, S_1 and S_2 can be two correlated equity indexes or a short and long interest rate. In the later case, the option can be structured so that it will knock out if the yield curve inverts and short rate exceeds the long rate during the option's life.

The pricing of FBOs is somewhat similar to quanto options (Babel and Eisenberg, 1993; Derman, Karasinski and Wecker, 1990; Reiner, 1991). We assume we live in the Black–Scholes world with two risky assets. The risk-neutral price processes for these two assets are

$$\frac{\mathrm{d}S_1}{S_1} = m_1\ \mathrm{d}t + \sigma_1\ \mathrm{d}z_1, \qquad \frac{\mathrm{d}S_2}{S_2} = m_2\ \mathrm{d}t + \sigma_2\ \mathrm{d}z_2, \tag{4.2.2}$$

with constant risk-neutral drifts and volatilities. The $\mathrm{d}z_1$ and $\mathrm{d}z_2$ are two standard Wiener processes correlated with the correlation coefficient ρ. If S_1 and S_2 are two foreign currencies, then the risk-neutral drifts are $m_1 = r - r_1,\ m_2 = r - r_2$, where r, r_1 and r_2 are domestic and two foreign risk-free rates, respectively.

A floating barrier call is defined by its payoff at expiration

$$\mathbf{1}_{\{S_1(t')>\Lambda S_2(t'),t\le t'\le T\}}\ \mathrm{Max}(S_{1T} - K, 0), \tag{4.2.3}$$

where $\mathbf{1}_{\{S_1(t')>\Lambda S_2(t'),t\le t'\le T\}}$ is the indicator functional on price paths $\{S_1(t'), S_2(t'),\ t \le t' \le T\}$ that is equal to one if $S_1(t')$ is greater than $\Lambda S_2(t'), S_1(t') > \Lambda S_2(t')$, at all times t' during the option's life, and zero otherwise.

To price these options, first introduce new variables $x_1 = \ln S_1$ and $x_2 = \ln S_2$:

$$\mathrm{d}x_1 = \mu_1\ \mathrm{d}t + \sigma_1\ \mathrm{d}z_1, \qquad \mu_1 = m_1 - \tfrac{1}{2}\sigma_1^2, \tag{4.2.4a}$$

$$\mathrm{d}x_2 = \mu_2\ \mathrm{d}t + \sigma_2\ \mathrm{d}z_1, \qquad \mu_2 = m_2 - \tfrac{1}{2}\sigma_2^2. \tag{4.2.4b}$$

Now let us introduce a cross-currency exchange rate

$$S_3 = S_1/S_2. \tag{4.2.5}$$

Its logarithm $x_3 = \ln S_3, x_3 = x_1 - x_2$, follows a process

$$\mathrm{d}x_3 = \mu_3\ \mathrm{d}t + \sigma_3\ \mathrm{d}z_3, \tag{4.2.6}$$

with drift rate and volatility

$$\mu_3 = \mu_1 - \mu_2, \qquad \sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \tag{4.2.7}$$

and dz_3 is a standard Wiener process correlated with the process dz_1 with the correlation coefficient ρ'

$$\rho' = \frac{\sigma_1 - \rho\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}. \tag{4.2.8}$$

In the variables S_1 and S_3, the problem reduces to pricing an *outside barrier option* with the payoff asset S_1 and the barrier asset S_3 with the constant fixed barrier level Λ and the payoff

$$\mathbf{1}_{\{S_3(t')>\Lambda, t\le t'\le T\}} \operatorname{Max}(S_{1T} - K, 0). \tag{4.2.9}$$

Outside barrier options were studied by Heynen and Kat (1994), Zhang (1995b) and Rich (1996).

A two-asset Lagrangian for the two-dimensional process is given by

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2(1-\rho'^2)}\left[\left(\frac{\dot{x}_1 - \mu_1}{\sigma_1}\right)^2\right. \\ & \left. + \left(\frac{\dot{x}_3 - \mu_3}{\sigma_3}\right)^2 - \frac{2\rho'(\dot{x}_1 - \mu_1)(\dot{x}_3 - \mu_3)}{\sigma_1\sigma_3}\right].\end{aligned} \tag{4.2.10}$$

If x_1 and x_3 were uncorrelated, the Lagrangian would reduce to the sum of two independent Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$. The correlation term makes the problem more interesting. The action functional can be re-written in the form

$$A = \int_t^T \mathcal{L}\, dt' = A_0 + \alpha\tau - \beta_1(x_{1T} - x_1) - \beta_3(x_{3T} - x_3), \tag{4.2.11}$$

where

$$A_0 = \int_t^T \mathcal{L}_0\, dt', \qquad \mathcal{L}_0 = \frac{1}{2(1-\rho'^2)}\left[\frac{\dot{x}_1^2}{\sigma_1^2} + \frac{\dot{x}_3^2}{\sigma_3^2} - \frac{2\rho'\dot{x}_1\dot{x}_3}{\sigma_1\sigma_3}\right], \tag{4.2.12}$$

and

$$\alpha = \frac{1}{2(1-\rho'^2)}\left[\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_3^2}{\sigma_3^2} - \frac{2\rho'\mu_1\mu_3}{\sigma_1\sigma_3}\right], \tag{4.2.13a}$$

$$\beta_1 = \frac{1}{\sigma_1^2(1-\rho'^2)}\left[\mu_1 - \frac{\rho'\sigma_1\mu_3}{\sigma_3}\right],$$

$$\beta_3 = \frac{1}{\sigma_3^2(1-\rho'^2)}\left[\mu_3 - \frac{\rho'\sigma_3\mu_1}{\sigma_1}\right]. \tag{4.2.13b}$$

Now the present value of a floating barrier call $C_\Lambda(S_1, S_2, t)$ is given by

$$\mathrm{e}^{-r\tau}\int_{\ln K}^{\infty}\int_{\ln\Lambda}^{\infty}(\mathrm{e}^{x_{1T}} - K)\ \mathrm{e}^{\beta_1(x_{1T}-x_1)+\beta_3(x_{3T}-x_3)-\alpha\tau}$$
$$\times\mathcal{K}_\Lambda(x_{1T}x_{3T}, T \mid x_1x_3, t)\ \mathrm{d}x_{3T}\ \mathrm{d}x_{1T}. \tag{4.2.14}$$

A transition probability density $\mathcal{K}_\Lambda$ is given by the two-asset path integral of the type (2.68) over all paths $\{x_1(t'), x_3(t'), t \leq t' \leq T\}$ such that $\{x_3(t') > \ln\Lambda,\ t \leq t' \leq T\}$. The path integration measure is defined by the action functional (4.2.12). It is calculated by introducing new uncorrelated variables

$$y_1 = x_1 - \frac{\rho'\sigma_1 x_3}{\sigma_3}, \qquad y_2 = x_3, \tag{4.2.15}$$

so that the Lagrangian $\mathcal{L}_0$ reduces to the sum of two independent terms

$$\mathcal{L}_0 = \frac{\dot{y}_1^2}{2\sigma_1^2(1-\rho'^2)} + \frac{\dot{y}_2^2}{2\sigma_3^2}. \tag{4.2.16}$$

Now, the path integral factorizes into a product of two independent factors which yield a normal density for the variable y_1 and a down-and-out density of the form (4.2.1) for the barrier variable y_2:

$$\begin{aligned}
&\mathcal{K}_\Lambda(x_{1T}x_{3T}, T \mid x_1x_3, t)\\
&\quad = \mathcal{K}(y_{1T}, T \mid y_1, t)\mathcal{K}_\Lambda(y_{2T}, T \mid y_2, t)\\
&\quad = \frac{1}{2\pi\sigma_1\sigma_3\tau\sqrt{1-\rho'^2}}\exp\left\{-\frac{(\sigma_3(x_{1T}-x_1) - \rho'\sigma_1(x_{3T}-x_3))^2}{2\sigma_1^2\sigma_3^2(1-\rho'^2)\tau}\right\}\\
&\quad\times\left(\exp\left\{-\frac{(x_{3T}-x_3)^2}{2\sigma_3^2\tau}\right\} - \exp\left\{-\frac{(x_{3T}+x_3-2\ln\Lambda)^2}{2\sigma_3^2\tau}\right\}\right).
\end{aligned} \tag{4.2.17}$$

Substituting this density back into Equation (4.2.14) and simplifying the integrals, we arrive at the pricing formula for the floating barrier call (recall that $S_3 = S_1/S_2$, and μ_3, σ_3 and ρ' are given by Equations (4.2.7–8)):

$$\begin{aligned}
C_\Lambda(S_1, S_2, t)\ =\ &\mathrm{e}^{-r_1\tau}S_1\left(N[d_2, d_4; \rho'] - \left(\frac{\Lambda S_2}{S_1}\right)^{\gamma_2}N[d_6, d_8; \rho']\right)\\
&-\mathrm{e}^{-r\tau}K\left(N[d_1, d_3; \rho'] - \left(\frac{\Lambda S_2}{S_1}\right)^{\gamma_1}N[d_5, d_7; \rho']\right),
\end{aligned} \tag{4.2.18}$$

where we have introduced the following notations

$$d_1 = \frac{\ln\left(\frac{S_1}{K}\right) + \mu_1\tau}{\sigma_1\sqrt{\tau}}, \qquad d_2 = d_1 + \sigma_1\sqrt{\tau},$$
$$d_3 = \frac{\ln\left(\frac{S_1}{\Lambda S_2}\right) + \mu_3\tau}{\sigma_3\sqrt{\tau}}, \qquad d_4 = d_3 + \rho'\sigma_1\sqrt{\tau}, \tag{4.2.19a}$$

$$d_5 = d_1 + \frac{2\rho' \ln\left(\frac{\Lambda S_2}{S_1}\right)}{\sigma_3\sqrt{\tau}}, \qquad d_6 = d_5 + \sigma_1\sqrt{\tau},$$
$$d_7 = d_3 + \frac{2\ln\left(\frac{\Lambda S_2}{S_1}\right)}{\sigma_3\sqrt{\tau}}, \qquad d_8 = d_7 + \rho'\sigma_1\sqrt{\tau}, \tag{4.2.19b}$$

$$\gamma_1 = \frac{2\mu_3}{\sigma_3^2}, \qquad \gamma_2 = \gamma_1 + \frac{2\rho'\sigma_1}{\sigma_3}, \tag{4.2.19c}$$

and $N[a, b; \rho]$ is the standard bivariate cumulative normal distribution function

$$N[a, b; \rho] = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(x^2 + y^2 - 2\rho xy\right)\right\} \mathrm{d}y \ \mathrm{d}x. \tag{4.2.20}$$

4.3. Ladder-like Barriers

Our next example illustrates the use of the Chapman–Kolmogorov semigroup property. Consider a down-and-out call with a time-dependent ladder-like barrier $B(t'), t \le t' \le T$, defined by

$$B(t') = \begin{cases} B_1 & \text{if } t \le t' < t^* \\ B_2 & \text{if } t^* \le t' \le T \end{cases}, \tag{4.3.1}$$

and we assume that $B_1 < B_2$. A knock-out provision of this type can be included into a long-term option or warrant if the underlying is expected to rise during the life of the contract. The present value of this path-dependent option at time T is given by

$$C_{B_1,B_2}(S, t) = \mathrm{e}^{-r\tau} \int_{\ln K}^{\infty} (\mathrm{e}^{x_T} - K)\ \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2\tau/2\sigma^2)} \mathcal{K}_{B_1,B_2}(x_T, T \mid x, t)\ \mathrm{d}x_T. \tag{4.3.2}$$

The density $\mathcal{K}_{B_1,B_2}$ is obtained from the Chapman–Kolmogorov semigroup property (2.23) by convolution of two standard down-and-out densities $\mathcal{K}_{B_1}$ and $\mathcal{K}_{B_2}$ of the form (2.4.1) for two time intervals $t \le t' < t^*$ and $t^* \le t' \le T$ (the integration on x^* is from b_2 to ∞ since $\mathcal{K}_{B_2}$ is equal to zero for $x \le b_2$ – the contract is already extinguished)

$$\mathcal{K}_{B_1,B_2}(x_T, T \mid x, t) = \int_{b_2}^{\infty} \mathcal{K}_{B_2}(x_T, T \mid x^*, t^*)\mathcal{K}_{B_1}(x^*, t^* \mid x, t)\ \mathrm{d}x^*, \tag{4.3.3}$$

where we introduced the following notations

$$\tau_1 = t^* - t, \qquad \tau_2 = T - t^*, \qquad b_1 = \ln B_1, \qquad b_2 = \ln B_2. \tag{4.3.4}$$

Substituting this back into Equation (4.3.2),

$$\begin{aligned}
&C_{B_1,B_2}(S,t)\\
&\quad= \frac{\mathrm{e}^{-r\tau}}{2\pi\sigma^2\sqrt{\tau_1\tau_2}} \int_{\ln K}^{\infty}\ \mathrm{d}x_T \int_{b_2}^{\infty}\ \mathrm{d}x^*(\mathrm{e}^{x_T} - K)\ \mathrm{e}^{(\mu/\sigma^2)(x_T - x) - (\mu^2\tau/2\sigma^2)}\\
&\qquad\times \left(\exp\left\{-\frac{(x_T - x^*)^2}{2\sigma^2\tau_1} - \frac{(x^* - x)^2}{2\sigma^2\tau_2}\right\}\right.\\
&\qquad\quad - \exp\left\{-\frac{(x_T - x^*)^2}{2\sigma^2\tau_1} - \frac{(x^* + x - 2b_2)^2}{2\sigma^2\tau_2}\right\}\\
&\qquad\quad - \exp\left\{-\frac{(x_T + x^* - 2b_1)^2}{2\sigma^2\tau_1} - \frac{(x^* - x)^2}{2\sigma^2\tau_2}\right\}\\
&\qquad\quad \left. + \exp\left\{-\frac{(x_T + x^* - 2b_1)^2}{2\sigma^2\tau_1} - \frac{(x^* + x - 2b_2)^2}{2\sigma^2\tau_2}\right\}\right).
\end{aligned} \tag{4.3.5}$$

Simplifying the integrals, we arrive at the pricing formula for the down-and-out call with ladder-like barrier (4.3.1):

$$\begin{aligned}
C_{B_1,B_2}(S,t) \ = \ S \Bigg(N[d_2, d_4; \rho] - \left(\frac{B_2}{S}\right)^{\gamma+2} N[d_6, d_8; \rho&\\
- \left(\frac{B_1}{S}\right)^{\gamma+2} N[d_{10}, -d_8; -\rho]&\\
+ \left(\frac{B_1}{B_2}\right)^{\gamma+2} N[d_{12}, -d_4; -\rho]\Bigg)&
\end{aligned}$$

$$-\mathrm{e}^{-r\tau}K\left(N[d_1,d_3;\rho]-\left(\frac{B_2}{S}\right)^{\gamma}N[d_5,d_7;\rho]\right.$$
$$-\left(\frac{B_1}{S}\right)^{\gamma}N[d_9,-d_7;-\rho]$$
$$\left.+\left(\frac{B_1}{B_2}\right)^{\gamma}N[d_{11},-d_3;-\rho]\right), \tag{4.3.6}$$

where we have introduced the following notations

$$\rho=\sqrt{\frac{\tau_2}{\tau}}, \qquad \gamma=\frac{2\mu}{\sigma^2}=\frac{2r}{\sigma^2}-1, \tag{4.3.7a}$$

$$\begin{aligned} d_1&=\frac{\ln\left(\frac{S}{K}\right)+\mu\tau}{\sigma\sqrt{\tau}}, \qquad d_2=d_1+\sigma\sqrt{\tau},\\ d_3&=\frac{\ln\left(\frac{S}{B_2}\right)+\mu\tau_2}{\sigma\sqrt{\tau_2}}, \qquad d_4=d_3+\sigma\sqrt{\tau_2}, \end{aligned} \tag{4.3.7b}$$

$$\begin{aligned} d_5&=\frac{\ln\left(\frac{B_2^2}{SK}\right)+\mu\tau}{\sigma\sqrt{\tau}}, \qquad d_6=d_5+\sigma\sqrt{\tau},\\ d_7&=\frac{\ln\left(\frac{B_2}{S}\right)+\mu\tau_2}{\sigma\sqrt{\tau_2}}, \qquad d_8=d_7+\sigma\sqrt{\tau_2}, \end{aligned} \tag{4.3.7c}$$

$$\begin{aligned} d_9&=\frac{\ln\left(\frac{B_1^2}{SK}\right)+\mu\tau}{\sigma\sqrt{\tau}}, \qquad d_{10}=d_9+\sigma\sqrt{\tau},\\ d_{11}&=\frac{\ln\left(\frac{B_1^2 S}{B_2^2 K}\right)+\mu\tau}{\sigma\sqrt{\tau}}, \qquad d_{12}=d_{13}+\sigma\sqrt{\tau}, \end{aligned} \tag{4.3.7d}$$

If we set $B_1=0$, this formula reduces to the pricing formula for partial barrier options of Heynen and Kat (1994). Setting $B_1=B_2$, it collapses to the standard formula for the down-and-out call. Using the same procedure one can obtain pricing formulas for ladder-like barriers with any finite number of steps through multi-variate normal probabilities.

5. Conclusion

In this paper we presented a brief overview of the path integral approach to options pricing. The path integral formalism constitutes a convenient and intuitive language for stochastic modeling in finance. It naturally brings together probability-based and PDE-based methodologies and is especially useful for obtaining closed-form solutions (when available) and analytical approximations for path-dependent problems. It also offers an interesting numerical framework that may yield some computational advantages for multi-dimensional models with general parameters, such as multi-factor term structure models, as well as path-dependent problems. In particular, in Linetsky (1996) we apply the methodology developed here to derived closed-form pricing formulas for a class of path-dependent derivatives contingent on occupation times.

To conclude, let us quote Barry Simon (1979): '*In part, the point of functional integration is a less cumbersome notation, but there is a larger point: like any successful language, its existence tends to lead us to different and very special ways of thinking.*'

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