Journal of Scientific Computing, Vol. 7, No. 4, 1992

Numerical Computation of $\mathbf{H}^\infty$ Optimal Performance

Hong Yang* and J. Michael Orszag†

Received October 12, 1992

We present new algorithms for computing the $\mathbf{H}^\infty$ optimal performance for a class of single-input/single-output (SISO) infinite-dimensional systems. The algorithms here only require use of one or two fast Fourier transforms (FFT) and Cholesky decompositions; hence the algorithms are particularly simple and easy to implement. Numerical examples show that the algorithms are stable and efficient and converge rapidly. The method has wide applications including to the $\mathbf{H}^\infty$ optimal control of distributed parameter systems. We illustrate the technique with applications to some delay problems and a partial differential equation (PDE) model. The algorithms we present are also an attractive approach to the solution of high-order finite-dimensional models for which use of state space methods would present computational difficulties.

KEY WORDS: $\mathbf{H}^\infty$ control; optimal performance; Kreĭn space; infinite-dimensional systems.

1. INTRODUCTION

Since the early 1980s control theory has been revolutionized by the introduction of methods that account for model uncertainty and unstructured disturbances and hence provide robust and stabilizing controllers with unstructured disturbance reduction. Classical control theory solves a problem for a single model or a probability distribution of disturbances. The idea of $\mathbf{H}^\infty$ control is to minimize the *weighted* effect of any of a class of disturbances on optimal plans. Whereas the basic intuition of the $\mathbf{H}^\infty$ control approach is simple, it is surprisingly difficult to formulate properly mathematically. This mathematical difficulty perhaps explains why the

* Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544.

† Department of Economics, University of Michigan, Ann Arbor, Michigan 48109.

854/7/4-1

simpler approach in which probability distributions are treated as fixed is so widely used and why the $\mathbf{H}^\infty$ approach was not developed until the 1980s. The $\mathbf{H}^\infty$ approach is most effective when there is substantial uncertainty about plant dynamics and the disturbances.

Most numerical and theoretical work on $\mathbf{H}^\infty$ control has focused on rational transfer functions. However, most real world transfer functions are unlikely to be rational. The problem is that traditional $\mathbf{H}^\infty$ computation methods rely on spectral factorization and hence are difficult to apply to infinite-dimensional systems. In this article we shall present some new algorithms to numerically compute the optimal performance for a general class of SISO infinite-dimensional systems. Our starting point is from the standard one-block and two-block problems, which correspond to weighted pure and mixed sensitivity minimization problems, respectively.

The algorithms presented here require only use of one or two fast Fourier transforms and Cholesky decompositions and hence are very efficient and easy to implement. The algorithms are also applicable to the finite-dimensional systems whose transfer functions are rational. The standard frequency domain method in $\mathbf{H}^\infty$ control is computationally complicated to implement due to high dimensionality and the related projections (Yang and Flamm, 1990). The standard method is also fragile because of "false zeros" related to the optimal performance when second- or higher-order poles occur in the related projections (Yang, 1992). An added advantage of the algorithms here is that computation does not increase as the orders of the plants and weights increase.

We introduce the $\mathbf{H}^\infty$ optimal control problem in the next section. Section 3 deals with the theoretic aspects of the algorithms. We present the algorithms formally in Sec. 4. Numerical examples and results are presented in Sec. 5. Some concluding remarks are made in Sec. 6.

2. $\mathbf{H}^\infty$ OPTIMAL CONTROL PROBLEM

To illustrate the most basic $\mathbf{H}^\infty$ optimal control problem we consider the general feedback model shown in Fig. 1. P is the plant and K is the compensator that we want to design, W_1 and W_2 are frequency-dependent weights on the disturbances d and d'. For clarity, we list the dimensions of input u, output y, etc., as follows:

$$\begin{array}{rlrl} y\colon & m\times 1, & u\colon & n\times 1 \\ d\colon & m\times 1, & d'\colon & m\times 1 \\ P\colon & m\times n, & K\colon & n\times m \\ W_1\colon & m\times m, & W_2\colon & m\times m \end{array}$$

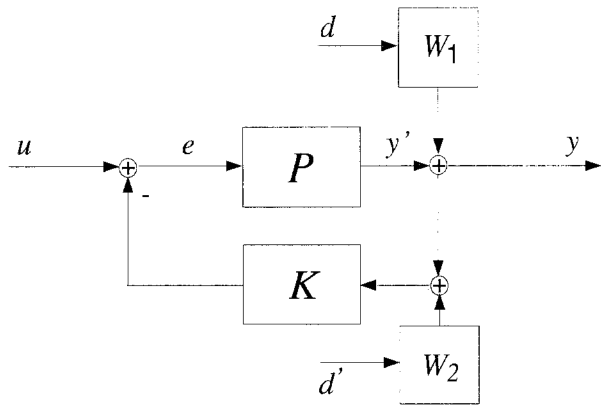

Fig. 1. A feedback system.

Without loss of generality we assume that W_1 and W_2 are square matrices, so they may be singular.

In the case where $W_1 = W_2 = I$, we see that the transfer matrix from the disturbance $d(s)$ to the output $y(s)$ is

$$S = (I + PK)^{-1}$$

and the transfer matrix from the disturbance d' to y, which is the same as the transfer matrix from d to y', is

$$(I + PK)^{-1} PK = I - S = PK(I + PK)^{-1} = PKS$$

(noting that for any matrix A with $(I + A)^{-1}$ existing, $A(I + A)^{-1} = (I + A)^{-1} A$). We also see that the transfer matrix from d to e, which is the same as the transfer matrix from d' to e, is

$$K(I + PK)^{-1} = KS$$

S is called the *sensitivity* and $I - S$ the *complementary sensitivity*. A small sensitivity over a frequency range of interest means good disturbance rejection and hence risk reduction, and it is known (Doyle and Stein, 1980; Safonov *et al.*, 1980) that a small complementary sensitivity means a good stability margin for multiplicative perturbations to the plant and a small KS means a good stability margin for additive plant perturbations. Here the term "small" refers to the $\mathbf{H}^\infty$ norm. In order to see why the $\mathbf{H}^\infty$ norm is used here, we list some of the properties of the $\mathbf{H}^\infty$ norm in the following remark. These properties are standard results, which can be found, for example, in Hoffman (1962) and Rosenblum and Rovnyak (1985).

Remark 1. (1) $\mathbf{H}^\infty$ is the space of transfer matrices of causal, $\mathbf{L}^2$-bounded input/bounded output stable, linear, time-invariant systems. Furthermore, the $\mathbf{H}^\infty$ norm of the transfer matrix of a system equals the system gain:

$$\|P\|_\infty = \sup\{\|Px\|_2 : x \in \mathbf{H}^2, \|x\|_2 = 1\}$$

where the space $\mathbf{H}^2$ is defined below.

(2) Definitions of the space $\mathbf{H}^\infty$ and its norm: For a matrix P analytic on the right half plane $\mathbb{C}^+$, define

$$\|P\|_\infty := \sup_{s \in \mathbb{C}^+} \sigma_{\max}[P(s)] = \operatorname{ess\,sup} \sigma_{\max}[P(j\omega)]$$

here $\sigma_{\max}$ denotes maximum singular value. The second equality follows from the maximal modulus principle and the existence of boundary value functions.

The space $\mathbf{H}^\infty$ and its norm: The space of matrices P analytic on the right half plane $\mathbb{C}^+$ with $\|P\|_\infty < +\infty$. $\|P\|_\infty$ is the $\mathbf{H}^\infty$ norm of P.

(3) Since we deal with causal systems, we need a mathematical framework for classifying functions analytic in the open right half plane. If a function $f(s)$ is analytic in the open right half plane, we say that $f(x)$ belongs to $\mathbf{H}^2$ provided that

$$\sup_{\alpha > 0} \int_{-\infty}^{\infty} |f(\alpha + j\omega)|^2 \, d\omega < \infty$$

The $\mathbf{H}^2$ norm of $f(s)$ is defined as

$$\|f\|_{\mathbf{H}^2} = \sup_{\alpha > 0} \left[\int_{-\infty}^{\infty} |f(\alpha + j\omega)|^2 \, d\omega \right]^{1/2}$$

It is known (Hoffman, 1962) that $\mathbf{H}^2$ can be considered as a subspace of $\mathbf{L}^2$. We shall denote the orthogonal complement of $\mathbf{H}^2$ in $\mathbf{L}^2$ by $\mathbf{H}^2_-$. The time domain version of $\mathbf{H}^2$ functions represent $\mathbf{L}^2$ signals (i.e., signals with finite energy) with support on the positive time axis, the time domain version of $\mathbf{H}^2_-$ functions represent $\mathbf{L}^2$ functions with support on the negative time axis. The time domain $\mathbf{H}^2$ spaces and frequency domain $\mathbf{H}^2$ spaces are connected by the Laplace transform.

(4) One can also define $\mathbf{H}^\infty$ space on the unit disk. For a function f on the disk, $f(x)$ is in $\mathbf{H}^\infty$ of the unit disk if and only if $f((s-1)/(s+1))$ is in $\mathbf{H}^\infty$ of the right half plane (Hoffman, 1962).

In greater generality, one can also add two $m \times m$-dimensional weighting matrices $W_{1,y}$ and $W_{2,y}$ to emphasize and deemphasize the importance of the outputs at different frequencies. They are in general nonsingular.

The $\mathbf{H}^\infty$ *optimal weighted pure sensitivity* problem is to find a stabilizing compensator K that minimizes

$$\| W_{1,y} S W_1 \|_\infty$$

The $\mathbf{H}^\infty$ *optimal weighted mixed sensitivity* problem is to find a stabilizing compensator K that minimizes

$$\left\| \begin{pmatrix} W_{1,y} S W_1 \\ W_{2,y}(I-S)\, W_2 \end{pmatrix} \right\|_\infty \tag{2.1}$$

Remark 2. In the single-input/single-output case, we can combine the weighting functions W_1 and $W_{1,y}$, W_2 and $W_{2,y}$, so (2.1) becomes

$$\left(\sup_\omega \{ |W_1(j\omega)\, S(j\omega)|^2 + |W_2(j\omega)[1 - S(j\omega)]|^2 \}\right)^{1/2}$$

Let

$$\mu_0 = \inf_{K \text{ stabilizing}} \left\| \begin{pmatrix} W_{1,y} S W_1 \\ W_{2,y}(I-S)\, W_2 \end{pmatrix} \right\|_\infty$$

μ_0 is called the $(\mathbf{H}^\infty)$ *optimal performance*. We define the $(\mathbf{H}^\infty)$ *optimal controller* as a compensator K which makes the right-hand side of (2.1) equal to μ_0. (The optimal performance and optimal controller for the pure sensitivity minimization problem are defined in the same way.)

In some circumstances we may want to define a $m \times m$ matrix W_3 which captures uncertainty about additive perturbations to the plant. We can define a corresponding $n \times n$ weighting matrix $W_{3,y}$ to emphasize or deemphasize the weights of disturbances at different frequencies. We can then define another type of $\mathbf{H}^\infty$ problem:

$$\mu_0 = \inf_{K \text{ stabilizing}} \left\| \begin{pmatrix} W_{1,y} S W_1 \\ W_{3,y} K S W_3 \end{pmatrix} \right\|_\infty$$

The optimal performance and optimal controllers are defined in the same way.

Solutions of most $\mathbf{H}^\infty$ optimal control problems come from the solutions of one-block or two-block problems which follow (nontrivially) from the original $\mathbf{H}^\infty$ optimization problems (see, for example, Francis, 1987, for the rational case and Flamm and Yang, 1990, and Yang and Flamm, 1990 for some irrational cases). The algorithms in this paper solve the one-block and two-block problems which are outlined below.

(P1): Given $g \in \mathbf{L}^\infty$, find

$$\mu_0 = \inf_{h \in \mathbf{H}^\infty} \|g - h\|_\infty \tag{2.2}$$

This problem corresponds to the *weighted pure sensitivity minimization problem.* It is a disturbance reduction problem which does not consider the stability margin. In general, controllers which solve the *pure sensitivity minimization problem* are improper and not robust to model misspecification.

(P2): Given $g \in \mathbf{L}^\infty$ and $f \in \mathbf{H}^\infty$, find

$$\mu_0 = \inf_{h \in \mathbf{H}^\infty} \left\| \begin{pmatrix} g - h \\ f \end{pmatrix} \right\|_\infty \tag{2.3}$$

This problem corresponds to the *weighted mixed sensitivity minimization problem.* Controllers which solve the *mixed sensitivity minimization problem* both reduce disturbances and achieve a good stability margin. Proper controllers can be obtained by properly choosing the complementary sensitivity weight (Flamm and Yang, 1990).

3. BASIC THEORY

We shall now develop some necessary theoretic background for the algorithms. To understand how the algorithms work, first we shall briefly summarize the Kreĭn space approach to $\mathbf{H}^\infty$ control theory. Since our main purpose is to introduce new algorithms for the computation of the optimal performance, we shall keep the notation and definitions to absolute minimum here. More details and related references can be found in Yang (1993), Francis *et al.* (1984), and Sarason (1985). All the results cited below can be found in these references too.

Hankel and Toeplitz operators associated with symbol g are defined as follows:

$$\mathscr{H}_g: \mathbf{H}^2 \to \mathbf{H}^2_- \qquad (h \mapsto \Pi_{\mathbf{H}^2_-} gh)$$

$$\mathscr{T}_g: \mathbf{H}^2 \to \mathbf{H}^2 \qquad (h \mapsto \Pi_{\mathbf{H}^2} gh)$$

where Π denotes the orthogonal projection.

Nehari theorem and its extensions say that

$$\|\mathscr{H}_g\| = \inf_{h \in \mathbf{H}^\infty} \|g - h\|_\infty$$

and

$$\left\| \begin{pmatrix} \mathscr{H}_g \\ \mathscr{T}_f \end{pmatrix} \right\| = \inf_{h \in \mathbf{H}^\infty} \left\| \begin{pmatrix} g - h \\ f \end{pmatrix} \right\|_\infty$$

Thus computing the optimal performance for (**P1**) is equivalent to computing the operator norm of the Hankel operator $\mathcal{H}_g$. Computing the optimal performance for (**P2**) is equivalent to computing the operator norm of the operator $\mathcal{A}(g, f) := \binom{\mathcal{H}_g}{\mathcal{T}_f}$.

First we consider the one-block case (**P1**). The space

$$\mathbf{K}_1 := \mathbf{L}^2 \oplus \mathbf{H}^2$$

equipped with the indefinite inner product

$$\left[\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\right] = \langle u_1, u_2 \rangle_{\mathbf{L}^2} - \langle v_1, v_2 \rangle_{\mathbf{H}^2} \tag{3.1}$$

will be called a Kreĭn space. The shift operator on $\mathbf{K}_1$ is defined as

$$S_1 \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} s_1 u \\ s_2 v \end{pmatrix}$$

where s_1 is the bilateral shift operator on $\mathbf{L}^2$ and s_2 is the unilateral shift operator on $\mathbf{H}^2$.

A vector $x \in \mathbf{K}_1$ is *positive*, *negative*, or *neutral* if $[x, x]$ is positive, negative, or zero, respectively. A subspace is *positive*, *negative*, or *neutral* if each vector in the subspace is positive, negative, or zero, respectively. A subspace with both positive and negative vectors is called *indefinite*.

For the adjoint operator $\mathcal{H}_g^*$ of the Hankel operator $\mathcal{H}_g$, let

$$\mathcal{G}(\mathcal{H}_g^*) = \left\{\begin{pmatrix} y \\ \mathcal{H}_g^* y \end{pmatrix} : y \in \mathbf{H}_-^2 \right\}$$

be the graph of $\mathcal{H}_g^*$.

The orthogonal companion of $\mathcal{G}(\mathcal{H}_g^*)$ in the Kreĭn space $\mathbf{K}_1$ with respect to the indefinite inner product $[\cdot, \cdot]$ defined in (3.1) is denoted as

$$\mathcal{M}_1 := \mathcal{G}(\mathcal{H}_g^*)^{[\perp]}$$

Now we define the subspace $\mathbf{L}_1$

$$\mathbf{L}_1 := \mathcal{M}_1 \cap (S_1 \mathcal{M}_1)^{[\perp]}$$

It has been shown (Ball and Helton, 1983; Sarason, 1985; Francis *et al.*, 1985) that to find the optimal sensitivity is the same as to characterize the set of all S_1-invariant maximal-negative subspaces of $\mathcal{M}_1$. In order to do so, we need to find a basis for the subspace $\mathbf{L}_1$.

Our basic idea comes from the following result regarding the subspace $\mathbf{L}_1$.

Lemma 1 (Sarason, 1985). $\dim \mathbf{L}_1 = 2$. Further if $\|\mathscr{H}_g\| < 1$ (regularity), then there exist two vectors $x_1, x_2 \in \mathbf{L}$ such that

$$[x_1, x_1] = 1 = -[x_2, x_2], \qquad [x_1, x_2] = 0$$

Lemma 1 says that if $\|\mathscr{H}_f\| < 1$, then $\mathbf{L}_1$ is indefinite. From Lemma 1, we get the following observation.

Observation 1. If $\|\mathscr{H}_g\| < 1$ (regularity), then $\mathbf{L}_1$ is indefinite. In other words, if $\mathbf{L}_1$ is not indefinite, then $\|\mathscr{H}_g\| \geqslant 1$.

The space $\mathbf{L}_1$ is crucial in our study. Yang (1993) studies the space $\mathbf{L}_1$ and gives the following characterization of $\mathbf{L}_1$.

Theorem 1 (Yang, 1993). A vector function $\binom{p}{r} \in \mathbf{L}_1$ if and only if

$$p^* \in \mathbf{H}^2 \tag{3.2}$$

$$p - gr \in \mathbf{H}^2 \tag{3.3}$$

$$r^* - gp^* \in \mathbf{H}^2 \tag{3.4}$$

where $p^*(z| = \overline{p(1/\bar{z})}$, $r^*(z| = \overline{r(1/\bar{z})}$.

We shall use Theorem 1 to compute the elements of $\mathbf{L}_1$.

Next we shall consider the two-block problem (**P2**). We define the Kreĭn space

$$\mathbf{K}_2 := \mathbf{L}^2 \oplus \mathbf{H}^2 \oplus \mathbf{H}^2$$

with indefinite inner product

$$\left[\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} \right] = \langle u_1, u_2 \rangle_{\mathbf{L}^2} + \langle v_1, v_2 \rangle_{\mathbf{H}^2} - \langle w_1, w_2 \rangle_{\mathbf{H}^2} \tag{3.5}$$

The shift operator on $\mathbf{K}_2$ is defined as

$$S_2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \triangleq \begin{pmatrix} s_1 u \\ s_2 v \\ s_2 w \end{pmatrix}$$

with s_1 the bilateral shift operator on $\mathbf{L}^2$ and s_2 the unilateral shift operator on $\mathbf{H}^2$.

Recall that

$$\mathscr{A}(g,f) := \begin{pmatrix} \mathscr{H}_g \\ \mathscr{T}_f \end{pmatrix} : \mathbf{H}^2 \to \mathbf{H}^2_- \oplus \mathbf{H}^2$$

We have

$$\mathscr{A}^*: \mathbf{H}^2_- \oplus \mathbf{H}^2 \to \mathbf{H}^2$$

$$\mathscr{A}^* \begin{pmatrix} h_- \\ h \end{pmatrix} = (\Pi_+ g^* + \Pi_+ f^*) \begin{pmatrix} h_- \\ h \end{pmatrix}$$
$$= \Pi_+ g^* h_- + \Pi_+ f^* h$$

The graph of $\mathscr{A}^*$ is:

$$\mathscr{G}(\mathscr{A}^*) = \left\{ \begin{pmatrix} h_- \\ h \\ \Pi_+ g^* h_- + \Pi_+ f^* h \end{pmatrix} : h_- \in \mathbf{H}^2_-, \, h \in \mathbf{H}^2 \right\}$$

Let $\mathscr{M}_2 = \mathscr{G}(\mathscr{A}^*)^{[\perp]}$, the orthogonal companion in $\mathbf{K}_2$ of $\mathscr{G}(\mathscr{A}^*)$ with respect to the indefinite inner product defined in (3.5). Similar to the one-block case, we need to find a basis for the subspace $\mathbf{L}_2 := \mathscr{M}_2 \cap (S_2 \mathscr{M}_2)^{[\perp]}$.

Corresponding to Lemma 1, Fagnani (1991) proved the following lemma.

Lemma 2 (Fagnani, 1991). $\dim \mathbf{L}_2 = 2$. Further if $\|\mathscr{A}(g,f)\| < 1$ (regularity), then there exist two vectors x_1, $x_2 \in \mathbf{L}_2$ such that

$$[x_1, x_2] = 1 = -[x_2, x_2], \qquad [x_1, x_2] = 0$$

We have the following.

Observation 2. If $\|\mathscr{A}(g,f)\| < 1$ (regularity), then $\mathbf{L}_2$ is indefinite. In other words, if $\mathbf{L}_2$ is not indefinite, then $\|\mathscr{A}(g,f)\| \geqslant 1$.

Yang (1993) gives the following characterization of $\mathbf{L}_2$:

Theorem 2 (Yang, 1993). A vector function

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \in \mathbf{L}_2 = \mathscr{M}_2 \cap (S_2 \mathscr{M}_2)^{[\perp]}$$

if and only if

$$p^* \in \mathbf{H}^2 \tag{3.6}$$

$$q = fr \tag{3.7}$$

$$(1 - f^*f)\, r^* - gp^* \in \mathbf{H}^2 \tag{3.8}$$

$$p - gr \in \mathbf{H}^2 \tag{3.9}$$

where $f^*(z) = \overline{f(1/\bar{z})}$, $p^*(z) = \overline{p(1/\bar{z})}$, $r^*(z) = \overline{r(1/\bar{z})}$.

We shall use Theorem 2 to compute the elements of $\mathbf{L}_2$.

4. ALGORITHMS

Based on the theory developed in Sec. 3, we shall present two algorithms for computing optimal performance for the weighted pure and mixed sensitivity design problems, assuming that we have arrived at (**P1**) and (**P2**). In this section we shall always use a caret (̂) to denote approximations.

We first consider the computation of the optimal performance μ_0 for (**P1**).

Our basic idea is the following: From (2.2) we know that $\mu_0 \leqslant \|g\|_\infty$. Beginning from $\mu = \|g\|_\infty$ we compute a vector $x = \binom{p_\mu}{r_\mu} \in \mathbf{L}_1$ for the function g/μ and search for the largest μ that makes $[\binom{p_\mu}{r_\mu}, \binom{p_\mu}{r_\mu}] = \langle p_\mu, p_\mu \rangle - \langle r_\mu, r_\mu \rangle = 0$. In the following we shall surpress the subscript μ for p_μ and r_μ for notation convenience.

In general, the $g(s)$ is defined on the imaginary axis. We use the map $s = (1+z)/(1-z)$ to map the imaginary axis to the unit circle.

Now we can write the functions g in terms of its Fourier series expansion

$$g(e^{j\theta}) = \sum_{n=-\infty}^{\infty} g_n e^{j\theta n}, \qquad \theta \in [0, 2\pi]$$

From (3.2) we can write p and r in terms of their Fourier series expansions as

$$p = \sum_{n=-\infty}^{0} p_n z^n$$

$$r = \sum_{n=0}^{\infty} r_n z^n$$

The conditions (3.3) and (3.4) mean

$$
\begin{aligned}
p-\frac{g}{\mu} r &= \sum_{n=-\infty}^{0} p_n z^n - \sum_{n=-\infty}^{\infty} \frac{g_n}{\mu} z^n \sum_{n=0}^{\infty} r_n z^n \\
&= \sum_{n=-\infty}^{0} p_n z^n - \sum_{n=-\infty}^{\infty} \left(\sum_{i+j=n} \frac{g_i}{\mu} r_j \right) z^n \\
&= \sum_{n=-\infty}^{0} \left(p_n - \sum_{i+j=n} \frac{g_i}{\mu} r_j \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i+j=n} \frac{g_i}{\mu} r_j \right) z^n \in \mathbf{H}^2
\end{aligned}
$$

and

$$
\begin{aligned}
r^* - \frac{g}{\mu} p^* &= \sum_{n=0}^{\infty} \bar{r}_n \left(\frac{1}{z}\right)^n - \sum_{n=-\infty}^{\infty} \frac{g_n}{\mu} z^n \sum_{n=-\infty}^{0} \bar{p}_n \left(\frac{1}{z}\right)^n \\
&= \sum_{n=-\infty}^{0} \bar{r}_{-n} z^n - \sum_{-\infty}^{\infty} \left(\sum_{i-j=n} \frac{g_i}{\mu} \bar{p}_j \right) z^n \\
&= \sum_{n=-\infty}^{\infty} \left(\bar{r}_{-n} - \sum_{i-j=n} \frac{g_i}{\mu} \bar{p}_j \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i-j=n} \frac{g_i}{\mu} \bar{p}_j \right) z^n \in \mathbf{H}^2
\end{aligned}
$$

Thus we see that (3.3) and (3.4) are equivalent to

$$
p_n - \sum_{i+j=n} \frac{g_i}{\mu} r_j = 0 \qquad \text{for} \quad n = -1, -2, \ldots \tag{4.1}
$$

$$
\bar{r}_{-n} - \sum_{i-j=n} \frac{g_i}{\mu} \bar{p}_j = 0 \qquad \text{for} \quad n = -1, -2, \ldots \tag{4.2}
$$

Suppose $\hat{g}_{-1}, \hat{g}_{-2}, \ldots, \hat{g}_{-m}$ are approximations of the Fourier coefficients $g_{-1}, g_{-2}, \ldots, g_{-m}$ for g. We approximate the conditions (4.1) and (4.2) by the following system of linear equations:

$$
\begin{aligned}
\hat{p}_{-1} - \frac{\hat{g}_{-2}}{\mu} \hat{r}_1 - \cdots - \frac{\hat{g}_{-m}}{\mu} \hat{r}_{m-1} &= \frac{\hat{g}_{-1}}{\mu} r_0 \\
&\vdots \\
\hat{p}_{-m+1} - \frac{\hat{g}_{-m}}{\mu} \hat{r}_1 &= \frac{\hat{g}_{-m+1}}{\mu} r_0 \\
\hat{p}_{-m} &= \frac{\hat{g}_{-m}}{\mu} r_0
\end{aligned}
$$

$$\hat{r}_1 - \frac{\bar{\hat{g}}_{-2}}{\mu}\hat{p}_{-1} - \cdots - \frac{\bar{\hat{g}}_{-m}}{\mu}\hat{p}_{-m+1} = \frac{\bar{\hat{g}}_{-1}}{\mu} p_0$$

$$\vdots$$

$$\hat{r}_{m-1} - \frac{\bar{\hat{g}}_{-m}}{\mu}\hat{p}_{-1} = \frac{\bar{\hat{g}}_{-m+1}}{\mu} p_0$$

$$\hat{r}_m = \frac{\bar{\hat{g}}_{-m}}{\mu} p_0$$

Write

$$A_1 = \begin{pmatrix} 1 & \cdots & 0 & 0 & -\frac{\hat{g}_{-2}}{\mu} & \cdots & -\frac{\hat{g}_{-m}}{\mu} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -\frac{\hat{g}_{-m}}{\mu} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ -\frac{\bar{\hat{g}}_{-2}}{\mu} & \cdots & -\frac{\bar{\hat{g}}_{-m}}{\mu} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\bar{\hat{g}}_{-m}}{\mu} & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$V_1 = \begin{pmatrix} r_0 \frac{\hat{g}_{-1}}{\mu} \\ \vdots \\ r_0 \frac{\hat{g}_{-m+1}}{\mu} \\ r_0 \frac{\hat{g}_{-m}}{\mu} \\ p_0 \frac{\bar{\hat{g}}_{-1}}{\mu} \\ \vdots \\ p_0 \frac{\bar{\hat{g}}_{-m+1}}{\mu} \\ p_0 \frac{\bar{\hat{g}}_{-m}}{\mu} \end{pmatrix} \qquad X_1 = \begin{pmatrix} \hat{p}_{-1} \\ \vdots \\ \hat{p}_{-m+1} \\ \hat{p}_{-m} \\ \hat{r}_1 \\ \vdots \\ \hat{r}_{m-1} \\ \hat{r}_m \end{pmatrix}$$

Then we have

$$A_1 X_1 = V_1 \tag{4.3}$$

A Fast Fourier transform can be used to get the approximation of the Fourier coefficients of g. The system of linear equations (4.3) is used to find the approximations for p and r. Here the matrix A_1 is Hermitian.

We now present an algorithm for the one block problem (**P1**) as follows:

Step 1. Use fast Fourier transform to get $\hat{g}_{-1}, \hat{g}_{-2}, \ldots, \hat{g}_{-m}$, the approximations of the Fourier coefficients $g_{-1}, g_{-2}, \ldots, g_{-m}$ for g.

Step 2. Pick r_0 and p_0 such that $|p_0| \neq |r_0|$. Starting from $\mu = \|g\|_{\infty}$, get the approximation $\{\hat{p}_{-1}, \hat{p}_{-2}, \ldots, \hat{p}_{-m}\}$ and $\{\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_m\}$, of p_μ and r_μ by solving (4.3). The subscript μ is used to remind the readers that the functions p and r depend on the value of μ.

Step 3. Search for the largest $\mu < \|g\|_{\infty}$ such that

$$\langle \hat{p}, \hat{p} \rangle - \langle \hat{r}, \hat{r} \rangle = \sum_{k=0}^{m} |\hat{p}_{-k}|^2 - \sum_{k=0}^{m} |\hat{r}_k|^2 = 0 \tag{4.4}$$

This μ is an estimate of the optimal performance μ_0 for (**P1**).

Next we shall extend the above algorithm to the two-block case (**P2**).

From (3.6) we know that $p^* \in \mathbf{H}^2$. Again we write the functions f, g, p and r in terms of their Fourier series expansions as

$$f = \sum_{n=0}^{\infty} f_n z^n$$

$$g = \sum_{n=-\infty}^{\infty} g_n z^n$$

$$r = \sum_{n=0}^{\infty} r_n z^n$$

$$p = \sum_{n=-\infty}^{0} p_n z^n$$

Then

$$f^* f = \sum_{n=0}^{\infty} \bar{f}_n \left(\frac{1}{z}\right)^n \sum_{n=0}^{\infty} f_n z^n = \sum_{n=-\infty}^{\infty} \left(\sum_{j-i=n} \bar{f}_i f_j \right) z^n$$

Let

$$F_0 = 1 - \sum_{j-i=0} \frac{\bar{f}_i}{\mu}\frac{f_j}{\mu} = 1 - \frac{1}{\mu^2}\sum_{i=0}^{\infty} \bar{f}_i f_i \tag{4.5}$$

$$F_n = -\sum_{j-i=n} \frac{\bar{f}_i}{\mu}\frac{f_j}{\mu}, \qquad n = \pm 1, \pm 2,... \tag{4.6}$$

Then

$$\left(1 - \frac{f^* f}{\mu^2}\right) = \sum_{n=-\infty}^{\infty} F_n z^n$$

We have

$$\begin{aligned}
&\left(1 - \frac{f^* f}{\mu^2}\right) r^* - \frac{g}{\mu} p^* \\
&= \sum_{n=-\infty}^{\infty} F_n z^n \sum_{n=0}^{\infty} \bar{r}_n \left(\frac{1}{z}\right)^n - \sum_{n=-\infty}^{\infty} \frac{g_n}{\mu} z^n \sum_{n=-\infty}^{0} \bar{p}_n \left(\frac{1}{z}\right)^n \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{i-j=n} F_i \bar{r}_j\right) z^n - \sum_{-\infty}^{\infty} \left(\sum_{i-j=n} \frac{g_i}{\mu} \bar{p}_j\right) z^n \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{i-j=n} \left(F_i \bar{r}_j - \frac{g_i}{\mu} \bar{p}_j\right)\right) z^n
\end{aligned}$$

Thus $(1 - f^*f/\mu^2)\, r^* - (g/\mu)\, p^* \in \mathbf{H}^2$ if and only if for $n = -1, -2,...,$

$$\sum_{i-j=n} \left(F_i \bar{r}_j - \frac{g_i}{\mu} \bar{p}_j\right) = 0 \tag{4.7}$$

Similarly, we have

$$\begin{aligned}
p - \frac{g}{\mu} r &= \sum_{n=-\infty}^{0} p_n z^n - \sum_{n=-\infty}^{\infty} \frac{g_n}{\mu} z^n \sum_{n=0}^{\infty} r_n z^n \\
&= \sum_{n=-\infty}^{0} p_n z^n - \sum_{n=-\infty}^{\infty} \left(\sum_{i+j=n} \frac{g_i}{\mu} r_j\right) z^n \\
&= \sum_{n=-\infty}^{0} \left(p_n - \sum_{i+j=n} \frac{g_i}{\mu} r_j\right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i+j=n} \frac{g_i}{\mu} r_j\right) z^n
\end{aligned}$$

So $p - (g/\mu) r \in \mathbf{H}^2$ if and only if for $n = -1, -2,...,$

$$p_n = \sum_{i+j=n} \frac{g_i}{\mu} r_j \tag{4.8}$$

It is easy to see that (4.7) and (4.8) are equivalent to

$$p_{-k}=\sum_{j=0}^{\infty} r_j \frac{g_{-k-j}}{\mu}, \qquad k=1,2,\ldots \tag{4.9}$$

$$\sum_{j=0}^{\infty} r_j \bar{F}_{-k+j}=\sum_{j=0}^{\infty} p_{-j} \frac{\bar{g}_{-k-j}}{\mu}, \qquad k=1,2,\ldots \tag{4.10}$$

From Theorem 2 we see that

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \in \mathbf{L}_2=\mathscr{M}_2 \cap (S_2 \mathscr{M}_2)^{[\perp]}$$

if and only if $q=fr$ and (4.9) and (4.10) hold.

We approximate (4.5), (4.6), (4.9), and (4.10) by

$$\hat{F}_0=1-\sum_{\substack{j-i=0,\\ i,j\leqslant m}} \frac{\bar{\hat{f}}_i}{\mu}\frac{\hat{f}_j}{\mu}=1-\frac{1}{\mu^2}\sum_{i=0}^{m} \bar{\hat{f}}_i \hat{f}_i$$

$$\hat{F}_n=-\sum_{\substack{j-i=n,\\ i,j\leqslant m}} \frac{\bar{\hat{f}}_i}{\mu}\frac{\hat{f}_j}{\mu}, \qquad n=-m,-m+1,\ldots,m-1$$

$$\hat{p}_{-k}=\sum_{j=0}^{m} \hat{r}_j \frac{\hat{g}_{-k-j}}{\mu}, \qquad k=1,2,\ldots,m \tag{4.11}$$

$$\sum_{j=0}^{m} \hat{r}_j \bar{\hat{F}}_{-k+j}=\sum_{j=0}^{m} \hat{p}_{-j} \frac{\bar{\hat{g}}_{-k-j}}{\mu}, \qquad k=1,2,\ldots,m \tag{4.12}$$

Let

$$A_2=\begin{bmatrix}
-1 & 0 & \cdots & 0 & \frac{\hat{g}_{-2}}{\mu} & \frac{\hat{g}_{-3}}{\mu} & \cdots & \frac{\hat{g}_{-1-m}}{\mu} \\
0 & -1 & \cdots & 0 & \frac{\hat{g}-3}{\mu} & \frac{\hat{g}_{-4}}{\mu} & \cdots & \frac{\hat{g}_{-2-m}}{\mu} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & -1 & \frac{\hat{g}_{-m-1}}{\mu} & \frac{\hat{g}_{-m-2}}{\mu} & \cdots & \frac{\hat{g}_{-2m}}{\mu} \\
\frac{\bar{\hat{g}}_{-2}}{\mu} & \frac{\bar{\hat{g}}_{-3}}{\mu} & \cdots & \frac{\bar{\hat{g}}_{-1-m}}{\mu} & -\bar{\hat{F}}_0 & -\bar{\hat{F}}_1 & \cdots & -\bar{\hat{F}}_{m-1} \\
\frac{\bar{\hat{g}}_{-3}}{\mu} & \frac{\bar{\hat{g}}_{-4}}{\mu} & \cdots & \frac{\bar{\hat{g}}_{-2-m}}{\mu} & -\bar{\hat{F}}_{-1} & -\bar{\hat{F}}_0 & \cdots & -\bar{\hat{F}}_{m-2} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\frac{\bar{\hat{g}}_{-m-1}}{\mu} & \frac{\bar{\hat{g}}_{-m-2}}{\mu} & \cdots & \frac{\bar{\hat{g}}_{-2m}}{\mu} & -\bar{\hat{F}}_{-m+1} & -\bar{\hat{F}}_{-m+2} & \cdots & -\bar{\hat{F}}_0
\end{bmatrix} \tag{4.13}$$

$$V_2 = \begin{bmatrix} -r_0 \dfrac{\hat{g}_{-1}}{\mu} \\ -r_0 \dfrac{\hat{g}_{-2}}{\mu} \\ \vdots \\ -r_0 \dfrac{\hat{g}_{-m}}{\mu} \\ r_0 \bar{\hat{F}}_{-1} - p_0 \dfrac{\bar{\hat{g}}_{-1}}{\mu} \\ r_0 \bar{\hat{F}}_{-2} - p_0 \dfrac{\bar{\hat{g}}_{-2}}{\mu} \\ \vdots \\ r_0 \bar{\hat{F}}_{-m} - p_0 \dfrac{\bar{\hat{g}}_{-m}}{\mu} \end{bmatrix}, \qquad X_2 = \begin{pmatrix} \hat{p}_{-1} \\ \hat{p}_{-2} \\ \vdots \\ \hat{p}_{-m} \\ \hat{r}_1 \\ \hat{r}_2 \\ \vdots \\ \hat{r}_m \end{pmatrix} \tag{4.14}$$

Then (4.11) and (4.12) are

$$A_2 X_2 = V_2 \tag{4.15}$$

We can solve the system of linear equations (4.15) to get approximations of the functions p_μ and r_μ. Here the subscript μ is used to remind the readers that p and r depend on the value of μ. We remark that by the definition of $\hat{F}_k$, one can see that A_2 is Hermitian.

We now present an algorithm for the two-block problem (P2) as follows:

Step 1. (a) Use the fast Fourier transform to get $\hat{g}_{-1}, \hat{g}_{-2},..., \hat{g}_{-m}$, the approximation of the Fourier coefficients $g_{-1}, g_{-2},..., g_{-m}$ for the $\mathbf{L}^\infty$ function g.

(b) Use the fast Fourier transform to get $\hat{f}_0, \hat{f}_1,..., \hat{f}_m$, the approximation of the Fourier coefficients $f_0, f_1,..., f_m$ for the $\mathbf{H}^\infty$ function f.

Step 2. Pick r_0 and p_0 such that $|p_0| \neq |r_0|$. Beginning from $\mu = \|(\substack{g\\f})\|_\infty$, get the approximation $\{\hat{p}_{-1}, p_{-2},..., \hat{p}_{-m}\}$ and $\{\hat{r}_1, \hat{r}_2,..., \hat{r}_m\}$, of p_μ and r_μ by solving (4.15).

Step 3. Search for the largest $\mu < \|(\substack{g\\f})\|_\infty$ that makes

$$\left[\begin{pmatrix} \hat{p} \\ \hat{f}\hat{r} \\ \hat{r} \end{pmatrix}, \begin{pmatrix} \hat{p} \\ \hat{f}\hat{r} \\ \hat{r} \end{pmatrix}\right] = \langle \hat{p}, \hat{p} \rangle + \langle \hat{f}\hat{r}, \hat{f}\hat{r} \rangle - \langle \hat{r}, \hat{r} \rangle = 0$$

This μ is an estimate of the optimal performance μ_0 for (P2).

Remark 3. In the first algorithm we can also make the upper-right and lower-left submatrices of A_1 full as we did for the matrix A_2. For cases in which the Fourier coefficients of the function g decay slowly, this should be done to improve accuracy of the approximations.

5. NUMERICAL EXAMPLES

To illustrate how the algorithms perform we present results of numerical solutions obtained from the algorithms for some infinite-dimensional problems and compare them to the analytic solutions for these problems. The first example is for the one-block problem with the function g continuous on the extended imaginary axis. The second example is for the one-block problem with the function g discontinuous at ∞. Example 3 is a two-block problem with both functions g and f continuous on the extended imaginary axis. The last example is a two-block problem where the irrational part is an infinite Blaschke product.

Example 1. For the first example, we consider a simple delay problem with

$$g = \frac{e^s}{1+s}$$

This is a weighted pure sensitivity minimization problem for a pure delay system with weight $1/(1+s)$ studied in Foias *et al.* (1986). The optimal performance is $\mu_0 \approx 0.44215$ as can be computed based on the theoretic results in Foias *et al.* (1986). We used 64 sampling points on the unit circle for the function g so that for each value of μ we need to solve a system of 128 simultaneous linear equations; since the structure of the matrix is complex Hermitian, systems of even 1024 equations can be solved (using Cholesky decompositions) in a matter of seconds on a fast workstation so that the method is computationally very attractive. Analytic bounds can be placed on the optimal performance, and computation need only be performed within the analytic bounds to select the largest zero which solves the equation (4.4). Figure 2 shows the numerical results when $p_0 = 1$, $r_0 = 2$. Figure 3 shows the numerical results for a much different choice of p_0 and r_0. Of course, the answer should not depend on the choice of the free parameters p_0 and r_0 as long as they are not chosen to be equal. The numerical experiments show that indeed the answer is independent of the choice of p_0 and r_0 even though the plots of the solutions look quite different. With 64 sampling points for the function g the numerical solutions achieved three-digit accuracy.

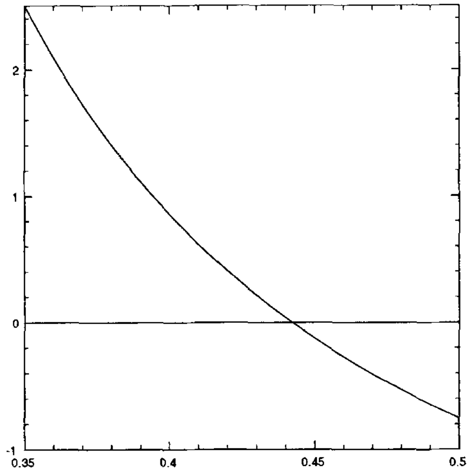

Fig. 2. Plot of $\langle \hat{p}, \hat{p} \rangle - \langle \hat{r}, \hat{r} \rangle$ for Example 1 with $p_0 = 1$, $r_0 = 2$.

Example 2. As a second example we investigate how the algorithm performs for

$$g = \frac{(1+s)\, e^{s}}{a+s}$$

This is a weighted pure sensitivity minimization problem for a pure delay system with weight $(1+s)/(a+s)$ studied in Flamm (1986). This problem

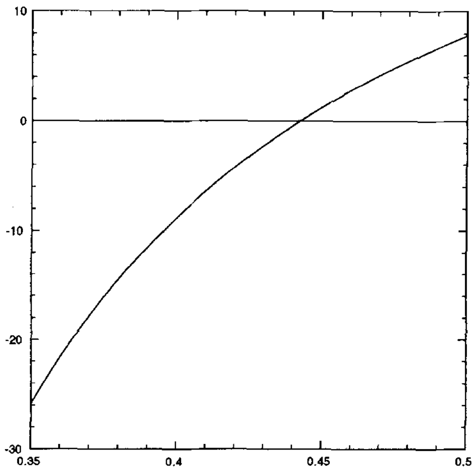

Fig. 3. Plot of $\langle \hat{p}, \hat{p} \rangle - \langle \hat{r}, \hat{r} \rangle$ for Example 1 with $p_0 = 5.0 - 4.0j$, $r_0 = 1.0 - 3.0j$.

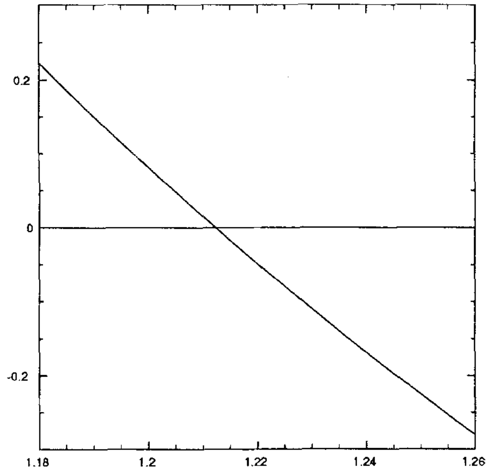

Fig. 4. Plot of $\langle \hat{p}, \hat{p} \rangle - \langle \hat{r}, \hat{r} \rangle$ for Example 2 with $p_0 = 1$, $r_0 = 2$.

is more difficult because g is not continuous on the extended imaginary axis, the related Hankel operator $\mathscr{H}_g$ is not compact. With $a = 0.5$ the theoretical result is an optimal performance $\mu_0 \approx 1.22$. The numerical results using the algorithm are shown in Figs. 4 and 5. Even though the problem has a singularity, with only 64 sampling points for the function g the algorithm gives the estimate of 1.214, which agrees well with the analytic solution. With more sampling points, accuracy of three or more digits was obtained.

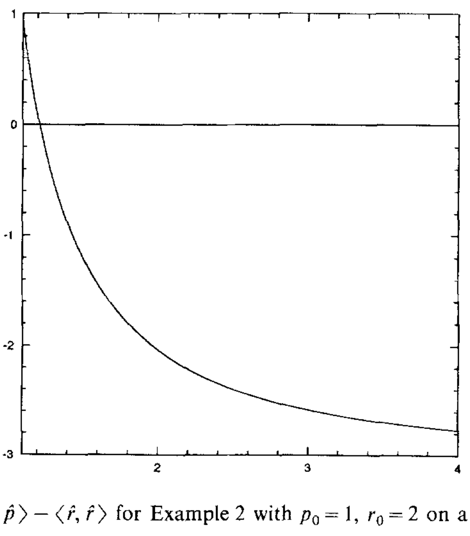

Fig. 5. Plot of $\langle \hat{p}, \hat{p} \rangle - \langle \hat{r}, \hat{r} \rangle$ for Example 2 with $p_0 = 1$, $r_0 = 2$ on a wider range of μ.

Example 3. We now show a simple example of the two-block problem. The functions g and f are

$$g(s)=\frac{e^{ds}}{(s+a)[(1+\varepsilon^2a^2)^{1/2}-\varepsilon^2a^2]}$$

$$f(s)=\frac{\varepsilon}{(1+\varepsilon^2a^2)^{1/2}+\varepsilon s}$$

This corresponds to a weighted mixed sensitivity minimization problem of a pure delay system with sensitivity weight $1/(1+s)$ and constant complementary sensitivity weight ε. Analytic solution can be obtained by using the results from Flamm and Yang (1990). Yang (1992) studies this example and show that "false zeros" can occur because of the occurrence of double poles in related projection formulas. This can result in a totally wrong estimate of the optimal performance if one is not careful. When the dimensions of the plant and weights are high, those "false zeros" can be very difficult to identify. The present algorithm does not produce "false zeros" and hence gives reliable results.

For a numerical example we choose $\varepsilon=1.1$, $a=0.9$, and $d=2.0$. With 64 sampling points for both g and f the numerical results are shown in Fig. 6. Single precision arithmetic was used for all calculations. With 512 or 1024 sampling points the numerical answer is correct to five decimal places. Even with only 16 sampling points the numerical answer is correct to four decimal places, and with 8 sampling points it is correct to three decimal places. Thus, even when very few sampling points are used on the functions, the numerical method produces remarkably good results. Numerical

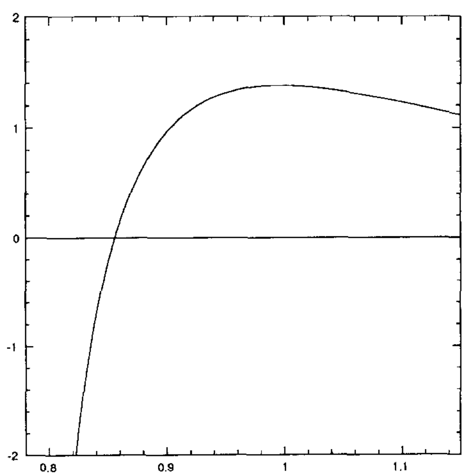

Fig. 6. Plot of $\langle \hat{p}, \hat{p}\rangle + \langle \hat{f}\hat{r}, \hat{f}\hat{r}\rangle - \langle \hat{r}, \hat{r}\rangle$ for Example 3 with $p_0=2.0-2.0j$, $r_0=-3.0$.

experiment also indicates that numerical solutions do not depend on the parameters p_0 and r_0.

Example 4. In this example we examine the Euler–Bernoulli beam model (Clough and Penzien, 1975) studied in Flamm (1990). With the sensitivity weight $1/(a+s)$ and the complementary sensitivity weight $\varepsilon(b+s)$, the corresponding g and f are (Flamm and Yang, 1990)

$$g = \frac{1}{\varepsilon(\xi_1 - s)(\xi_2 - s)(a+s)\, N_i(s)}$$

$$f = \frac{b+s}{(\xi_1 + s)(\xi_2 + s)}$$

where $N_i(s)$ is an infinite Blaschke product. The zeroes of $N_i(s)$ have been found explicitly in Flamm (1990). Finite Blaschke product approximation is used to evaluate $N_i(s)$. We compare the analytic solution given in Flamm and Yang (1990).

Parameters used were $a = 0.9$, $b = 0.8$, $\varepsilon = 1.1$. Using the formulas in Flamm and Yang (1990), ξ_1 and ξ_2 are computed as $\xi_1 = 0.970740716 + 0.46619474j$, $\xi_2 = 0.970740716 - 0.46619474j$. Single precision computation with only 64 sampling points for both functions g and f yielded an optimal performance of $\mu_0 = 0.862345$. These computations agree well with the analytic solution which gives $\mu_0 \approx 0.86236933$. The results are shown in Fig. 7. Again the answer does not depend on different choices of p_0 and r_0.

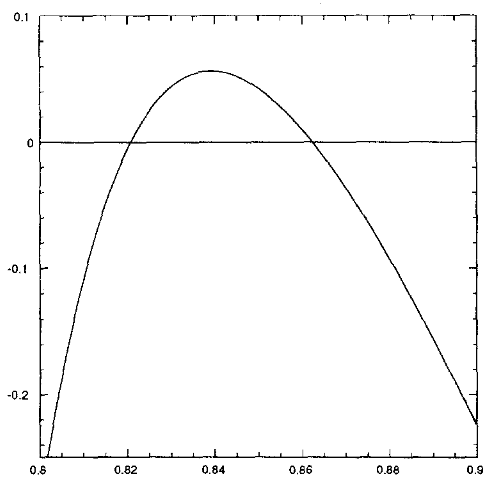

Fig. 7. Plot of $\langle \hat{p}, \hat{p} \rangle + \langle \hat{fr}, \hat{fr} \rangle - \langle \hat{r}, \hat{r} \rangle$ for Example 4 with $p_0 = 2.0$, $r_0 = -3.0$.

6. CONCLUSION

This article has presented two simple numerical algorithms for the computation of the optimal performances for the $\mathbf{H}^\infty$ weighted pure and mixed sensitivity minimization problems. The algorithms use FFT and fast matrix solvers for complex Hermitian matrices and hence are computationally very attractive. The algorithms are very easy to implement; they do not require the computation of varies projections as in the analytic frequency domain approach. Therefore the difficulties associated with the projections such as high dimensionality and complexity of formulation with high-order poles are not present in our algorithms. Our method treats the irrational and rational cases in a unified way. The computation does not increase as the order of the model and weights increase. Applications of the numerical methods are presented to problems involving delays as well as to a PDE model. Comparison with the analytic results shows that even with a small number of sampling points the algorithms give very accurate estimates of the optimal performances both in the continuous and the discontinuous cases. The algorithms presented here are also applicable to the solution of $\mathbf{H}^\infty$ control problems with rational transfer functions where the high order of models in some practical problems presents numerical problems.

The strength of the algorithms is highlighted by the ease at which they can be extended to be applied to $\mathbf{H}^\infty$ design for MIMO systems and model reduction such as are required in spacecraft design problems as well as multi-input/single-output problems such as occur in portfolio choice in financial economics (Orszag and Yang, 1992; Yang and Orszag, 1992). In addition, we believe that the algorithms can also be used to compute suboptimal controllers. Most $\mathbf{H}^\infty$ controllers tend to be of fairly high order so the approach here may not only speed computation but also eliminate some numerical errors in controller design. We also believe that the approach here is one that can be adapted for spectral prediction of time series with irrational spectral densities. Since rational spectral densities can be estimated well in the time domain, the time series literature has moved away from frequency domain analysis. The present study raises new possibilities for the spectral analysis of time series which are not feasible in the time domain.

ACKNOWLEDGMENTS

This work was partially supported by AFOSR grant No. AFOSR-91-0231 and ONR grant No. N00014-90-J-1315.

REFERENCES

Ball, J. A., Helton, J. W. (1983). A Beurling–Lax theorem for the Lie group $u(m, n)$ which contains most classical interpolation theory, *J. Operator Theory* **9**(1), 107–142.

Clough, R. W., and Penzien, J. (1975). *Dynamics of Structures*, McGraw-Hill, New York.

Doyle, J. C., and Stein, G. (1980). Multivariable feedback design: Concepts for a classical/modern synthesis, *IEEE Trans. Automatic Control* **AC-26**(1), 4–16.

Fagnani, F. (1991). An operator-theoretic approach to the mixed-sensitivity minimization problem, *Syst. Control Lett.* **17**, 227–235.

Flamm, D. S. (1986). Control of Delay Systems for Minimax Sensitivity, Technical Report No. LIDS-TH-1560, Massachusetts Institute of Technology Laboratory for Information and Decision Systems.

Flamm, D. S. (1990). A Model of a Damped Flexible Beam, ISS Report No. 54, Department of Electrical Engineering, Princeton University.

Flamm, D. S. and Yang, H. (1990). Optimal mixed sensitivity for general distributed plants, to appear in *IEEE Trans. Automatic Control.*

Foias, C., Tannenbaum, A., and Zames, G. (1986). Weighted sensitivity minimization for delay systems, *IEEE Trans. Automatic Control* **AC-31**(8), 763–766.

Francis, B., Helton, J. W., and Zames, G. (1984). $\mathbf{H}^\infty$-optimal feedback controllers for linear multivariable systems, *IEEE Trans. Automatic Control* **AC-29**(10), 888–900.

Francis, B. A. (1987). *A Course in* $\mathbf{H}^\infty$ *Control Theory*, Springer-Verlag, New York.

Hoffman, K. (1962). *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, New Jersey.

Orszag, J. M., and Yang, H. (1992). Portfolio Choice with Knightian Uncertainty, Department of Economics, University of Michigan.

Rosenblum, M., and Rovnyak, J. (1985). *Hardy Classes and Operator Theory*, Oxford University Press, Oxford.

Safonov, M. G., Laub, A. J., and Hartmann, G. (1980). Feedback properties of multivariable systems: The role and use of the return difference matrix, *IEEEE Trans. Automatic Control* **AC-26**(1), 47–65.

Sarason, D. (1985). Operator-theoretic aspects of the Nevanlinna–Pick interpolation problem, in *Operator and Function Theory*, pages 279–314, 1985.

Yang, H. (1993). $\mathbf{H}^\infty$ optimal compensators for a class of infinite dimensional systems, submitted to *American Control Conference* 1993.

Yang, H. (1992). Frequency domain method for $\mathbf{H}^\infty$ optimal mixed sensitivity design, manuscript.

Yang, H., and Flamm, D. S. (1990). Mixed sensitivity design for a class of multivariable infinite dimensional systems. Submitted to *IEEE Transactions on Automatic Control.*

Yang, H., and Orszag, J. M. (1992). Spectral and nonparametric optimization methods for portfolio choice. Princeton University Program in Applied and Computational Mathematics, October 1992.