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AN ANALYSIS OF TRUSS DISPLACEMENTS

(Donald Adam)

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NOMENCLATURE

Symbols are defined when they appear in the text. Those symbols which are frequently used are collected here for reference.

i,u	subscript identifying joints in the truss
$\bar{R}_j$	position vector to joint j
$\bar{L}_{ji}$	length vector directed from joint j to joint i
$\bar{u}_{ji}$	unit vector directed from joint j to joint i
d_{ji}	elongation of the bar connecting joints j and i
$\bar{e}_k$	unit coordinate vectors $(k = 1,2,3)$
$\bar{v}_k$	arbitrary linearly independent vectors $(k = 1,2,3)$
$\bar{Z}_j$, $\bar{Y}_j$	displacement vectors
$\bar{C}_j$	corrective displacement vector
$\bar{H}$	rotation vector
$\bar{n}$	vector normal to a fixed plane
$\bar{u}$	vector normal to a fixed direction in a fixed plane
a^k_{ji}	components of $\bar{u}_{ji}$ relative to the unit coordinate vectors
y^k_j	components of $\bar{Y}_j$ relative to the unit coordinate vectors
$[A]$	in Chapter III a third-order geometric matrix
$\{y\}$	in Chapter III a third-order column vector
$[G]$, $[A]$	in Chapter IV a geometric matrix of order 3J
$\{y\}$	in Chapter IV a column vector of order 3J
$[G]_{ji}$, $[A]_{ji}$	third-order submatrix in $[G]$ or $[A]$
$\{y\}_j$	third-order column vector in $\{y\}$
$\{d\}$	column vector of bar elongations of order 3J
$\{d\}_j$	third-order column vector in $\{d\}$

NOMENCLATURE (CONT'D)

$[B]$	transformation matrix
$\{q\}$	load vector
$\{p\}$	bar force vector
$[f]$	bar flexibility matrix
$[F]$	joint flexibility matrix
Δ	appearing before a quantity denotes the error in that quantity
$[H]$	matrix derived from calculated displacements (dimension L)
$\{\Delta u\}$	a column vector containing the errors in the components of the unit vectors $\bar{u}_{ji}$
$\{\Delta y\}$	error in the displacement vector
ϵ_d	uniform error in bar elongations
ϵ_u	uniform error in components of the unit vectors
$\{e_d\}$, $\{e_u\}$	column vector with unity in every position
σ	standard deviation
$\{\sigma_y\}$	standard deviation of the calculated displacements
$[M]$	matrix derived from the bar force vector (dimension F)
$[\ ]^T$	transpose of a matrix
$[\ ]^{-1}$	inverse of a matrix
$\|\|\ \|\|$	determinant
$[\|\ \|]$	matrix containing the absolute value of the elements in another matrix
$[*]$	matrix containing the squared elements from another matrix
$\{\ \}^T$	transpose of a column vector
$\{\|\ \|\}$	column vector containing absolute values of the elements in another column vector

CHAPTER I

INTRODUCTION

The bothersome problem of calculating truss displacements has been given considerable attention in textbooks as well as in published papers.[1-8] Most of the standard texts present at least two methods for calculating truss displacements. One of the classical methods, for example, virtual work or Castigliano's theorem, is usually discussed first.

The classical methods can be used to calculate displacements whether the truss is a space or plane truss. When the displacement of but one or at most a few joints is required, these methods can be applied without great inconvenience. However, more often than not one is interested in the displacements of several or all the joints. It is well known that in such cases the classical methods are not practical because of the excessive numerical work involved.

The Williot-Mohr construction, the second method which is invariably discussed, yields the displacements of all joints in the truss. The main objection to the Williot-Mohr construction is that, because it is graphical, the accuracy of the solution is limited and depends on the care taken in making the construction. Because of the additional constructions involved, the problem of accuracy is probably much more acute in the graphical solution for space trusses.[4] Chu[5] and Cornish[6] have proposed algebraic solutions of the graphical construction for plane trusses. Use of the algebraic methods is complicated by multiple-sign conventions and the virtually necessary condition that the graphical construction be made first.

In the case of plane trusses, a number of special methods or variations on methods for particular cases are available. For example, the method of elastic weights[1] is intended primarily for the calculation of the vertical components of the joint displacements. The method can be adjusted to obtain horizontal displacements; however, the additional work involved may require as much time as the method of virtual work.

Aside from the classical methods and the graphical solution proposed by Ewell,[4] the displacement problem for space trusses has apparently remained untouched.

This dissertation presents a vector method for analyzing displacements in simple, compound, and complex trusses. Although the method is presented for space trusses, it is also applicable to plane trusses. The method of solution is such that the displacements are obtained in a step-by-step fashion, and in each step one has to solve but three equations in three unknowns. Thus, numerical solution for the displacements poses no problem. That the proposed method offers a practical solution of the problem is demonstrated by means of nontrivial examples.

A step-by-step solution for the displacements is possible only because the displacements, considered as a whole, are governed by a special system of simultaneous equations. When the displacements are needed for several sets of bar elongations, it is sometimes convenient to obtain a general or "matrix," solution of the special system of equations. Thus the matrix formulation and solution of the problem is also considered in detail. The matrix version of the displacement problem explains the marked similarities which exist between the methods of displacement and stress analysis for trusses.

In general, an exact solution for the displacements is possible only in principle. This is true not only for the methods presented in this report but also for any of the so-called exact methods (e.g., virtual work). The accuracy of the calculated displacements is limited due to the effect of rounding-off errors, which arise in the process of solving the equations. The initial or inherent errors, which arise when approximate numbers are used to represent the constants in the problem, also limit the accuracy of the solution. Both types of errors are considered. Round-off errors are discussed only with respect to minimizing their effect. The inherent errors, which, in general, determine the accuracy of the solution, are discussed in considerable detail, and methods for estimating bounds on the inherent errors are presented.

CHAPTER II

DISPLACEMENT ANALYSIS

The truss-displacement problem as it is considered here, is purely geometric. The problem can be stated as follows: Given a compatible set of elongations of the truss members, find the displacements.

The initial step in solving the problem is to develop the relationship between the displacements at the ends of a member and the elongation of the member. In this study it is assumed that the elongation of a bar is small compared to the length of the member. The effect of the assumption is to linearize the displacement problem. The assumption is justified for the materials and truss proportions used in conventional construction.

The displacement problem, especially in the case of space trusses, is well suited for analysis by the methods of vector algebra. The vector analysis approach to the problem is employed in this study.[9] The displacement elongation relationship is obtained from a consideration of a typical member in the truss. With reference to Figure 1 let:

$\bar{R}_j$ = position vector of joint j relative to a convenient pole.

$\bar{Y}_j$ = displacement vector at joint j.

$\bar{L}_{ji}$ = length vector directed from joint j to joint i when the truss is in the undistorted configuration.

$\bar{L}'_{ji}$ = length vector directed from joint j to joint i when the truss is in the distorted configuration.

$\bar{u}_{ji}$ = a unit vector directed from joint j to joint i.

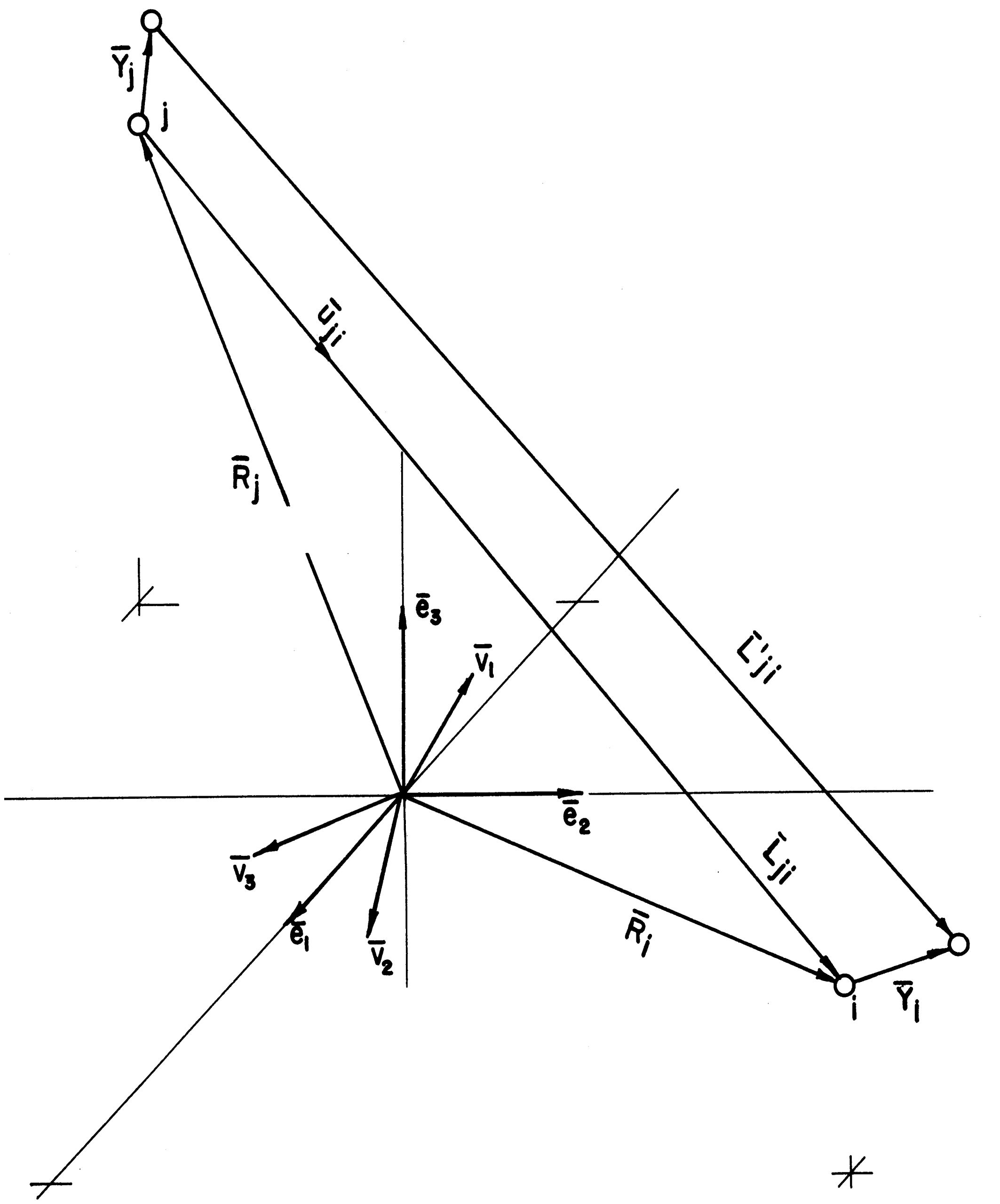


Figure 1. Joint Displacements.

$\bar{e}_k$, k=1,2,3 = a right-handed triad of unit coordinate vectors.

$\bar{v}_t$, t=1,2,3 = a set of three linearly independent vectors.

d_{ji} = change in length of the member connecting joint j and i.

d_{ji} is positive if the length of the member increases.

The displacement and length vectors are connected by the relationship

$$\bar{Y}_j - \bar{Y}_i = -(\bar{L}'_{ji} - \bar{L}_{ji}) \tag{2.1}$$

Since the change in length of the member is assumed to be small compared to the length of the member, the elongation is

$$(\bar{L}'_{ji} - \bar{L}_{ji}) \cdot \bar{u}_{ji} = d_{ji} \tag{2.2}$$

Combining Equations (2.1) and (2.2), one obtains

$$(\bar{Y}_j - \bar{Y}_i) \cdot \bar{u}_{ji} = -d_{ji} \tag{2.3}$$

Equation (2.3) expresses the relationship between the elongation of the member and the displacement at each end of the member, and is the fundamental equation used in the analysis of truss displacements. When the vector quantities are expressed in terms of their components relative to the unit coordinate vectors, Equation (2.3) reduces to the deformation equation used in the analysis of statically indeterminate trusses by the displacement method.[10]

It should be noted that Equation (2.3) is symmetric, that is, the roles of $\bar{Y}_j$ and $\bar{Y}_i$ are interchanged simply by interchanging the subscripts j and i, and that Equation (2.3) can be used to determine the increase in the length of a line connecting any two points in the truss, when the displacement of each point is known.

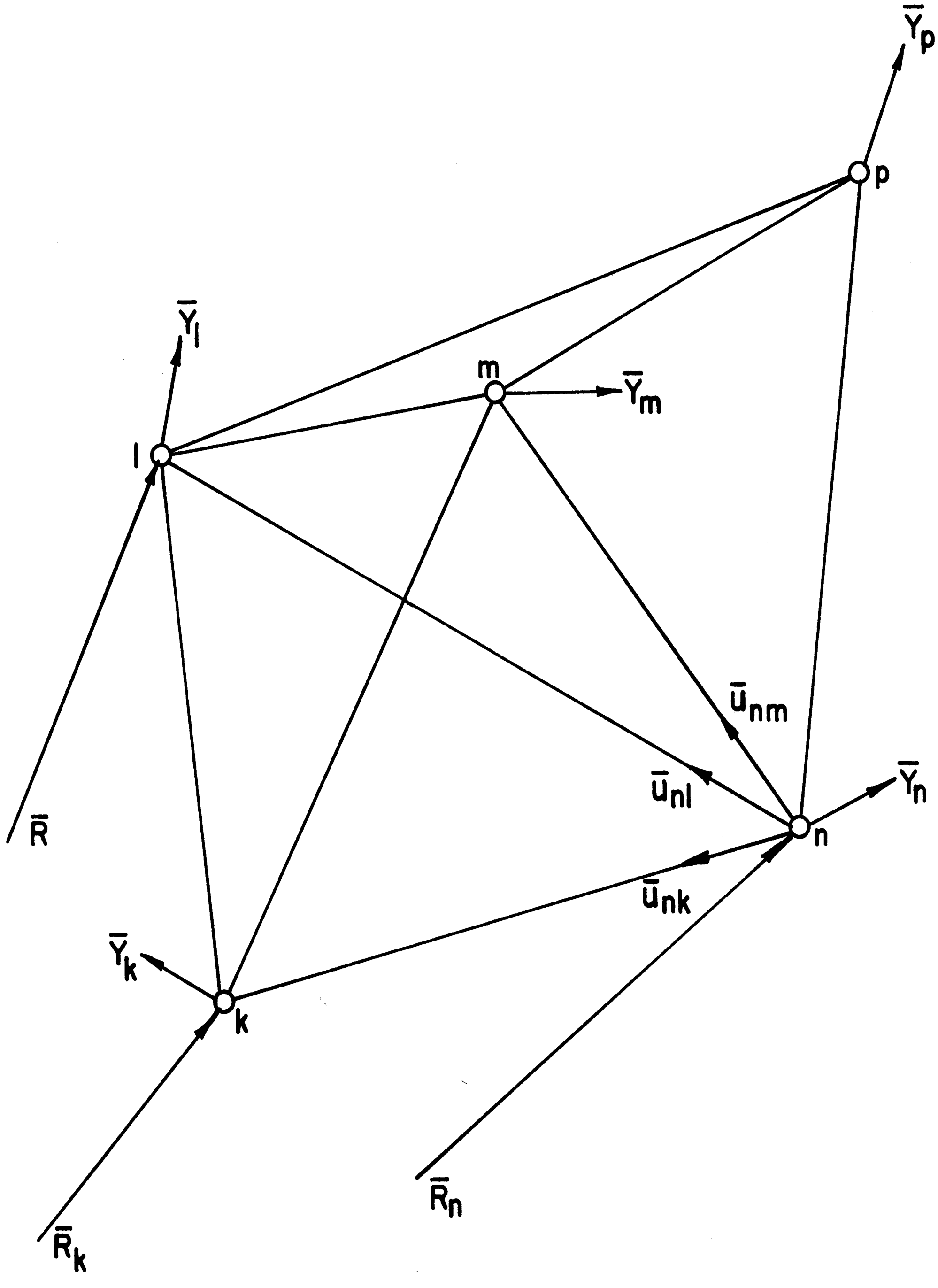


Figure 2. Simple Truss.

In the truss-deflection problem the bar elongations are given and the problem is to determine the joint displacements. The joint displacements must be such that: (a) Equation (2.3) is satisfied for every bar of the truss, and (b) the conditions of constraint at joints connecting the truss to its foundation are satisfied. For a stable truss, conditions (a) and (b), are both necessary, and sufficient to guarantee that the displacements are unique.

Anticipating the results presented in Chapter IV, it may be noted here that the displacement and stress-analysis problems are virtually identical. Thus, it is convenient to consider the various types of trusses in the order in which they are discussed in the stress-analysis problem. Simple trusses are considered first.

Simple Trusses

Consider a typical portion of a simple truss as shown in Figure 2. The displacements at joints k, ℓ, and m are assumed to be known, and one wants to find the displacement at joint n. To find the displacement at joint n, it should first be recalled that the rule for forming a simple space truss excludes the possibility that joint n lies in the plane of joints k, ℓ, and m, so that the unit vectors $\overline{u}_{nk}$, $\overline{u}_{n\ell}$, and $\overline{u}_{nm}$ are linearly independent.

Applying Equation (2.3) to bars nk, nℓ, and nm, one obtains

$$\begin{aligned} \overline{Y}_n \cdot \overline{u}_{nk} &= \overline{Y}_k \cdot \overline{u}_{nk} - d_{nk} = b_{nk} \\ \overline{Y}_n \cdot \overline{u}_{n\ell} &= \overline{Y}_\ell \cdot \overline{u}_{n\ell} - d_{n\ell} = b_{n\ell} \\ \overline{Y}_n \cdot \overline{u}_{nm} &= \overline{Y}_m \cdot \overline{u}_{nm} - d_{nm} = b_{nm} \end{aligned} \tag{2.4}$$

Equations (2.4) are three independent equations involving $\overline{Y}_n$. One can express $\overline{Y}_n$ as a linear combination of any three nonzero linearly independent vectors as

$$\overline{Y}_n = y_{n1}\overline{v}_1 + y_{n2}\overline{v}_2 + y_{n3}\overline{v}_3 \tag{2.5}$$

In Equation (2.5) the vectors $\overline{v}_t$ (t=1,2,3) are assumed to be known and the y_{nt} are unknown scalars to be determined. The details of the procedures for determining the y_{nt} are discussed in Chapter III; however, it is apparent that by combining Equations (2.4) and (2.5) one obtains three simultaneous equations from which the y_{nt} and therefore $\overline{Y}_n$ can be determined. With $\overline{Y}_n$ now known, Equation (2.3) can be applied to bars pn, pm, and pℓ to determine $\overline{Y}_p$. This process can be repeated to obtain all the remaining unknown joint displacements, as long as the truss is a simple truss.

Frequently the displacements of the three joints required to start the solution are not known in advance. Virtual work or other methods could be used to obtain these displacements. However, if the truss contains a great number of joints, application of the classical methods can be quite tedious and time-consuming.

In an alternative approach, the joint displacement problem is broken down into two separate problems as in the graphical determination of joint displacements for plane trusses. The advantage in this approach is that the actual displacements of the joints required to start the solution do not have to be known in advance.

The first problem consists of finding a set of displacements such that Equation (2.3) is satisfied for every pair of joints in the truss. These displacements completely define the distorted configuration of the truss. These displacements are the relative joint displacements and correspond to those obtained from the Williot diagram for plane trusses.

It is not required that the relative joint displacements satisfy the conditions of constraint at joints connecting the truss to its foundation. Thus, the second problem consists of determining a correction to each relative displacement such that Equation (2.3) is still satisfied for every pair of joints, and such that the restraining conditions are satisfied. In effect, the second problem consists of determining a rigid-body displacement for the distorted truss such that the restraining conditions are satisfied. The corrective displacements correspond to those obtained from the Mohr correction diagram for plane trusses.

Relative Displacements

The most general type of displacement which can be given to a rigid body is equivalent to a pure translation of a point in the body, and a rotation about a line passing through the point. Six quantities, are required to determine the displacement, three to determine the translation, and three to determine the rotation. Thus if the procedure is to yield the correct solution, one can introduce at most six assumptions in the determination of the relative displacements. Furthermore these assumptions must be such that Equation (2.3) can be satisfied for every pair of joints. A logical way to introduce the assumptions is to proceed in a manner similar to that used in the construction of the Williot diagram for plane trusses. First, it is assumed that joints k, ℓ, and m are constrained to remain in the plane defined by these joints in the undistorted truss. Second, it is assumed that joints k and ℓ are constrained to remain on the line connecting these joints in the undistorted truss. Finally, it is assumed that the position of joint k is fixed.

With these assumptions,

$$\overline{Z}_k = 0 \;, \tag{2.6}$$

where $\overline{Z}$ is used to designate the relative displacement. The relative displacement of joint ℓ is

$$\overline{Z}_\ell = - d_{\ell k}\overline{u}_{\ell k} \tag{2.7}$$

Taking

$$\overline{Z}_m = z_{mk}\overline{u}_{mk} + z_{m\ell}\overline{u}_{m\ell} \tag{2.8}$$

and using Equation (2.3) for bars mk and mℓ, one obtains the following simultaneous equations to determine z_{mk} and $z_{m\ell}$

$$z_{mk} + z_{m\ell}\,(\overline{u}_{m\ell} \cdot \overline{u}_{mk}) = - d_{mk}$$

$$z_{mk}\,(\overline{u}_{m\ell} \cdot \overline{u}_{mk}) + z_{m\ell} = \overline{Z}_\ell \cdot \overline{u}_{m\ell} - d_{m\ell} \tag{2.9}$$

The relative displacement of the remaining joints are determined by the procedure given for the actual displacements.

Corrective Displacements

The actual displacement of joint i is

$$\overline{Y}_i = \overline{Z}_i + \overline{C}_i \;, \tag{2.10}$$

where $\overline{C}_i$ is the corrective displacement vector. Since the relative displacements are such that Equation (2.3) is satisfied for every pair of joints in the truss, the corrective displacements must be such that

$$(\overline{C}_j - \overline{C}_i) \cdot \overline{u}_{ji} = 0 \tag{2.11}$$

for every pair of joints in the truss. If $\overline{C}_j$ is regarded as given, then all possible $\overline{C}_i$ which can satisfy Equation (2.11) are given by

$$\overline{C}_i = \overline{C}_j + \overline{H} \times (\overline{R}_i - \overline{R}_j), \tag{2.12}$$

where $\overline{H}$ is an arbitrary vector with dimensionless components. Equation (2.12) is general and is applicable between any pair of joints in the truss. Thus, if a particular value is assigned to j, say j = q, and $\overline{C}_q$ and $\overline{H}$ are determined, then the remaining corrections are completely determined from Equation (2.12). Equation (2.12) is linear in $\overline{C}_j$ and $\overline{H}$, and it follows that the correction at i due to each of these terms can be determined separately and the results added to obtain $\overline{C}_i$. It should be noted that the portion of $\overline{C}_i$, due to $\overline{C}_j$, is a pure translation. That portion of $\overline{C}_i$ due to $\overline{H} \times (\overline{R}_i - \overline{R}_j)$ is also a translation, but is due to a pure infinitesimal rotation of the truss about a line in the direction $\frac{\overline{H}}{|\overline{H}|}$ passing through joint j.

There are no restrictions on the way that a truss can be attached to its foundation, as long as the reactions provided by the constraints are such that the equations of equilibrium can be satisfied for arbitrary loading of the truss. Full constraint of a truss to its foundation is guaranteed if the assumptions introduced in determining the relative displacements are true for three of the joints connecting the truss to its foundation. That is, the truss if fully constrained to its foundation if:

(1) At least three joints, not all on the same line, remain in the same plane regardless of the distortion of the truss.

(2) One of these joints is constrained to move in a fixed direction in the plane.

(3) One of the joints is fixed in position.

It is to be noted that the first condition must be satisfied regardless of how the truss is attached to its foundation, that is, this is a necessary condition. The second and third conditions are sufficient, but not necessary to complete the constraint; however, these are the conditions which are frequently imposed in conventional construction.

If $j = 0$ is the fixed point and the pole for the position vector of the remaining joints, then from Equation (2.10) and (2.12),

$$\overline{Y}_i = \overline{Z}_i + \overline{C}_0 + \overline{H} \times \overline{R}_i \tag{2.13}$$

$\overline{C}_0$ is determined by setting $i = 0$.

Thus

$$\overline{C}_0 = - \overline{Z}_0 \tag{2.14}$$

For the determination of $\overline{H}$, let

$i = r$ designate the joint constrained to move in the fixed direction.

$i = s$ designate the third joint constrained to remain in the fixed plane.

$\overline{n}$ = the unit vector normal to the plane containing joints 0, r, and s.

$\overline{u}$ = the unit vector in the plane 0, r, s normal to the fixed direction.

From conditions (2) and (3) in the foregoing discussion,

$$\overline{Y}_r \cdot \overline{n} = 0$$
$$\overline{Y}_r \cdot \overline{u} = 0 \qquad (2.15)$$
$$\overline{Y}_s \cdot \overline{n} = 0$$

From Equations (2.13), (2.14), and (2.15)

$$\overline{H} \cdot \overline{R}_r \times \overline{n} = (\overline{Z}_0 - \overline{Z}_r) \cdot \overline{n}$$
$$\overline{H} \cdot \overline{R}_r \times \overline{u} = (\overline{Z}_0 - \overline{Z}_r) \cdot \overline{u} \qquad (2.16)$$
$$\overline{H} \cdot \overline{R}_s \times \overline{n} = (\overline{Z}_0 - \overline{Z}_s) \cdot \overline{n}$$

Equations (2.16) are three independent equations involving $\overline{H}$. One can expand $\overline{H}$ as

$$\overline{H} = h_1\overline{v}_1 + h_2\overline{v}_2 + h_3\overline{v}_3 \qquad (2.17)$$

When Equations (2.16) and (2.17) are combined, one obtains three simultaneous equations from which the h_t $(t=1,2,3)$ and therefore $\overline{H}$ can be determined. The geometry of the corrective displacements, at joints 0, r, and s, is shown in Figure 3.

Compound and Complex Trusses

Compound and complex trusses are distinguished from the simple truss by the fact that one eventually comes to a point, where the available information is insufficient to continue with the solution for the remaining unknown joint displacements. One possible simple situation is shown in Figure 4. The displacements of joints a, b, and c and all bar elongations are assumed to be given and the remaining joint displacements

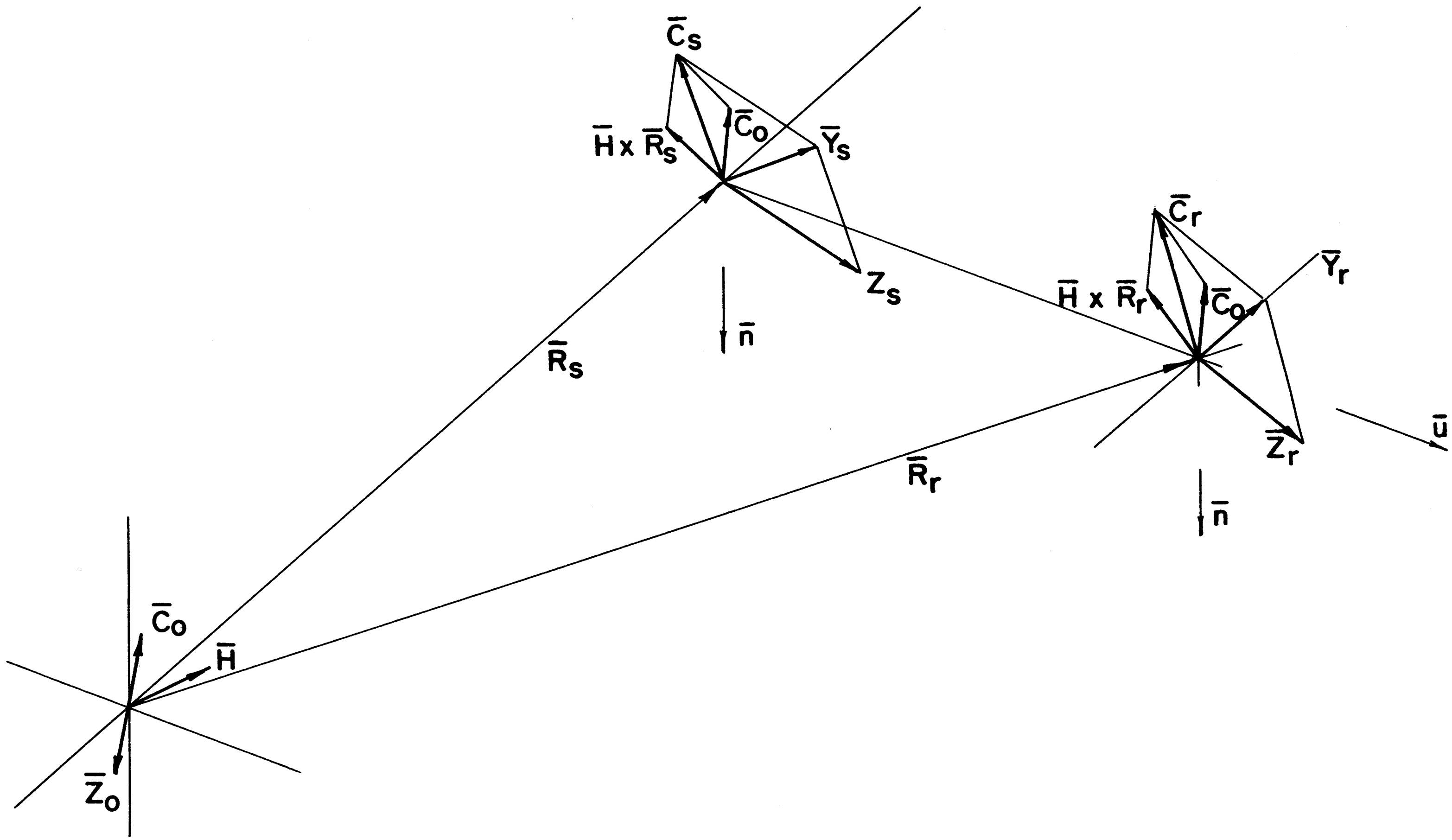


Figure 3. Corrective Displacements.

are to be determined. With the given information one can find, in order, the displacements of joints d, e, f, g, h, n, and m. Beyond this point the available information is insufficient to continue with the solution.

The compound truss as a transition between the simple and complex trusses possesses neither the simplicity of the former nor the generality of the latter. Because of its transitional nature, it is convenient to treat a given compound truss as a special problem. In the general case the compound truss is considered and treated as a complex truss.

The truss shown in Figure 4 is chosen as a special case to illustrate one way of computing the displacements of a compound truss. The truss is broken up into two parts as shown in Figure 5. Using the procedure outlined for the determination of relative joint displacements, one calculates relative displacements for the right half of the truss. Unless an extremely fortunate guess is made, the relative displacements of joints m, n, and p will not be such that Equation (2.3) is satisfied for bars ph, mg, and md, and $\overline{Z}_n = \overline{Y}_n$. Therefore, corrective displacements must be determined for the right-hand section of the truss. Since the relative displacements of the right-hand section of the truss already satisfy Equation (2.3), the correction is a rigid-body displacement. With joint n as the pole, the correction displacements are

$$\overline{C}_s = \overline{C}_n + \overline{H} \times \overline{R}_s \ , \qquad (2.18)$$

where the subscript s applies only to the joints of the right-hand section of the truss. One determines $\overline{C}_n$ from the condition that

$$\overline{Y}_n = \overline{Z}_n + \overline{C}_n \qquad (2.19)$$

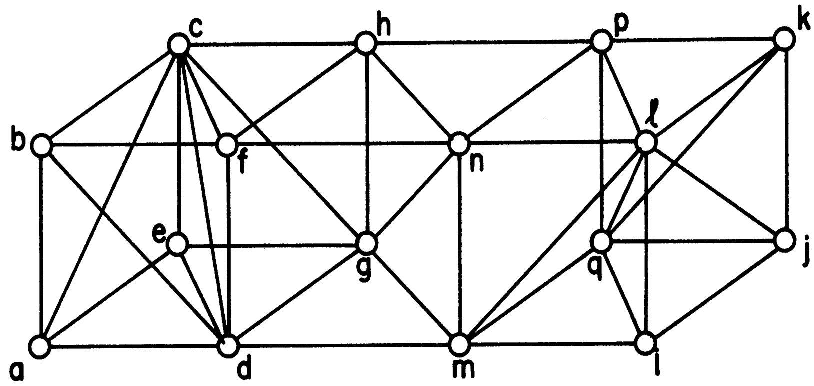


Figure 4. Compound Truss.

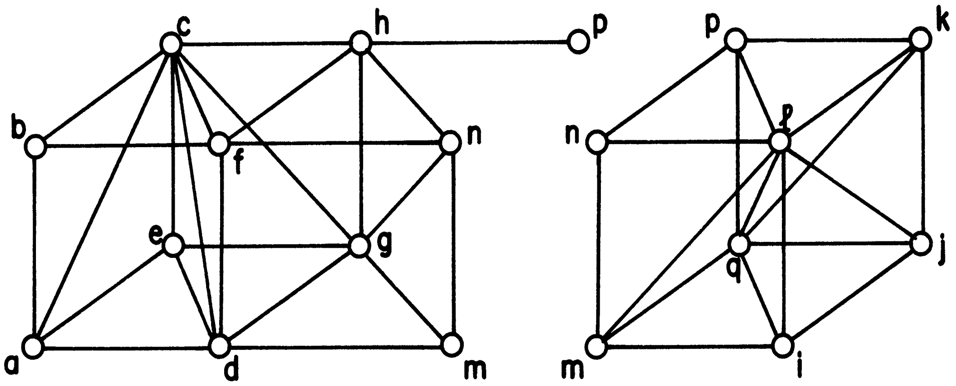


Figure 5. Compound Truss of Figure 4 Separated into Two Parts.

The displacement of joints m and p are

$$\overline{Y}_m = \overline{Z}_m + \overline{C}_n + \overline{H} \times \overline{R}_m$$
$$\overline{Y}_p = \overline{Z}_p + \overline{C}_n + \overline{H} \times \overline{R}_p \tag{2.20}$$

Combining Equations (2.20) and (2.3) for bars ph, mg, and md, one obtains

$$\overline{H} \cdot \overline{R}_p \times \overline{u}_{ph} = - d_{ph} - (\overline{Z}_p - \overline{Y}_h + \overline{C}_n) \cdot \overline{u}_{ph}$$
$$\overline{H} \cdot \overline{R}_m \times \overline{u}_{mg} = - d_{mg} - (\overline{Z}_m - \overline{Y}_g + \overline{C}_n) \cdot \overline{u}_{mg} \tag{2.21}$$
$$H \cdot R_m \times u_{md} = - d_{md} - (Z_m - Y_d + C_n) \cdot u_{md}$$

Equations (2.21) are three simultaneous equations involving $\overline{H}$. Thus, $\overline{H}$ can be determined as indicated for the case of simple trusses.

In treating the truss of Figure 4 as a compound truss, one makes use of the fact that the joint displacement vectors are linear functions of the bar elongations. With reference to Figure 4, it is seen that if the elongation of a fictitious bar connecting joints m and p were known, it would be possible to determine the displacements of joints p, ℓ, q, i, k and j, in that order, by the procedure for simple trusses. These displacements could be determined without knowing the elongation of bar jℓ. Furthermore, the displacements of joints j and ℓ would be such that Equation (2.3) is satisfied for the bar jℓ. For an arbitrarily assumed value of x, the elongation of the fictitious bar mp, the displacements of the joints p, ℓ, q, i, k, and j would not be correct. However, because the displacements of these joints are linear functions of the bar elongations, one can write for the displacement of any joint s

$$\overline{Y}_s = (\overline{Y}_s)_{x=0} + x\overline{Y}_s' \tag{2.22}$$

In Equation (2.22) the prime is used to denote differentiation with respect to x,

x = actual but as yet unknown elongation of the fictitious bar connecting joints m and p,

$\overline{Y}'_s = (\overline{Y}_s)_{x=1} - (\overline{Y}_s)_{x=0}$, the constant rate of change of $\overline{Y}_s$ with respect to x.

It is to be noted that since the actual bar elongations are held constant their contribution to $\overline{Y}_s$ is the same for $x = 1$ and $x = 0$ Therefore $\overline{Y}'_s$ is simply the displacement at s for $x = 1$ and zero elongation for all other bars.

The displacements given in Equation (2.22) will satisfy Equation (2.3) for every value of x, and for every bar except bar $j\ell$. The requirement that Equation (2.3) be satisfied for bar $j\ell$ is used to determine x. Thus

$$[(\overline{Y}_j)_{x=0} - (\overline{Y}_\ell)_{x=0} + x(\overline{Y}'_j - \overline{Y}'_\ell)] \cdot \overline{u}_{j\ell} = - d_{j\ell} \qquad (2.23)$$

The actual displacements are found by solving Equation (2.23) for x and substituting the result into Equation (2.22).

The idea of using the elongation of a fictitious or substitute bar can be extended to the general case of a complex truss where more than one substitute bar is introduced. Before generalizing the method, it is convenient to restate some of the well-known facts about space trusses.

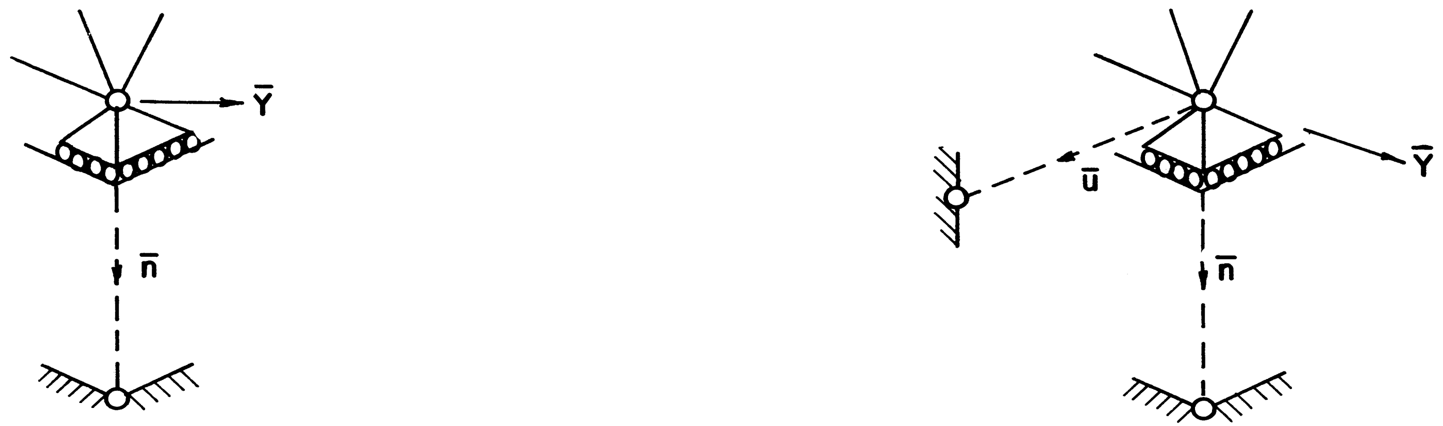
Consider a general space truss which is stable and statically determinate. A number of joints, a minimum of three, connect the truss to its foundation. Without loss of generality it can be assumed that all the foundation joints are completely constrained, that is, the displacement at each of these joints is zero for an arbitrary load applied to the joint. For the statically determinate truss, the number of bars is given by

$$B = 3J \ , \tag{2.24}$$

where J is the number of joints excluding the joints connecting the truss to its foundation.

If some of the foundation joints are supported on rollers as shown in Figure 6, then Equation (2.24) is not correct. The equation would underestimate the number of bars, by two for each roller with two degrees of freedom, and by one for each roller with one degree of freedom. However, as far as either the stress or displacement analysis is concerned, one can consider Equation (2.24) valid.

The effect of a two-degree-of-freedom roller can be duplicated, in every respect, by a rigid bar of finite length, which is pin-connected to a second foundation. Similarly, the action of a single-degree-of-freedom roller can be duplicated by two rigid bars. If the roller supports are replaced by the equivalent rigid bars, and for the analysis the rigid bars are considered to be members of the truss, then all joints connecting the truss to its foundation would be fully constrained. By adding the rigid bars, the number of joints not directly connected to the foundation is increased by one for each roller joint. When the deficit indicated by Equation (2.24) is taken into account along with the number of (rigid) bars and joints added to the truss, it is seen that the equation is valid.



(a) Two degrees of freedom

$\bar{Y} \cdot \bar{n} = 0$

(b) One degree of freedom

$\bar{Y} \cdot \bar{n} = \bar{Y} \cdot \bar{u} = 0$

Figure 6. Roller Supports and Equivalent Rigid Bar Supporting Systems.

Since the truss is assumed to be stable, one can always find three different foundation joints, which do not all lie on the same line. Using the three joints one can always construct at a second truss, which has the same number of joints J, and the same geometric layout as the given truss. Furthermore, we know that the second truss can be a simple and statically determinate truss since Equation (2.24) also applies to any simple truss which is pin-connected to its foundation.

With the foregoing information in mind, one can now proceed to obtain the joint displacements of complex trusses in the following way. The given truss is replaced by a substitute truss, which is a simple truss and has the same geometric layout as the given truss. The substitute truss contains as many bars as possible from the given truss (this is desirable only from a computational point of view). Each of the remaining bars is a substitute bar having an unknown elongation.

Let

i,k = subscripts referring to the substitute bars $k=1,2,\ldots,N$

x_k = elongation of the k^{th} substitute bar

ℓ = index referring to an actual bar in the substitute truss $\ell=1,2,\ldots,3J-N$

x_ℓ = elongation of the ℓ^{th} actual bar in the substitute truss

s = index referring to a general joint

$\bar{Z}_s$ = displacement of joint s of the substitute truss

$\bar{Z}_{s,0}$ = displacement of joint s of the substitute truss for all $x_k = 0$ and all $x_\ell = d_\ell$

$\overline{Y}_s$ = displacement of joint s in the given truss

pq = paired values of s designating a bar from the given truss which is not among the bars in the substitute truss. All together there are N such bars.

It is assumed that the joints in the substitute truss are numbered in the same way as in the given truss.

The displacements in the substitute truss can be calculated by the method for simple trusses for any assumed values of the x_k. One encounters no difficulties in calculating the displacements in the substitute truss since all its foundation joints have zero displacement. Since the displacements are linear functions of the bar elongations, one can write for the substitute truss

$$\overline{Z}_s = \overline{Z}_{s,0} + \sum_{k=1}^{N} \overline{Z}_{s,k} x_k \ , \tag{2.25}$$

where

$$Z_{s,k} = \frac{\partial \overline{Z}_s}{\partial x_k} = (Z_s)_{x_k = 1},\ x_j = 0 \quad \text{for} \quad j \neq k, \text{ and all } x_\ell = 0$$

The displacements given by Equation (2.25) are such that Equation (2.3) is satisfied for every bar of the substitute truss, and for every bar, except the bars pq, of the given truss, for all values of x_k. Furthermore, since x_k is nothing more than the change of the distance between certain pairs of joints in the given truss, the displacements in the two trusses will be the same, i.e., $\overline{Z}_s = \overline{Y}_s$, when the correct values are assigned to x_k. The requirement that Equation (2.3) be satisfied for the bars pq is used to determine x_k. Substituting

Equation (2.25) into Equation (2.3) one obtains for each of the bars pq

$$\sum_{k=1}^{N} (\overline{Z}_{p,k} - \overline{Z}_{q,k}) \cdot \overline{u}_{pq} x_k = - d_{pq} - (\overline{Z}_{p,0} - \overline{Z}_{q,0}) \cdot \overline{u}_{pq} \qquad (2.26)$$

Since pq takes on N different combinations, Equation (2.26) actually represents a set of N simultaneous linear equations in N unknowns.

Applications of the methods presented in this chapter are demonstrated by means of examples given in Appendix A.

CHAPTER III

SOLUTION OF THE EQUATIONS

In Chapter II it was shown how the displacement problem can be solved in a step-by-step fashion. In each step of the solution three equations are written, which involve but one unknown displacement vector. The three equations may easily be solved for the components of the displacement vector, but it is worthwhile to discuss briefly how the solution of the equations may be obtained.

In Equations (2.4) the letter subscripts serve no other useful purpose than to identify the joints being considered. If this is kept in mind, no confusion arises when these equations are written in the simpler form:

$$\begin{aligned} \overline{Y} \cdot \overline{u}_1 &= \overline{Y}_1 \cdot \overline{u}_1 - d_1 = b_1 \\ \overline{Y} \cdot \overline{u}_2 &= \overline{Y}_2 \cdot \overline{u}_2 - d_2 = b_2 \\ Y \cdot u_3 &= Y_3 \cdot u_3 - d_3 = b_3 \end{aligned} \tag{3.1}$$

Similarly, no confusion arises by dropping the subscript n in Equation (2.5). Thus,

$$\overline{Y} = y_1\overline{v}_1 + y_2\overline{v}_2 + y_3\overline{v}_3 \tag{3.2}$$

Equations (3.1) have to be solved as many times as there are unknown joint displacements. It is in no way difficult to solve the equations for any given case, but for computation it is desirable to have a general solution or "plug-in" type of formula. Since there are only three equations to be solved at one time, the general solution may easily be obtained.

In most instances one wants the displacement vector in terms of the unit coordinate vectors. Thus

$$\bar{v}_t = \bar{e}_t \tag{3.3}$$

The unit vectors $\bar{u}_i$ $(i = 1,2,3)$ in terms of unit coordinate vectors are

$$\bar{u}_i = a_{i1}\bar{e}_1 + a_{i2}\bar{e}_2 + a_{i3}\bar{e}_3 , \tag{3.4}$$

where a_{i1}, a_{i2}, a_{i3} are the direction cosines of $\bar{u}_i$ relative to the unit coordinate vectors.

When Equations (3.1), (3.2), (3.3), and (3.4) are combined, one obtains the set of equations

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 &= b_1 \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 &= b_2 \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 &= b_3 , \end{aligned} \tag{3.5}$$

which are written more compactly in matrix form[11] as

$$[A]\{y\} = \{b\} \tag{3.6}$$

The solution of Equation (3.6) is

$$\{y\} = [A]^{-1}\{b\} , \tag{3.7}$$

where

$$[A]^{-1} = \frac{1}{||A||}\begin{bmatrix} (a_{22}a_{33} - a_{32}a_{23}), & (a_{13}a_{32} - a_{12}a_{33}), & (a_{12}a_{23} - a_{22}a_{13}) \\ (a_{31}a_{23} - a_{21}a_{33}), & (a_{11}a_{33} - a_{31}a_{13}), & (a_{13}a_{21} - a_{11}a_{23}) \\ (a_{21}a_{32} - a_{31}a_{22}), & (a_{31}a_{12} - a_{11}a_{32}), & (a_{11}a_{22} - a_{31}a_{12}) \end{bmatrix} \tag{3.8}$$

and $||A||$ is the determinate of $[A]$.

In many cases the foregoing solution is satisfactory; however, a direct solution is not necessary, and sometimes not the most desirable, since in the direct solution it is not usually possible to take advantage of any special relationship which may exist among the unit vectors $\bar{u}_i$. The following solution permits special geometric relationships among the $\bar{u}_i$ to be taken into account.

The displacement vector $\bar{Y}$, is expressed in terms of the $\bar{u}_i$ as

$$\bar{Y} = w_1\bar{u}_1 + w_2\bar{u}_2 + w_3\bar{u}_3 \tag{3.9}$$

The cosine of the angle between two unit vectors is denoted as

$$\begin{aligned} \bar{u}_1 \cdot \bar{u}_2 &= \cos\alpha_{12} = \ell_{12} \\ \bar{u}_2 \cdot \bar{u}_3 &= \cos\alpha_{23} = \ell_{23} \\ \bar{u}_3 \cdot \bar{u}_1 &= \cos\alpha_{31} = \ell_{31} \end{aligned} \tag{3.10}$$

Combining Equations (3.1), (3.9), and (3.10), and taking into account the fact that $\bar{u}_i \cdot \bar{u}_i = 1$, one obtains

$$\begin{aligned} w_1 + \ell_{12}w_2 + \ell_{31}w_3 &= b_1 \\ \ell_{12}w_2 + w_2 + \ell_{23}w_3 &= b_2 \\ \ell_{31}w_1 + \ell_{23}w_2 + w_3 &= b_3 \end{aligned} \tag{3.11}$$

In matrix notation, Equation (3.11) can be written as

$$[L]\,\{w\} = \{b\} \tag{3.12}$$

The solution is

$$\{w\} = [L]^{-1}\{b\}\ , \tag{3.13}$$

where

$$[L]^{-1} = \frac{1}{||L||} \begin{bmatrix} (1-\ell_{23}^2), & (\ell_{31}\ell_{23} - \ell_{12}), & (\ell_{12}\ell_{23} - \ell_{31}) \\ (\ell_{31}\ell_{23} - \ell_{12}), & (1 - \ell_{31}^2), & (\ell_{12}\ell_{31} - \ell_{23}) \\ (\ell_{12}\ell_{23} - \ell_{31}), & (\ell_{12}\ell_{31} - \ell_{23}), & (1 - \ell_{12}^2) \end{bmatrix} \tag{3.14}$$

The components of $\bar{Y}$ in the direction of the unit coordinate vectors are obtained by substituting $\bar{u}_i$ from Equation (3.4), and w_i from Equation (3.13) into Equation (3.9).

To illustrate the advantage of this method when there is a special geometric condition, consider a case in which the unit vectors $\bar{u}_i$ are mutually perpendicular, but not parallel to the unit coordinate vectors.* In the first solution $[A]^{-1}$ will, in general, have all non-zero elements. On the other hand, $[L]^{-1}$ reduces to the identity matrix since $\ell_{12} = \ell_{23} = \ell_{31} = 0$.

The second method of solution is also quite useful if the truss has a series of joints wherein the unit vectors have the same relative orientation. In this case the matrix of the coefficients in Equation (3.11) will be the same at these joints regardless of the orientation of the unit vectors with respect to the unit coordinate vectors. As a result, the inverse of the matrix can be established for the first of the series of joints, and then be treated as a constant matrix in solving for the displacements at the remaining joints in the series.

In a third method of solution the displacement vector is expanded in terms of cross products of the unit vectors as

$$\bar{Y} = \sum_{k=1}^{3} x_k \bar{v}_k \tag{3.15}$$

where

$$\bar{v}_1 = \bar{u}_2 \times \bar{u}_3$$

$$\bar{v}_2 = \bar{u}_3 \times \bar{u}_1$$

$$\bar{v}_3 = \bar{u}_1 \times \bar{u}_2$$

When Equations (3.15) and (3.1) are combined, one obtains

$$x_1 = \frac{b_1}{U}, \quad x_2 = \frac{b_2}{U}, \quad x_3 = \frac{b_3}{U} \tag{3.16}$$

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* Actually for the case cited here $[A]^{-1} = [A]^T$ and $[L] = [L]^{-1} = [I]$.

where

$$U = \bar{u}_1 \cdot \bar{u}_2 \times \bar{u}_3$$

The components of the displacement vector relative to the unit coordinate vectors are obtained by replacing the $\bar{v}_k$ in Equation (3.15) by their expansions in terms of the unit coordinate vectors. In terms of the unit coordinate vectors, the $\bar{v}_k$ can be written as

$$\bar{v}_k = \sum_{\ell=1}^{3} c_{k\ell}\bar{e}_\ell \qquad (3.17)$$

The components of the displacement vector in the directions of the unit coordinate vectors are then given by

$$y_\ell = \sum_{k=1}^{3} c_{k\ell}x_k \qquad (3.18)$$

CHAPTER IV

MATRIX VERSION OF THE DISPLACEMENT PROBLEM

For the most part the vector methods outlined in Chapter II, and the methods of solution given in Chapter III, are satisfactory when the displacements are wanted for a few different sets of bar elongations. When the displacements are wanted for several or many sets of bar elongations, it is sometimes convenient to formulate and solve the problem in terms of matrices.

In many examples of linear systems, the matrix solution of the problem is simple in principle but difficult in application. The difficulty usually arises because one is interested in the general solution of the linear system, and this invariably involves the inversion of a matrix.

Generally the matrix to be inverted does not possess properties which can be used to reduce the number of arithmetic operations involved in inverting the matrix. For large matrices the number of operations performed in the inversion process is proportional to the cube of the order of the matrix.[12] In contrast to the general case, the matrix associated with the displacement problem takes a form which substantially reduces the number of arithmetic operations in the inversion process.

The matrix version of the displacement problem can take different forms according to the formulation of the problem. The simplicity of the vector method suggests that a matrix formulation of the displacement problem along the same lines would be appropriate. As it turns out, the matrix equivalent of the vector method, which is considered here, leads to a particularly simple system of equations.

The matrix method can be developed for any truss, regardless of the arrangement of the truss members. However, in general, variations in the method may be required when complex arrangements of members are encountered. No particular classification of such structures will be used here, but the reader should be able to make his own adaptation from the following discussion.

The matrix and vector methods are very closely related. In fact, the two are identical except that in the matrix approach numerical solution of the problem is postponed until the displacement equations have been written for every joint (bar) in the truss.

We consider first the case of a simple statically determinate truss which is pin-connected to its foundation.

Let j and i be indices which identify the joints in the truss. Let $j = 1, 2, \ldots, m$ identity the joints connecting the truss to its foundation. Let $j = m + 1, m + 2, m + 3, \ldots, m + J$ identify the remaining joints in the truss according to the order in which they would be considered in using the vector method. In general, the order in which the joints are considered is not unique. For convenience the displacement-elongation equation [Equation (2.3)] is rewritten as

$$-\overline{Y}_i \cdot \overline{u}_{ji} + \overline{Y}_j \cdot \overline{u}_{ji} = -d_{ji} \qquad (4.1)$$

For systematic use of Equation (4.1), let j apply to the joint at which the displacement vector would be unknown in using the vector method. Let $\alpha < \beta < \gamma$ be the three values of $i (i < j)$ that define the joints and bars which must be considered to find $\bar{Y}_j$ by the vector method. The equations for the three bars at any joint are then given as

$$\begin{aligned} -\bar{Y}_\alpha \cdot \bar{u}_{j\alpha} + \bar{Y}_j \cdot \bar{u}_{j\alpha} &= -d_{j\alpha} \\ -\bar{Y}_\beta \cdot \bar{u}_{j\beta} + \bar{Y}_j \cdot \bar{u}_{j\beta} &= -d_{j\beta} \\ -\bar{Y}_\gamma \cdot \bar{u}_{j\gamma} + \bar{Y}_j \cdot \bar{u}_{j\gamma} &= -d_{j\gamma} \end{aligned} \qquad \begin{aligned} &j = m+1, m+2, \ldots m+J \\ &\alpha < \beta < \gamma < j \end{aligned} \tag{4.2}$$

To eliminate the possibility of identifying a bar in more than one way, we require that the paired indices identifying the bar and its elongation always be taken in the order indicated in Equation (4.2). We also require that the three equations at any joint always be written according to the natural order of the indices identifying the bars.

Expanding the displacement and unit vectors in terms of the unit coordinate vectors, one obtains

$$\bar{Y}_j = \sum_{\ell=1}^{3} y_j^{\ell} e_{\ell} \tag{4.3}$$

$$\bar{u}_{ji} = \sum_{m=1}^{3} a_{ji}^{m} \bar{e}_m \qquad i = \alpha, \beta, \gamma \tag{4.4}$$

Substituting into Equation (4.2) one gets

$$\begin{aligned} \sum_{\ell=1}^{3} -a_{j\alpha}^{\ell} y_\alpha^{\ell} + \sum_{\ell=1}^{3} a_{j\alpha}^{\ell} y_j^{\ell} &= -d_{j\alpha} \\ \sum_{\ell=1}^{3} -a_{j\beta}^{\ell} y_\beta^{\ell} + \sum_{\ell=1}^{3} a_{j\beta}^{\ell} y_j^{\ell} &= -d_{j\beta} \\ \sum_{\ell=1}^{3} -a_{j\gamma}^{\ell} y_\gamma^{\ell} + \sum_{\ell=1}^{3} a_{j\gamma}^{\ell} y_j^{\ell} &= -d_{j\gamma} \end{aligned} \tag{4.5}$$

$$j = m+1, m+2, \ldots m+J \qquad \alpha < \beta < \gamma < j$$

In general Equations (4.5) are three equations in twelve unknowns. Matrix notation will be used to facilitate treatment of the equations. Let the matrix

$$[A]_{j\alpha} = \begin{bmatrix} -a^1_{j\alpha} & -a^2_{j\alpha} & -a^3_{j\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A]_{j\beta} = \begin{bmatrix} 0 & 0 & 0 \\ -a^1_{j\beta} & -a^2_{j\beta} & -a^3_{j\beta} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A]_{jj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a^1_{j\gamma} & -a^2_{j\gamma} & -a^3_{j\gamma} \end{bmatrix}$$

$$[A]_{jj} = -[A]_{j\alpha} - [A]_{j\beta} - [A]_{j\gamma}$$

Let the displacement vectors be

$$\{y\}_\alpha = \begin{Bmatrix} y^1_\alpha \\ y^2_\alpha \\ y^3_\alpha \end{Bmatrix}, \quad \{y\}_\beta = \begin{Bmatrix} y^1_\beta \\ y^2_\beta \\ y^3_\beta \end{Bmatrix}, \quad \{y\}_\gamma = \begin{Bmatrix} y^1_\gamma \\ y^2_\gamma \\ y^3_\gamma \end{Bmatrix}, \quad \{y\}_j = \begin{Bmatrix} y^1_j \\ y^2_j \\ y^3_j \end{Bmatrix},$$

and let the bar elongation vector be

$$\{d\}_j = \begin{Bmatrix} d_{j\alpha} \\ d_{j\beta} \\ d_{j\gamma} \end{Bmatrix}$$

Equations (4.5) can then be written as

$$[A]_{j\alpha}\{y\}_\alpha + [A]_{j\beta}\{y\}_\beta + [A]_j \{y\}_\gamma + [A]_{jj}\{y\}_j = -\{d\}_j \qquad (4.6)$$

$$j = m+1,\ m+2, \ldots m+J$$

$$\alpha < \beta < \gamma < j$$

Recalling that α, β, γ are only the specific values of i associated with the displacement at joint j we define a set of matrices $[A]_{ji}$ $i = 1,2, \ldots. j$ such that

$$\begin{aligned} [A]_{ji} &= 0 && \text{if } i \neq \alpha, \beta, \gamma, j \\ &= [A]_{j\alpha} && i = \alpha \\ &= [A]_{j\beta} && i = \beta \\ &= [A]_{j\gamma} && i = \gamma \\ &= [A]_{jj} && i = j \end{aligned}$$

Equation (4.6) can then be written as

$$\sum_{i=1}^{j} [A]_{ji}\{y\}_i = -\{d\}_j \qquad j = m+1, m+2, \ldots m+J \qquad (4.7)$$

The vector $\{d\}_j$ defines fully the bars to be considered at any joint. Since the truss is pinned to its foundation $\{y\}_i = 0$ for $1 \leq i \leq m$. If we keep this information in mind, no confusion arises when we replace j by t, where $t = 1,2,\ldots J$. Equation (4.7) then becomes

$$\sum_{i=1}^{t} [A]_{ti}\{y\}_i = -\{d\}_t \qquad t = 1,2,\ldots J \qquad (4.8)$$

and the complete system of matrix equations governing the displacements takes the particularly simple form

$$\begin{bmatrix} [A]_{11} & [0] & \cdot & \cdot & [0] \\ [A]_{21} & [A]_{22} & [0] & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [A]_{t1} & \cdot & [A]_{tt} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & [0] \\ [A]_{J1} & \cdot & \cdot & \cdot & [A]_{JJ} \end{bmatrix} \begin{bmatrix} \{y\}_1 \\ \{y\}_2 \\ \cdot \\ \{y\}_t \\ \cdot \\ \cdot \\ \{y\}_J \end{bmatrix} = - \begin{bmatrix} \{d\}_1 \\ \{d\}_2 \\ \cdot \\ \{d\}_t \\ \cdot \\ \cdot \\ \{d\}_J \end{bmatrix} \tag{4.9}$$

It should be observed that the form of the matrix equation as indicated in Equation (4.9) occurs only in the case of a simple truss which is pin-connected to its foundation.

Equation (4.9) can also be written as

$$[A]\{y\} = -\{d\} \tag{4.10}$$

Let the inverse of [A] be [C] so that the solution of Equation (4.10) is

$$\{y\} = -[C]\{d\} \tag{4.11}$$

The matrix [C] has the same form as [A], that is, [C] is essentially a lower triangle matrix. If [C] is partitioned in exactly the same way as [A], then the submatrices of [C] are

$$\begin{aligned} [C]_{tt} &= [A]_{tt}^{-1} && t = 1,2,\ldots J \\ [C]_{tm} &= -[C]_{tt} \sum_{\ell=m}^{t-1} [A]_{t\ell}[C]_{\ell m} && t = 2,3,\ldots J\text{-}1,\ m < t \\ [C]_{tm} &= 0 && m > t \end{aligned} \tag{4.12}$$

In general none of the submatrices $[C]_{tm}$ $(m \leq t)$ is identically zero.

It is to be noted that the order in which the joints and bars are considered is such that all the $[A]_{tt}$ are nonsingular. Thus we are assured that the matrix [C] exists and is unique. It is also important to note that, in any one row of submatrices in [A], there are at most four submatrices which are not identically zero, and that three of the nonzero submatrices have but one row different from zero. Therefore the actual arithmetic operations to be carried out in calculating [C] is far less than is implied by Equation (4.12). If all the submatrices in [A] were nonzero and J is large, then the number of arithmetical operations would be on the order of $14J^2$.

In a haphazard formulation of the displacement problem where the bars are considered in an arbitrary order, which is different from the foregoing development, Equation (4.10) would be transformed to

$$[A]^*\{y\} = \{d\}^* \tag{4.13}$$

in which the asterisk signifies that the elements in $\{d\}$, and the corresponding rows of [A] have been deranged. Knowing that $[A]^*$ (and $\{d\}^*$) can be converted to [A] (and $\{d\}$) by rearranging its rows, one recognizes immediately that inverting $[A]^*$ should involve no more arithmetic than inverting [A]. In fact, $[A]^{*-1}$ would differ from $[A]^{-1}$ only in the arrangement of the columns. On the other hand, if one did not know that the difference between $[A]^*$ and [A] is only superficial, one could very easily and falsely conclude from the appearance of $[A]^*$ that inversion of $[A]^*$ is a hopeless task,[≠] and that a general matrix solution for

≠ $[A]^*$ would have the appearance of a general matrix except that it would have a large number of zero elements. However, the logical conclusion drawn from the appearance of $[A]^*$ would, nevertheless, be that the number of arithmetic operations required to invert the matrix is on the order of $27J^3$ when J is large.

the displacements is not practical. Thus, while the fact that the displacement problem can be put into matrix form is not in itself particularly significant, the fact that the matrix involved can be made to take a simple form which is readily inverted is of significance.

In general, a matrix solution for the relative displacements in a simple truss is not very satisfactory if one is interested in the actual displacements. The reason for this is that the relative displacements would have to be corrected for each set of bar elongations for which the displacements are desired. There are, however, two cases where a matrix solution for the relative displacements is worthwhile: (1) when one is interested in the relative displacements only, as in an investigation of secondary stresses in trusses; and (2) when the elements in the inverse of the geometric matrix, which governs the relative displacements, can be given in a general analytical form. For the most part analytic inversion of the geometric matrix is practical only for certain specialized types of plane trusses.

In the matrix solution for the relative displacements, rigid bars, as described in Chapter II, can be used to simulate the artificial rigid constraints of the fixed point, fixed line, and fixed plane. The rigid bars are considered to be actual members of the truss. The problem is then identical to that of finding the actual displacements in a simple truss, which is pin-connected to its foundation.

In the following discussion complex trusses are treated almost solely from a matrix point of view. Where possible, the features which tie the matrix and vector approaches together are pointed out.

A development of the matrix method for complex trusses does not follow readily from a direct consideration of the vector method. The difficulties which arise can be traced to the fact that in the vector method we are interested in particular solutions, whereas in thematrix approach a general solution is desired.

A matrix solution would not be of practical significance unless the matrix relating the displacements and bar elongations is such that it can be inverted readily. In the preceding discussion it was shown that the matrix associated with a simple truss could be made to take a form such that inversion was not difficult. In Chapter II it was shown how the displacements in a complex truss could be obtained by finding the displacement in a simple substitute truss in which the elongation of certain bars were unknown. A compatibility relation [Equation (2.26)] was then set up to determine what the unknown bar elongations had to be, so that the displacements in the substitute and given truss would be the same. The approach used in Chapter II was based on the fact that the displacements and bar elongations are linearly related. The use of the substitute truss, a compatibility relation, and the linearity property suggests that the matrix solution of the problem would be simplified by the use of a linear transformation.

Consider a general truss which is stable, statically determinate, and pin-connected to its foundation. If we apply Equation (4.1) only once to each bar in the truss and replace the vectors by their scalar equivalents there results a system of equations:

$$[G]\{y\} = -\{d\} \tag{4.14}$$

For a stable truss [G] is nonsingular and

$$\{y\} = -[G]^{-1}\{d\} \tag{4.15}$$

In general, [G] will be such that direct inversion of the matrix is not

We next consider a simple truss which is pin-connected to its foundation. Let the number of joints in the simple truss be the same as for the general truss. Let the disposition of the joints in a simple truss be identical to that of the general truss. Let the joints and bars in the simple truss be identified in the manner outlined in the development of the matrix method for simple trusses. For the simple truss we have that

$$[A]\{z\} = -\{d\}^* \qquad (4.16)$$

and

$$\{z\} = -[A]^{-1}\{d\}^* \qquad (4.17)$$

in which $\{z\}$ is the displacement vector and $\{d\}^*$ is the bar elongation vector.

Means of identifing the joints and bars in the general truss were not stipulated. It is now convenient to number the joints in the general truss in the same way as the simple truss, i.e., $j' = j = 1,2,\ldots m, m+1,\ldots m+J$ identifies the same joint in both trusses.

Any joint $j, (j > m)$ in the general truss is connected to several (at least three) other joints. Of the several bars at any joint we want to consider only the three bars $j\alpha', j\beta', j\gamma'$ where $\alpha' < \beta' < \gamma'$ are the three smallest numbers identifying the joints connected to joint j by a bar. In general α', β', or γ' may be either greater or smaller than j.

In applying Equation (4.1) to bars $j\alpha'$, $j\beta'$, $j\gamma'$ (in that order), the unit vectors must be directed from joint j to joints α', β', γ', respectively. We also stipulate that Equation (4.1) be applied to each bar in the natural order indicated by the paired indices identifying the bars. The effect of the foregoing conditions is to limit to one the number of acceptable ways in which the expanded form of Equation (4.14) can be written.

For convenience it is assumed that the unit coordinate vectors have the same origin and orientation for both trusses. None of the conditions restricts the form of the general truss.

The system for identifying and ordering the joints and bars in the two trusses is such that, if a bar connects joints j and α in the general truss, and a bar connects the same joints in the simple truss, then that row of [G] and that row of [A], which result from applying Equation (4.1) to the bar $j\alpha$ in the general and simple trusses, respectively, are identical. Furthermore, if the row is the kth row in [G], it is also the kth row in [A].

For each truss the bar elongation vectors are, in general, arbitrary; however, if we require that the displacement vector for the two trusses be identical, then by setting $\{z\} = \{y\}$ one obtains from Equations (4.15) and (4.17)

$$\{d\}^* = [A][G]^{-1}\{d\} \tag{4.18}$$

or

$$\{d\} = [G][A]^{-1}\{d\}^* \tag{4.19}$$

Define $[B] = [A][G]^{-1}$ as the matrix which transforms $\{d\}$ into $\{d\}^*$; then

$$\{d\}^* = [B]\{d\} \tag{4.20}$$

If the two trusses have no bars in common, that is, if we cannot find a pair of joints ji such that there is a bar connecting these joints in both trusses, then we can say very little about the matrix [B], or what $\{d\}^*$ is in terms of $\{d\}$. On the other hand, if the arrangement of the bars in the two trusses is identical, then [G] = [A] so that [B] is the identity matrix and $\{d\}^* = \{d\}$.

Between the two extreme cases one can expect that the two trusses may have a certain number of bars in common. For each such bar a row in [G] and the same row in [A] are identical. If one of the bars is bar $j\alpha$, as in the previous discussion, then the kth row in [G] and [A] are identical, and from the definition of $[G]^{-1}$, it is concluded that the kth row in [B] has zero in every position except for the diagonal element which is one. From Equation (4.20) one would then have that $d^*_{j\alpha} = d_{j\alpha}$. Knowing what the rows of [B] must be when the two trusses have bars in common, and knowing that $[A]^{-1}$ can be computed with comparative ease, one can find the solution for the general truss in the following way, which avoids direct inversion of [G].

Given the general truss, we construct a simple truss through the same joints and in such a way that it has as many bars as possible in common with the general truss. Combining Equations (4.17) and (4.20), one obtains for the displacements of the general truss

$$\{y\} = -[A]^{-1}[B]\{d\} \equiv -[G]^{-1}\{d\} \tag{4.21}$$

From Equation (4.21) it is seen that we can find $[G]^{-1}$ (which is what we really want) if [B] is known. Fortunately, it is already known that, for every bar which the two trusses have in common, there is a row in [B] which is a row from the identity matrix. The total number of rows, N, in [B] which are not rows of the identity matrix, is the same as the number of substitute bars which would be used in finding the displacements by the vector method. It is assumed that N is small compared to 3J, which is the order of the matrices involved in analyzing the truss. To find the remaining elements of [B], premultiply Equation (4.21) by [G] and take into account Equation (4.14). The result is

$$[I]\{d\} = [G][A]^{-1}[B]\{d\} \tag{4.22}$$

Since $\{d\}$ is an arbitrary displacement vector, Equation (4.22) is valid only if

$$[I] = [G][A]^{-1}[B] \qquad (4.23)$$

It is already known what the rows of [B] will be for each row of [G] which is identical to the same row in [A]. Thus it is necessary to consider only the case where the corresponding rows of [G] and [A] are different, and the same rows of [B] which contain unknown elements.

For simplicity in showing how the unknown elements in [B] are determined, we assume a special case where all but the first, second, and fourth rows of [G] are identical to the rows of [A]. We also ignore the facts that $[A]^{-1}$ is essentially a lower triangle matrix, and that any one row in [G] can have at most six nonzero elements. It is known that we can delete from [G] all but the first second and fourth rows. The expansion of Equation (4.23), with [C] replacing $[A]^{-1}$, would appear as follows.

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdot & \cdot & g_{1n} \\ g_{21} & g_{22} & g_{23} & \cdot & \cdot & g_{2n} \\ g_{41} & g_{42} & g_{43} & \cdot & \cdot & g_{4n} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & c_{2n} \\ c_{31} & c_{32} & \cdot & \cdot & c_{3n} \\ c_{41} & c_{42} & \cdot & \cdot & c_{4n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdot & \cdot & c_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdot & \cdot & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdot & \cdot & b_{2n} \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ b_{41} & b_{42} & b_{43} & & \cdot & b_{4n} \\ 0 & 0 & 0 & & \cdot & 0 \\ 0 & 0 & & & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

In the expansion the customary method of identifying the elements in a matrix has been used. This was done only to conserve space. It should be kept in mind that from the numbering system set up for the bars and joints, the first three elements in the first row of [G] are the components of the unit vector from the first joint ($j = m+1$ or $t = 1$) to some other joint in the general truss. If the first joint happened to be connected to, say, the fifth joint ($t = 5$), then the actual expansion of the first row is

$$[g_{1m}] = [[a_{15}^1 \; a_{15}^2 \; a_{15}^3],[0\;0\;0],[0\;0\;0],[0\;0\;0],[-a_{15}^1 \; -a_{15}^2 \; -a_{15}^3],[0\;0\;0]..[000]]$$

Similarly, the first three elements in the second row of [G] are the components of the unit vector from the first joint to some other joint in the truss beyond the fifth. If the second joint ($j = m+2$ or $t = 2$) were connected to the first joint in the general truss, then

$$[g_{4m}] = [[-a_{21}^1 \; -a_{21}^2 \; -a_{21}^3],[0\;0\;0],[0\;0\;0],[a_{21}^1 \; a_{21}^2 \; a_{21}^3],[0\;0\;0]...[0\;0\;0]]$$

Let the elements in the product of the first two matrices in the expansion be defined by the letter h with appropriate subscripts; then the expansion becomes

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ h_{41} & h_{42} & h_{43} & \cdots & h_{4n} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & b_{24} & \cdots & b_{2n} \\ 0 & 0 & 1 & 0 & & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & & b_{4n} \\ 0 & 0 & 0 & 0 & & 0 \\ & & & & & \vdots \\ 0 & 0 & 0 & 0 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & 0 & .. & 0 \\ 0 & 0 & 0 & 1 & .. & 0 \end{bmatrix}$$

Multiplying each row of [H] by the first column of [B], and setting the product equal to the first column of the reduced identity matrix, one obtains

$$h_{11}b_{11} + h_{12}b_{21} + h_{14}b_{41} = 1$$

$$h_{21}b_{11} + h_{22}b_{21} + h_{24}b_{41} = 0$$

$$h_{41}b_{11} + h_{42}b_{21} + h_{43}b_{41} = 0 \quad ,$$

which can be solved for b_{11}, b_{21}, b_{41}. The remaining elements in [B] are determined in a like manner.

In the development of the methods for analyzing displacements no consideration was given to the possible sources which cause the bars to elongate. The most probable source, of course, is a system of loads which may be applied to the truss. In this case the bar forces must be found first. It is now convenient to consider the problem of stress analysis and its relation to the problem of displacement analysis.

It can be observed that there are some marked similarities between the methods of analyzing joint displacements, and the methods of analyzing stresses in statically determinate trusses. For example, in a simple truss one usually finds the bar forces by the method of joints. In parallel with this, a method of joints is used to find the displacements in simple trusses. Similarly, in a compound truss, one usually uses the method of sections and joints to find the bar forces, and here again, a method of sections and joints is used to find the displacements. To analyze the stresses in complex trusses one may be forced to use Henneberg's method, which, in parallel with the method outlined for complex trusses, involves a consideration of the stresses in substitute bars.(1)

The displacement and stress analysis problems are much more intimately related than one would suspect from the superficial parallel existing in the methods of analysis.

To show the relation between the displacement and stress problems, define:

$\{q\}$ = load vector (dimension F).

$\{p\}$ = bar force vector (dimension F). An element in $\{p\}$ is positive if the bar is in tension.

$[R]^{-1}$ = dimensionless matrix which transforms the loads into bar forces.

$[f]$ = diagonal matrix, with positive elements, which transforms bar forces into bar elongations (dimension LF^{-1})

The arrangement of the elements in the load vector is the same as the arrangement of the elements in the displacement vector, that is $\{q\} = [q_1^1, q_1^2, q_1^3, \ldots, q_t^1, q_t^2, q_t^3, \ldots, q_J^3]$. In the displacement problem for a general truss,

$$[G]\{y\} = -\{d\} \tag{4.14}$$

Collectively the equations of equilibrium for each joint in the truss are

$$[R]\{p\} + \{q\} = 0 \tag{4.24}$$

so that the bar forces are

$$\{p\} = -[R]^{-1} \{q\} \tag{4.25}$$

The matrix $[f]$ transforms the bar forces into bar elongations and therefore

$$\{d\} = [f] \{p\} \tag{4.26}$$

Combining Equations (4.14), (4.24), and (4.25) and solving for $\{y\}$, one obtains

$$\{y\} = [G]^{-1}[f][R]^{-1}\{q\} \tag{4.27}$$

or

$$\{y\} = [F] \{q\} \tag{4.28}$$

where $[F]$ is the flexibility, or deflection-force, for the joints in the truss. Since the vectors $\{y\}$ and $\{q\}$ are of the same dimension, it is known that $[F]$ must be a square matrix. It is also known that, if the truss exhibits linear elastic behavior, then [F] is also symmetric and nonsingular. The nonsingularity of [F] requires that [G] as well as [R] be nonsingular and therefore a square matrix.* It has been shown that [F] is also given by:[13]

$$[F] = [R]^{-1T}[f][R]^{-1} \tag{4.29}$$

Inverting [F] in Equation (4.29), and premultiplying by [F] as defined by Equations (4.27) and (4.28) one obtains

$$[I] = [G]^{-1}[R]^{T} \tag{4.30}$$

or

$$[G] = [R]^{T} \tag{4.31}$$

Thus, the stress analysis problem given by Equation (4.24) can also be written as

$$[G]^{T}\{p\} = -\{q\}, \tag{4.32}$$

and the solution as

$$\{p\} = -[G]^{-1T}\{q\} \tag{4.33}$$

Since the displacement and stress analysis problems differ only in that the former is governed by [G] and the latter by $[G]^T$, the two problems are virtually identical. Therefore it should not be surprising that the methods of analysis are closely related.

* The nonsingularity of [G] was, of course, tacitly assumed throughout this chapter.

CHAPTER V

ERROR ANALYSIS

It is not generally possible to obtain an exact solution of the displacement problem, even in the linearized form considered here, because in the numerical work rounded numbers are used which are only approximations of the actual numbers which they represent. Recognizing that the solution is not exact, one cannot avoid asking how large the errors are. In general, the question cannot be answered exactly, but by means of an error analysis it is possible to estimate the expected error.

In the error analysis it is assumed that the displacement-elongation relationship, Equation (2.3), is exact. The problem then becomes one of investigating the errors in the solution of a system of simultaneous linear equations. General discussions of the problem of errors, error analysis, and the improvement of approximate solutions are available in the literature.[11,14,15] In the following discussion the more general methods are adapted to the displacement problem.

Two kinds of errors must be considered. First, there is the inherent error which arises when the constants in the problem are replaced by approximate numbers, which are then used in the numerical solution. In the displacement problem, rounded, and therefore approximate, numbers are almost invariably used to represent the components of the unit vectors. Inherent errors are considered in greater detail later in this chapter. Secondly, there is the round-off error which occurs in the numerical solution.

In the proposed method of solving the displacement problem, the joints are considered in a particular sequence. The sequence is such that the calculated displacements at three joints are used in solving for the displacement at a fourth joint. The method is, therefore, essentially a marching or step-by-step method of solution. The advantage in solving only three simultaneous equations at each step is partially offset by the fact that errors in the calculated displacement at one joint are carried over to succeeding joints. The alarming feature here is not so much that the errors are carried forward, but that their effects may be magnified and in extreme cases may go unnoticed. It therefore becomes important to minimize the accumulation of errors. In the error analysis which follows, the approximate numbers which represent the components of the unit vectors and the bar elongations used in the process of computing the displacements are considered to be exact. The errors generated in the solution are considered due solely to the process of rounding-off.

Round-off errors accumulate algebraically. It may be expected that these errors will, in part, cancel one another; however, it cannot be taken for granted that this will actually occur.

The existence of errors due to round-off is apparent when the computed solution fails to satisfy the original equations. At each stage in the displacement analysis the system of equations to be solved (in the direct solution) is

$$[A]\,\{y\} = \{b\} \tag{3.5}$$

The three elements in $\{b\}$ depend on the calculated displacements at other joints. Thus it is clear that past errors will be carried over, even if an exact solution of the equations is possible in the current step. Little can be done to remedy this situation, but the errors can be reduced in the current step.[14]

Let $\{\bar{y}\}$ be the calculated displacement which is in error due to round-off. Let $\{\bar{b}\}$ be such that, when the calculated displacement vector is substituted into Equation (3.6), there results

$$[A]\,\{\bar{y}\} = \{\bar{b}\} \tag{5.1}$$

Let the error $\{\delta y\}$ in the calculated displacement vector be such that

$$\{y\} = \{\bar{y}\} + \{\delta y\} \tag{5.2}$$

and let the residual $\{\delta b\}$ be such that

$$\{b\} = \{\bar{b}\} + \{\delta b\} \tag{5.3}$$

Substituting these quantities into Equation (3.6) and subtracting Equation (5.1), one obtains

$$[A]\,\{\delta y\} = \{\delta b\} \tag{5.4}$$

An exact solution of this system of equations would yield corrections such that all computational errors are eliminated. Usually this cannot be done so that the calculated corrections are in error. In general, however, the calculated corrections together with the original solution yield an improved displacement vector. If the residual for the improved displacement vector is not sufficiently small, the process can be repeated. One cannot be assured that this process converges, either rapidly or at all; however, the method is usually satisfactory.(14) An alternative to this procedure is to retain as many significant figures as possible in going through the initial solution. Since only three equations in three unknowns are being solved at any one time, it appears that the alternative procedure is preferable.

The computed displacements are based on approximate input data. The accuracy of the solution depends, therefore, on the initial or inherent error incurred in using approximate numbers. An investigation of the inherent errors permits one to estimate the errors which can be expected as the result of the initial approximation.

The customary approach is used in investigating the inherent errors.(14) Let $\bar{u}_{ji}$ and d_{ji} be the approximate quantities used in solving for the displacement vector $\bar{Y}_j$. Let $\Delta\bar{u}_{ji}$, Δd_{ji} and $\Delta\bar{Y}_j$ be the errors such that

$$\bar{u}_{ji} + \Delta\bar{u}_{ji} = \text{true unit vector}$$

$$d_{ji} + \Delta d_{ji} = \text{true bar elongation}$$

$$\bar{Y}_j + \Delta\bar{Y}_j = \text{true displacement vector}$$

For the true quantities the displacement-elongation relationship is

$$(\bar{Y}_j - \bar{Y}_i) \cdot (\bar{u}_{ji} + \Delta\bar{u}_{ji}) + (\Delta\bar{Y}_j - \Delta\bar{Y}_i) \cdot (\bar{u}_{ji} + \Delta\bar{u}_{ji}) = - d_{ji} - \Delta d_{ji} \qquad (5.5)$$

whereas for the error quantities, the displacement-elongation relationship is

$$(\overline{Y}_j - \overline{Y}_i) \cdot \overline{u}_{ji} = - d_{ji} \tag{2.3}$$

Expanding Equation (5.5), subtracting Equation (2.3), and rearranging terms, one gets

$$(\Delta\overline{Y}_j - \Delta\overline{Y}_i) \cdot \overline{u}_{ji} = - \Delta d_{ji} - (\overline{Y}_j - \overline{Y}_i) \cdot \Delta\overline{u}_{ji} \tag{5.6}$$

In obtaining Equation (5.6), the second-order error term $(\Delta\overline{Y}_j - \Delta\overline{Y}_i) \cdot \Delta\overline{u}_{ji}$ was assumed to be negligible compared to the remaining terms on the right-hand side of the equation. This is the assumption customarily used in investigating the inherent errors in a linear system. It should be kept in mind that any error computed on the basis of Equation (5.6) is itself in error to a degree consistent with the assumption used in obtaining the equation.

The error equation is basically the same as the original displacement-elongation equation, and if $\Delta\overline{u}_{ji}$ and Δd_{ji} were known, one could consider the right-hand side of Equation (5.6) as being equivalent to a bar elongation, and use the methods of Chapter II to find $\Delta\overline{Y}_j$. Unfortunately, the $\Delta\overline{u}_{ji}$ and Δd_{ji} are not known exactly. In fact, about all that can be said about $\Delta\overline{u}_{ji}$ is that the absolute value of each of its components does not exceed a certain positive number α^{ℓ}_{ji}. Similarly, it is usually known only that Δd_{ji} is less than a positive number δ_{ji}. In general, the positive number is taken to be the error incurred in rounding-off the elements in the input data to the significant figures used in solving the problem. Because of the variability of these terms, it is possible to determine $\Delta\overline{Y}_j$ only to within certain limits.

Matrix notation is used in the discussion of error limits. The matrix formulation of the error equations in this problem differs somewhat from the usual case. Because of this, it is convenient to consider the matrix form of a set of simultaneous equations. Let the set of equations be given by

$$[K]\{x\} = \{b\} \tag{5.7}$$

The error equation is obtained by the same process used to obtain Equation (5.6). When the higher-order terms are neglected, the error equation is

$$[K]\{\Delta x\} = \{\Delta b\} - [\Delta K]\{x\} \tag{5.8}$$

If the foregoing procedure is applied to the equations for a general truss, the error equation becomes

$$[G]\{\Delta y\} = -\{\Delta d\} - [\Delta G]\{y\} \tag{5.9}$$

In obtaining Equation (5.8) it is assumed that the elements in $[\Delta K]$ are independent of one another, and that each element can be and is varied by a small amount. This is not the case in the displacement problem. From the matrix formulation of the displacement problem, it is known that in any one row of third-order submatrices in $[G]$ there are at most four submatrices which are not identically zero. Therefore, the elements in these submatrices cannot be varied. In addition, it is also known that any one row in the submatrix $[G]_{tt}$

is always identical to the same row in one of the other submatrices, which is not identically zero. Thus it is clear, that the simple error equation, Equation (5.9), cannot be applied to the problem at hand. The fact that the matrix formulation of the error problem does not take the form of Equation (5.9) is also indicated by the basic equation, Equation (5.6), which governs the errors in the displacement problem.

The correct error equation (in terms of matrices) for the truss problem is found from a consideration of Equation (5.6), and of how the matrix version of the displacement problem was formulated. Consider joint j and the three connecting joints α, β, γ, as defined in Chapter IV. Temporarily assume that $\Delta\bar{u}_{ji}$ is identically zero for all bars in the truss. Equation (5.6) is then identical to Equation (2.3) except that $\Delta\bar{Y}_j$ and Δd_{ji} replace $\bar{Y}_j$ and d_{ji}. When Equation (5.7) (with $\Delta\bar{u}_{ji} = 0$) is written for the three bars $j\alpha$, $j\beta$, $j\gamma$, one obtains the matrix equation

$$[G]_{jj}\,\{\Delta y\}_j + [G]_{j\alpha}\,\{\Delta y\}_\alpha + [G]_{j\beta}\,\{\Delta y\}_\beta + [G]_{j\gamma}\,\{\Delta y\}_\gamma = -\{\Delta d\}_j \quad (5.10)$$

where the matrices are as defined in Chapter IV, and the elements in the vector $\{\Delta y\}_j$ are the components of the error vector $\Delta\bar{Y}_j$ in the directions of the unit coordinate vectors. The elements in the vector $\{\Delta d\}_j$ are the errors in the bar elongations of bars $j\alpha$, $j\beta$, $j\gamma$. Since $\Delta\bar{u}_{ji}$ actually has some value, the right-hand side of Equation (5.10) must be modified to take this into account. Let the difference between $\bar{Y}_j$ and $\bar{Y}_i$ be defined by the vector $\bar{Y}_{ji}$. Then the terms to be added to the right-hand side of Equation (5.10) for bars $j\alpha$, $j\beta$, and $j\gamma$, respectively, are

$$- \overline{Y}_{j\alpha} \cdot \Delta\overline{u}_{j\alpha} \qquad\qquad - \Delta\overline{u}_{j\alpha} \cdot \overline{Y}_{j\alpha}$$

$$- \overline{Y}_{j\beta} \cdot \Delta\overline{u}_{j\beta} \text{ or equivalently} - \Delta\overline{u}_{j\beta} \cdot \overline{Y}_{j\beta}$$

$$- \overline{Y}_{j\gamma} \cdot \Delta\overline{u}_{j\gamma} \qquad\qquad - \Delta\overline{u}_{j\gamma} \cdot \overline{Y}_{j\gamma}$$

It is convenient to set these terms up in matrix form. As in Chapter IV, we define matrices

$$[\Delta U]_{j\alpha} = \begin{bmatrix} \Delta a^1_{j\alpha} & \Delta a^2_{j\alpha} & \Delta a^3_{j\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\Delta U]_{j\beta} = \begin{bmatrix} 0 & 0 & 0 \\ \Delta a^1_{j\beta} & \Delta a^2_{j\beta} & \Delta a^3_{j\beta} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Delta a^1_{j\gamma} & \Delta a^2_{j\gamma} & \Delta a^3_{j\gamma} \end{bmatrix}$$

in which the non-zero elements are the components of the errors in the unit vectors, and vectors

$$\{y\}_{j\alpha} = \begin{Bmatrix} y^1_{j\alpha} \\ y^2_{j\alpha} \\ y^3_{j\alpha} \end{Bmatrix}, \quad \{y\}_{j\beta} = \begin{Bmatrix} y^1_{j\beta} \\ y^2_{j\beta} \\ y^3_{j\beta} \end{Bmatrix}, \quad \{y\}_{j\gamma} = \begin{Bmatrix} y^1_{j\gamma} \\ y^2_{j\gamma} \\ y^3_{j\gamma} \end{Bmatrix}$$

With the foregoing definitions, the terms to be added to the right-hand side of Equation (5.10) are

$$[\Delta U]_{j\alpha} \{y\}_{j\alpha} + [\Delta U]_{j\beta} \{y\}_{j\beta} + [\Delta U]_{j\gamma} \{y\}_{j\gamma}$$

By defining additional matrixes $[\Delta U]_{ji}$ such that

$$[\Delta U]_{ji} = 0 \quad i \neq \alpha, \beta, \gamma\,, \quad i = 1, 2, \ldots m, m + 1, \ldots m + J,$$

and noting that $\{y\}_{jj}$ is zero, one can write the complete error equation at joint j as

$$\sum_i [G]_{ji} \{\Delta y\}_i = - \{\Delta d\}_j - \sum_i [\Delta U]_{ji} \{y\}_{ji} \tag{5.11}$$

The error equation for the entire system is derived by extending Equation (5.11) to include every joint in the truss. Thus, we obtain

$$[G] \{\Delta y\} = - \{\Delta d\} - [\Delta U] \{\eta\}, \tag{5.12}$$

where $[\Delta U]$ is a matrix of order 3J by 9J which is formed from the $[\Delta U]_{ji}$. In expanded form the product $[\Delta U] \{\eta\}$ is

$$\begin{bmatrix}
[\Delta U]_{1\alpha} & [\Delta U]_{\alpha\beta} & [\Delta U]_{1\gamma} & [0] & [0] & [0] & \cdots & & & [0] \\
[0] & [0] & [0] & [\Delta U]_{2\alpha} & [\Delta U]_{2\beta} & [\Delta U]_{2\gamma} & & & & \cdot \\
 & & & & & & & & & \cdot \\
 & & & & & & & & & \vdots \\
 & & & & & & & & & [0] \\
[0] & & & & & \cdots & [0] & [\Delta U]_{J\alpha} & [\Delta U]_{J\beta} & [\Delta U]_{J\gamma}
\end{bmatrix}
\begin{Bmatrix}
\{y\}_{1\alpha} \\ \{y\}_{1\beta} \\ \{y\}_{1\gamma} \\ \{y\}_{2\alpha} \\ \{y\}_{2\beta} \\ \{y\}_{2\gamma} \\ \vdots \\ \{y\}_{J\alpha} \\ \{y\}_{J\beta} \\ \{y\}_{J\gamma}
\end{Bmatrix}$$

An alternate and more useful form of Equation (5.12) is derived by noting that the individual products $[\Delta U]_{ji}\,\{y\}_{ji}$ can also be written as (for $i = \alpha$)

$$[\Delta U]_{j\alpha}\{y\}_{j\alpha} = \begin{bmatrix} y^1_{j\alpha} & y^2_{j\alpha} & y^3_{j\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta a^1_{j\alpha} \\ \Delta a^2_{j\alpha} \\ \Delta a^3_{j\alpha} \end{Bmatrix}$$

Thus, by defining matrices $[H]_{j\alpha}$ and vectors $\{\Delta a\}_{j\alpha}$, etc., in the same general way as $[\Delta U]_{j\alpha}$ and $\{y\}_{j\alpha}$ were defined, one obtains for the alternate form of Equation (5.11)

$$\sum_i [G]_{ji}\,\{\Delta y\}_i = -\,\{\Delta d\}_j - \sum_i\,[H]_{ji}\,\{\Delta a\}_{ji}, \tag{5.13}$$

and the final equation for the whole truss becomes

$$[G]\,\{\Delta y\} = -\,\{\Delta d\} - [H]\,\{\Delta u\}, \tag{5.14}$$

where $\{\Delta u\}$ is the column vector formed from the $\{\Delta a\}_{ji}$. The advantage in the formulation given by Equation (5.14) is that the components of the errors in the unit vectors appear as a column vector, and is pre-multiplied by a matrix which is constant for a given set of displacements. This makes the problem of investigating the errors for different values of the elements in $\{\Delta u\}$, if desired, relatively simple. The solution for the errors in the displacement vector is

$$\{\Delta y\} = -\,[G]^{-1}\,\{\Delta d\} - [G]^{-1}\,[H]\,\{\Delta u\} \tag{5.15}$$

The customary approach in error analysis is to investigate the upper-bound or extreme-limit error. To obtain the upper bound, one replaces all negative signs in [G] and in the product $[G]^{-1}$ [H] by positive signs. The absolute values of the maximum errors in the bar elongations and in the components of the unit vectors are used in the $\{\Delta d\}$ and $\{\Delta u\}$ vectors. Thus the extreme limits for the errors in the displacement vector are given by

$$\{|\Delta y|\} = [|G|]^{-1}\{|\Delta d|\} + \left[\left|[G]^{-1}[H]\right|\right]\{|\Delta u|\}, \tag{5.16}$$

which in view of the nature of the matrix [H] is identical to

$$\{|\Delta y|\} = [|G|]^{-1}\left[\{|\Delta d|\} + [|H|]\{|\Delta u|\}\right] \tag{5.17}$$

In Equations (5.16) and (5.17) the bars are used to signify that one is to use the absolute values of the elements in the matrices and vectors. Equation (5.16) is used when the errors in $\{\Delta d\}$ and $\{\Delta u\}$, respectively, are different for each bar in the truss. If every element in the error vector $\{\Delta d\}$ has the same absolute value $\epsilon_d > 0$, and if every element in the vector $\{\Delta u\}$ has the same value $\epsilon_u > 0$, some labor can be saved in evaluating the extreme limits of $\{\Delta y\}$. In this case the extreme-limit error is given by

$$\{|\Delta y|\} = [|G|]^{-1}\left[\{e_d\}\,\epsilon_d + \epsilon_u\,[|H|]\{e_u\}\right], \tag{5.18}$$

where $\{e_d\}$ and $\{e_u\}$ are column vectors with unity in every position. It should be noted that the only function of $\{e_d\}$ and $\{e_u\}$ is to sum the elements in the rows of the matrices. If it is further assumed that ϵ_d and ϵ_u are numerically equal, then the extreme limit becomes

$$\{|\Delta y|\} = \epsilon[|G|]^{-1}[\{e_d\} + [|H|]\{e_u\}] \tag{5.19}$$

From the definition of [H], it follows that Equation (5.19) can be reduced to

$$\{|\Delta y|\} = \epsilon[|G|]^{-1}\{\bar{\eta}\}, \tag{5.20}$$

where the three consecutive elements in $\{\bar{\eta}\}$ corresponding to any joint, j, in the truss are

$$\bar{\eta}_{j\alpha} = 1 + \sum_{\ell} |y^{\ell}_{j\alpha}|$$

$$\bar{\eta}_{j\beta} = 1 + \sum_{\ell} |y^{\ell}_{j\beta}|$$

$$\bar{\eta}_{j\gamma} = 1 + \sum_{\ell} |y^{\ell}_{j\gamma}|$$

In investigating the extreme limits one should use whichever of the three error-estimate equations is most appropriate for the particular problem. However, in most investigations one generally uses Equation (5.20), whether or not it is appropriate, because it affords an estimate with the least amount of labor. In such cases ϵ is taken to be the largest error in $\{\Delta d\}$ or $\{\Delta u\}$, as the case may be. In this application one obtains an estimate which is in excess of the more precise extreme error computed from Equation (5.17).

The extreme-limit approach provides an estimate of the inherent errors in the system. However, the errors in $\{\Delta d\}$ and $\{\Delta u\}$ are combined in the most unfavorable way, and the possibility that the errors may be at least partially self-canceling is automatically excluded. Thus, it is to be expected that the extreme-limit error $\{|\Delta y|\}$ will in general overestimate the expected error. A more realistic error estimate is made by using the methods of the theory of probability.
For the problem at hand, investigating error bounds on the basis of the theory of probability involves only slightly more of computation than that required in the investigation of the extreme errors.

Those results from the theory of probability which are pertinent to the problem at hand are summarized as follows. Let $F = F(x_1, x_2, \ldots x_n)$ be a function of n independent variables x_i. Let Δx_i be a small variation such that the second number and higher-order terms in a Taylor series expansion of F can be neglected; then

$$\Delta F = \sum_i \frac{\partial F}{\partial x_i} \Delta x_i \tag{5.21}$$

or

$$\Delta F = \sum_i k_i \Delta x_i \ , \tag{5.22}$$

where $k_i = \partial F / \partial x_i$.

In neglecting the higher-order terms, ΔF becomes a linear function of the independent variables. If the x_i are assumed to be measured quantities, and the error distribution law for Δx_i is known, it is possible to compute the standard deviation $\sigma(x_i)$, and variance, $\sigma^2(x_i)$, for errors in x_i. If the mean value of each x_i is used to calculate the function F and the partial derivatives, the standard deviation, $\sigma(F)$, for errors in F is related to the $\sigma(x_i)$ by the formula[16]

$$\sigma^2(F) = \sum_i (k_i)^2 \sigma^2(x_i) \qquad (5.23)$$

If the Δx_i are normally distributed (i.e., if the error distribution follows the normal distribution law) and the errors Δx_i are small, the errors in F are also normally distributed. It has been observed that, when i is sufficiently large, the errors in F frequently follow the normal distribution law very closely, even if the errors in the x_i do not.[16]

In applying the foregoing information to the truss-displacement problem, it is assumed that the bar elongations and the components of the unit vectors are independent variables. It is also assumed that errors in the components of the displacement vector are normally distributed. In this case the probable error is approximately 2 $\sigma/3$, there is a 95% probability that the error lies within $\pm$ 2σ, and a 99.7% probability that the error lies within $\pm$ 3σ. From the previous discussion it appears that the assumption of a normal distribution law for errors in the displacement vector is more than justified in this case since $i = 12$ J.

It should be observed that the coefficients appearing in Equation (5.23) are the square of the coefficients, which relate ΔF to the Δx_i in Equation (5.22). The foregoing relationships apply in the case of a set of simultaneous equations. However, in this case, the coefficients corresponding to the k_i are not easily obtained. To show this, consider Equation (5.8), the solution of which is

$$\{\Delta x\} = - [K]^{-1} \{\Delta b\} - [K]^{-1} [\Delta K] \{x\} \qquad (5.24)$$

To compute the variance of the solution vector one must first find the coefficients corresponding to k_i for each element of $\{\Delta b\}$ and of $\{\Delta K\}$. The coefficients for the elements in $\{\Delta b\}$ are simply the elements of $[K]^{-1}$. On the other hand, because $[\Delta K]$ is premultiplied by $[K]^{-1}$ and postmultiplied by $\{x\}$, a complete expansion of the product is required to find the coefficient for each element in $[\Delta K]$. Furthermore, the elements in $[\Delta K]$ would have to be considered in a general form to obtain the correct results.

In contrast to the more general case of a linear system, the equation for the errors in the displacement vector takes the simple form

$$\{\Delta y\} = - [G]^{-1} \{\Delta d\} - [G]^{-1}[H] \{\Delta u\}, \qquad (5.15)$$

in which the error elements in $[G]$ appear in the vector $\{\Delta u\}$. Because of this, the coefficients for the elements in $\{\Delta u\}$ are simply the elements in the product $[G]^{-1}[H]$.

Let $[G^*]$ and $[H^*]$ be matrices the elements of which are the squares of the elements in $[G]$ and $[H]$, respectively. Let the variance vectors be $\{\sigma_y^2\}$, $\{\sigma_d^2\}$, and $\{\sigma_u^2\}$. Taking into account the properties of the matrix $[H]$, one obtains the variance equation

$$\{\sigma_y^2\} = [G^*][\{\sigma_d^2\} + [H^*] \{\sigma_u^2\}] \qquad (5.25)$$

It should be noted that the labor required to calculate $\{\sigma_y^2\}$ is about the same as that required to compute the extreme-limit error by Equation (5.16).

In a problem, where the only "factual" information about the errors in the input data consists of knowing the round-off error, one must assume a distribution law to compute the standard deviations of the input data. Fox[17] assumes that the error is uniformly distributed in the interval enclosed by the round-off error in the input data. On the other hand, McCalley[18] assumes that the errors are normally distributed, and that the round-off error is twice the standard deviation.

Let ϵ designate the round-off error; then

$\sigma_F = \frac{\epsilon}{\sqrt{3}}$ by Fox's assumption

$\sigma_M = \frac{\epsilon}{2}$ by McCalley's assumption

Under the same conditions Fox's assumption will lead to a slightly greater standard deviation for errors in the displacement vector. Experience indicates that the normal distribution assumption is more realistic; however, neither the uniform or normal distribution assumption is based on factual data. Since Fox's assumption leads to a slightly greater standard deviation, it is used here.

If the round-off error for each bar elongation has the same value, ϵ_d, and the round-off error for each component of the unit vectors has the constant value ϵ_u, then

$$\{\sigma_d^2\} = \frac{1}{3}\{e_d\}\,\epsilon_d^2,$$

and

$$\{\sigma_u^2\} = \frac{1}{3}\{e_u\}\,\epsilon_u^2,$$

and Equation (5.25) reduces to

$$3\{\sigma_y^2\} = [G*]^{-1}[\{e_d\}\,\epsilon_d^2 + [H*]\,\{e_u\}\,\epsilon_u^2]. \tag{5.26}$$

If $\epsilon_d = \epsilon_u = \epsilon$ this result becomes

$$3\{\sigma_y^2\} = \epsilon^2[G*]^{-1}[\{e_d\} + [H*]\{e_u\}], \tag{5.27}$$

which reduces to

$$3\{\sigma_y^2\} = \epsilon^2[G*]^{-1}\{\omega\}, \tag{5.28}$$

where the three consecutive elements in ω corresponding to any joint, j, in the truss are

$$\omega_{j\alpha} = 1 + \sum_{\ell} y_{j\alpha}^{\ell\,2}$$

$$\omega_{j\beta} = 1 + \sum_{\ell} y_{j\beta}^{\ell\,2}$$

$$\omega_{j\gamma} = 1 + \sum_{\ell} y_{j\gamma}^{\ell\,2}$$

In the foregoing analysis it was assumed that errors in the bar elongations are independent of errors in the components of the unit vectors. If the bar deformations are due to loads on the truss, this is not true. The reason is that the same approximate components of the unit vectors, used in calculating the displacements, would also be used to calculate the bar forces, and then the bar elongations. A separate analysis can be made to take this into account. However, the result would be a special case of the following analysis in which it is considered that the loads and flexibilities of the members as well as the components of the unit vectors are represented by approximate members.

We must first find the errors in the bar forces due to errors in the applied loads and geometry. In Chapter IV it was shown that for a statically determinate truss, the bar forces are related to the loads by the equation

$$[G]^T\{p\} = - \{q\} \tag{4.32}$$

Let

$\{\Delta p\}$ = error in bar force vector

$\{\Delta q\}$ = error in load vector

$[\delta G]^T$ = error in geometry matrix

From the discussion dealing with the elements in [G] it is clear that only the independent elements in $[G]^T$ can be varied arbitrarily. It should also be noted that the error in the non-zero element g_{rs} of $[G]^T$ is the error in the element g_{sr} of [G]. To indicate this situation, we have used the symbol δ to define the errors in the geometry matrix.

Neglecting the higher-order error term $[\delta G]\{\Delta p\}$, one finds that

$$[G]^T\{\Delta p\} = - \{\Delta q\} - [\delta G]^T\{p\} \tag{5.29}$$

In Appendix B it is shown that the second product on the right-hand side of Equation (5.29) can be replaced by [M] $\{\Delta u\}$ so that Equation (5.29) becomes

$$[G]^T\{\Delta p\} = -\{\Delta q\} - [M]\{\Delta u\}, \tag{5.30}$$

where [M] is a matrix formed from the vector $\{p\}$, and $\{\Delta u\}$ is the same as in the previous discussion. The dimension of the matrix [M] is 3J x 9J. The solution of Equation (5.30) is

$$\{\Delta p\} = -[G]^{-1T}\{\Delta q\} - [G]^{-1T}[M]\{\Delta u\} \tag{5.31}$$

Let $[L] = [G]^{-1T}[M]$; then

$$\{\Delta p\} = [G]^{-1T}\{\Delta q\} - [L]\{\Delta u\} \tag{5.32}$$

By analogy with the previous error analysis, we have for the extreme-limit errors

$$\{|\Delta p|\} = [|G|]^{-1T}\{|\Delta q|\} + [|L|]\{|\Delta u|\} \tag{5.33}$$

For a uniform error, ϵ_q, in the applied loads, and a uniform error, ϵ_u, in the components of the unit vectors

$$\{|\Delta p|\} = \epsilon_q[|G|]^{-1T}\{e_q\} + \epsilon_u[|L|]\{e_u\} \tag{5.34}$$

The variance relationships for errors in $\{p\}$ are

$$\{\sigma^2_p\} = [G*]^{-1T}\{\sigma^2_q\} + [L*]\{\sigma^2_u\}, \tag{5.35}$$

and for uniform errors

$$3\{\sigma_p^2\} = \epsilon_q^2[G^*]^{-1T}\{e_q\} + \epsilon_u^2[L^*]\{e_u\} \tag{5.36}$$

The bar elongations are given by

$$\{d\} = [f]\ \{p\} \tag{4.26}$$

and the first-order error relationship is readily found to be

$$\{\Delta d\} = [\Delta f]\{p\} + [f]\{\Delta p\} \tag{5.37}$$

When Equation (5.31) is used to eliminate $\{\Delta p\}$, the result is

$$\{\Delta d\} = [\Delta f]\{p\} - [f][G]^{-1T}\{\Delta q\} - [f][G]^{-1T}[M]\{\Delta u\} \tag{5.38}$$

$[\Delta f]$ like $[f]$ is a diagonal matrix. Because of this, the product $[\Delta f]\{p\}$ can be replaced by $[P]\{\Delta f\}$ where $[P]$ is diagonal. The diagonal elements in $[P]$ are the corresponding elements of $\{p\}$ and the elements in $\{\Delta f\}$ are the corresponding diagpnal elements of $[\Delta f]$. Let

$$[N] = [f][G]^{-1T}$$

$$[R] = [N][M];$$

then the extreme-limit error is given by

$$\{|\Delta d|\} = [|P|]\{|\Delta f|\} + [|N|]\{|\Delta q|\} + [|R|]\{|\Delta u|\} \tag{5.39}$$

For uniform errors ϵ_f, ϵ_q, ϵ_u, we have

$$\{|\Delta d|\} = \epsilon_f[|P|]\{e_f\} + \epsilon_q[|N|]\{e_q\} + \epsilon_u[|R|]\{e_u\} \tag{5.40}$$

The variance relationship is

$$\{\sigma_d^2\} = [P^*]\{\sigma_f^2\} + [N^*]\{\sigma_q^2\} + [R^*]\{\sigma_u^2\}, \tag{5.41}$$

and for uniform errors

$$3\{\sigma_d^2\} = \epsilon_f^2[P^*]\{e_f\} + \epsilon_q^2[N^*]\{e_q\} + \epsilon_u^2[R^*]\{e_u\} \tag{5.42}$$

The flexibility matrix for the truss is, from Equations (4.29) and (4.30),

$$[F] = [G]^{-1}[f][G]^{-1T} \tag{5.43}$$

Combining Equations (5.38) and (5.15) to eliminate $\{\Delta d\}$, and taking into account the foregoing relationship, one gets the result

$$\{\Delta y\} = - [G]^{-1}[P]\{\Delta f\} + [F]\{\Delta q\} + \Big[[F][M] - [G]^{-1}[H]\Big]\{\Delta u\}, \tag{5.44}$$

which is correct for first-order errors. It is important to note that possible errors in the geometry of the truss have a twofold effect on the displacements. The first term in the brackets, i.e., $[F][M]\{\Delta u\}$, is the error induced by an equivalent error $[M]\{\Delta u\}$ in the loads. Similarly, the second term is the error induced by an equivalent error $[H]\{\Delta u\}$ in the bar elongations. Let

$$[T] = - [G]^{-1}[P]$$

$$[S] = [F][M] - [G]^{-1}[H];$$

then

$$\{\Delta y\} = - [T]\{\Delta f\} + [F]\{\Delta q\} + [S]\{\Delta u\} \tag{5.45}$$

The extreme-limit error is

$$\{|\Delta y|\} = [|T|]\{|\Delta f|\} + [|F|]\{|\Delta q|\} + [|S|]\{|\Delta u|\} \tag{5.46}$$

For uniform errors ϵ_f, ϵ_q, and ϵ_u

$$\{|\Delta y|\} = \epsilon_f[|T|]\{e_f\} + \epsilon_q[|F|]\{e_q\} + \epsilon_u[|S|]\{e_u\} \tag{5.47}$$

The variance equation is

$$\{\sigma_y^2\} = [T^*]\{\sigma_f^2\} + [F^*]\{\sigma_q^2\} + [S^*]\{\sigma_u^2\}, \qquad 5.48)$$

which for uniform errors becomes

$$3\{\sigma_y^2\} = \epsilon_f^2[T^*]\{e_f\} + \epsilon_q^2[F^*]\{e_q\} + \epsilon_u^2[S^*]\{e_u\} \tag{5.49}$$

By setting $\epsilon_p = \epsilon_q = 0$, one obtains the result for the inherent errors when the components of the unit vectors are represented by approximate numbers and the elongations are due to applied loads.

All the formulas for estimating the inherent errors use the inverse of the matrix [G]. In a practical application inversion of the matrix may not be warranted. When this situation occurs in the more general case of a system of simultaneous equations, Miline's method(19) is often used to obtain an estimate of the bounds for the inherent errors. In the Milne procedure each element in the right-hand side of the error equation, Equation (5.7), is replaced by its maximum posible value, that is, in Equation (5.7) the term

$$\{\Delta b\} - [\Delta K]\{x\}$$

is replaced by

$$\{\epsilon\} = \{|\Delta b|\} + [\,|\Delta K|\,]\,\{|x|\}$$

The system of equations

$$[K]\{\Delta x\} = \{\epsilon\} \tag{5.50}$$

is then solved as a new problem. The only difference is that in solving these equations the signs of the elements in the $\{\epsilon\}$ vector are adjusted so that, in the solution process, all subtractions are replaced by additions (or vice versa). Solution of the equations yields the inherent error bounds. Solution for the error bounds in this way is relatively simple; however, the bounds so obtained are usually greater than the more precise bounds which use the inverse of the matrix [K].

The feature of the Milne process which makes its use desirable is that solution of the main problem and solution for the errors can be obtained simultaneously.

Milne's procedure can always be used to estimate bounds for the errors in the truss-displacement problem. However, the advantage of solving for the displacements by the methods described in Chapter II and estimating the errors simultaneously is realized only in the case of a simple truss which is pin-connected to its foundation. In general the process of computing relative and corrective displacements cannot be used in estimating error bounds. Thus, with the exception of the foregoing special case, one would have to solve the error equations for the truss, that is, the system of equations

$$[G]\{\Delta y\} = \{\epsilon\}, \tag{5.51}$$

by one of the standard methods such as the Gauss-Doolittle method, Crout's method, etc.

Example

The simple plane truss shown in Figure 7 is used to provide a numerical example of the error calculations. The plane truss is used to avoid a large amount of arithmetic which adds nothing to an understanding of the ideas presented in this chapter or Chapter IV. For a plane truss all quantities in the direction of $\bar{e}_3$ are zero, and the order of the matrices involved are correspondingly reduced. In this example it is assumed that $\{\Delta q\}$ and $[\Delta f] = 0$. The dimensions of the truss are such that the elements in the matrix [G] are exact numbers. However, for illustrative purposes it is assumed that the elements are subject to small errors. For this truss

$$\{q\}^T = [q_3^1 \quad q_3^2 \quad q_4^1 \quad q_4^2 \quad q_5^1 \quad q_5^2]$$

$$= [0 \quad -24 \quad 0 \quad 0 \quad 0 \quad -24] \text{ kips}$$

$$[f] = 16 \times 10^{-4} [I] \text{ in kip}^{-1}$$

$$[G] = \frac{1}{5} \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 0 \\ -4 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 4 & 3 & -4 & -3 \end{bmatrix}$$

$$[G]^{-1} = \frac{1}{3} \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 4 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 4 & -5 & 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 & -3 & 0 \\ 8 & -5 & -4 & 3 & 4 & -5 \end{bmatrix}$$

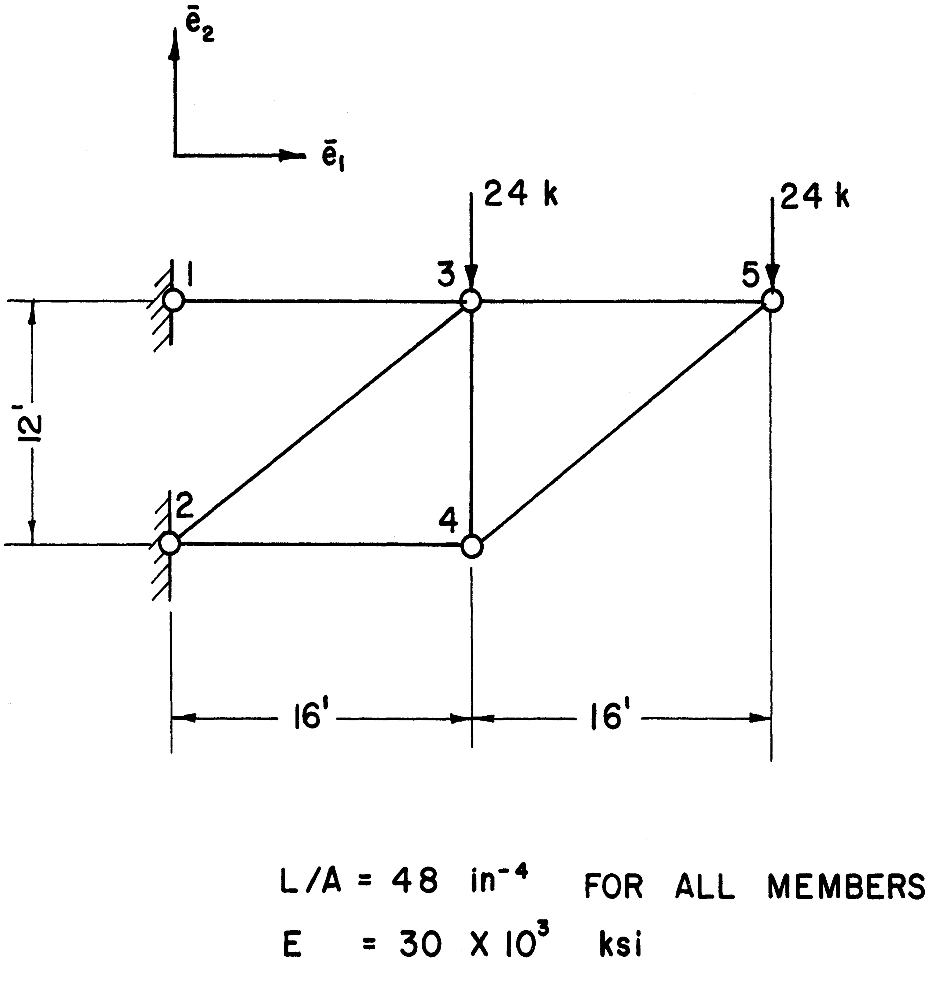

Figure 7. Plane Truss.

$$[F] = \frac{16x10^{-4}}{9}\begin{bmatrix} 9 & -12 & 0 & -12 & 9 & -24 \\ -12 & 41 & 0 & 41 & -12 & 57 \\ 0 & 0 & 9 & 0 & 0 & 12 \\ -12 & 41 & 0 & 50 & -12 & 66 \\ 9 & -12 & 0 & -12 & 18 & -36 \\ -24 & 57 & 12 & 66 & -36 & 155 \end{bmatrix} \text{ in kip}^{-1}$$

Equation (4.33) yields for the bar forces

$$\{P\}^T = [p_{31} \quad p_{32} \quad p_{42} \quad p_{43} \quad p_{53} \quad p_{54}]$$

$$= [96 \quad -80 \quad -32 \quad 24 \quad 32 \quad -40] \text{ kip}$$

The bar elongation vector is

$$\{d\}^T = [d_{31} \quad d_{32} \quad d_{42} \quad d_{43} \quad d_{53} \quad d_{54}]$$

$$= 16x10^{-4} [96 \quad -80 \quad -32 \quad 24 \quad 32 \quad -40] \text{ in}$$

Substituting $[G]^{-1}$ and $\{d\}$ in Equation (4.21) or $[F]$ and $\{q\}$ in Equation (4.28), one gets

$$\{y\}^T = [y_3^1 \quad y_3^2 \quad y_4^1 \quad y_4^2 \quad y_5^1 \quad y_5^2]$$

$$= \frac{16x10^{-4}}{3} [288 \quad -784 \quad -96 \quad -856 \quad 288 \quad -1696] \text{ in}$$

The vector $\{\Delta u\}$ is

$$\{\Delta u\}^T = [\Delta a_{31}^1 \quad \Delta a_{31}^2 \quad \Delta a_{32}^1 \quad \Delta a_{32}^2 \quad \Delta a_{42}^1 \quad \Delta a_{42}^2$$

$$\Delta a_{43}^1 \quad \Delta a_{43}^2 \quad \Delta a_{53}^1 \quad \Delta a_{53}^2 \quad \Delta a_{54}^1 \quad \Delta a_{54}^2]$$

The matrix [M] is formed from the bar forces and in this example (see Appendix B for a derivation of [M])

$$[M] = \begin{bmatrix} 96 & 0 & -80 & 0 & 0 & 0 & -24 & 0 & -32 & 0 & 0 & 0 \\ 0 & 96 & 0 & -80 & 0 & 0 & 0 & -24 & 0 & -32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -32 & 0 & 24 & 0 & 0 & 0 & 40 & 0 \\ 0 & 0 & 0 & 0 & 0 & -32 & 0 & 24 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & -40 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & -40 \end{bmatrix} \text{ kip}$$

Premultiplying [M] by $[G]^{-1T}$, one gets

$$[L] = \frac{1}{3}\begin{bmatrix} -288 & 384 & 240 & -320 & 0 & -128 & 72 & 0 & 0 & 128 & 120 & -160 \\ 0 & -480 & 0 & 400 & 0 & 160 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 96 & 0 & -72 & 0 & 0 & -128 & -120 & 160 \\ 0 & 0 & 0 & 0 & 0 & -96 & 0 & 72 & 0 & 96 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -96 & 128 & 120 & -160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -160 & 0 & 200 \end{bmatrix} \text{kip}$$

For a uniform error ϵ_u Equation (5.34) yields for the upper bound of the error in the bar forces

$$\{|\Delta p|\}^T = [\,|\Delta p_{31}| \;\; |\Delta p_{32}| \;\; |\Delta p_{42}| \;\; |\Delta p_{43}| \;\; |\Delta p_{53}| \;\; |\Delta p_{54}|\,]$$

$$= [613 \quad 347 \quad 192 \quad 88 \quad 168 \quad 120]\ \epsilon_u \text{ kips}$$

From Equation (5.36) the variance of $\{p\}$ is found to be

$$3\{\sigma_p^2\} = \frac{10^4}{27}[46.84 \quad 41.60 \quad 70.78 \quad 23.62 \quad 65.60 \quad 65.60]\epsilon_u^2 \text{ kip}^2$$

There is a 95.5% probability that $\{\Delta p\}$ lies within $2\{\sigma_p\}$, which is readily found to be

$$\{\Delta p\}_{2\sigma}^T = 2\{\sigma_p\}^T = [263 \quad 248 \quad 103 \quad 59 \quad 99 \quad 99]\ \epsilon_u \text{ kip}$$

Since [f] is just the identity matrix multiplied by 16×10^{-4} in kip^{-1} the matrix [R] is

$$[R] = 16 \times 10^{-4}\ [G]^{-1T}[M] = 16 \times 10^{-4}\ [L] \text{ in kip,}$$

so that

$$\{|\Delta d|\} = 16 \times 10^{-4}\ \{|\Delta p|\} \text{ in,}$$

and

$$\{\Delta d\}_{2\sigma} = 16 \times 10^{-4}\{\Delta p\}_{2\sigma} \text{ in}$$

The matrix [H] is

$$[H] = \frac{16\text{x}10^{-4}}{3}\begin{bmatrix} 288 & -784 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 288 & -784 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -96 & -856 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -384 & -72 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -912 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 384 & -840 \end{bmatrix} \text{in}$$

Premultiplying [H] by $[G]^{-1}$ one gets

$$[G]^{-1}[H] = \frac{16\text{x}10^{-4}}{9}\begin{bmatrix} -864 & 2352 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1152 & -3136 & -1440 & 3920 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 288 & 2568 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1152 & -3136 & -1440 & 3920 & 0 & 0 & -1152 & -216 & 0 & 0 & 0 & 0 \\ -864 & 2352 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2736 & 0 & 0 \\ 2304 & -6272 & -1440 & 3920 & 384 & 3424 & -1152 & -216 & 0 & -3648 & -1920 & 4200 \end{bmatrix} \text{in}$$

Premultiplying [M] by [F], one gets

$$[F][M] = \frac{16\text{x}10^{-4}}{9}\begin{bmatrix} 864 & -1152 & -720 & 960 & 0 & 384 & -216 & 0 & 0 & -384 & -360 & 480 \\ -1152 & 3936 & 960 & -3280 & 0 & -1312 & 288 & 0 & 0 & 512 & 480 & -640 \\ 0 & 0 & 0 & 0 & -288 & 0 & 216 & 0 & 0 & 384 & 360 & -480 \\ -1152 & 3936 & 960 & -3280 & 0 & -1600 & 288 & 216 & 0 & 800 & 480 & -640 \\ 864 & -1152 & -720 & 960 & 0 & 384 & -216 & 0 & 288 & -768 & -720 & 960 \\ -2304 & 5472 & 1920 & -4560 & -384 & -2112 & 864 & 216 & -384 & 3136 & 1920 & -3560 \end{bmatrix} \text{in}$$

The matrix [S] is the difference between the foregoing products. Thus

$$
[S] = \frac{16\text{x}10^{-4}}{9}\begin{bmatrix}
1728 & -3504 & -720 & 960 & 0 & 384 & -216 & 0 & 0 & -384 & -360 & 480 \\
-2304 & 7072 & 2400 & -7200 & 0 & -1312 & 288 & 0 & 0 & 512 & 480 & -640 \\
0 & 0 & 0 & 0 & -576 & -2568 & 216 & 0 & 0 & 384 & 360 & -480 \\
-2304 & 7072 & 2400 & -7200 & 0 & -1600 & 1440 & 432 & 0 & 800 & 480 & -640 \\
1728 & -3504 & -720 & 960 & 0 & 384 & -216 & 0 & 288 & -3504 & -720 & 960 \\
-4608 & 11744 & 3360 & -8480 & 768 & -5536 & 2016 & 432 & -384 & 6784 & 3840 & 7760
\end{bmatrix} \text{ in}
$$

From Equation (5.47) the extreme-limit error for the displacement vector is calculated to be

$$\{|\Delta y|\}^T = \frac{16\text{x}10^{-4}}{3} \; [2912 \quad 7402 \quad 1528 \quad 8122 \quad 4328 \quad 18{,}904]\,\epsilon_u \;\text{ in,}$$

and from Equation (5.48)

$$3\{\sigma_y^2\}^T = \frac{256\text{x}10^{-2}}{81}\,[17.406 \quad 115.628 \quad 7.480 \quad 119.022 \quad 30.699 \quad 398.976]\,\epsilon_u^2 \;\text{ in}^2,$$

from which

$$\{\Delta y\}^T_{2\sigma} = \frac{16\text{x}10^{-4}}{3} \; [1606 \quad 4139 \quad 1053 \quad 4199 \quad 2133 \quad 7688]\; \epsilon_u \text{ in.}$$

For this particular truss the calculations to be carried out in Milne's method can be stated in terms of matrices. Let $\{\Delta p\}_M$, $\{\Delta d\}_M$, $\{\Delta y\}_M$ denote the errors calculated by Milne's procedure, then

$$\{\Delta p\}_M = [\,|G|\,]^{-1T}[\,|M|\,]\{e_u\}\epsilon_u$$

$$\{\Delta d\}_M = [\,|f|\,]\{\Delta p\}_M$$

$$\{\Delta y\}_M = [\,|G|\,]^{-1}[\{\Delta d\}_M + [\,|H|\,]\{\epsilon_u\}\;\epsilon_u]$$

for this particular truss. Substituting numerical values into these equations, one gets

$$\{\Delta p\}_M^T = [933 \quad 667 \quad 192 \quad 168 \quad 120]\ \epsilon_u \ \text{kips}$$

$$\{\Delta d\}_M^T = 16 \times 10^{-4} \{\Delta p\}_M \ \epsilon_u \ \text{in}$$

$$\{\Delta y\}_M^T = \frac{16 \times 10^{-4}}{3} [3872 \quad 10{,}281 \quad 1528 \quad 11{,}242 \quad 5288 \quad 22{,}970]\ \epsilon_u \ \text{in}$$

With the aid of a desk calculator there is no difficulty in calculating the components of the unit vectors to four-decimal-place accuracy. With four place accuracy, $\epsilon_u \leq 5 \times 10^{-5}$. Using this value for ϵ_u, one obtains the error estimates displayed in lines 2 to 4 in Table I. A comparison of the entries in Table I shows that the errors are completely negligible compared to the displacements.

TABLE I

ESTIMATED DISPLACEMENT ERRORS*, $\epsilon_u = 5 \times 10^{-5}$

	joint	3		4		5	
Line	direction	1	2	1	2	1	2
1	$\{y\}$	298.00	-784.00	-96.00	-856.00	288.00	-1696.00
2	$\{\lvert\Delta y\rvert\}$	.14	.37	.08	.41	.22	.94
3	$\{\Delta y\}_{2\sigma}$	.08	.21	.05	.21	.11	.38
4	$\{\Delta y\}_M$	.19	.51	.08	.56	.26	1.15

* All entries to be multiplied by $\frac{16 \times 10^{-4}}{3}$ in.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The problem of analyzing displacements in space trusses has been considered in this study.

A vector method for calculating displacements was presented in Chapter II. The method was formulated on the assumptions that the displacements are small, and that the axial deformations of the truss members were given. With the assumption of small displacements, the methods lead to an exact solution of the problem. Compound and complex as well as simple trusses were considered.

Solution for the displacements is obtained in a step-by-step fashion. For the most part only three equations in three unknowns must be solved in each step. Application of the method was demonstrated by examples which, although simple, are by no means trivial and clearly demonstrate that the proposed method is practical.

It was noted that solution of the three equations was not difficult. However, for computation it was considered desirable to have a general solution or "plug-in" type of formula. Formulas for three methods of solution were given in Chapter III.

A step-by-step solution for the displacements is possible only because the displacements, considered as a whole, are governed by a special system of simultaneous equations. When the displacements are needed for several sets of bar elongations, it is sometimes worthwhile to find a general solution of the equations. The equations were derived in a matrix formulation of the problem in Chapter IV. Practical methods for inverting the matrix of coefficients were given.

From the matrix formulation it was shown that the stress and displacement analysis problems are virtually identical. This result, combined with the experienced gained in working the example problems, has convinced the writer that, in general, the labor required to calculate the displacements at every joint in a truss by the proposed method is about the same as that required to calculate the displacement at one joint in one direction by the classical methods.

In general, an exact solution for the displacements is possible only in principle. This is true not only for the methods presented in this work but also for any of the so-called exact methods.

Inherent errors were considered in detail in Chapter V. Formulas for estimating the inherent errors under different assumptions were given. It was shown that estimation of the inherent errors on the assumption of a uniform or normal probability distribution does not demand much computational work besides that needed to estimate the extreme limit errors. It was pointed out that Milne's method of error analysis,[19] which overestimates the errors, can be used to advantage only in the case of a simple truss, which is pin-connected to its foundation. A simple method for estimating the error bounds for other types of trusses could not be found.

A simple example was chosen to illustrate computation of the inherent errors, in which, the errors were found to be completely negligible. The example does not permit any general conclusions relative to the errors which may be expected in a general case, but it shows that a considerable amount of numerical work is required to obtain a close estimate of the inherent errors.

APPENDIX A

EXAMPLES

Example I. A simple truss as shown in Figure 1-A is loaded at joint 1 with a vertical load of 10 kips. Joint 0 is fixed in the horizontal plane. Joints 2 and 4 are constrained to remain in the horizontal plane. Joint 4 is also constrained to remain on a line at an angle of 45° to $\bar{e}_1$. Find the displacement of all the joints if $\frac{L}{AE} = 10^{-4}$ ft kip^{-1} for all members of the truss.

The basic data for the truss is given in Table I-A. The bars of the truss are identified in Column 2. The components of the length vector for each bar are given in Columns 3 to 5. The origin of each length vector is given by the first of the two indices required to identify the length vector. Columns 7 to 9 contain the components of the unit vectors. The entries in Column 10 are the bar forces, and, when multiplied by 10^{-4}, are the corresponding bar elongations. Columns 12 to 14 contain the components of the position vector to the joints identified in Column 11.

To require both a translational and rotational correction, it was assumed that

(1) joints 4, 8, and 3 are constrained to remain in the plane defined by these joints in the undistorted truss;

(2) joint 3 is constrained to remain on the line connecting joints 3 and 4;

(3) joint 4 is fixed.

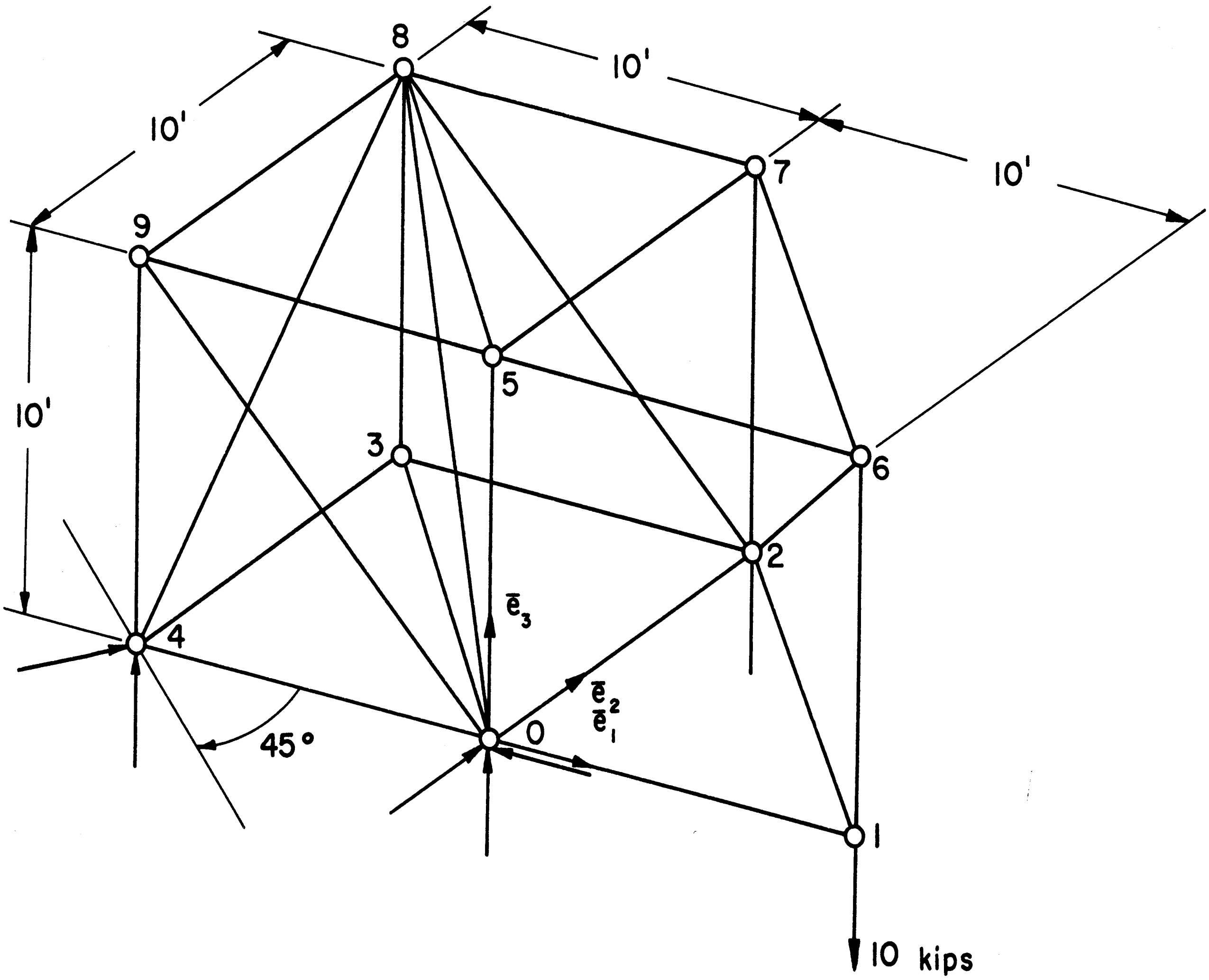

Figure 1-A. Truss for Example I.

TABLE I-A

BASIC DATA FOR EXAMPLE I

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
	Bar	$\bar{L}_{ji}$	(ft)		L_{ji}		$\bar{u}_{ji}$		d_{ji}		$\bar{R}_i$	(ft)	
Line	j,i	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	ft	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	10^{-4}ft	i	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
1	0,1	10	0	0	10	1	0	0	0	1	10	0	0
2	0,2	0	10	0	10	0	1	0	10	2	0	10	0
3	0,3	-10	10	0	$10\sqrt{2}$	$-\sqrt{2}/2$	$\sqrt{2}/2$	0	$20\sqrt{2}$	3	-10	10	0
4	0,4	-10	0	0	10	-1	0	0	0	4	-10	0	0
5	0,5	0	0	10	10	0	0	1	0	5	0	0	10
6	0,8	-10	10	10	$10\sqrt{3}$	$-\sqrt{3}/3$	$\sqrt{3}/3$	$\sqrt{3}/3$	$-30\sqrt{3}$	6	10	0	10
7	0,9	-10	0	10	$10\sqrt{2}$	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	$10\sqrt{2}$	7	0	10	10
8	1,2	-10	10	0	$10\sqrt{2}$	$-\sqrt{2}/2$	$\sqrt{2}/2$	0	0	8	-10	10	10
9	1,6	0	0	10	10	0	0	1	10	9	-10	0	10
10	2,3	-10	0	0	10	-1	0	0	-20				
11	2,6	10	-10	10	$10\sqrt{3}$	$\sqrt{3}/3$	$-\sqrt{3}/3$	$\sqrt{3}/3$	$-10\sqrt{3}$				
12	2,7	0	0	10	10	0	0	1	0				
13	2,8	-10	0	10	$10\sqrt{2}$	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	$10\sqrt{2}$				
14	3,4	0	-10	0	10	0	-1	0	-20				
15	3,8	0	0	10	10	0	0	1	0				
16	4,8	0	10	10	$10\sqrt{2}$	0	$\sqrt{2}/2$	$\sqrt{2}/2$	$20\sqrt{2}$				
17	4,9	0	0	10	10	0	0	1	-10				
18	5,6	10	0	0	10	1	0	0	0				
19	5,7	0	10	0	10	0	1	0	-10				
20	5,8	-10	10	0	$10\sqrt{2}$	$-\sqrt{2}/2$	$\sqrt{2}/2$	0	$10\sqrt{2}$				
21	5,9	-10	0	0	10	-1	0	0	-10				
22	6,7	-10	10	0	$10\sqrt{2}$	$-\sqrt{2}/2$	$\sqrt{2}/2$	0	$10\sqrt{2}$				
23	7,8	-10	0	0	10	-1	0	0	10				
24	8,9	0	-10	0	10	0	-1	0	0				

Relative Displacements

Since joint 4 is assumed fixed, $\overline{Z}_4 = 0$. To find the relative displacement of joint 3, set $\ell = 3$, $k = 4$ in Equation (2.7). Substituting d_{34} and $\overline{u}_{34}$ in Equation (2.7), one gets

$$\overline{Z}_3 = - (-20)(-\overline{e}_2)10^{-4}\text{ft} = (-20\overline{e}_2)10^{-4}\text{ft}$$

For the relative displacement of joint 8, one sets $m = 8$, $\ell = 3$, and $k = 4$ in Equation (2.9).

$$\begin{aligned}
d_{mk} &= d_{84} = d_{48} = 20\sqrt{2}\ (10^{-4}\text{ft}) \\
d_{m\ell} &= d_{83} = d_{38} = 0 \\
\overline{u}_{m\ell} &= \overline{u}_{83} = -(\overline{e}_3) \\
\overline{u}_{mk} &= \overline{u}_{84} = - [(\sqrt{2}/2)\overline{e}_2 + (\sqrt{2}/2)\overline{e}_3] \\
\overline{u}_{83} \cdot \overline{u}_{84} &= \sqrt{2}/2 \\
\overline{Z}_3 \cdot \overline{u}_{83} &= (-20\overline{e}_2) \cdot (\overline{e}_3)10^{-4}\text{ft} = 0
\end{aligned}$$

Substituting the foregoing numerical values in Equation (2.9), one gets

$$\begin{aligned}
z_{84} + (\sqrt{2}/2)z_{83} &= - 20\sqrt{2}\ (10^{-4}\text{ft}) \\
(\sqrt{2}/2)z_{84} + z_{83} &= 0
\end{aligned}$$

The solution is

$$\begin{aligned}
z_{84} &= - 40\sqrt{2}\ (10^{-4}\text{ft}) \\
z_{83} &= 40\ (10^{-4}\text{ft})
\end{aligned}$$

From Equation (2.8), one gets

$$\begin{aligned}
\overline{Z}_8 &= z_{84}\overline{u}_{84} + z_{83}\overline{u}_{83} \\
\overline{Z}_8 &= - 40\sqrt{2}\ [-(\sqrt{2}/2)\overline{e}_2 - (\sqrt{2}/2)\overline{e}_3\ ]\ (10^{-4}\text{ft}) \\
&\qquad + 40(-\overline{e}_3)(10^{-4}\text{ft}) \\
\overline{Z}_8 &= (40\overline{e}_2)(10^{-4}\text{ft})
\end{aligned}$$

Using the relative displacements at joints 4, 3, and 8, one can now determine the relative displacement at joint 0. For the relative displacement at joint 0, Equation (2.4) becomes

$$\bar{Z}_0 \cdot \bar{u}_{04} = \bar{Z}_0 \cdot \bar{u}_{04} - d_{04} = b_{04}$$
$$\bar{Z}_0 \cdot \bar{u}_{03} = \bar{Z}_3 \cdot \bar{u}_{03} - d_{03} = b_{03}$$
$$\bar{Z}_0 \cdot \bar{u}_{08} = \bar{Z}_8 \cdot \bar{u}_{08} - d_{08} = b_{08}$$

Inserting appropriate quantities from Table I-A one obtains

$$b_{04} = 0,\ b_{03} = (-10\alpha\sqrt{2} - 20\alpha\sqrt{2})10^{-4}\text{ft} = -30\alpha\sqrt{2}\ (10^{-4}\text{ft})$$
$$b_{08} = \left(\frac{40\alpha\sqrt{3}}{3} + 30\alpha\sqrt{3}\right)10^{-4}\text{ft} = \frac{130\alpha\sqrt{3}}{3}\ (10^{-4}\text{ft})$$

In the solution for $\bar{Z}_0$, take

$$\bar{u}_{04} = \bar{u}_1,\ \bar{u}_{03} = \bar{u}_2,\ \bar{u}_{08} = \bar{u}_3,\ \bar{Z}_0 = \bar{Z}$$
$$b_{04} = b_1,\ b_{03} = b_2,\ b_{08} = b_3$$

For the direction solution

$$\bar{Z} = z_1\bar{e}_1 + z_2\bar{e}_2 + z_3\bar{e}_3$$

From Equation (3.5), the matrix [A] is found to be

$$[A] = \begin{bmatrix} -1 & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{bmatrix},\ \text{and}\ |A| = -\sqrt{6}/6$$

$$[A]^{-1} = \sqrt{6}\begin{bmatrix} \sqrt{6}/6 & 0 & 0 \\ \sqrt{6}/6 & -\sqrt{3}/3 & 0 \\ 0 & \sqrt{3}/3 & -\sqrt{2}/2 \end{bmatrix}$$

The solution for $\{z\}$ is found from Equation (3.8):

$$\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} = \sqrt{6}\begin{bmatrix} \sqrt{6}/6 & 0 & 0 \\ \sqrt{6}/6 & -\sqrt{3}/3 & 0 \\ 0 & \sqrt{3}/3 & -\sqrt{2}/2 \end{bmatrix}\begin{Bmatrix} 0 \\ -30\sqrt{2} \\ \frac{130\sqrt{3}}{3} \end{Bmatrix} (10^{-4}\text{ft})$$

$$= \begin{Bmatrix} 0 \\ -60 \\ 190 \end{Bmatrix} \quad (10^{-4}\text{ft})$$

Thus

$$\bar{Z} = \bar{Z}_0 = (-60\bar{e}_2 + 190\bar{e}_3)(10^{-4}\text{ft})$$

If, instead of a direct solution, one took

$$\bar{Z} = w_1\bar{u}_1 + w_2\bar{u}_2 + w_3\bar{u}_3 \ ,$$

then in Equation (3.11)

$$[L] = \begin{bmatrix} 1 & \sqrt{2}/2 & \sqrt{3}/3 \\ \sqrt{2}/2 & 1 & \sqrt{6}/3 \\ \sqrt{3}/3 & \sqrt{6}/3 & 1 \end{bmatrix} \quad ,$$

and from Equation (3.14), one gets

$$[L]^{-1} = 6 \begin{bmatrix} 1/3 & -\sqrt{2}/6 & 0 \\ -\sqrt{2}/6 & 2/3 & -\sqrt{6}/6 \\ 0 & -\sqrt{6}/6 & 1/2 \end{bmatrix}$$

From Equation (3.13) the solution for $\{w\}$ is found to be

$$\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = 6 \begin{bmatrix} 1/3 & -\sqrt{2}/6 & 0 \\ -\sqrt{2}/6 & 2/3 & -\sqrt{6}/6 \\ 0 & -\sqrt{6}/6 & 1/2 \end{bmatrix} \begin{Bmatrix} 0 \\ -30\sqrt{2} \\ \dfrac{130\sqrt{3}}{3} \end{Bmatrix} \quad (10^{-4}\text{ft})$$

$$= \begin{Bmatrix} 60 \\ -250\sqrt{2} \\ 190\sqrt{3} \end{Bmatrix} \quad (10^{-4}\text{ft})$$

Thus,

$$\bar{Z}_0 = (60\bar{u}_1 - 250\sqrt{2}\,\bar{u}_2 + 190\sqrt{3}\,\bar{u}_3)(10^{-4}\text{ft})$$

Since

$$\bar{u}_1 = \bar{u}_{04} = -\bar{e}_1$$

$$\bar{u}_2 = \bar{u}_{03} = -\sqrt{2}/2\,\bar{e}_1 + \sqrt{2}/2\,\bar{e}_2$$

$$\bar{u}_3 = \bar{u}_{08} = -\sqrt{3}/3\,\bar{e}_1 + \sqrt{3}/3\,\bar{e}_2 + \sqrt{3}/3\,\bar{e}_3 ,$$

the foregoing result reduces to

$$\bar{Z}_0 = (-60\,\bar{e}_2 + 190\,\bar{e}_3)(10^{-4}\text{ft}) ,$$

which is identical to the previous value calculated for $\bar{Z}_0$.

Continued application of one of the methods used to obtain $\bar{Z}_0$ yields the remaining relative displacements. Columns 3 to 5 of Table II-A contain the components of the relative displacement for each joint, which is identified in Column 2. Column 1 of the table shows the order in which the relative displacements were calculated.

Corrective Displacements

$\overline{Z}_0$, $\overline{Z}_2$, and $\overline{Z}_4$ do not satisfy the conditions of constraint at the corresponding joints. Thus, corrective displacements are required. Since joint 0 is fixed, Equation (2.14) yields

$$\overline{C}_0 = - \overline{Z}_0 = (+60\ \overline{e}_2 - 190\ \overline{e}_3)(10^{-4}\text{ft})$$

Since joints 0, 4, and 2 remain in the horizontal plane

$$\overline{n} = \pm (\overline{e}_3)$$

For joint 4, the fixed direction is $\pm [(\sqrt{2}/2)\overline{e}_1 - (\sqrt{2}/2)\overline{e}_2]$ so that

$$\overline{u} = \pm [(\sqrt{2}/2)\overline{e}_1 + (\sqrt{2}/2)\overline{e}_2]$$

Using the positive signs on $\overline{n}$ and $\overline{u}$

$$\overline{R}_4 \times \overline{n} = (+10\overline{e}_2)\text{ft.}, \quad (\overline{Z}_0 - \overline{Z}_4)\cdot \overline{n} = 190\ (10^{-4}\text{ft})$$

$$\overline{R}_4 \times \overline{u} = (-5\sqrt{2}\overline{e}_3)\text{ft.}, \quad (\overline{Z}_0 - \overline{Z}_4)\cdot \overline{u} = -30\sqrt{2}\ (10^{-4}\text{ft})$$

$$\overline{R}_2 \times \overline{n} = (10\overline{e}_1)\text{ft.}, \quad (\overline{Z}_0 - \overline{Z}_2)\cdot \overline{n} = 230\ (10^{-4}\text{ft})$$

Taking

$$\overline{H} = h_1\overline{e}_1 + h_2\overline{e}_2 + h_3\overline{e}_3$$

and putting numerical values in Equation (2.16), one gets

$$10h_2 = 190\ (10^{-4}), \qquad h_2 = 19\ (10^{-4})$$

$$-5\sqrt{2}h_3 = -30\sqrt{2}\ (10^{-4}), \quad h_3 = 6\ (10^{-4})$$

$$10h_1 = 230\ (10^{-4}), \qquad h_1 = 23\ (10^{-4})$$

TABLE II-A

DISPLACEMENTS FOR EXAMPLE I

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
		$\bar{Z}_i$ $(10^{-4}ft)$			$\bar{H} \times \bar{R}_i$ $(10^{-4}ft)$			$\bar{Y}_i = [\bar{Z}_i + \bar{C}_0 + \bar{H} \times \bar{R}_i](10^{-4}ft)$		
	i	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
4	0	0	- 60	190	0	0	0	0	0	0
10	1	0	- 30	-300	0	60	-190	0	90	-680
7	2	- 20	- 50	- 40	- 60	0	230	- 80	10	0
2	3	0	- 20	0	- 60	- 60	420	- 60	- 20	230
1	4	0	0	0	0	- 60	190	0	0	0
6	5	-230	-210	190	190	-230	0	- 40	-380	0
9	6	-230	-480	-290	190	-170	-190	- 40	-590	-670
8	7	10	-220	- 40	130	-230	230	140	-390	0
3	8	0	40	0	130	-290	420	130	-190	230
5	9	-220	40	- 10	190	-290	190	- 30	-190	- 10

Thus,

$$\bar{H} = (23\bar{e}_1 + 19\bar{e}_2 + 6\bar{e}_3)\ (10^{-4})$$

The corrections due to $\bar{H}$ are entered in Columns 6 to 8 of Table II-A. Columns 9 to 11 contain the components of the displacements for each joint in the truss. The vertical displacement at joint 1 is -680 (10^{-4}ft) $\bar{e}_3$, which is what one would obtain by using virtual work.

<u>Example II</u>. The compound truss shown in Figure 2-Aa is loaded at joint 9 with a vertical load of 10 kips. Take $\frac{L}{AE} = 10^{-4}$ft kip^{-1} for all members. Find the displacement at all joints using the method for compound trusses. Check the results by using the method for complex trusses. The basic data for the truss is given in Table III-A. Columns 12 to 14 of Table III-A contain the position vector to the joints indicated in Column 11, when the pole is at joint 7.

The displacements at joints 7 and 4 are found by the method for simple trusses. It is not possible to determine immediately the displacement at joint 6. Therefore the displacement at each of the remaining joints are obtained by using the method outlined for compound trusses.

The truss is separated at joints 4, 7, and 6 as indicated in Figure 6-Ab and relative displacements are computed for the right-hand section of the truss. Columns 2 to 4 of Table IV-A contain the components of the relative displacements for the joints indicated in Column 1. Joints 8, 9, and 11 were used to start the solution for the relative displacements. The procedure is identical to that used in Example I.

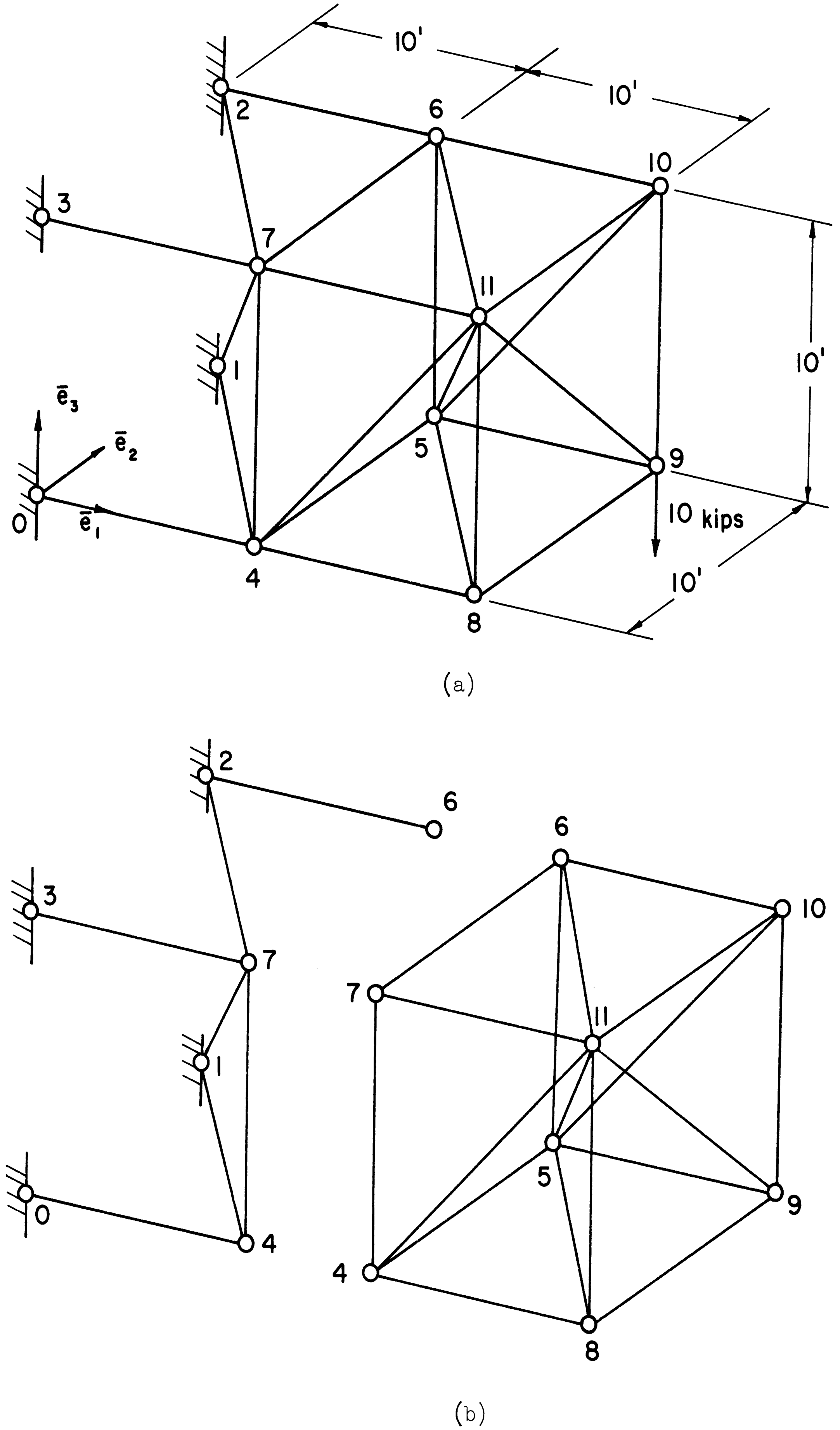

Figure 2-A. Truss for Example II.

TABLE III-A

BASIC DATA FOR EXAMPLE II

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
	Bar	$\bar{L}_{ji}$ (ft)			L_{ji}		$\bar{u}_{ji}$		d_{ji}		$\bar{R}_s$ (ft)		
Line	j,i	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	ft	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	10^{-4}ft	s	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
1	0,4	10	0	0	10	1	0	0	-20	4	0	0	-10
2	1,4	10	-10	0	$10\sqrt{2}$	$\sqrt{2}/2$	$-\sqrt{2}/2$	0	$10\sqrt{2}$	5	0	10	-10
3	1,7	10	-10	10	$10\sqrt{3}$	$\sqrt{3}/3$	$-\sqrt{3}/3$	$\sqrt{3}/3$	$-10\sqrt{3}$	6	0	10	0
4	2,6	10	0	0	10	1	0	0	0	7	0	0	0
5	2,7	10	-10	0	$10\sqrt{2}$	$\sqrt{2}/2$	$-\sqrt{2}/2$	0	0	8	10	0	-10
6	3,7	10	0	0	10	1	0	0	20	9	10	10	-10
7	4,5	0	10	0	10	0	1	0	-10	10	10	10	0
8	4,7	0	0	10	10	0	0	1	10	11	10	0	0
9	4,8	10	0	0	10	1	0	0	0				
10	4,11	10	0	10	$10\sqrt{2}$	$\sqrt{2}/2$	0	$\sqrt{2}/2$	$-10\sqrt{2}$				
11	5,6	0	0	10	10	0	0	1	0				
12	5,8	10	-10	0	$10\sqrt{2}$	$\sqrt{2}/2$	$-\sqrt{2}/2$	0	0				
13	5,9	10	0	0	10	1	0	0	0				
14	5,10	10	0	10	$10\sqrt{2}$	$\sqrt{2}/2$	0	$\sqrt{2}/2$	$-10\sqrt{2}$				
15	5,11	10	-10	10	$10\sqrt{3}$	$\sqrt{3}/3$	$-\sqrt{3}/3$	$\sqrt{3}/3$	$10\sqrt{3}$				
16	6,7	0	-10	0	10	0	-1	0	10				
17	6,10	10	0	0	10	1	0	0	10				
18	6,11	10	-10	0	$10\sqrt{2}$	$\sqrt{2}/2$	$-\sqrt{2}/2$	0	$-10\sqrt{2}$				
19	7,11	10	0	0	10	1	0	0	10				
20	8,9	0	10	0	10	0	1	0	0				
21	8,11	0	0	10	10	0	0	1	0				
22	9,10	0	0	10	10	0	0	1	10				
23	9,11	0	-10	10	$10\sqrt{2}$	0	$-\sqrt{2}/2$	$\sqrt{2}/2$	0				
24	10,11	0	-10	0	10	0	-1	0	0				

A comparison of $\overline{Y}_7$ (line 1 in Table IV-A) and $\overline{Z}_7$ indicates that corrective displacements are required. From Equation (2.19) it is readily found that

$$\overline{C}_7 = (30\overline{e}_1 + 120\overline{e}_2 - 60\overline{e}_3)\ (10^{-4})\ \text{ft}$$

TABLE IV-A

DISPLACEMENTS FOR EXAMPLE II,
TREATED AS A COMPOUND TRUSS

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	$\overline{Z}_s$ (10^{-4}ft)			$\overline{C}_s = [\overline{C}_7 + \overline{H} \times \overline{R}_s](10^{-4}\text{ft})$			$\overline{Y}_s$ (10^{-4}ft)		
s	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$
7							20	20	- 30
4							-20	-40	- 40
8	0	0	0	-20	- 90	-110	-20	-90	-110
9	0	0	0	20	- 90	-280	20	-90	-280
11	0	0	0	30	80	-110	30	80	-110
5	0	0	-30	20	- 50	-230	20	-50	-260
10	-60	0	10	70	80	-280	10	80	-270
4'	0	10	20	-20	- 50	- 60	-20	-40	- 40
6'	-70	- 90	-30	70	120	-230	0	30	-260
7'	-10	-100	30	30	120	- 60	20	20	- 30

A corrective rotation is also required in this case. The data needed to determine $\bar{H}$ from Equations (2.21) are

$$\bar{R}_6 \times \bar{u}_{62} = 10\bar{e}_3 \text{ ft}$$
$$\bar{R}_4 \times \bar{u}_{41} = (5\sqrt{2}\,\bar{e}_1 + 5\sqrt{2}\,\bar{e}_2) \text{ ft}$$
$$\bar{R}_4 \times \bar{u}_{40} = 10\bar{e}_2 \text{ ft}$$

$$\bar{Z}_6 - \bar{Y}_2 + \bar{C}_7 = (-40\bar{e}_1 + 30\bar{e}_2 - 90\bar{e}_3)\ 10^{-4}\text{ft}$$
$$\bar{Z}_4 - \bar{Y}_1 + \bar{C}_7 = (30\bar{e}_1 + 130\bar{e}_2 - 40\bar{e}_3)\ 10^{-4}\text{ft}$$
$$\bar{Z}_4 - \bar{Y}_0 + \bar{C}_7 = (30\bar{e}_1 + 130\bar{e}_2 - 40\bar{e}_3)\ 10^{-4}\text{ft}$$

$$-d_{62} - (\bar{Z}_6 - \bar{Y}_2 + \bar{C}_7)\cdot\bar{u}_{62} = (0 - 40)\ 10^{-4}\text{ft}$$
$$-d_{41} - (\bar{Z}_4 - \bar{Y}_1 + \bar{C}_7)\cdot\bar{u}_{41} = (-10\sqrt{2} - 50\sqrt{2})\ 10^{-4}\text{ft}$$
$$-d_{40} - (\bar{Z}_4 - \bar{Y}_0 + \bar{C}_7)\cdot\bar{u}_{40} = (+20 + 30)\ 10^{-4}\text{ft}$$

Taking $\bar{H} = h_1\bar{e}_1 + h_2\bar{e}_2 + h_3\bar{e}_3$ and entering numerical values in Equation (2.21), one obtains

$$10h_3 = -40 \times 10^{-4}, \quad h_3 = -4 \times 10^{-4}$$
$$5\sqrt{2}\,h_1 + 5\sqrt{2}\,h_2 = -60\sqrt{2} \times 10^{-4}, \quad h_1 = -17 \times 10^{-4}$$
$$10h_2 = 50 \times 10^{-4}, \quad h_2 = 5 \times 10^{-4}$$

Therefore,

$$\bar{H} = (-17\bar{e}_1 + 5\bar{e}_2 - 4\bar{e}_3)\ (10^{-4})$$

The correction at joint 8 due to $\bar{H}$ is

$$\bar{H} \times \bar{R}_8 = (-50\bar{e}_1 - 210\bar{e}_2 - 50\bar{e}_3)\ 10^{-4}\text{ft} ,$$

and the total correction at joint 8 is

$$C_7 + \bar{H} \times \bar{R}_8 = (-20\bar{e}_1 - 90\bar{e}_2 - 110\bar{e}_3)\ 10^{-4}\text{ft} .$$

The components of the total correction at each joint are given in Columns 5 to 7 of Table IV-A. The components of the displacements of all joints are given in Columns 8 to 10.

As in the first example the actual entries in Column 10 of Table IV-A are the bar forces for the applied loading. By means of the principle of virtual work, it is readily verified that the vertical displacement of joint 9 is $-280\bar{e}_3$ (10^{-4}ft).

In treating Example II as a complex truss, the bar connecting joints 9 and 11 is removed and a substitute bar inserted between joints 4 and 6, thereby forming a simple truss. The components of the displacement at each joint for zero elongation of the fictitious bar are given in Columns 2 to 4 of Table V-A. The components of $\bar{Y}'_s$ are given in Columns 5 to 7 of Table V-A.

Equation (2.23) rewritten for this example is

$$x(\bar{Y}'_9 - \bar{Y}'_{11}) \cdot \bar{u}_{9,11} = -d_{9,11} - [(\bar{Y}_9)_{x=0} - (\bar{Y}_{11})_{x=0}] \cdot \bar{u}_{9,11}$$

Substituting the appropriate numerical values from Tables III-A and V-A one obtains

$$x = 0 - (75\sqrt{2})\ 10^{-4}\text{ft} = -\ 75\sqrt{2}\ 10^{-4}\text{ft}$$

Substituting the above value for x, and the data from Columns 2 to 7 of Table V-A into Equation (2.22) yields the displacements given in Columns 8 to 10.

It is seen that both methods yield the same result, as was expected. For this particular example it turns out that fewer arithmetic operations are required when the compound truss is treated as a complex truss, but it is apparent that this will not always be the case.

TABLE V-A

DISPLACEMENTS FOR EXAMPLE II,
TREATED AS A COMPLEX TRUSS

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	($\overline{Y}_s$)	(10^{-4}ft)			$\overline{Y}'_s$		$\overline{Y}_s$ (10^{-4}ft)		
s	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$
7	20	20	- 30	0	0	0	20	20	- 30
4	- 20	-40	- 40	0	0	0	-20	-40	- 40
6	0	30	-110	0	0	$\sqrt{2}$	0	30	-260
5	-130	-50	-110	$-\sqrt{2}$	0	$\sqrt{2}$	20	-50	-260
8	- 20	60	-110	0	$\sqrt{2}$	0	-20	-50	-110
9	-130	60	-280	$-\sqrt{2}$	$\sqrt{2}$	0	20	-90	-280
10	10	80	-270	0	0	0	10	80	-270
11	30	80	-110	0	0	0	30	80	-110

Example III. The bar force for each member of the complex truss loaded as shown in Figure 3-A* is given in Columns 5 and 10 of Table VI-A. The components of the unit vector along each bar (directed as indicated by the paired numbers in Columns 1 and 6 which identify the bar) are given in Columns 2 to 4 and 7 to 9 of Table VI-A. Assume that $\frac{L}{AE} = 10^{-4}$ft kip^{-1} for every member of the truss and compute the displacement of all joints.

The substitute truss used in this example is shown in Figure 4-A in which substitute members are indicated with dashed lines (there are several other obvious choices for the substitute truss). The calculation of $\overline{Z}_{s,0}$, $\overline{Z}_{s,k}$ poses no problem since the substitute truss is a simple

* The layout of this truss was taken from Figure 269 of Reference 1.

truss. The displacements for $x_k = 0$ are given in Columns 2 to 4 of Table VII-A. Columns 5 to 7, 8 to 10, and 11 to 13 contain the components of $\overline{Y}_{s,k}$ for k = 1,2, and 3, respectively. Substituting the appropriate numerical quantities from Tables VI-A and VII-A into Equation (2.26), one obtains

$$
\begin{array}{llcll}
(11,15) & 0x_1 - 2\sqrt{2}\, x_2 + 0x_3 & = & [0 + (-5)]\ 10^{-4}\text{ft} & = 5\ (10^{-4}\text{ft}) \\
(11,16) & (\sqrt{10}/2)x_1 - 2x_2 + 0x_3 & = & -\ [-\ 5\sqrt{2} + (17.5\sqrt{2})]10^{-4}\text{ft} & = 12.5\sqrt{2}\ (10^{-4}\text{ft}) \\
(12,16) & \sqrt{5}\, x_1 + 0x_2 - 2\sqrt{2}\, x_3 & = & [5 + 0]\ 10^{-4}\text{ft} & = 5\ (10^{-4}\text{ft})
\end{array}
$$

The solution is

$$
\begin{aligned}
x_1 &= -6\sqrt{5}\ (10^{-4}\text{ft}) \\
x_2 &= -1.25\sqrt{2}\ (10^{-4}\text{ft}) \\
x_3 &= -6.25\sqrt{2}\ (10^{-4}\text{ft})
\end{aligned}
$$

For joint 9, Equation (2.25) yields

$$
\begin{aligned}
\overline{Z}_9 = \overline{Y}_9 = -\ [&(220\overline{e}_1 + 30\overline{e}_2 + 15\overline{e}_3) \\
&- 6\sqrt{5}(0\overline{e}_1 + 0\overline{e}_2 + 0\overline{e}_3) \\
&- 1.25\sqrt{2}(-\sqrt{2}\ \overline{e}_1 - \sqrt{2}\ \overline{e}_2 + \sqrt{2}\ \overline{e}_3) \\
&- 6.25\sqrt{2}(\sqrt{2}\ \overline{e}_1 + 0\overline{e}_2 + 0\overline{e}_3)]10^{-4}\text{ft}
\end{aligned}
$$

$$
\overline{Y}_9 = (210\overline{e}_1 + 32.5\overline{e}_2 + 12.5\overline{e}_3)10^{-4}\text{ft}
$$

It can be verified, by virtual work, that the displacement of joint 9 in the direction of the applied load is $(32.5)10^{-4}$ft. The components of the displacement of each joint in the truss are given in Columns 14 to 16 of Table VII-A.

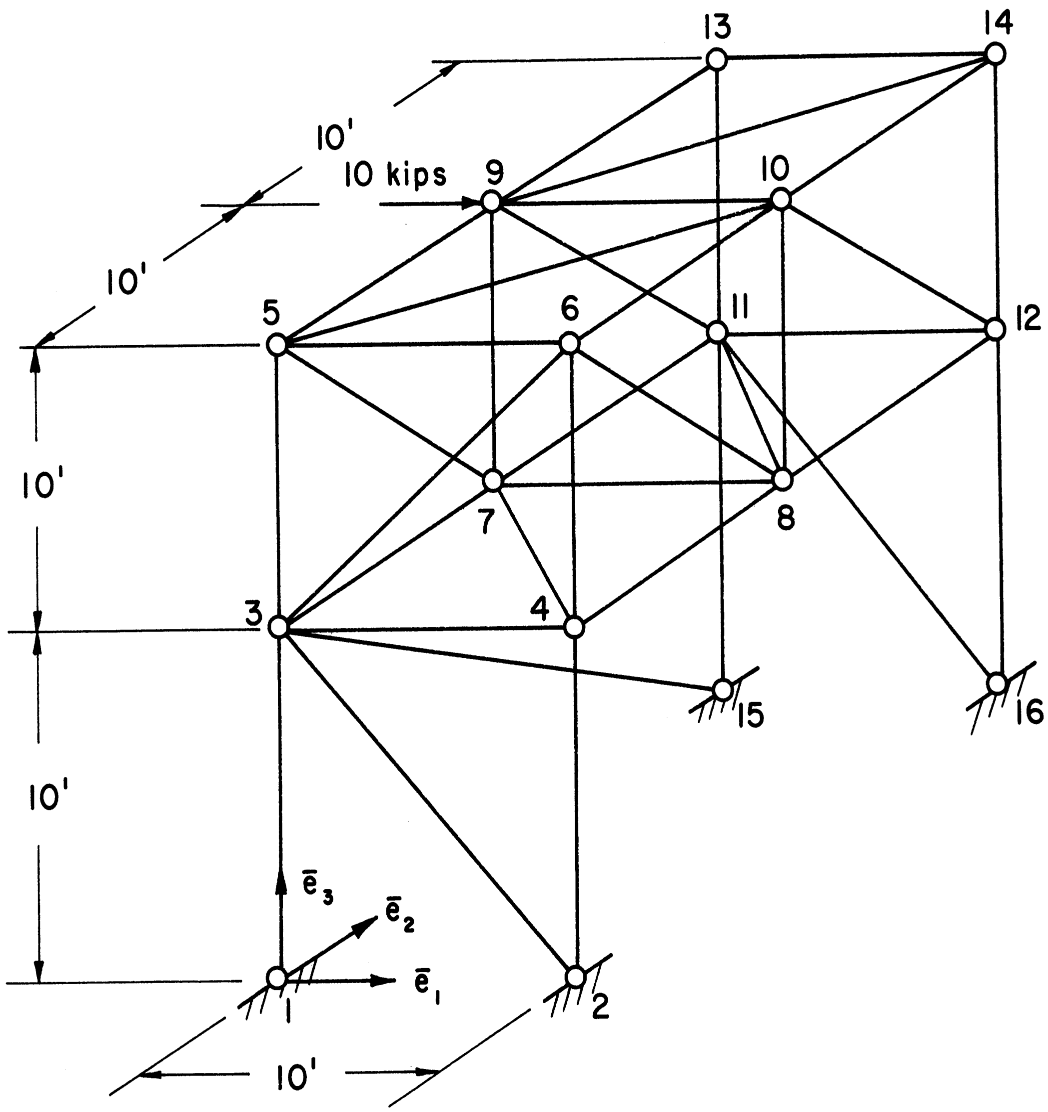

Figure 3-A. Truss for Example III.

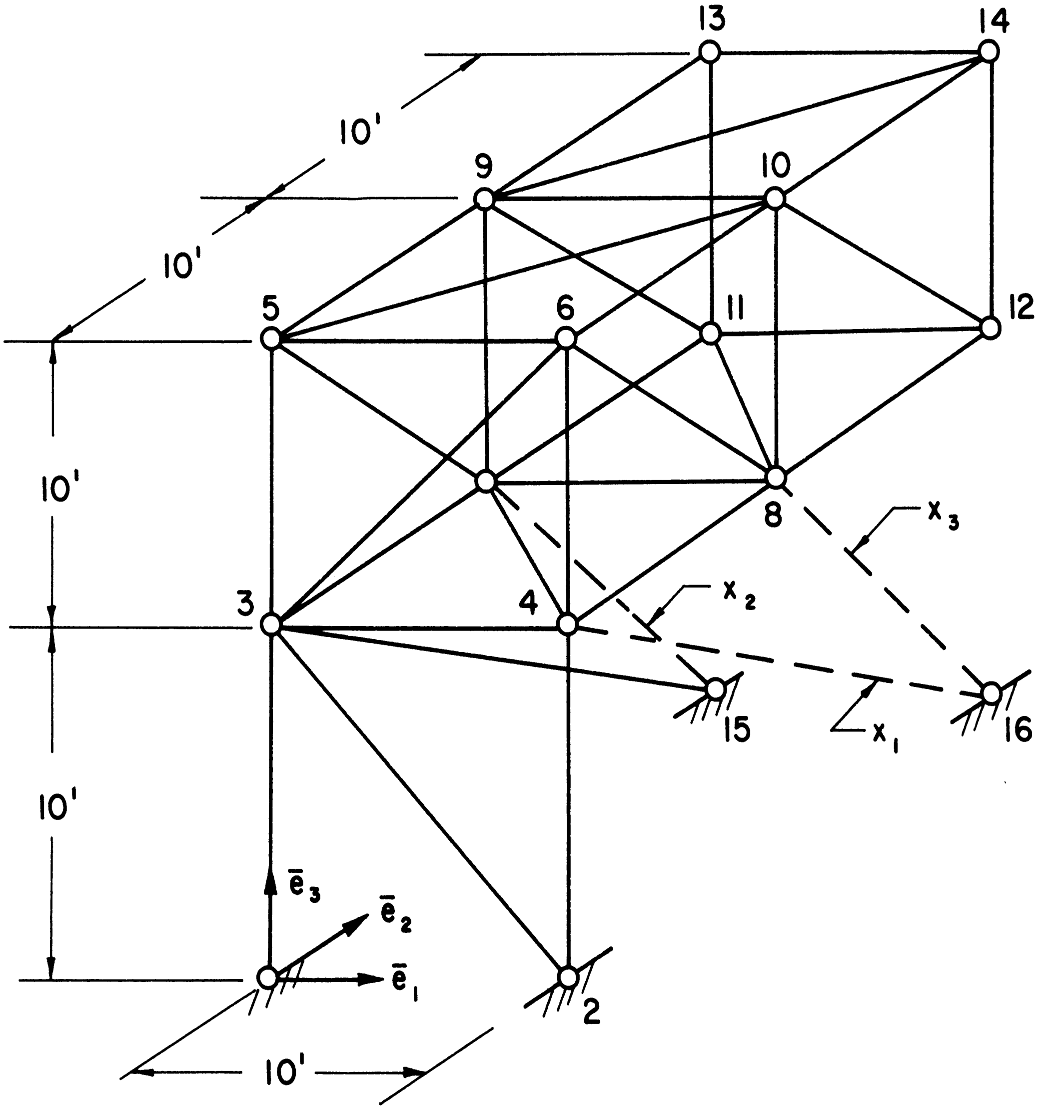

Figure 4-A. Substitute Truss for Example III.

TABLE VI-A

DATA FOR EXAMPLE III

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Bar		$\bar{u}_{ji}$		p_{ji}	Bar		$\bar{u}_{ji}$		p_{ji}
j,i	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	kip	j,i	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	kip
3,1	0	0	-1	20	7,9	0	0	1	5
3,2	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	$-5\sqrt{2}$	7,11	0	1	0	0
3,4	1	0	0	-5	8,10	0	0	1	-5
3,5	0	0	1	5	8,11	$-\sqrt{2}/2$	$\sqrt{2}/2$	0	$5\sqrt{2}$
3,6	$\sqrt{2}/2$	0	$\sqrt{2}/2$	$10\sqrt{2}$	8,12	0	1	0	-5
3,7	0	1	0	0	9,10	1	0	0	-10
3,15	0	$2\sqrt{5}/5$	$-\sqrt{5}/5$	0	9,11	0	$\sqrt{2}/2$	$-\sqrt{2}/2$	$-5\sqrt{2}$
4,2	0	0	-1	-15	9,13	0	1	0	0
4,6	0	0	1	-15	9,14	$\sqrt{2}/2$	$\sqrt{2}/2$	0	0
4,7	$-\sqrt{2}/2$	$\sqrt{2}/2$	0	$5\sqrt{2}$	10,12	0	$\sqrt{2}/2$	$-\sqrt{2}/2$	$5\sqrt{2}$
4,8	0	1	0	-5	10,14	0	1	0	0
5,6	1	0	0	-10	11,12	1	0	0	0
5,7	0	$\sqrt{2}/2$	$-\sqrt{2}/2$	$-5\sqrt{2}$	11,13	0	0	1	0
5,9	0	1	0	-5	11,15	0	0	-1	0
5,10	$\sqrt{2}/2$	$\sqrt{2}/2$	0	$10\sqrt{2}$	11,16	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	$-5\sqrt{2}$
6,8	0	$\sqrt{2}/2$	$-\sqrt{2}/2$	$5\sqrt{2}$	12,14	0	0	1	0
6,10	0	1	0	-5	12,16	0	0	-1	5
6,8	1	0	0	-5	13,14	1	0	0	0

TABLE VII-A

DISPLACEMENT DATA FOR EXAMPLE III

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
	$\bar{Z}_{s,0}$ (10^{-4}ft)			$\bar{Z}_{s,1}$			$\bar{Z}_{s,2}$			$\bar{Z}_{s,3}$			$\bar{Z}_s = \bar{Y}_s$ (10^{-4}ft)		
s	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
3	30	10	20	0	0	0	0	0	0	0	0	0	30	10	20
4	25	- 7.5	-15	0	$-\sqrt{5}/2$	0	0	0	0	0	0	0	25	7.5	-15
5	110	35	20	0	0	0	0	$-\sqrt{2}$	0	0	0	0	110	37.5	25
6	100	-40	-30	0	0	0	0	0	0	0	$-\sqrt{2}$	0	100	-37.5	-30
7	32.5	10	10	$\sqrt{5}/2$	0	0	0	0	$\sqrt{2}$	0	0	0	17.5	10	7.5
8	27.5	-12.5	-12.5	$\sqrt{5}/2$	$-\sqrt{5}/2$	$-\sqrt{5}/2$	0	0	0	0	0	$\sqrt{2}$	12.5	2.5	-10
9	220	30	15	0	0	0	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	0	0	210	32.5	12.5
10	210	-45	-17.5	0	0	$-\sqrt{5}/2$	$-\sqrt{2}$	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	200	-32.5	-15
11	40	10	5	$\sqrt{5}$	0	0	0	0	$\sqrt{2}$	0	0	0	10	10	2.5
12	40	-17.5	0	$\sqrt{5}$	$-\sqrt{5}/2$	$-\sqrt{5}$	0	0	0	0	0	$2\sqrt{2}$	10	- 2.5	5
13	295	30	5	0	0	0	$-2\sqrt{2}$	$-\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$	0	0	275	32.5	0
14	295	-45	0	0	0	$-\sqrt{5}$	$-2\sqrt{2}$	0	0	$2\sqrt{2}$	$-\sqrt{2}$	$2\sqrt{2}$	275	-32.5	5
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

APPENDIX B

THE MATRIX [M]

Small variations (or errors) in the geometry of the truss and applied loads affect the bar forces. When higher-order terms are neglected, the variation or error in the bar forces is governed by the equation

$$[G]^T\{\Delta p\} = -\{\Delta q\} - [\delta G]^T\{p\} \qquad (5.29)$$

The elements in $[\delta G]^T$ are the errors in the components of the unit vectors $\bar{u}_{ji}$, and the elements in $\{p\}$ are the bar forces due to the applied loads. An alternate form of the product $[\delta G]^T\{p\}$ is needed for the error analysis. The form required is one in which the elements of $[\delta G]$, which define the errors in the geometry, are transformed into a column vector. The transformation is simplified by treating the product $[\delta G]^T\{p\}$ as an equivalent error, $\{\Delta q\}*$, in the load vector. In expanded form

$$[\delta G]^T\{p\} = \begin{bmatrix} [\delta G]^T_{11} & [\delta G]^T_{21} & [\delta G]^T_{31} & \cdots & [\delta G]^T_{t1} & \cdots & [\delta G]^T_{J1} \\ [\delta G]^T_{12} & [\delta G]^T_{22} & [\delta G]^T_{32} & \cdots & [\delta G]^T_{t2} & & \\ & & & & [\delta G]^T_{ts} & & \\ & & & & [\delta G]^T_{tt} & & \\ & & & & & & \\ [\delta G]^T_{1J} & & & \cdots & [\delta G]^T_{tJ} & \cdots & [\delta G]^T_{JJ} \end{bmatrix} \begin{bmatrix} \{p\}_1 \\ \{p\}_2 \\ \{p\}_3 \\ \vdots \\ \{p\}_t \\ \vdots \\ \{p\}_J \end{bmatrix} = \begin{bmatrix} \{\Delta q\}^*_1 \\ \{\Delta q\}^*_2 \\ \{\Delta q\}^*_3 \\ \vdots \\ \{\Delta q\}^*_t \\ \vdots \\ \{\Delta q\}^*_J \end{bmatrix} \qquad (B.1)$$

in which $\{\Delta q\}^*_t$ is the equivalent error in the loads at a joint due to errors in the geometry of the truss. For that value of $t(t = 1,2,\ldots J)$ which corresponds to joint $j(t = j-m)$

$$\{p\}_j = \{p\}_j = \begin{Bmatrix} p_{j\alpha} \\ p_{j\beta} \\ p_{j\gamma} \end{Bmatrix} \tag{B.2}$$

where $p_{j\alpha}$, $p_{j\beta}$, $p_{j\gamma}$ are the forces in bars $j\alpha$, $j\beta$, and $j\gamma$.

From the developments in Chapter IV it is known that, in any one column of submatrices of $[G]^{-T}$, there are at most four submatrices which are not identically zero. The elements in the null submatrices cannot be varied. The nonzero matrices for that value of t corresponds to joint j, and those values of s corresponding to joints α, β, γ, are

$$[G]^T_{j\alpha} = \begin{bmatrix} -a^1_{j\alpha} & 0 & 0 \\ -a^2_{j\alpha} & 0 & 0 \\ -a^3_{j\alpha} & 0 & 0 \end{bmatrix} \qquad [G]^T_{j\beta} = \begin{bmatrix} 0 & -a^1_{j\beta} & 0 \\ 0 & -a^2_{j\beta} & 0 \\ 0 & -a^3_{j\beta} & 0 \end{bmatrix}$$

$$[G]^T_{j\gamma} = \begin{bmatrix} 0 & 0 & -a^1_{j\gamma} \\ 0 & 0 & -a^2_{j\gamma} \\ 0 & 0 & -a^3_{j\gamma} \end{bmatrix} \qquad [G]^T_{jj} = \begin{bmatrix} a^1_{j\alpha} & a^1_{j\beta} & a^1_{j\gamma} \\ a^2_{j\alpha} & a^2_{j\beta} & a^2_{j\gamma} \\ a^3_{j\alpha} & a^3_{j\beta} & a^3_{j\gamma} \end{bmatrix} \tag{B.3}$$

The null columns of $[G]^T_{j\alpha}$, etc., cannot be varied. Thus the corresponding error matrices become

$$[\delta G]^T_{j\gamma} = \begin{bmatrix} -\Delta a^1_{j\alpha} & 0 & 0 \\ -\Delta a^2_{j\alpha} & 0 & 0 \\ -\Delta a^3_{j\alpha} & 0 & 0 \end{bmatrix} \qquad [\delta G]^T_{j\beta} = \begin{bmatrix} 0 & -\Delta a^1_{j\beta} & 0 \\ 0 & -\Delta a^2_{j\beta} & 0 \\ 0 & -\Delta a^3_{j\beta} & 0 \end{bmatrix}$$

$$[\delta G]^T_{j\gamma} = \begin{bmatrix} 0 & 0 & -\Delta a^1_{j\gamma} \\ 0 & 0 & -\Delta a^2_{j\gamma} \\ 0 & 0 & -\Delta a^3_{j\gamma} \end{bmatrix} \qquad [\delta G]^T_{jj} = \begin{bmatrix} \Delta a^1_{j\alpha} & \Delta a^1_{j\beta} & \Delta a^1_{j\gamma} \\ \Delta a^2_{j\alpha} & \Delta a^2_{j\beta} & \Delta a^2_{j\gamma} \\ \Delta a_{j\alpha} & \Delta a_{j\beta} & \Delta a_{j\gamma} \end{bmatrix}$$

$$[\delta G]^T_{ji} = 0 \quad i \neq \alpha,\ \beta,\ \gamma,\ j$$

The contribution of errors in the components of $\bar{u}_{j\alpha}$, to the equivalent load error $\{\Delta q\}^*_\alpha$ at joint α, is

$$[\delta G]^T_{j\alpha}\{p\}_j = \begin{bmatrix} -\Delta a^1_{j\alpha} & 0 & 0 \\ -\Delta a^2_{j\alpha} & 0 & 0 \\ -\Delta a^3_{j\alpha} & 0 & 0 \end{bmatrix} \begin{Bmatrix} p_{j\alpha} \\ p_{j\beta} \\ p_{j\gamma} \end{Bmatrix} \tag{B.5}$$

which can also be written as

$$[\delta G]_{j\alpha}\{p\}_j = [M]_{j\alpha}\{\Delta u\}_j \tag{B.6}$$

where the matrix $[M]_{j\alpha}$ is defined by

$$[M]_{j\alpha} = \begin{bmatrix} -p_{j\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p_{j\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p_{j\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the vector $\{\Delta u\}_j$ is defined by

$$\{\Delta u\}^T_j = [\Delta a^1_{j\alpha}\ \Delta a^2_{j\alpha}\ \Delta a^3_{j\alpha}\ \Delta a^1_{j\beta}\ \Delta a^2_{j\beta}\ \Delta a^3_{j\beta}\ \Delta a^1_{j\gamma}\ \Delta a^2_{j\gamma}\ \Delta a^3_{j\gamma}]$$

If we define additional matrices

$$[M]_{j\beta} = \begin{bmatrix} 0 & 0 & 0 & -p_{j\beta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p_{j\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p_{j\beta} & 0 & 0 & 0 \end{bmatrix}$$

$$[M]_{j\gamma} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -p_{j\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{j\gamma} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_{j\gamma} \end{bmatrix}$$

$$[M]_{jj} = -[M]_{j\alpha} - [M]_{j\beta} - [M]_{j\gamma} ,$$

and let

$$
\begin{aligned}
[M]_{jr} &= 0 && \text{for } r \neq \alpha,\ \beta,\ \gamma,\ j \\
&= [M]_{j\alpha} && r = \alpha \\
&= [M]_{j\beta} && r = \beta \\
&= [M]_{j\gamma} && r = \gamma \\
&= [M]_{jj} && r = j
\end{aligned}
$$

then the contribution of errors in the components of $\bar{u}_{j\alpha}$, $\bar{u}_{j\beta}$, $\bar{u}_{j\gamma}$ to the equivalent load error at any joint, $r(r = m+1,\ m+2, \ldots, m+J)$ is given by

$$[\delta G]_{jr}\{p\}_j = [M]_{jr}\{\Delta u\}_j \qquad r = m+1,\ m+2, \ldots, m+J \qquad (B.7)$$

and the total equivalent load error is

$$\{\Delta q\}^*_r = \sum_{j=m+1}^{m+J} [\delta G]^T_{jr}\{p\}_j = \sum_{j=m+1}^{m+J} [M]_{jr}\{\Delta u\}_j \qquad r = m+1,\ m+2, \ldots, m+J \qquad (B.8)$$

which is the same as

$$\{\Delta q\}^*_s = \sum_{t=1}^{J} [\delta G]^T_{ts}\{p\}_s = \sum_{t=1}^{J} [M]_{ts}\{\Delta u\}_s \qquad s = 1,2,\ldots,J \qquad (B.9)$$

or equivalently

$$[\delta G]^T\{p\} = [M]\{\Delta u\} \quad . \qquad (B.10)$$

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