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Research and Development Technical Report ECOM-0138-14-T

PARTIAL-PERIOD CORRELATION OF PSEUDO-RANDOM BINARY SEQUENCES

Technical Report No. 202

by

J. L. Daws, Jr.

March 1970

ECOM

UNITED STATES ARMY ELECTRONICS COMMAND FORT MONMOUTH, N.J.

Contract No. DAAB 07-68-C-0138

COOLEY ELECTRONICS LABORATORY
Department of Electrical Engineering
The University of Michigan
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PARTIAL-PERIOD CORRELATION OF PSEUDO-RANDOM BINARY SEQUENCES

C. E. L. Technical Report No. 202

Contract No. DAAB 07-68-C-0138
DA Project No. 1 HO 211101 A04 01 02

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ABSTRACT

The partial-period, or short-time, correlation between linear maximal binary sequences of the same length is studied from a number of viewpoints. Partial-period correlation is defined to include all cases in which the correlation time base is less than a full period of the sequences involved. The purpose of this study of the partial-period correlation function is to determine how this correlation function depends on the particular sequences used.

It is shown that the study of the partial-period correlation is directly related to the study of weight distributions of linear codes and the standard coding theory bounds are applied to the correlation problem. The coding theory bounds are used to determine, in general, how the peak values of correlation behave as the correlation interval is decreased, and as a means of establishing a range of possible correlation peaks. Peak values of the partial-period correlation function for selected pairs of sequences are compared with the bounds and for the cases examined the peak values are not close to the bounds,

except at the end points.

The moments of the distribution of correlation values are calculated. The first two moments of this distribution depend only on the length of the sequences and the number of digits in the correlation. The higher order moments depend on the particular sequences through the characteristic polynomials. The moments provide a method for comparing the correlation distributions for pairs of sequences.

The correlation function is calculated in terms of the Fourier series coefficients of the periodic sequences. In order to accomplish this it was necessary to study the Fourier series of linear maximal binary sequences, and it is found that, for the characteristic phase position of the sequences, the Fourier series coefficients for all linear maximal sequences of the same length are related in a well defined manner. The frequency domain description of the correlation function leads to extremely complex equations; however, they do provide another technique for examining the structure of the partial-period correlation function.

The correlation ambiguity matrix is introduced as a technique for displaying the partial-period correlation functions. The correlation ambiguity matrix represents the actual correlation function a system must use if it is operating in a partial-period environment. The difference matrix, in which the elements are found by subtracting the mean value along a diagonal from the elements of the correlation ambiguity matrix, is also introduced as a possible tool in the study of partial-period correlation functions. This matrix is a direct method of displaying how the full period correlation function is modified as the correlation interval is decreased.

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LIST OF SYMBOLS

a_i	element of a sequence of binary digits taken from the set $\{0, 1\}$, the sequence dependent part of the ith Fourier coefficient
A	number of agreements in the correlation of two sequences
b_i	element of a sequence of binary digits taken from the set $\{0, 1\}$, the sequence dependent part of the ith Fourier coefficient
B_{xj}, B_{yj}, B_{xyj}	number of j-term polynomials divisible by the characteristic polynomial of the sequence $\{x_i\}$, $\{y_i\}$ or $\{x_i y_i\}$
$c_i(\tau)$	element of a sequence generated by mod-2 adding two maximal $\{0, 1\}$ sequences, i.e., $c_i(\tau) = a_{i+\tau} \oplus b_i$
C_j	coset
$C_s, C_{s_x s_y}$	correlation value as a function of the starting point in the sequences; element of correlation ambiguity matrix
$\overline{C^p}$	pth moment of correlation distribution
d_0	minimum distance between code vectors for a linear code
D	number of disagreements in the correlation of two sequences
e	number of errors in a code vector
f_0	fundamental frequency in the frequency spectrum of a linear maximal binary sequence, $f_0 = 1/T_0$
f(X)	characteristic polynomial for the sequence $\{x_i\}$

$F(j, p, k)$	coefficient in the expansion of pth order products into jth order products in the calculation of the pth moment of the correlation distribution for k digits
G	generator matrix for linear code
$h(\,)$	impulse response of a filter
I_j	$j \times j$ identity matrix
k	number of digits in partial period correlation; number of elements in a linear code word
L	length of linear maximal sequence, $L = 2^n - 1$
L_T	total number of partial period correlation values
m_i	coset leader of the coset C_i
n	number of stages in a shift-register generator
$N_{w \leq e}$	number of k-tuples of weight equal to or less than e
N_{xj}	number of j-term polynomials of the form $x^d + x^c + \ldots + 1$ which are divisible by the characteristic polynomial of the sequence $\{x_i\}$
P	matrix contained in the generator matrix for a linear code in which the elements are the check symbols of the code
PR	pseudo-random
r	number of redundant symbols, i.e., check symbols, in a linear code. $r = k - 2n$
R	rate of a code, $R = \frac{2n}{k}$
RADA	random access discrete address
$s(\,)$	stored reference signal

LIST OF SYMBOLS (Cont.)

S_j	sequence corresponding to the coset C_j
s, s_x, s_y	starting point in sequence relative to the characteristic phase position
t_c	clock period for sequence waveform
T_0	period of the sequence waveform, $T_0 = L t_c$
$V_{s_x s_y}$	element of the difference matrix
W_s	Hamming weight of a subsequence of k-digits
$\overline{W}$	average Hamming weight
$\overline{W^p}$	pth moment of the distribution of weights
x_i	element of a sequence of binary digits taken from the set $\{1, -1\}$, value of ith pulse in sequence waveform
$x(\,)$	sequence waveform
y_i	element of a sequence of binary digits taken from the set $\{1, -1\}$
$y(\,)$	sequence waveform
$z_i(\tau)$	element of the product sequence; $z_i(\tau) = x_{i+\tau} y_i$
$z(t, \tau)$	product sequence waveform $x(t+\tau)\, y(t)$
α_r	Fourier coefficient for linear maximal sequence waveform $x(t)$
β_r	Fourier coefficient for linear maximal sequence waveform $y(t)$
γ_r	Fourier coefficient for shortened sequence
β_r^*	complex conjugate of β_r

LIST OF SYMBOLS (Cont.)

$\delta(\ )$	impulse function
$\theta_{a_r}, \theta_{b_r}, \theta_r$	Fourier coefficient phase angle
$\rho_{xy}(\tau)$	normalized full period cross-correlation function
$\rho_{xy}(k, \tau, s)$	normalized partial period cross-correlation function
$\rho_{s_x s_y}$	partial period cross-correlation value
$\rho_{xy}\|_{max}$	maximum cross-correlation value
$\rho_x(\tau)$	normalized full period autocorrelation function
$\rho_x(k, \tau, s)$	normalized partial period autocorrelation function
$\rho_x\|_{max}$	maximum value of autocorrelation
$\overline{\rho}$	average value of correlation function
τ	relative phase shift between sequences
$\phi(\ )$	Euler ϕ-function
$(\ ,\ )$	greatest common divisor
$\{\ \}$	sequence; coset; set
$\|\ \|$	magnitude
$[\cdot]$	greatest integer equal to or less than
$*$	convolution
$\oplus$	mod-2 sum
$\binom{k}{i}$	binomial coefficient
(k, 2n)	linear code 2n information bits, k total bits

CHAPTER I

INTRODUCTION

1.1 The Problem

During the past several years, there has been a steady increase in the use of digital techniques in communication systems. This increasing interest in digital communications techniques has, at least in part, been brought about by the rapidly increasing use of large scale digital computers and digital data processing. The number of data sources has increased tremendously and accompanying this growth is the increasing need for intercommunications which can most readily be accomplished by digital techniques. Another reason for the increasing interest in digital communications has been the rapid improvement in digital component technology. The reliability and speed of digital components has increased while the size has been steadily shrinking. As noted by Golomb (Ref. 1), the requirements for space communications, i.e., reliability over long missions, precision in the data gathered, and efficient use of the channel available, make digital communications very attractive, if not necessary, for this application. In addition to transmission of information from inherently digital sources, digital communication systems may be used to transmit analog information after it has been sampled and quantized. Digital communications techniques may offer some advantages over analog techniques for the transmission of this "analog" information including ease in multiplexing with other digital signals for transmission over a common trunk, ease with which error correcting coding may be used to improve performance, and, in particular for military communications, the ease in adding varying degrees of security. An additional advantage of digital communications is its ready adaptability to the random access system concept.

In digital communications it is necessary for the receiver to determine the message sequence which was transmitted where each element of the sequence is drawn from a finite set of symbols or waveforms. In the most common case the message sequence will be made up of binary elements and the receiver must make a decision between two alternatives for each element of the message sequence. One receiver technique for implementing the required decision making function is that of correlation detection. For correlation detection it is desirable to use waveforms which are mutually uncorrelated to represent the possible signals and the search for uncorrelated, or orthogonal, digital signals has led to the study of pseudo-random (PR) sequences. PR sequences are digital sequences which appear, based on certain tests, to have many of the properties of a random sequence but which can be reproduced by a deterministic process. In this study the sequences considered are binary sequences, i.e., sequences of 1's and 0's or -1's and +1's. The sequences which are considered here all satisfy the randomness properties of Golomb listed below: (see Refs. 1 and 2)

(1) The balance property: in each period of the sequence, the difference between the number of 1's and the number of 0's is at most 1. In fact, for the linear maximal sequences studied, there will always be one more 1 than 0.

(2) The run property: a run is defined to be a series of like digits within the sequence (either 1's or 0's in the binary case), the length of which is the number of digits involved. In any periodic binary sequence there must be an equal number of runs of 1's as of 0's and for the pseudo-random case, one-half of the runs of each type in a period of the sequence are of length 1, one-fourth of length 2, $(\frac{1}{2})^k$ of length k, as long as these numbers have any meaning.

(3) The correlation property: in the term-by-term comparison of the sequence with a cyclic shift of itself, the number of agreements will differ from the number of disagreements by, at most, one.

One particularly simple method of generating the sequences which satisfy these conditions involves the use of a shift-register generator (see Refs. 1, 2, and 4). A shift register is a set of binary storage elements which have been connected in such a way that the contents of the ith stage can be shifted by a clock pulse into the (i+1)st stage. One configuration for a shift-register generator consists of an n-stage shift register with a feedback network which computes a logical function of the contents of the n stages and uses the result to feed the first stage of the shift register. The feedback is necessary to generate any sequence longer than n digits since the basic shift register would be empty after n clock pulses. For sequences of interest in this study the feedback function is linear (the modulo-2 sum of selected stages), and the sequences generated are termed linear shift register sequences. This form of shift-register generator has been called a simple shift-register generator. For certain choices of the feedback stages of the shift register a sequence of length $L = 2^n - 1$, the maximum possible length for a generator of this type, is generated. Such sequences are termed linear maximal binary sequences. The generation of linear maximal binary sequences has been thoroughly covered in the literature which is discussed in the next section.

The randomness properties listed above apply when the entire period of the sequence is considered, i.e., the entire L digits. If these sequences are used as a signal waveform in a system with correlation detection, the correlation integration must be carried out over an integral number of periods of the sequence to insure that the autocorrelation properties implied by randomness property number 3 are obtained. In some situations, when the sequence period is very long, this may be impractical because of the time required. In these cases it would be useful to know the effect of partial period autocorrelation. For a given value of n there exists a set of feedback relationships which result in the generation of different linear maximal sequences of the same length. It is possible that the environment in which a system making use of sequence properties must operate will contain other signals from the set generated by linear maximal shift-register generators of the same length. In this case the cross-correlation between pairs of sequences from the set becomes important because of the interference involved. For example, in a random access discrete address (RADA) system the sequences from the set might be used for addressing purposes, i.e., the particular station called might depend on the sequence used by the transmitter. In a code multiplexing situation, as proposed for a satellite communications system, the sequences would be used as a signal waveform and the separation of the signals would depend on the cross-correlation properties of the sequences. The full period cross-correlation has been studied (see Gold, Ref. 6) however, as noted previously for the autocorrelation case, the system parameters may be such that correlation over the entire period of the sequences involved will be impractical. In order to determine system performance in this situation it is necessary to consider the partial period cross-correlation.

The objective of the research in this thesis is the study of the partial period correlation function where the signals involved are linear maximal binary sequences of the same length. Partial period, or short-time, correlation is meant to include all cases in which the correlation is carried out over less than the full period of the sequences involved. The partial period autocorrelation is a special case in which the sequences involved are identical. Since the full period correlation functions have been tabulated by Gold (Ref. 6) for generators of 13 stages and less it is the function of this study to determine how these cross-correlation functions are modified as the integration time of the correlator is decreased.

1.2 Survey of the Literature

During the past several years a great many reports and papers have been written concerning the generation, properties, and uses of PR sequences. In this section the literature pertinent to the problem studied here is summarized.

The theory of sequence generation is discussed in a number of papers which were published in the mid-1950's. It should be noted that the mathematical study of linear recurrence relations goes back much further, however the implementation and practical application of sequence generators was delayed until this time. Some of the better known papers and reports have recently been published in book form (Refs. 2 and 3). In particular, Golomb's report, "Sequences with Randomness Properties", which was first published in 1955 is included in Ref. 2 and papers by Elspas, Zierler, and others, on sequence generation are included in Ref. 3. These papers, along with the report by Birdsall and Ristenbatt (Ref. 4) consider primarily the generation of linear sequences and develop the mathematical tools necessary to analyze these sequences. Elspas (Ref. 3) has extended the work with binary sequences to p-nary sequences where p is any prime integer. Golomb, in the section on linear sequences, and Birdsall and Ristenbatt have used the simple shift-register generator to generate the sequences and have discussed in detail the properties of these sequences and generators. Hoopes and Randall (Ref. 5) have discussed four additional generators; the modular shift-register generator, the simple complement register generator, the modular complement register generator, and a hybrid generator which contains both shift and complement stages. A given sequence may be generated by any of these generators and the relationships between these generators are given in Ref. 5. The generation of binary sequences has also been considered in books which are primarily concerned with the application of the sequences. For example, Golomb (Ref. 1) has discussed sequence generation and the uses of sequences in space communications. Peterson (Ref. 7), in addition to a chapter on linear switching circuits, has two chapters which cover the mathematics needed to analyze sequences. The emphasis in this book (Ref. 7) is in showing how linear sequences may be used in error correcting codes. Berkowitz (Ref. 8) has a chapter (Chapter 4 by M. P. Ristenbatt) which contains the basic shift register theory along with the application of sequences to radar problems. The generation of linear maximal sequences is not considered in detail in this report and the interested reader is referred to any one of these references for additional information concerning the generation of sequences.

Since the problem to be studied here is concerned with correlation properties of PR sequences, references to previous studies of these properties are of particular interest. It should be noted that many of the references cited above on sequence generation will also include the full period correlation properties of the sequences involved. Titsworth (Ref. 9) has used correlation to compare PR sequences to Markov chains. It is shown that periodic sequences which exhibit exact Markovian character do not exist and that the only nontrivial sequences with two level correlation matrices are binary pseudonoise sequences. Judge (Ref. 10) has investigated the use of binary codes for multiplexing. Gill (Refs. 11 and 12) is primarily concerned with the autocorrelation properties and how they are affected by synchronization error. Persons (Ref. 13) has extended Gill's work to include time displacements of more than one digit of the sequence. In the references cited above the sequences have been binary and the entire period of the sequence has been used.

Gilchriest (Ref. 14) has analyzed the effect of filtering the received PR sequence by either an RC high-pass or an RC low-pass filter before correlation with the locally generated PR reference sequence. Both the case of identical sequences and different PR sequences are covered and a number of curves are included to show the effect of different filter time constants. Roberts and Davis (Ref. 15) investigate the effect of a simple low-pass filter being used to smooth a linear maximal sequence. The autocorrelation function, power spectrum, and the amplitude probability distribution of the output are discussed. Gupta and Painter (Ref. 16) have presented a technique for the determination of the behavior of the cross-correlation function between a PR sequence and the signal at any point of interest in a linear system driven by the sequence.

While the references cited above (Refs. 9 - 16) are generally related to the area of correlation of PR sequences the following references are specifically related to the problem studied here and will be discussed in more detail in Chapter II. Stalder and Cahn (Ref. 17) have calculated a lower bound on the peak value of the auto- and cross-correlation functions for two digital sequences of the same length. The bounds found depend upon the correlation being taken over a full period of the sequences involved. As noted previously, Gold (Refs. 6 and 36) has studied thoroughly the full period cross-correlation properties of linear maximal PR sequences. In this study the full period cross-correlation between all linear maximal sequences of a given length has been tabulated for sequences generated by shift-register generators of 13, or less, stages. (The tabulated correlation functions are contained in Ref. 36.) This report (Ref. 6) contains an excellent theoretical discussion of sequence generation and the cross-correlation problem. A method is developed for finding pairs of sequences for which the maximum absolute value of the cross-correlation is minimum, and shows that this is the "best" cross-correlation that can be achieved. ("Best" in the sense that the maximum absolute value of cross-correlation as a function of time shift for this pair of sequences is equal to or less than the maximum absolute value of cross-correlation for any other pair of sequences of the same length.)

The partial period, or short time, autocorrelation properties have been discussed in a few papers. Tausworthe (Ref. 18) has calculated the mean and variance of the partial period autocorrelation for linear maximal binary sequences. Lindholm (Ref. 19) and Mattson and Turyn (Ref. 20) have investigated the distribution of the weights of m-tuples of a linear maximal sequence. The weight is the Hamming weight in a sequence of m digits taken from a linear maximal sequence of length L where L is larger than m. Because of the shift-and-add property of linear maximal sequences this is equivalent to finding the distribution of the autocorrelation values over m digits. One of the most important results of this paper (Ref. 19) is the fact that the 3rd moment of the distribution of weights is the lowest order moment which depends on the particular sequence involved (i.e., depends on the sequence law). The 3rd moment then can be used to weed out those sequences which have skewed weight distributions and consequently do not have good randomness properties as far as subsequences (m-tuples) are concerned. Additional work along this line has been done by Cooper and Lord (Ref. 21) and Schmandt, et al. (Ref. 22).

1.3 Areas of Investigation and Format

The study of the partial period correlation function is divided into two major areas of investigation. First, it is shown that the partial period correlation for linear maximal binary sequences is closely related to the study of linear codes, and that coding theory may be useful in determining properties of the correlation function. The standard coding theory bounds have been applied to the code words which are generated in the cross-correlation process. The moments of the distribution of correlation values which result from the partial period correlation are studied. The second major area of investigation is the frequency domain description of the partial period correlation function. For this description of the partial period correlation function the Fourier series coefficients for linear maximal binary sequences are calculated and the relationship between the coefficients for maximal sequences of the same length is shown.

In Chapter II the background information concerning correlation and sequence properties, which is used in the remaining chapters, is reviewed. In Chapter III the standard coding theory bounds are applied to the problem studied here and the results compared with computer search experimental bounds. In Chapter IV the moments of the distribution of correlation values are calculated. These results are compared with moments of k-tuples taken from a random sequence. In Chapter V the Fourier coefficients for PR sequences are calculated and the correlation function is given as a function of the Fourier series coefficients. In Chapter VI the correlation ambiguity diagram is introduced along with a discussion of how it may be used as a tool for partial period correlation situations. Finally, the conclusions of this research are given in Chapter VII.

CHAPTER II

CORRELATION AND SEQUENCES

The purpose of this chapter is to present a review of the known properties of the correlation function for linear maximal binary sequences along with those properties of the sequences which will be used for this study. The relationship between partial period correlation and linear codes will be discussed. The characteristic phase position of the linear maximal sequence will be defined and the coset function assignments which lead to the characteristic sequences will be considered. The information in this chapter is, for the most part, well known, however it is essential for the understanding of the remaining chapters and has been included for the sake of completeness.

2.1 General Correlation Properties of Linear Maximal Binary Sequences

In the theory of sequence generation the sequences have usually been considered to be sequences of 1's and 0's. With this convention the maximal sequence and all its phase shifts form an Abelian group in which the group operation is term-by-term addition, modulo-2. An equivalent description of binary sequences can be given in terms of sequences of elements taken from the set $\{1, -1\}$ in which the maximal sequence with all its phase shifts forms an Abelian group in which the group operation is term-by-term multiplication. In order to show how these two conventions for digital sequences are related let $\{a_i\} = \{a_0, a_1, a_2, \ldots\}$ be a linear maximal sequence whose elements are taken from the set $\{0, 1\}$. Then the linear maximal sequence $\{x_i\} = \{x_0, x_1, x_2, \ldots\}$ whose elements are from the set $\{1, -1\}$ is related to the sequence $\{a_i\}$ as follows:

$$x_i = 1 - 2\,a_i \tag{2.1}$$

Since these sequences are periodic the full period correlation can be calculated over one period. The correlation of two digital sequences is given by

$$\rho_{a,b} = \frac{A - D}{A + D} = \frac{L - 2D}{L} \tag{2.2}$$

where

$\{a_i\}$ and $\{b_i\}$ are periodic binary sequences of common period L and

A = number of bit-by-bit agreements/period

D = number of bit-by-bit disagreements/period

With this definition of correlation it can readily be seen how the correlation is calculated for the $\{0, 1\}$ sequences and the $\{1, 1\}$ sequences. With the $\{0, 1\}$ sequences the bit-by-bit mod-2 sum of the sequences will result in a 1 whenever the bits disagree. The correlation between the sequence $\{a_i\}$ and $\{b_i\}$, where these sequences are $\{0,1\}$ sequences is as follows:

$$\rho_{a,b}(\tau) = 1 - \frac{2}{L} \sum_{i=0}^{L-1} a_{i+\tau} \oplus b_i \tag{2.3}$$

$\oplus \Longrightarrow$ addition - mod-2

L = length of the sequence $(L = 2^n - 1)$

For the $\{1,-1\}$ sequences it should be noted that bit-by-bit multiplication will result in a $+1$ if the digits have the same value and a -1 if the digits have a different value and therefore, if $\{x_i\}$ and $\{y_i\}$ are $\{1,-1\}$ sequences, the correlation over the full period can be expressed as

$$\rho_{xy}(\tau) = \frac{1}{L} \sum_{i=0}^{L-1} x_{i+\tau}\, y_i \tag{2.4}$$

If the sequence $\{x_i\}$ is related to the sequence $\{a_i\}$ by the relationship given in Eq. 2.1 above and the sequence $\{y_i\}$ is related to the sequence $\{b_i\}$ by the same relationship then Eqs. 2.3 and 2.4 will give identical results for the correlation function. Therefore either definition of the sequence can be used to study the correlation function. In the remaining chapters of this report those sequences designated by letters from the beginning of the alphabet will be $\{0,1\}$ sequences and those sequences designated by letters from the end of the alphabet will be $\{1, -1\}$ sequences. In either case the period of the sequence is assumed to be $L (L = 2^n - 1)$. In the definitions and properties of correlation functions which follow the $\{1, -1\}$ sequences will be used.

The full period autocorrelation for the sequence $\{x_i\}$ is defined by

$$\rho_x(\tau) = \frac{1}{L} \sum_{i=0}^{L-1} x_{i+\tau} x_i \qquad (2.5)$$

When $\tau = 0$ the product inside the summation is always equal to 1 and it can be seen that $\rho_x(0) = 1$. Since $x_{i+\ell L} = x_i$ for all integer values of ℓ it can be seen that $\rho_x(\ell L)$ will also be equal to one. For all other values of τ the product inside the summation will, term-by-term, be equivalent to the sequence $\{x_i\}$ at some other phase shift due to the shift-and-multiply property of $\{1, -1\}$ linear maximal sequences. In this case then the summation will be just the summation of the terms in the sequence and the correlation will be equal to $-1/L$. This is the well-known two-level autocorrelation function of linear maximal binary sequences, i.e.,

$$\rho_x(\tau) = \begin{cases} 1 & \text{for} \quad \tau = 0 \quad \text{mod } L \\ -\frac{1}{L} & \text{for} \quad \tau \neq 0 \quad \text{mod } L \end{cases} \qquad (2.6)$$

for $\{x_i\}$ a linear maximal sequence.

The value of the correlation function for non-integer values of τ can be found by joining the points given above by a straight line.

The partial period, or short time, autocorrelation is defined by

$$\rho_x(k, \tau, s) = \frac{1}{k} \sum_{i=0}^{k-1} x_{i+s+\tau}\, x_{i+s} \tag{2.7}$$

s = starting point in the sequence

k = number of digits in the partial period correlation.

The values of k of interest are in the range $2n \leq k < L$. It should be noted that the short time correlation is a function of the particular starting point in the sequence since k digits can be chosen from the sequence starting at L different points. The average value of the short time autocorrelation for a given sequence $\{x_i\}$ can be found by taking the average over all starting points in the sequence.

$$\rho_x(k, \tau) = \frac{1}{L} \sum_{s=0}^{L-1} \left(\frac{1}{k} \sum_{i=0}^{k-1} x_{i+s+\tau}\, x_{i+s} \right)$$

$$= \frac{1}{k} \sum_{i=0}^{k-1} \left(\frac{1}{L} \sum_{s=0}^{L-1} x_{i+s+\tau}\, x_{i+s} \right) \tag{2.8}$$

$$\rho_x(k, \tau) = \rho_x(\tau)$$

Thus for a given value of τ, the average value over the starting points for the short time autocorrelation is just equal to the full period autocorrelation. The final property of the autocorrelation function which may be of interest is the sum of the values of the autocorrelation function over one full period, i.e.,

$$\begin{aligned} \sum_{\tau=0}^{L-1} \rho_x(\tau) &= \sum_{\tau=0}^{L-1} \left(\frac{1}{L} \sum_{i=0}^{L-1} x_{i+\tau} x_i \right) \\ &= \frac{1}{L} \sum_{i=0}^{L-1} x_i \sum_{\tau=0}^{L-1} x_{i+\tau} \\ &= \frac{1}{L} \left[(\text{number of } +1\text{'s}) - (\text{number of } -1\text{'s})\right]^2 \\ &= \frac{1}{L} \left[L - 2\,(\text{number of } -1\text{'s})\right]^2 \end{aligned} \tag{2.9}$$

Equation 2.9 is valid for any periodic $\{1, -1\}$ sequence. For a linear maximal binary sequence it is known that there are always $(L+1)/2$ positions which have value -1 and this equation reduces to:

$$\sum_{\tau=0}^{L-1} \rho_x(\tau) = \frac{1}{L} \left[L - 2\left(\frac{L+1}{2}\right)\right]^2 = \frac{1}{L} \tag{2.10}$$

Thus the sum of the values, at integer shifts, of the autocorrelation function of a linear maximal sequence is seen to be a function of only the period of the sequence.

The full period cross-correlation between two linear maximal sequences was given previously in Eq. 2.4 and is repeated below.

$$\rho_{xy}(\tau) = \frac{1}{L} \sum_{i=0}^{L-1} x_{i+\tau} y_i \tag{2.4}$$

The cross-correlation between linear maximal sequences is not so amenable to analysis as the autocorrelation, however, as noted previously, Gold (Ref. 6) has studied this problem and his results will be given later. The partial period, or short time, cross-correlation is defined by

$$\rho_{xy}(k, \tau, s) = \frac{1}{k} \sum_{i=0}^{k-1} x_{i+s+\tau} y_{i+s} \tag{2.11}$$

s = the starting point in the sequences

k = number of digits in the partial period correlation.

Here again, as with the partial period autocorrelation function, there are L starting points in the short time cross-correlation for a given value of τ, and the partial period cross-correlation must also be a function of the starting point s. The partial period cross-correlation may be averaged over s to find the mean value as a function of starting point and, as shown in Eq. 2.12, this mean value is again the full period cross-correlation value.

$$\rho_{xy}(k,\tau) = \frac{1}{L}\sum_{s=0}^{L-1}\left(\frac{1}{k}\sum_{i=0}^{k-1} x_{i+s+\tau}\, y_{i+s}\right)$$

$$= \frac{1}{k}\sum_{i=0}^{k-1}\rho_{xy}(\tau) = \rho_{xy}(\tau) \tag{2.12}$$

The sum of the full period cross-correlation values can be found as shown in Eq. 2.13.

$$\sum_{\tau=0}^{L-1}\rho_{xy}(\tau) = \sum_{\tau=0}^{L-1}\left(\frac{1}{L}\sum_{i=0}^{L-1} x_{i+\tau}\, y_i\right)$$

$$= \frac{1}{L}\sum_{i=0}^{L-1} y_i \sum_{\tau=0}^{L-1} x_{i+\tau} \tag{2.13}$$

For linear maximal sequences the sum is 1 and Eq. 2.13 will reduce to:

$$\sum_{\tau=0}^{L-1}\rho_{xy}(\tau) = \frac{1}{L} \tag{2.14}$$

The final property of the full period correlation functions which will be shown at this point is a relationship between the full period cross-correlation between linear maximal sequences and the full period autocorrelation for the same sequences, i.e.,

$$\sum_{\tau=0}^{L-1}\rho_{xy}^2(\tau) = \sum_{\tau=0}^{L-1}\rho_x(\tau)\,\rho_y(\tau) \tag{2.15}$$

This relationship may be easily verified by direct substitution of the definition of cross-correlation into the left-hand side of the equation and regrouping the terms. For linear maximal sequences of period L the autocorrelation function for both $\{x_i\}$ and $\{y_i\}$ is given in Eq. 2.6 above and Eq. 2.15 reduces to:

$$\sum_{\tau=0}^{L-1} \rho_{xy}^2(\tau) = 1 + (L-1)\left(\frac{1}{L^2}\right) = \frac{L^2 + L - 1}{L^2} \qquad (2.16)$$

As noted previously Stalder and Cahn (Ref. 17) have calculated a lower bound on the peak value of the auto- and cross-correlation functions for two digital sequences of the same length. The calculation of this bound makes use of Eq. 2.15 above and the results are shown as follows:

Let $\rho_{xy}\big|_{max}$ be the largest cross-correlation peak and $\rho_x\big|_{max}$ and $\rho_y\big|_{max}$ be the largest autocorrelation peaks for non-zero phase shift. Then it is shown that

$$\sqrt{\rho_{xy}\big|_{max}^{\,2} + \left(\rho_x\big|_{max}\right)\left(\rho_y\big|_{max}\right)} > \sqrt{\frac{1}{L}} \qquad (2.17)$$

The derivation of this bound makes use of the full period correlation and hence when the sequences $\{x_i\}$ and $\{y_i\}$ are linear maximal binary sequences it is known that $\rho_x\big|_{max} = \rho_y\big|_{max} = -1/L$ and

Eq. 2.17 can be rewritten as shown below.

$$\rho_{xy}\Big|_{max} > \sqrt{\frac{1}{L} - \frac{1}{L^2}} = \frac{1}{L}\sqrt{L-1} \qquad (2.18)$$

For linear maximal sequences,

$$\rho_{xy}\Big|_{max} > \frac{\sqrt{2^n - 2}}{2^n - 1} \approx \frac{2^{\frac{n}{2}}}{2^n - 1} \qquad (2.19)$$

Gold (Ref. 6) has shown that sequences can be found such that the full period cross-correlation function satisfies the inequality

$$|\rho_{xy}(\tau)| \leq \begin{cases} \dfrac{2^{\frac{n+1}{2}} + 1}{2^n - 1} ; & \text{n odd} \\ \\ \dfrac{2^{\frac{n+2}{2}} + 1}{2^n - 1} ; & \text{n even and not a multiple of 4} \end{cases} \qquad (2.20)$$

In addition to proving the existence of such sequences, Gold has shown, given one sequence, how to find another sequence for which the cross-correlation satisfies this inequality. In addition it is shown that this is the best cross-correlation function that can be achieved, i.e., the cross-correlation function with the smallest peaks.

It should be noted that both of the bounds discussed above depend upon the correlation being taken over the full period of the sequences involved. The problem to be studied here is concerned with the properties of the partial period cross-correlation function (Eq. 2.11) in which the sequences are linear maximal binary sequences. The set of numbers which will be generated by Eq. 2.11 for all possible starting points and all possible shift values, τ, is therefore the set of correlation values of interest and the set of correlation values which will be studied further here.

2.2 Linear Codes

In the calculation of the correlation between two $\{0, 1\}$ binary sequences the initial operation is a term-by-term mod-2 sum of the two sequences. The sequence of digits which are generated by this mod-2 sum are then further processed, as shown in Eq. 2.3, to compute the correlation value. It can be shown (see for example, Birdsall and Ristenbatt, Ref. 4) that the mod-2 sum of two different linear maximal sequences will generate a family of non-maximal sequences. It is these non-maximal sequences which are then further processed to calculate the correlation function. A block diagram of the operations which generate this family of non-maximal sequences is shown in Fig. 2.1. The sequences $\{a_i\}$ and $\{b_i\}$ are the linear maximal sequences and the family of non-maximal sequences

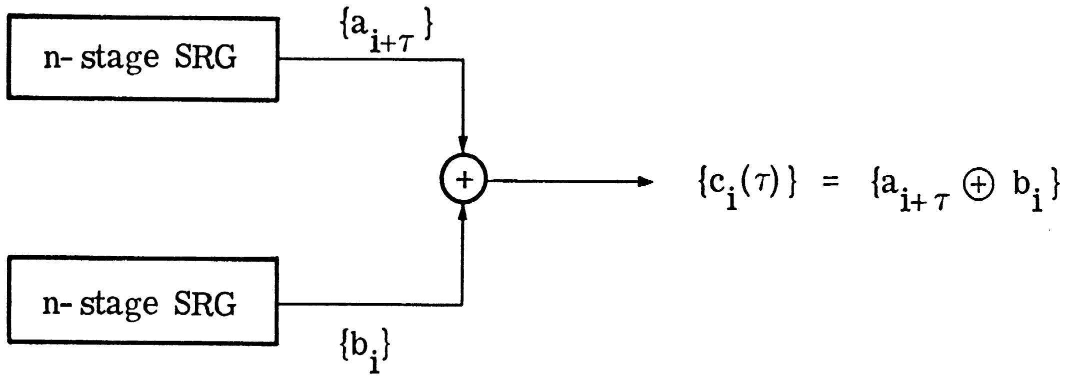

Fig. 2.1. Non-maximal sequence generator

are the sequences $\{c_i(\tau)\}$. [For the $\{1, -1\}$ sequence the operation would be multiplication instead of mod-2 addition however, as noted previously, the results are equivalent.] It should be noted at this point that a shift-register generator which is capable of producing a linear maximal sequence will also, if the initial conditions of the shift register are all zeros, produce a sequence of all zeros. Therefore the family of sequences which is produced by the non-maximal sequence generator shown in Fig. 2.1 will include the original two sequences. Since it has been assumed that the original two linear maximal sequences are generated by n-stage shift-register generators, it is known that the period of these sequences will be $2^n - 1$. The family of non-maximal sequences will have the following properties:

1. There are $2^n + 1$ sequences, in the family of sequences,

of length $2^n - 1$ digits.

2. The family of sequences can be generated by an equivalent shift register generator whose characteristic polynomial is the product of the original two characteristic polynomials. That is,

$$f(x) = \sum_{i=0}^{2n} C_i x^i = \left(\sum_{j=0}^{n} A_i x^i \right) \left(\sum_{k=0}^{n} B_i x^i \right) \qquad (2.21)$$

where all coefficients are reduced modulo-2.

3. Since the characteristic polynomial for the equivalent generator is of degree 2n, the equivalent generator is a 2n-stage shift-register generator.

4. All 2n-tuples, except the all zero 2n-tuple, will appear once and only once in the family of sequences. (The entire family of sequences will possess the balance property, randomness property number 1.)

5. The mod-2 sum of any two members of the family is also a member of the family, i.e., the family is closed under bit-by-bit modulo-2 addition.

6. The mod-2 sum of one member of the family with a shifted version of itself is a member of the family but not, in general, the same member. That is, the shift-and-add property of linear maximal sequences does not hold for the non-maximal sequences.

If we consider each member of the family of sequences to be a vector of dimension L $(L = 2^n - 1)$ and add to this set of vectors all the possible phase shifts of the original family of sequences along with the zero vector then the resulting set of vectors satisfies all of the axioms for a vector space. In order for the set of vectors to be a vector space it must be an Abelian group under addition. Since in this case the vectors considered are binary vectors the addition is mod-2 addition and the properties 5 and 6 above show that the set is an Abelian group in which the operation is modulo-2 addition and the zero vector is the identity element. Since the field elements consist of 0 and 1 the remaining axioms required for the set of vectors to be a vector space are trivially satisfied. The vector space so defined will be a subspace of the space of all L-dimensional binary vectors. The subspace is of dimension 2n and therefore contains 2^{2n} vectors. These are precisely the requirements which must be satisfied by a set of vectors in order for that set of vectors to be called a linear code (see, for example, Peterson, Ref. 7).

It can also be shown that the set of all possible k-tuples $(2n \leq k \leq L)$ is a linear code. In the partial period correlation case we will be operating on all possible k-tuples out of the family of sequences in order to determine the correlation function. With the vectors defined as above for the vector space it can be seen that any 2n-tuple will be the first 2n digits in one of the vectors in the vector space. Since k must be equal to or greater than 2n all possible k-tuples from the family of non-maximal sequences can be achieved by deleting the last L - k digits of the vectors which make up the vector space described above. The new vector space will also have dimension 2n and, of course, also 2^{2n} vectors. (This includes the zero vector.) Therefore the set of all possible k-tuples from the family of non-maximal sequences can be considered to be a linear code. (Also commonly called a group code.) This code is equivalent to what is normally called an (k, 2n) code in which there are k total digits in the code word and 2n information digits. (The usual notation for a code of this type is (n, k) and it is hoped that this change in the usual notation will not cause confusion.)

Since the set of all k-tuples is a linear code it is possible to describe all possible k-tuples in terms of the generator matrix. The generator matrix (G) for the linear code can be formed in the following manner. The rows of the generator matrix are the basis vectors for the vector space.

$$G = [I_{2n}\ P] \qquad (2.22)$$

I_{2n} is a 2n x 2n Identity Matrix

P is a 2n x (k - 2n) matrix formed by using the recurrence relation for the equivalent 2n-stage register.

Any k-tuple which can be generated in the cross-correlation process is now a linear combination of a subset of the 2n rows of the generator matrix.

2.3 Characteristic Phase of the Linear Maximal Sequence

The partial period correlation values depend on the starting point in the sequences and the relative phase shift between the sequences. In order to have a reference point from which to work it is necessary to define a phase position in such a way that the zero starting point and zero relative phase shift between sequences will be well defined. For this purpose the characteristic phase position has been used in this study. (See for example, Golomb, Ref. 2, Gold, Ref. 6 or Lindholm, Ref. 19.) The characteristic phase position of the sequence is defined in terms of the cyclotomic cosets. The purpose of this section then will be to define the cyclotomic cosets and using these cyclotomic cosets show how the characteristic phase position is defined for linear maximal binary sequences.

The cyclotomic cosets are made up of the integers in the range 0 to $L-1$. (For this discussion L will always represent the length of a linear maximal binary sequence and therefore will be of the form $2^n - 1$.) The cyclotomic cosets are formed as follows: Let $\phi(L)$ denote the number of integers less than L which are relatively prime to L. [$\phi(L)$ is the Euler ϕ-function.] These integers form a group under the operation multiplication mod-L. The set $(1, 2, 4, \ldots, 2^{n-1})$ is a subgroup containing n elements. The proper cosets are formed by multiplying any element of the group by the elements in the subgroup defined above. Since there are $\phi(L)$ relatively prime numbers in the range of interest and there are n integers in each coset there will be $\phi(L)/n$ different proper cosets. In addition to the proper cosets, the cyclotomic cosets contain at least one improper coset. The improper cosets are generated by multiplying the elements of the subgroup by integers which are not relatively prime to L. When L is prime there is only one improper coset which is generated by the integer 0. The improper cosets contain n, or some factor of n, elements. The entire set of cosets constitute the cyclotomic cosets modulo-L.

The cyclotomic cosets are very useful in the theory of linear maximal binary sequence generation. One of the very important properties of the proper cosets involves the sampling of sequences. If a linear maximal binary sequence of length L is sampled at a rate corresponding to one of the integers in the original subgroup, i.e., $(1, 2, 4, \ldots, 2^{n-1})$, the resulting sequence will be the same linear maximal sequence at some phase shift with respect to the original sequence. If the sequence is sampled at a rate corresponding to one of the elements of a proper coset the resulting sequence will be one of the other linear maximal sequences of the same length. Sampling at rates corresponding to different elements in the same coset results in the generation of the same sequence and, therefore, given one linear maximal sequence of length L, all of the $\phi(L)/n$ maximal sequences can be generated by sampling. (One sequence for each proper coset.) Sampling at a rate corresponding to an element of an improper coset will result in a sequence whose length is less than L.

It has also been shown (see Ref. 2, 6, or 19) that every linear maximal binary sequence has a phase shift such that the value of each digit, x_i, in the sequence depends only on the coset containing i (modulo-L). Thus for each sequence there is a unique assignment of +1's and -1's to the cyclotomic cosets which defines the sequence for one phase position. It is this phase position which has been called the characteristic phase position. [Methods for determining the correct assignment of +1's and -1's to the cosets have been discussed by Golomb (Ref. 2) and Gold (Ref. 6) and are not considered here.] As a simple example of this the cyclotomic cosets and the coset assignments for the linear maximal sequences which can be generated by four stage shift register generators are shown in Fig. 2.2. As shown in this figure, there are 8 integers less than 15 which are relatively prime to 15 and these integers fall into the subgroup and the one proper coset. The remaining seven integers make up the three improper cosets. The coset assignments are shown to

$n = 4$ $L = 2^n - 1 = 15$ $\phi(L) = 8$

1 2 4 7 8 11 13 14 are relatively prime to L

0 3 5 6 9 10 12 have factors in common with L

		[4, 1, 0]	[4, 3, 0]
Improper coset	{0}	1	1
Subgroup	{1, 2, 4, 8}	-1	1
Proper coset	{7, 14, 13, 11}	1	-1
Improper coset	{5, 10}	1	1
Improper coset	{3, 6, 9, 12}	-1	-1

i=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Feedback
1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	1	[4, 1, 0]
1	1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	[4, 3, 0]

Fig. 2.2. Cyclotomic cosets and coset assignments for $n = 4$

the right of the cosets and one period of each sequence in the characteristic phase position is shown at the bottom of the figure.

For some values of n it is possible to arrange the cosets in such a way that each coset leader is some fixed integer m times the previous coset leader. In order for this to be possible it is necessary, but not sufficient, that $(m, L) = 1$. It should be noted that it will never be possible to find an m such that multiplication of the elements of the coset by m will transform a proper coset into an improper coset and therefore another necessary condition for all of the cosets to be related in this fashion is that all of the cosets be proper cosets. (And hence that L be prime.) When proper cosets and improper cosets are both present in the cyclotomic cosets it may be possible to find a value of m which will lead to the proper cosets being related in this fashion and the improper cosets forming a separate group related in this fashion. When the cosets are arranged in this way the coset assignment of +1's and -1's for any maximal sequence is a cyclic permutation of the assignment for the original sequence (on the proper cosets). This can be seen by considering the case in which it is possible to arrange the cosets such that each coset is m times the previous coset and therefore such that $L = 2^n - 1$ is prime and all cosets are proper except the zero coset. The cosets then will be: (all integers reduced mod-L)

$$C_0 = \{0\}$$

$$C_1 = \{1,\ 2,\ 4,\ \ldots,\ 2^{n-1}\}$$

$$C_2 = \{m,\ 2m,\ 4m,\ \ldots,\ 2^{n-1}\ m\}$$

$$C_3 = \{m^2,\ 2m^2,\ 4m^2,\ \ldots,\ 2^{n-1}\ m^2\}$$

$$\vdots$$

$$C_{\phi(L)/n} = \left\{ m^{\frac{\phi(L)}{n} - 1},\ 2m^{\frac{\phi(L)}{n} - 1},\ \ldots,\ 2^{n-1}\ m^{\frac{\phi(L)}{n} - 1} \right\} \qquad (2.23)$$

Consider the sequence $\{x_i\}$ which, for the characteristic phase position, is constant on the cosets, i.e., has an assigned value of +1, or -1, for $i = 1, 2, 4, \ldots, 2^{n-1}$, etc., then the other maximal sequences can be achieved by sampling $\{x_i\}$ at sample rates corresponding to the coset leaders $\left(m, m^2, \ldots, m^{[\phi(L)/n] - 1}\right)$. Let the sequence $\{y_i\} = \{x_{mi}\}$ be the sequence generated by sampling every mth digit of the sequence $\{x_i\}$. Then the digits in the sequence $\{y_i\}$ corresponding to values of i in C_1 are those digits from the sequence $\{x_i\}$ corresponding to i an element of C_2, the digits in $\{y_i\}$ for i an element of C_2 correspond to the digits from $\{x_i\}$ for i an element of C_3 and so on. The digits in $\{y_i\}$ for i an element of $C_{\phi(L)/n}$ will correspond to the digits in $\{x_i\}$ for $i = m^{\phi(L)/n},\ 2m^{\phi(L)/n},\ \ldots,\ 2^{n-1}\ m^{\phi(L)/n}$ (reduced mod-L). For L prime $\phi(L) = L-1$ and hence:

$$m^{\phi(L)/n} = m^{(L-1)/n} \tag{2.24}$$

If, as assumed previously, the entire set of proper cosets is related by m, then the only remaining digits in $\{x_i\}$ are those for which i is an element of C_1 and therefore:

$$m^{(L-1)/n} = 2^p \tag{2.25}$$

The criterion which must be satisfied by m in order to make it possible to define the cosets as above then is that $\phi(L)/n$ must be equal to the smallest integer j such that:

$$m^j = 2^p \qquad (\text{mod } L) \tag{2.26}$$

If there is an integer $j < \phi(L)/n$ such that Eq. 2.26 is satisfied, then the cosets will be divided into two or more sets of cosets which will exhibit the cyclic coset assignment described above. The improper cosets will also lead to subsets of cosets which will exhibit this property. It should be noted at this point that the digit corresponding to $i=0$ will be identical for the characteristic phase position of all maximal sequences of a given length. This is the digit associated with the improper coset C_0, which consists of this element alone, and remains invariant under the sampling technique. (This digit has been called the absolute zero term by Golomb, Ref. 2.)

The coset assignments for the linear maximal sequences which may be generated by n stage shift-register generators for $n = 5$ through 8 are shown in Figs. 2.3 through 2.6. These examples of the coset assignments show three of the possible configurations. For $n=5$ and $n=7$ the period of the sequence is prime (these are Mersenne prime periods) and hence all the cosets except C_0 are proper cosets. For the values of m which were chosen for these cases the coset assignments are cyclic permutations of the assignment for the original sequence. For $n=6$ and $m=5$ there is one set of proper cosets, i.e., C_1 through C_6, for which the assignments permute cyclically. The improper cosets are C_0 and C_7 through C_{12} and are divided into several subsets by $m=5$, i.e., C_7 and C_8 are related by a factor of 5 as are C_9 and C_{10}. C_0, C_{11}, and C_{12} are all transformed into themselves by $m=5$. It should be noted that the assignment for C_7 and C_8 permute as do the assignments for C_9 and C_{10} although C_9 and C_{10} are both assigned 1 so that the permutation is not of any consequence. The reamining cosets are necessarily constant. In the remaining case, $n=8$ and $m=7$, the proper cosets are divided into two subsets. The cyclic permutation of coset assignments is also divided into two subsets, i.e., one set for the sequences S_1 through S_8 and another set for the sequences S_9 through S_{16}. Thus it can be seen that the coset assignments are all related although the relationship may be rather complicated for certain values of n and m.

$n = 5 \quad m = 3$

Coset	S_1	S_2	S_3	S_4	S_5	S_6	Feedback
$C_0 = \{0\}$	-1	-1	-1	-1	-1	-1	
$C_1 = \{1, 2, 4, 8, 16\}$	1	1	-1	1	-1	-1	$S_1 = [5, 2, 0]$
$C_2 = \{3, 6, 12, 24, 17\}$	1	-1	1	-1	-1	1	$S_2 = [5, 4, 3, 2, 0]$
$C_3 = \{9, 18, 5, 10, 20\}$	-1	1	-1	-1	1	1	$S_3 = [5, 4, 2, 1, 0]$
$C_4 = \{27, 23, 15, 30, 29\}$	1	-1	-1	1	1	-1	$S_4 = [5, 3, 0]$
$C_5 = \{19, 7, 14, 28, 25\}$	-1	-1	1	1	-1	1	$S_5 = [5, 3, 2, 1, 0]$
$C_6 = \{26, 21, 11, 22, 13\}$	-1	1	1	-1	1	-1	$S_6 = [5, 4, 3, 1, 0]$

Fig. 2.3. Coset assignment and feedback connections $n = 5$

$n = 6 \quad m = 5$

Coset		S_1	S_2	S_3	S_4	S_5	S_6
C_0	$= \{0\}$	1	1	1	1	1	1
C_1	$= \{1, 2, 4, 8, 16, 32\}$	-1	-1	1	1	-1	-1
C_2	$= \{5, 10, 20, 40, 17, 34\}$	-1	1	1	-1	-1	-1
C_3	$= \{25, 50, 37, 11, 22, 44\}$	1	1	-1	-1	-1	-1
C_4	$= \{62, 61, 59, 55, 47, 31\}$	1	-1	-1	-1	-1	1
C_5	$= \{58, 53, 43, 23, 46, 29\}$	-1	-1	-1	-1	1	1
C_6	$= \{38, 13, 26, 52, 41, 19\}$	-1	-1	-1	1	1	-1
C_7	$= \{3, 6, 12, 24, 48, 33\}$	-1	1	-1	1	-1	1
C_8	$= \{15, 30, 60, 57, 51, 39\}$	1	-1	1	-1	1	-1
C_9	$= \{9, 18, 36\}$	1	1	1	1	1	1
C_{10}	$= \{27, 54, 45\}$	1	1	1	1	1	1
C_{11}	$= \{21, 42\}$	-1	-1	-1	-1	-1	-1
C_{12}	$= \{7, 14, 28, 56, 49, 35\}$	1	1	1	1	1	1

$S_1 = [6, 1, 0]$

$S_2 = [6, 5, 2, 1, 0]$

$S_3 = [6, 5, 3, 2, 0]$

$S_4 = [6, 5, 0]$

$S_5 = [6, 5, 4, 1, 0]$

$S_6 = [6, 4, 3, 1, 0]$

Proper Cosets: $C_1, C_2, C_3, C_4, C_5, C_6$

Improper Cosets: $C_0, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}$

Fig. 2.4. Coset assignment and feedback connections $n = 6$

$n = 7 \quad m = 3$

Coset	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	Feedback
$C_0 = \{0\}$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	
$C_1 = \{1, 2, 4, 8, 16, 32, 64\}$	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	$S_1 = [7, 3, 0]$
$C_2 = \{3, 6, 12, 24, 48, 96, 65\}$	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	1	$S_2 = [7, 3, 2, 1, 0]$
$C_3 = \{9, 18, 36, 72, 17, 34, 68\}$	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	1	-1	$S_3 = [7, 5, 4, 3, 2, 1, 0]$
$C_4 = \{27, 54, 108, 89, 51, 102, 77\}$	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	$S_4 = [7, 6, 4, 1, 0]$
$C_5 = \{81, 35, 70, 13, 26, 52, 104\}$	-1	-1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	-1	$S_5 = [7, 1, 0]$
$C_6 = \{116, 105, 83, 39, 78, 29, 58\}$	-1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	$S_6 = [7, 5, 3, 1, 0]$
$C_7 = \{94, 61, 122, 117, 107, 87, 47\}$	1	-1	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	$S_7 = [7, 5, 4, 3, 0]$
$C_8 = \{28, 56, 112, 97, 67, 7, 14\}$	-1	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	$S_8 = [7, 6, 5, 4, 2, 1, 0]$
$C_9 = \{84, 41, 82, 37, 74, 21, 42\}$	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	$S_9 = [7, 6, 5, 2, 0]$
$C_{10} = \{63, 126, 125, 123, 119, 111, 95\}$	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	$S_{10} = [7, 4, 0]$
$C_{11} = \{62, 124, 121, 115, 103, 79, 31\}$	1	1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	$S_{11} = [7, 6, 5, 4, 0]$
$C_{12} = \{93, 59, 118, 109, 91, 55, 110\}$	1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	$S_{12} = [7, 6, 5, 4, 3, 2, 0]$
$C_{13} = \{38, 76, 25, 50, 100, 73, 19\}$	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	$S_{13} = [7, 6, 3, 1, 0]$
$C_{14} = \{57, 114, 101, 75, 23, 46, 92\}$	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	$S_{14} = [7, 6, 0]$
$C_{15} = \{69, 11, 22, 44, 88, 49, 98\}$	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	$S_{15} = [7, 6, 4, 2, 0]$
$C_{16} = \{33, 66, 5, 10, 20, 40, 80\}$	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	$S_{16} = [7, 4, 3, 2, 0]$
$C_{17} = \{15, 30, 60, 120, 113, 99, 71\}$	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	1	$S_{17} = [7, 6, 5, 3, 2, 1, 0]$
$C_{18} = \{45, 90, 53, 106, 85, 43, 86\}$	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	-1	1	1	1	-1	$S_{18} = [7, 5, 2, 1, 0]$

Fig. 2.5. Coset assignment and feedback connections $n=7$

n = 8 m = 7

		S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
C_0	= {0}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C_1	= {1, 2, 4, 8, 16, 32, 64, 128}	1	1	1	-1	-1	1	1	-1	1	1	1	-1	-1	-1	-1	1
C_2	= {7, 14, 28, 56, 112, 224, 193, 131}	1	1	-1	-1	1	1	-1	1	1	1	-1	-1	-1	-1	1	1
C_3	= {49, 98, 196, 137, 19, 38, 76, 152}	1	-1	-1	1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1
C_4	= {88, 176, 97, 194, 133, 11, 22, 44}	-1	-1	1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1	1
C_5	= {106, 212, 169, 83, 166, 77, 154, 53}	-1	1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1
C_6	= {232, 209, 163, 71, 142, 29, 58, 116}	1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1
C_7	= {94, 186, 121, 242, 229, 203, 151, 47}	1	-1	1	1	1	-1	-1	1	-1	1	1	1	1	-1	-1	-1
C_8	= {146, 41, 82, 164, 73, 146, 37, 74}	-1	1	1	1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1
C_9	= {127, 254, 253, 251, 247, 239, 223, 141}	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1
C_{10}	= {124, 248, 241, 227, 199, 143, 31, 62}	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1	1
C_{11}	= {103, 206, 157, 59, 118, 236, 217, 179}	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1	1	1
C_{12}	= {211, 167, 79, 158, 61, 122, 244, 233}	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1	1	1	1
C_{13}	= {202, 149, 43, 86, 172, 89, 178, 101}	-1	-1	-1	1	1	1	1	-1	-1	1	1	-1	1	1	1	-1
C_{14}	= {139, 23, 46, 92, 184, 113, 226, 197}	-1	-1	1	1	1	1	-1	-1	1	1	-1	1	1	1	-1	-1
C_{15}	= {208, 161, 67, 134, 13, 26, 52, 104}	-1	1	1	1	1	-1	-1	-1	1	-1	1	1	1	-1	-1	1
C_{16}	= {91, 182, 109, 218, 181, 107, 214, 173}	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	-1	-1	1	1
C_{17}	= {3, 6, 12, 24, 48, 96, 192, 129}	-1	-1	1	1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	1
C_{18}	= {21, 42, 84, 168, 81, 162, 69, 138}	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1
C_{19}	= {147, 39, 78, 156, 57, 114, 228, 201}	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1
C_{20}	= {18, 36, 72, 144, 33, 66, 132, 9}	1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	-1
C_{21}	= {63, 126, 252, 249, 243, 231, 207, 159}	1	-1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1
C_{22}	= {186, 117, 234, 213, 171, 87, 174, 93}	-1	-1	-1	-1	-1	1	1	1	-1	1	1	1	-1	-1	-1	-1
C_{23}	= {99, 198, 141, 27, 54, 108, 216, 177}	-1	-1	-1	-1	1	1	1	-1	1	1	1	-1	-1	-1	-1	-1
C_{24}	= {189, 123, 246, 237, 219, 183, 111, 222}	-1	-1	-1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	1
C_{25}	= {5, 10, 20, 40, 80, 160, 65, 130}	-1	-1	-1	1	-1	-1	-1	1	-1	1	-1	-1	-1	1	-1	-1
C_{26}	= {35, 70, 140, 25, 50, 100, 200, 145}	-1	-1	1	-1	-1	-1	1	-1	1	-1	-1	-1	1	-1	-1	-1
C_{27}	= {175, 95, 190, 125, 250, 245, 235, 215}	-1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	1
C_{28}	= {230, 205, 155, 55, 110, 220, 185, 115}	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1
C_{29}	= {15, 30, 60, 120, 240, 225, 195, 135}	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
C_{30}	= {105, 210, 165, 75, 150, 45, 90, 180}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
C_{31}	= {17, 34, 68, 136}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C_{32}	= {119, 238, 221, 187}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C_{33}	= {51, 102, 204, 153}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
C_{34}	= {85, 170}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

S_1 = [8, 4, 3, 2, 0]
S_2 = [8, 6, 5, 3, 0]
S_3 = [8, 6, 5, 2, 0]
S_4 = [8, 7, 6, 5, 2, 1, 0]
S_5 = [8, 7, 2, 1, 0]
S_6 = [8, 7, 3, 2, 0]
S_7 = [8, 7, 5, 3, 0]
S_8 = [8, 6, 4, 3, 2, 1, 0]
S_9 = [8, 6, 5, 4, 0]
S_{10} = [8, 5, 3, 2, 0]
S_{11} = [8, 6, 3, 2, 0]
S_{12} = [8, 7, 6, 3, 2, 1, 0]
S_{13} = [8, 7, 6, 1, 0]
S_{14} = [8, 6, 5, 1, 0]
S_{15} = [8, 5, 3, 1, 0]
S_{16} = [8, 7, 6, 5, 4, 2, 0]

Proper Cosets
$C_1 - C_{16}$

Improper Cosets
C_0, $C_{17} - C_{34}$

Fig. 2.6. Coset assignment and feedback connections n = 8

The characteristic phase position of the sequence then is defined in terms of the assignment of digits to the cosets and in the remainder of this study it will be assumed that when the cross-correlation is taken for $\tau = 0$ with starting point equal to 0 that the sequences involved will be in the characteristic phase position.

CHAPTER III

CODING THEORY BOUNDS

In the study of error correcting codes, a great deal of effort has been devoted to finding bounds on the minimum distance between code words and in designing codes with a specified minimum distance. The minimum distance is of interest because it is useful in determining the error correcting capability of the code. For binary codes the distance between code words is the Hamming distance. The Hamming weight of a vector is defined to be the number of non-zero elements of the vector and the Hamming distance between two vectors is the number of elements that differ. For binary vectors, therefore, the Hamming distance between the vectors is the number of 1's in the mod-2 sum of the vectors and since the mod-2 sum of the two vectors in the vector space is another vector in the space, the minimum distance is equal to the minimum weight of the non-zero vectors in the space. (See Peterson, Ref. 7.)

It has been shown that in the partial period correlation calculation the k-tuples operated on to compute the correlation value form a linear code. The correlation value is calculated using Eq. 3.1. The summation in Eq. 3.1 is the Hamming distance between the vectors $\{a_i\}$ and $\{b_i\}$ for the given values of starting point, s, and relative phase shift, τ. The vectors in the vector space

$$\rho_{ab}(k, \tau, s) = 1 - \frac{2}{k} \sum_{i=0}^{k-1} \left(a_{i+s+\tau} \oplus b_{i+s} \right) \tag{3.1}$$

described previously are the vectors generated by the term inside the summation sign for all possible values of starting point and relative phase shift. Therefore it is seen that the minimum weight vector leads to the maximum correlation, and the coding theory bounds on minimum distance lead to bounds on the maximum value of the partial period correlation function. The lower bounds on minimum distance correspond to upper bounds on the maximum value of correlation, the upper bounds on minimum distance lead to a lower bound on the maximum value of correlation.

In this chapter the coding theory bounds are examined and compared with the actual bounds for the partial period correlation between selected pairs of sequences. The coding theory bounds provide a general picture of the behavior of the partial period correlation function as the correlation interval is decreased.

The coding theory bounds discussed in this chapter are, for the most part, standard coding theory bounds and are covered thoroughly in the literature. (See for example Peterson, Ref. 7, Berlekamp, Ref. 23, and the chapter by Solomon, Chapter 6, in Ref. 24.)

3.1 Plotkin Bound (Ref. 25)

The simplest bound is the average distance bound which states that the minimum distance between code words must be equal to or less than the average weight of the code words. (This bound is the Plotkin low-rate average distance bound discussed in Berlekamp, Ref. 23.) The average weight of the code words in a (k, 2n) linear code can be shown to be equal to: (see Peterson, Ref. 7)

$$\overline{W} = \left(\frac{2^{2n}}{2^{2n} - 1}\right) \frac{k}{2} \tag{3.2}$$

Therefore the minimum distance must satisfy the inequality

$$d_0 \leq k\left(\frac{2^{2n-1}}{2^{2n}-1}\right) \tag{3.3}$$

Using Eq. 3.1, it can be seen that the maximum value of correlation must satisfy the inequality

$$\begin{aligned} \rho_{max} &\geq 1 - \frac{2}{k}\left[k\left(\frac{2^{2n-1}}{2^{2n}-1}\right)\right] \\ &= 1 - \frac{2^{2n}}{2^{2n}-1} = -\frac{1}{2^{2n}-1} \end{aligned} \tag{3.4}$$

It should be pointed out again that with the notation used here the quantity k refers to the length of the code word and the quantity

2n refers to the number of information symbols in the code. It can be seen that as the number of information symbols in the linear code increases, the right hand side of Eq. 3.4 approaches zero.

The Plotkin bound discussed by Peterson, Ref. 7, makes use of the average distance bound above to show that for the code with maximum minimum distance

$$2^{2n} \leq 2^{k-2d_0+1} \; (2d_0)$$

or (3.5)

$$2n \leq k - 2d_0 + 2 + \log_2 d_0 \; ; \qquad k \geq 2d_0 - 1$$

Since the interest for the problem being considered here is in finding the limits on the minimum distance this equation can be rewritten as follows:

$$2d_0 - \log_2 d_0 - 2 \leq k - 2n \; ; \qquad k \geq 2d_0 - 1 \tag{3.6}$$

If the length of the code, k, and the number of information symbols, 2n, are known, this inequality provides a method of finding an upper bound on the minimum distance and thus provides a lower bound on the maximum value of correlation.

This equation as it stands cannot be easily manipulated to show the correlation bound. Peterson has made use of the fact that for large values of d_0 the terms involving $\log_2 d_0$ and the constant term are negligible in order to arrive at a simpler asymptotic bound. When this approximation is valid Eq. 3.6 reduces to

$$2d_0 \leq k - 2n$$

Therefore the asymptotic bound (large d_0) on the maximum value of correlation is

$$\rho_{max} \geq \frac{k - 2d_0}{k} = \frac{2n}{k} \tag{3.7}$$

For the partial period correlation problem the range of interest for k $(2n \leq k \leq L)$ will lead to small values of d_0 and it will be necessary to use the inequality as shown in Eq. 3.6. It is possible to simplify this inequality somewhat by noting the fact that the number of check digits, $(k - 2n)$, must be equal to or greater than a quantity which is a function of the minimum distance in the code. Since both the number of check digits and the minimum distance must be integers it is possible to make use of a piecewise linear approximation to the quantity on the left hand side of the inequality. For integer values the bound may be rewritten as shown below.

$$2d_0 - [\log_2 d_0 + 2] \leq k - 2n \tag{3.8}$$

where

$[r]$ = greatest integer equal to or less than r .

The equations for the bound for various values of minimum distance and the corresponding range of values for the number of check digits are shown in Table 3.1.

Min. Distance	$[\log_2 d_0 + 2]$	Bound	Check Digits
$d_0 = 1$	2	$2d_0 - 2 = k - 2n$	$k - 2n = 0$
$2 \leq d_0 \leq 3$	3	$2d_0 - 3 = k - 2n$	$1 \leq k - 2n \leq 3$
$4 \leq d_0 \leq 7$	4	$2d_0 - 4 = k - 2n$	$4 \leq k - 2n \leq 10$
$8 \leq d_0 \leq 15$	5	$2d_0 - 5 = k - 2n$	$11 \leq k - 2n \leq 25$
$16 \leq d_0 \leq 31$	6	$2d_0 - 6 = k - 2n$	$26 \leq k - 2n \leq 56$
$32 \leq d_0 \leq 63$	7	$2d_0 - 7 = k - 2n$	$57 \leq k - 2n \leq 119$
$64 \leq d_0 \leq 127$	8	$2d_0 - 8 = k - 2n$	$120 \leq k - 2n \leq 246$

Table 3.1. Plotkin bound

As an example of the use of this bound consider the following case.

Let $k = 50$, $n = 7$

Then $k - 2n = 36$

From Table 3.1 it can be seen that at the bound

$$2d_0 = 42$$

Therefore $d_0 \leq 21$

$$\rho_{max} \geq \frac{k - 2d_0}{k} = \frac{50 - 42}{50} = 0.16 \ .$$

Therefore in terms of the partial period correlation function it is seen that the Plotkin bound states that, for the cross-correlation between two 7-stage maximal sequences over 50 digits, the peak correlation value must always exceed 0.16.

3.2 The Hamming Bound

One of the most basic bounds on error correcting codes is the Hamming bound (Ref. 26). This bound is also known as the volume bound or the sphere packing bound (Refs. 7, 23, 24). In order to find the Hamming bound it is assumed that the code is of length k and that it corrects e errors. For a linear binary code to correct e errors the minimum distance between code words must be $d_0 \geq 2e+1$. The bound then shows the maximum possible number of information symbols.

If a code is to correct e errors then each code word must have all words which differ from it in e, or less, positions associated with it. The number of such words is:

$$N_{w \le e} = \sum_{i=0}^{e} \binom{k}{i}$$

$$N_{w \le e} = \text{the number of k-tuples of weight} \le e \qquad (3.9)$$

In the codes which are being considered here there are 2^{2n} code words and since each code word must have $N_{w \le e}$ k-tuples associated with it, the product of these two is the number of distinct k-tuples necessary to correct e errors and must be equal to or less than the total number of k-tuples possible. That is:

$$2^{2n} \left(\sum_{i=0}^{e} \binom{k}{i} \le 2^k \right) \qquad (3.10)$$

This bound is tight for high rates ($2n/k$ close to 1), i.e., there exists a set of codes which achieve this bound with equality; however, it is weak at low rates (Ref. 23). Therefore, this bound is expected to be more appropriate for small values of k in the range of interest.

The equation for this bound can be rewritten as follows:

$$2^{k-2n} \ge \sum_{i=0}^{e} \binom{k}{i}$$

or (3.11)

$$k - 2n \ge \log_2 \left(\sum_{i=0}^{e} \binom{k}{i} \right)$$

This inequality can be used in individual cases to give a bound on e and, with the bound on e, find a bound on d_0. Using the values from the previous example (k = 50 and n = 7) we can calculate the Hamming bound as follows:

$$36 \geq \log_2\left(\sum_{i=0}^{e} \binom{50}{i}\right) = \log_2\left[1 + 50 + \frac{50 \cdot 49}{2} + \dots + \frac{50 \cdot 49 \cdots (50-e+1)}{e!}\right]$$

Using tables of binomial coefficients it can be seen that

$$e \leq 11$$

Therefore

$$d_0 = 2e+1 \leq 23$$

$$\rho_{max} \geq \frac{50 - 46}{50} = 0.08$$

The bound on the maximum value of correlation found using the Hamming bound is somewhat below the previous bound calculated using the Plotkin bound. The rate of the code for this example is $R = 2n/k = 0.28$. As noted previously, the Hamming bound is more appropriate for rates close to 1 and, therefore is not expected to be a particularly good bound for the example code parameters.

This bound is very difficult to use because the calculation of the sum of the binomial coefficients becomes quite tedious for large values of k and e . The bound can be simplified somewhat by considering the limits on the sum of the binomial coefficients. It can be shown (see for example Ref. 7 or Ref. 27) that

$$\binom{k}{e} < \sum_{i=0}^{e} \binom{k}{i} < \frac{k-e}{k-2e} \binom{k}{e}; \qquad e < \frac{k}{2} \tag{3.12}$$

Therefore the bound can be written as:

$$k - 2n > \log_2 \binom{k}{e} \tag{3.13}$$

The use of this simplified bound leads to the same result for the example parameters. For any code word length, k, and number of information symbols, 2n, the largest values of e which satisfy the two inequalities will never differ by more than 1.

For large values of k it has been shown (see for example Ref. 7 or Ref. 23) that an asymptotic equation for the Hamming bound can be obtained. It is shown that the following inequalities are valid:

$$\begin{aligned} 1 - \frac{2n}{k} &\geq B_H(k) = \frac{1}{k} \log_2 \left(\sum_{i=0}^{e} \binom{k}{i} \right) \\ \lim_{k \to \infty} B_H(k) &= H\left(\frac{e}{k}\right) \end{aligned} \tag{3.14}$$

where

$$H(m) = -m \log_2(m) - (1 - m) \log_2(1 - m) \ .$$

The function $H(x)$ is commonly called the entropy function and is tabulated in Ref. 27. The asymptotic equation for the Hamming bound becomes

$$\frac{2n}{k} \leq 1 - \lim_{k \to \infty} B_H(k) = 1 - H\left(\frac{e}{k}\right) \tag{3.15}$$

Then $\delta\left(\frac{2n}{k}\right)$ is defined to be the value of $\frac{e}{k}$ for equality. That is

$$\frac{2n}{k} = 1 - H\left(\delta\left(\frac{2n}{k}\right)\right) \tag{3.16}$$

Therefore

$$\frac{e}{k} \leq \delta\left(\frac{2n}{k}\right)$$

$$e \leq k\,\delta\left(\frac{2n}{k}\right)$$

The function $\delta(R)$ is shown in Fig. 3.1 and, using the parameters from the previous example, the value of the asymptotic bound is shown below.

for $k = 50$, $n = 7$ $\delta\left(\frac{2n}{k}\right) = 0.199$

$$e \leq 50(0.199) = 9.95$$

$$d_0 = 2e+1 \leq 20.9$$

Therefore $\rho_{max} > \frac{50 - 2(20)}{50} = 0.20$.

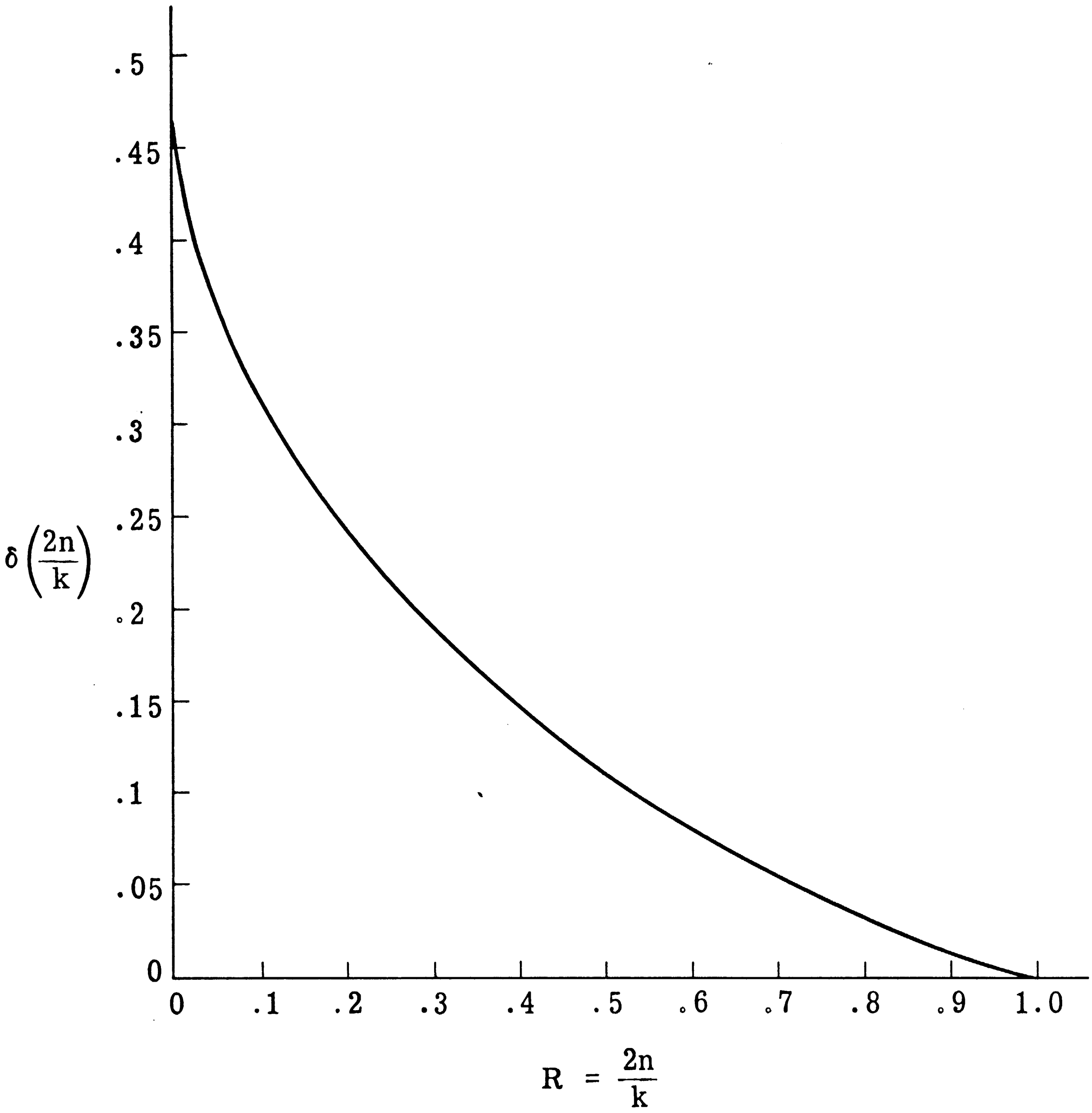

Fig. 3.1. Asymptotic Hamming bound

The asymptotic bound is seen to be somewhat higher than the bound calculated from the inequality (Eq. 3.11) directly and is also seen to be somewhat higher than the Plotkin bound from before. It should be noted, however, that the asymptotic bound is obtained as a limit for large k and, therefore, should become more appropriate as the length of the code word increases.

3.3 Elias Bound

It has been noted previously that the Hamming bound is appropriate at high rates (2n/k close to 1) and that the average distance bound provides a better bound at low rates. For medium rates the Elias bound is considered by Berlekamp (Ref. 23) in which the volume bound and average distance bound are combined in order to obtain a stronger bound. The asymptotic form of the Elias bound, as given by Berlekamp (Ref. 23), is shown below.

$$d_0 < 2k\,\delta\left(\frac{2n}{k}\right)\left[1 - \delta\left(\frac{2n}{k}\right)\right] + \epsilon \qquad (3.17)$$

for large k and any $\epsilon > 0$

For the example which has been used previously the value of minimum distance can be seen to be:

for $k = 50$, $n = 7$, $\frac{2n}{k} = 0.28$

From the curve (Fig. 3.1) $\delta(0.28) = 0.199$

Therefore $d_0 < 2(50)\,(0.199)\,(1 - 0.199) + \epsilon$

$$d_0 < 15.9$$

$$\rho_{max} \geq \frac{50 - 30}{50} = 0.4$$

It is apparent that this bound provides a much larger bound than the two considered previously, however it should be pointed out again that this is an asymptotic bound and therefore is appropriate for large values of k and medium rates.

3.4 The Griesmer Bound

The Griesmer Bound (Ref. 24, 28 or 29) is a bound which is tight for low rates, i.e., there exist low rate codes which satisfy the bound with equality. The statement of the bound is as follows:

Given 2n and d_0, then for a (k, 2n) code to have minimum weight d_0,

$$k \geq d_0 + d_1 + d_2 + \ldots + d_{2n-1} \tag{3.18}$$

where

$$d_{i+1} = \left[\frac{d_i + 1}{2}\right]$$

$[r]$ = greatest integer $\leq r$.

Calculations using this bound are somewhat cumbersome, however some simplification is possible. For example since

$d_1 = [(d_0+1)/2]$ will, for d_0 an even number, be equal to d_1 for the preceding odd value of d_0, the value of the summation for an even d_0 will be 1 greater than the value of the sum for the preceding odd d_0. That is

$$\text{Let} \qquad S_{d_0} = \sum_{i=0}^{2n-1} d_i \,, \quad \text{then}$$

$$S_{2\ell} = S_{2\ell-1} + 1 \tag{3.19}$$

Therefore it is only necessary to consider the summation for odd values of d_0.

The summation S_{d_0} can be simplified as follows: (in the following discussion it will be assumed that d_0 is odd)

$$S_{d_0} = d_0 + \left[\frac{d_0 + 1}{2}\right] + \left[\frac{d_1 + 1}{2}\right] + \ldots + \left[\frac{d_{2n-2} + 1}{2}\right] \tag{3.20}$$

for d_0 odd, $\frac{d_0+1}{2}$ is an integer and the brackets may be removed from the d_1 term. Then

$$d_1 = \frac{d_0 + 1}{2}$$

$$d_2 = \left[\frac{d_1 + 1}{2}\right] = \left[\frac{d_0 + 3}{4}\right]$$

Since d_0 is an odd integer, $d_0 + 3$ is even and $(d_0 + 3)/4$ will differ from an integer by at most $1/2$. In the calculation of d_3, shown in

Eq. 3.21 below, removing the integer restriction on the term $[(d_0 + 3)/4]$ will result in an error of at most 1/4 in the value of $(d_2 + 1)/2$ and thus will not change the value of d_3. That is

$$d_3 = \left[\frac{d_2 + 1}{2}\right] = \left[\frac{\left[\frac{d_0 + 3}{4}\right] + 1}{2}\right] \tag{3.21}$$

$$\frac{d_0 + 3}{4} \leq \left[\frac{d_0 + 3}{4}\right] + \frac{1}{2}$$

$$\frac{\frac{d_0 + 3}{4} + 1}{2} = \frac{d_0 + 7}{8} \leq \frac{\left[\frac{d_0 + 3}{4}\right] + \frac{3}{2}}{2} \tag{3.22}$$

$$\frac{\left[\frac{d_0 + 3}{4}\right] + \frac{3}{2}}{2} - \frac{\left[\frac{d_0 + 3}{4}\right] + 1}{2} = \frac{1}{4}$$

Since the value of an integer divided by 2 will either by an integer or an integer plus 1/2 it is apparent that the quantity on the left hand side of Eq. 3.22 may be substituted for the quantity inside the brackets in Eq. 3.21 with no change in the value of d_3.

Continuing in this fashion the summation can be written

$$S_{d_0} = d_0 + \left[\frac{d_0 + 1}{2}\right] + \left[\frac{d_0 + 3}{4}\right] + \ldots + \left[\frac{d_0 + 2^{2n-1} - 1}{2^{2n-1}}\right] \tag{3.23}$$

where the quantities inside the brackets will be larger than the corresponding quantities shown in Eq. 3.20 above but where the difference will never be large enough to change the value of the integer.

Therefore

$$S_{d_0} \doteq \sum_{i=0}^{2n-1} \left[\frac{d_0 + 2^i - 1}{2^i}\right]$$

$$= 2n + \sum_{i=0}^{2n-1} \left[\frac{d_0 - 1}{2^i}\right] \tag{3.24}$$

The bound then becomes

$$k - 2n \geq \sum_{i=0}^{2n-1} \left[\frac{d_0 - 1}{2^i}\right] \tag{3.25}$$

When the denominator (2^i) exceeds the value of the numerator $(d_0 - 1)$ the value of the term will be zero as will the remaining terms for higher values of i. Since the minimum distance d_0 must be less than $k/2$ there will always be enough terms in the summation to include all the non-zero terms of the form $[(d_0 - 1)/2^i]$. Table 3.2 gives values of the right hand side of the inequality as a function of d_0. It should be recalled at this point that the value of the right hand side of inequality (3.25) for even values of d_0 can be calculated using Eq. 3.19 above.

For the example used previously, $k = 50$ and $n = 7$, the Griesmer bound is

$$k - 2n = 50 - 14 \geq \sum_{i=0}^{2n-1} \left[\frac{d_0 - 1}{2^i}\right]$$

$$36 > \sum_{i=0}^{2n-1} \left\lceil \frac{d_0 - 1}{2^i} \right\rceil \qquad \text{for } d_0 = 20$$

$$d_0 \leq 20$$

$$\rho_{max} \geq \frac{50 - 40}{50} = 0.20$$

d_0	$\sum_{i=0}^{2n-1} \left\lceil \frac{d_0 - 1}{2^i} \right\rceil$
3	3
5	7
7	10
9	15
11	18
13	22
15	25
17	31
19	34
21	38
23	41
25	46
27	49
29	53
31	56
33	63
35	66
37	70
39	73
41	78
43	81
45	85
47	88
49	94
51	97
53	101
55	104
57	109
59	112
61	116

Table 3.2. Griesmer bound

It can be seen that this bound is somewhat higher than the Plotkin bound and the Hamming bound but below the Elias bound for this example. As noted previously, the Griesmer bound is more appropriate for low rate codes.

3.5 Bounds Assuming a Binomial Model

It has been noted previously that the k-tuples which are operated on to compute the partial period correlation values form a linear code. The vectors which form this linear code can be generated by a 2n stage generator whose characteristic polynomial is the product of the characteristic polynomials of the original two sequences. Since the equivalent generator has 2n stages, all possible 2n tuples will be generated, as noted in the properties of the k-tuples (Section 2.2). Therefore the weight distribution for these vectors for $k = 2n$ will be binomial. The binomial distribution includes the all zero vector which must be included in order that the code space be a vector space, however the all zero vector is not considered in minimum distance calculations. The mean value of the binomial distribution is equal to $k/2$ (for $k = 2n$) and the extreme values, i.e., the maximum and minimum, are $\frac{k}{2} - n$ and $\frac{k}{2} + n$. If we assume that the range of the distribution will remain constant at 2n as k is increased and that the distribution is symmetrical about the mean value then we have the following bounds:

$$
\begin{aligned}
d_{min} &\leq \frac{k}{2} - n \\
\rho_{max} &\geq \frac{1}{k}(k - 2d_{min}) = \frac{2n}{k} \\
d_{max} &\geq \frac{k}{2} + n \\
\rho_{min} &\leq \frac{1}{k}(k - 2d_{max}) = -\frac{2n}{k}
\end{aligned}
\qquad (3.26)
$$

It should be noted at this point that this has been called a bound although no attempt has been made, nor is any proof known, in which it is shown that these bounds, based on a binomial model, do indeed bound the correlation values. The results of making use of the binomial model to calculate the bounds, as shown in Eq. 3.26, are compared with some actual bounds which were calculated, and which will be discussed in a later section, and the bounds are consistent in all those cases considered. It should also be noted that the bound on ρ_{max} in Eq. 3.26 is equal to the asymptotic Plotkin bound of Eq. 3.7.

The results of applying the parameters from the example used previously to this bound are shown in Eq. 3.27.

For $k = 50$, $n = 7$

$$\frac{2n}{k} = 0.28$$

$$\rho_{max} \geq 0.28 \qquad (3.27)$$

$$\rho_{min} \leq -0.28$$

It can be seen from the bounds given in Eq. 3.27 that the binomial bound leads to a better value than either the Hamming bound or the Plotkin bound and has the additional advantage of giving some information concerning the negative peak value of correlation. It should be noted that this bound implies a correlation of 1 for $k = 2n$ which is not possible unless one considers the all zero k-tuple to be included. The all zero k-tuple is not usually considered in the study of weight distributions since it will always lead to a correlation value of 1. Therefore this assumption should be used with caution for values of k close to $2n$.

3.6 The Varsharmov-Gilbert Bound

All of the bounds which have been discussed up to this point provide upper bounds on the minimum distance. The upper bound on the minimum distance leads to a lower bound on the maximum value of correlation possible. The Varsharmov-Gilbert bound provides a lower bound on the minimum distance and therefore will lead to an upper bound on the maximum value of partial period correlation. The bound presented here is a refinement of a bound of Gilbert's

(Ref. 30). [Berlekamp (Ref. 23) discussed the Gilbert bound.] In Peterson (Ref. 7) the Varsharmov-Gilbert bound is considered and it is this bound which will be used here. The bound was found by Varsharmov and also by Sacks (Ref. 31).

The relationships which lead to the V-G bound are most easily discussed by considering the parity check matrix for the linear code. It has been noted previously that the k-tuples which are used in the calculation of the partial period correlation form a vector space of dimension 2n. It can be shown that the null space of this vector space V has dimension k - 2n and that we can define a matrix H of rank k - 2n whose row space is this null space V'. (See for example, Peterson, Ref. 7.) The matrix H whose row space is the null space of the vector space V can be defined as shown in Eq. 3.28.

$$H = \left[-P^T \; I_{k-2n} \right] \tag{3.28}$$

P^T is the transpose of the matrix P (Eq. 2.22)

I_{k-2n} is a (k-2n) x (k-2n) identity matrix.

Then a vector v is in the vector space V if and only if it is orthogonal to every row of the matrix H, i.e., $v \cdot H^T = 0$. When the vector space V defines a linear code the matrix H is called the parity check matrix of V.

It can be seen from the definition of the parity check matrix given above that a code vector of weight d implies a linear dependence relation among d columns of the matrix H. Therefore if it is possible to find a parity check matrix for which no set of $d-1$ or fewer columns is linearly dependent, the corresponding linear code will have minimum distance at least equal to d. The V-G bound is found by showing how to construct a parity check matrix such that no set of $d-1$ or fewer columns is linearly dependent.

With a $(k, 2n)$ code the number of parity check digits is $r = k - 2n$ and the parity check matrix is an $r \times k$ matrix. The upper bound on r is found by considering ways of constructing the parity check matrix. If each column of the parity check matrix is chosen in such a way that it is not equal to any of the previously chosen columns or any linear combination of the previously chosen columns then the addition of that column to the parity check matrix will not lead to a linear dependence relationship among the columns. If we wish to construct a code of minimum distance d then the k-th column must be chosen in such a way that it is not equal to any linear combination of $d-2$ or fewer columns out of the $k-1$ previous columns. Since the column vectors are r-tuples the maximum possible number of non-zero column vectors is $2^r - 1$. There are

$$\binom{k-1}{1} + \binom{k-1}{2} + \dots + \binom{k-1}{d-2}$$

linear combinations of $d-2$ or fewer columns out of $k-1$ total columns. If all of the linear combinations are distinct and do not include all possible k-tuples it is possible to add one of the k-tuples which is not included in the set of linear combinations and insure that the parity check matrix will be made of columns such that no combination of $d-1$ or fewer columns will be equal to zero. This leads to the following inequality:

$$2^r - 1 > \binom{k-1}{1} + \binom{k-1}{2} + \dots + \binom{k-1}{d_0-2}$$

or (3.29)

$$2^r > \sum_{i=0}^{d_0-2} \binom{k-1}{i}$$

The minimum integer r that satisfies this inequality is an upper bound on the number of check digits necessary to construct a code of minimum distance d_0. In the same way for a given number of check digits, $r = k - 2n$, the value of d_0 found using the inequality will lead to a lower bound on the minimum distance for the code with maximum-minimum distance. Since the codes of interest for this particular problem are not codes with maximum-minimum distance this bound does not necessarily provide a bound on the values achieved for selected sequences. It has been included here for comparison purposes only.

The value specified by the bound can be worked out for specific cases and the value, for the parameters used in the previous examples, is shown below.

For $k = 50$ and $n = 7$

$r = k - 2n = 36$

$$2^{36} > \sum_{i=0}^{d_0-2} \binom{49}{i}$$

Using tables of binomial coefficients it can be seen that

$$d_0 - 2 \leq 11$$

$$d_0 \leq 13$$

Therefore the lower bound on the minimum distance for codes with maximum-minimum distance is $d_0 = 13$. Then

$$\rho_{max} \leq \frac{50 - 26}{50} = 0.48$$

As before the sum of the binomial coefficients can be approximated by using the last term, i.e.,

$$k - 2n > \log_2 \binom{k-1}{d-2} \tag{3.30}$$

This will give the same result as above for the parameters of this example. For the asymptotic results it can be shown (see for example Berlekamp, Ref. 23) that the bound becomes

$$\frac{d_0}{k} \geq \delta\left(\frac{2n}{k}\right) \tag{3.31}$$

where

$\delta\left(\frac{2n}{k}\right)$ is the function defined previously in Section 3.3 and shown in Fig. 3.1.

As before, it should be recalled that the asymptotic bound is found by assuming k very large and consequently Eq. 3.31 should become more appropriate as the length of the code word increases. Inserting the parameters from the example it is found that

$$d_0 \geq 50\,(0.199) = 9.95$$

and

$$\rho_{max} \leq \frac{50 - 20}{50} = 0.60$$

The V-G bound is the only lower bound on minimum distance of the bounds discussed. From the examples given above it can be seen that there is a wide discrepancy between the bound calculated using Eq. 3.29 and the asymptotic bound calculated using Eq. 3.31. This discrepancy can be attributed to the particular example used, i.e., the length of the code does not warrant the use of the asymptotic bound.

3.7 Actual Bounds on the Partial Period Correlation

In order to compare the bounds discussed previously with the actual values of the partial period correlation function, a computer program was written to calculate the partial period correlation, for selected sequences, as a function of k. Using this program the two sequences are generated and the correlation is calculated for all possible values of starting point and phase shift. The positive and negative peak values of the partial period correlation function are then printed out and may be used for comparison purposes with the bounds calculated from the coding theory bounds. The actual bounds have been calculated for pairs of sequences generated by 5-stage through 9-stage shift-register generators and examples of the results are shown in Figs. 3.2 through 3.6. The pairs of sequences, for which the peak values of the partial period correlation function are shown in these figures, were chosen because the peak value of the full period correlation was as small as possible for the set of all pairs of linear maximal sequences generated by shift register generators of the same length. (These are the "preferred sequences" of Gold, Ref. 6, with the exception of the example for $n = 8$, for which no preferred sequences are specified.) It can be seen from these curves that, at least for the longer sequences, the peak value of the partial period correlation drops very rapidly as k is increased from the initial value $(k_0 = 2n-1)$. The general character of the

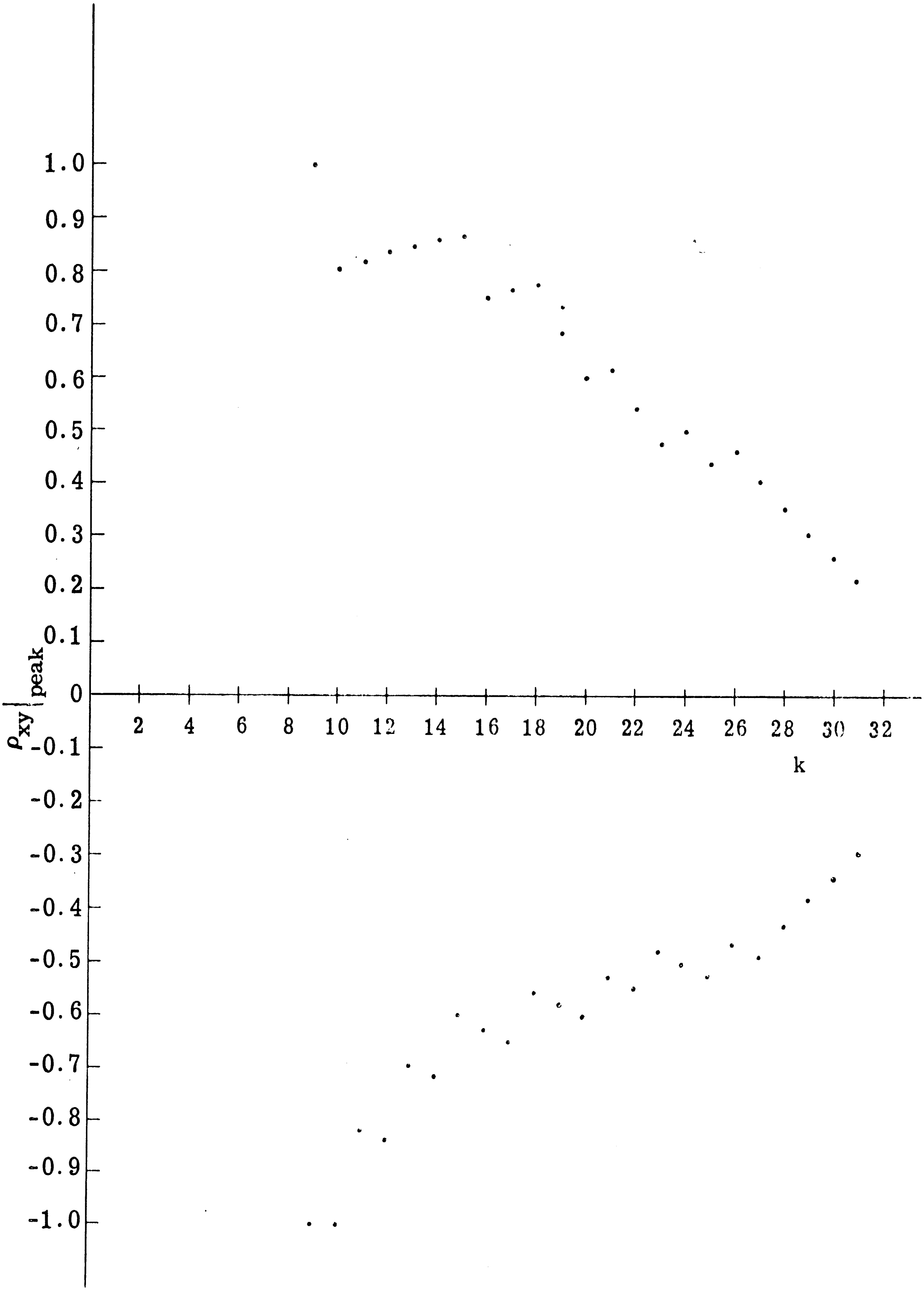

Fig. 3.2. Peak values of the partial period correlation between [5, 3, 0] and [5, 4, 3, 1, 0]

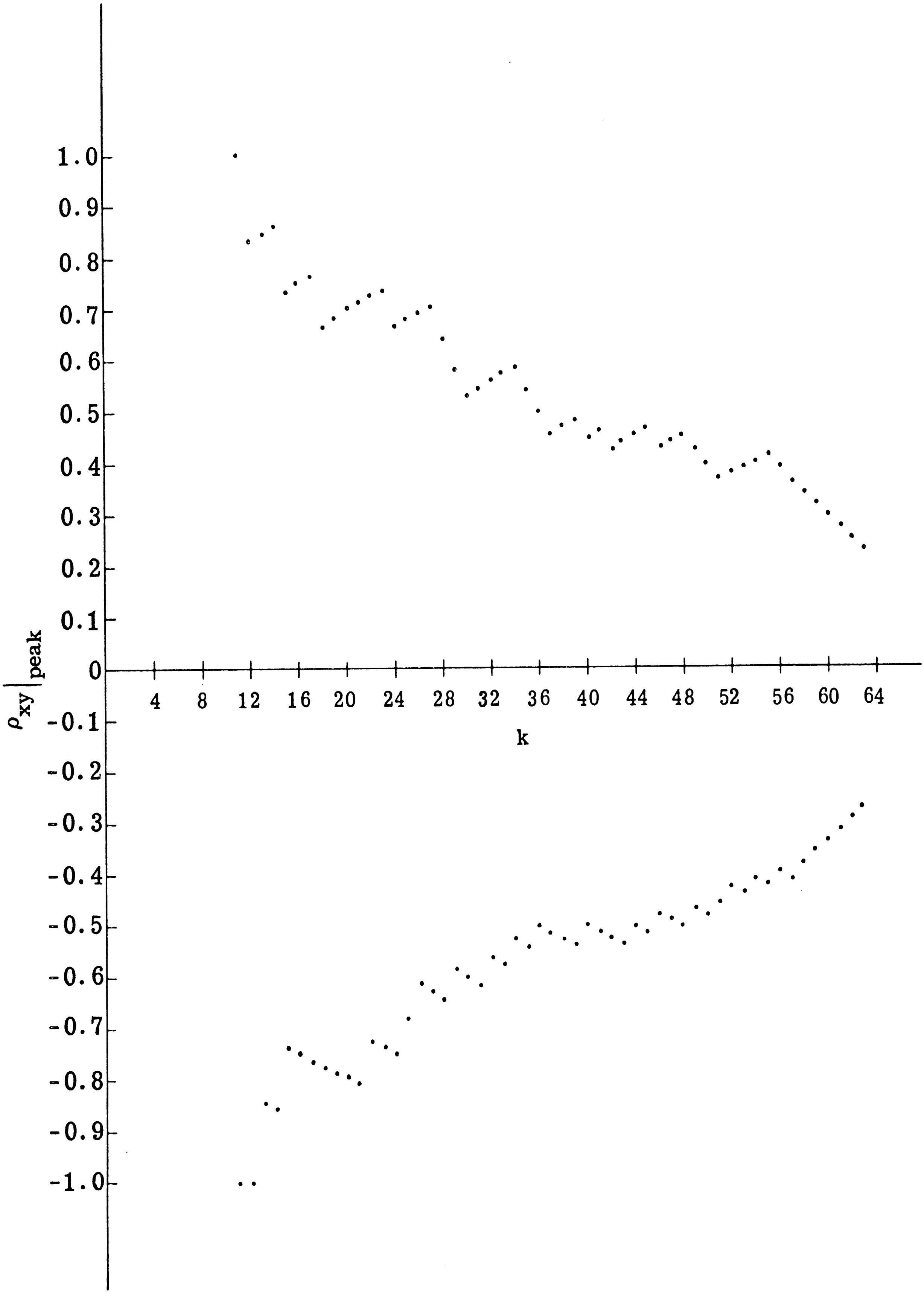

Fig. 3.3. Peak values of the partial period correlation between [6, 5, 0] and [6, 5, 4, 1, 0]

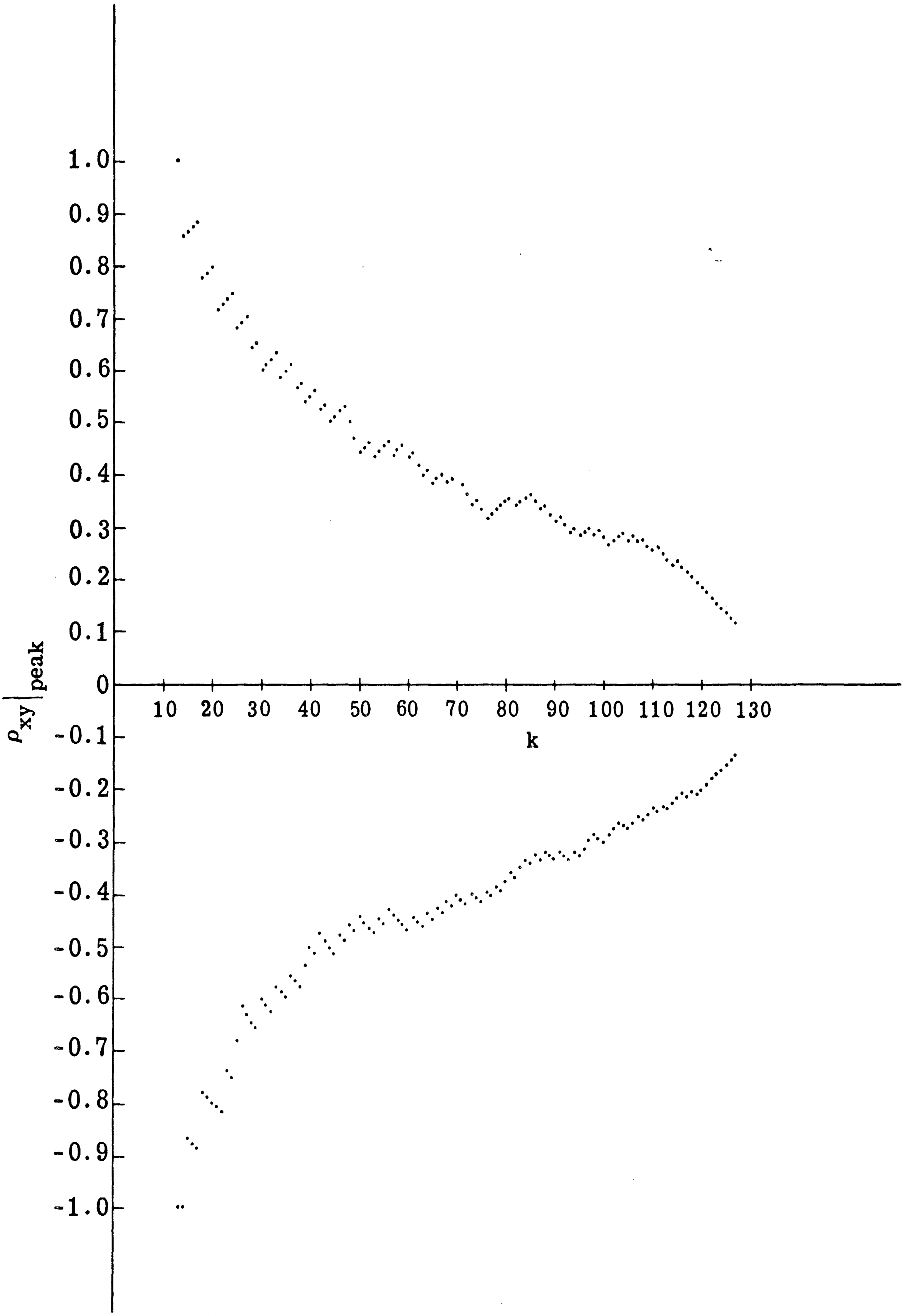

Fig. 3.4. Peak values of the partial period correlation between [7, 4, 0] and [7, 6, 5, 4, 3, 2, 0]

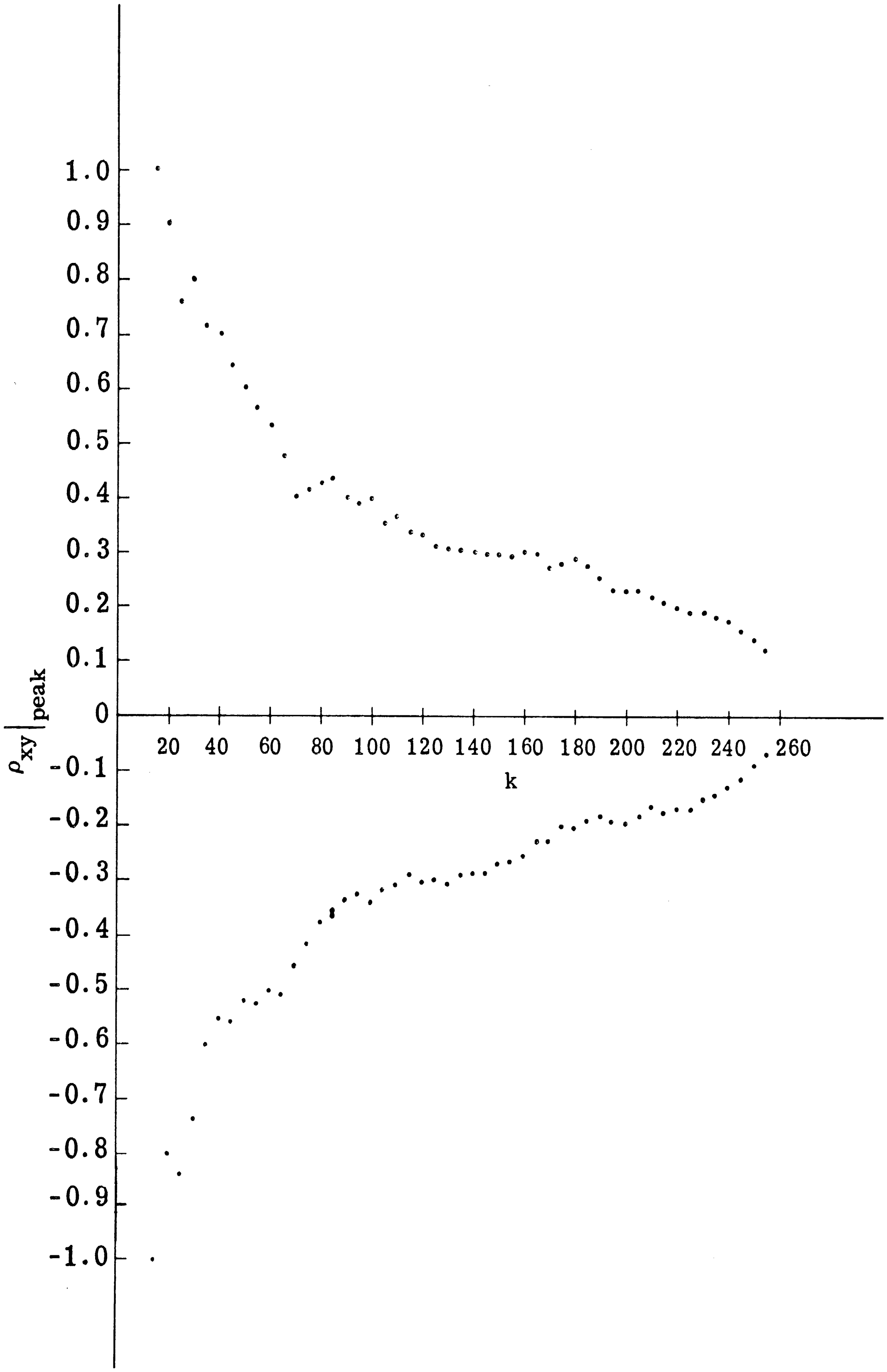

Fig. 3.5. Peak values of the partial period correlation between [8, 6, 5, 4, 0] and [8, 6, 5, 3, 0]

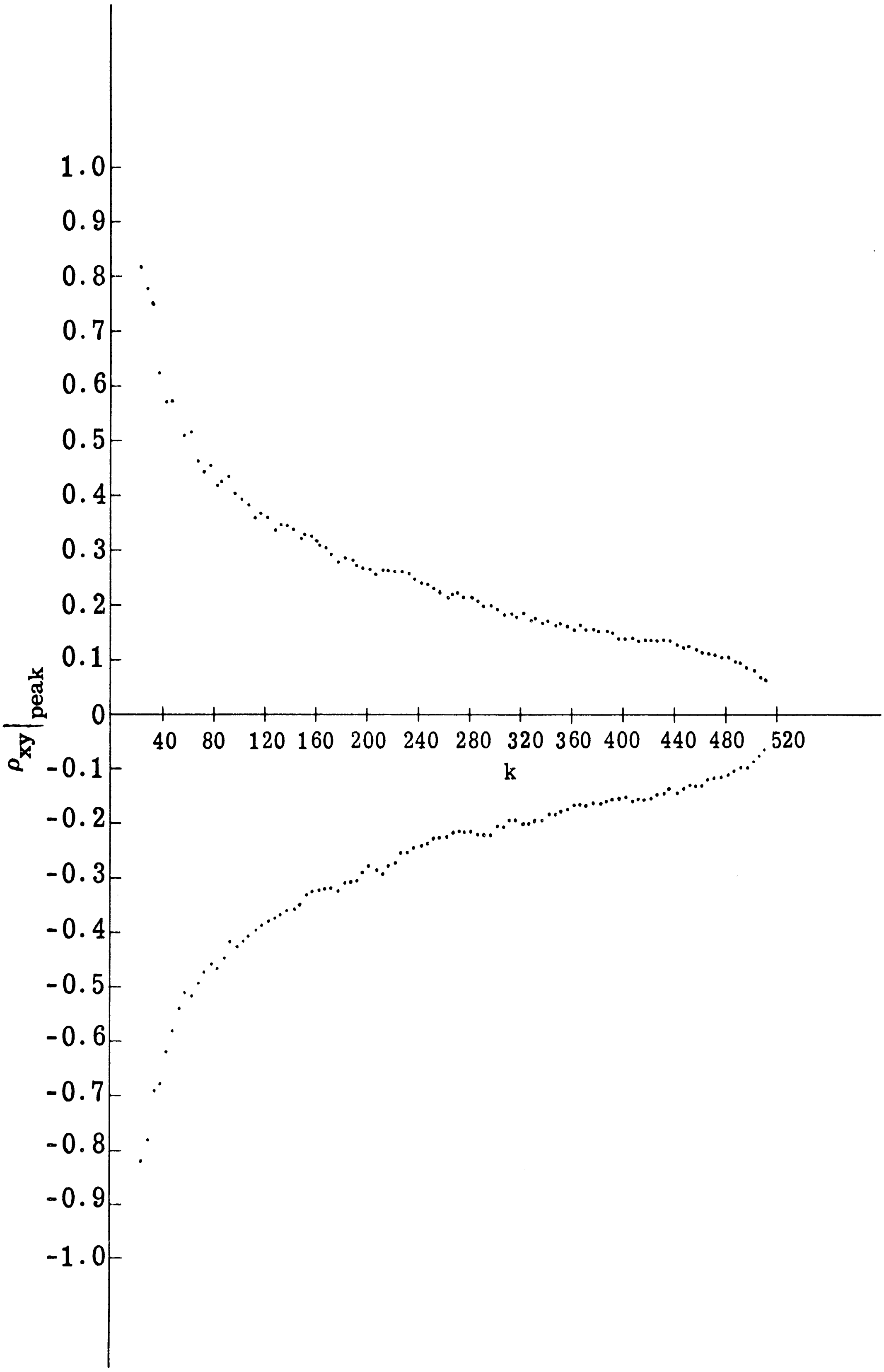

Fig. 3.6. Peak values of the partial period correlation between [9, 5, 0] and [9, 8, 6, 5, 3, 2, 0]

limiting curves appears to be somewhat exponential until k becomes very nearly equal to L, at which time the peak value of the partial period correlation drops at a higher rate to the full period correlation.

Some comments about the fine structure of these limiting values of the partial period correlation function seem appropriate at this point. Consider the vectors which are used to calculate the partial period correlation to be $\{0, 1\}$ k-tuples. The correlation is calculated using

$$\rho_{xy}(k, \tau, s) = \frac{1}{k}\left[k - 2\,(\text{number of 1's})\right] \qquad (3.32)$$

In determining the peak values the number of 1's in the k-tuple which leads to the peak correlation cannot decrease as k is incremented. Therefore if k is incremented by 1, the number of 1's can increase by 1, resulting in a decrease in the positive peak value, or remain the same, resulting in an increase in the positive peak value. In order to calculate the negative peak value the k-tuple with the highest weight is used and again the number of 1's in the k-tuple cannot decrease as k is incremented.

Let D_k be the number of 1's in the k-tuple of minimum weight. Then

$$\rho_{max}(k) = 1 - 2\frac{D_k}{k}$$

$$\rho_{max}(k+1) = 1 - 2\frac{D_{k+1}}{k+1}$$

Therefore

$$\begin{aligned}\Delta \rho_{max}(k) &= \rho_{max}(k+1) - \rho_{max}(k) \\ &= \left(1 - \frac{2D_{k+1}}{k+1}\right) - \left(1 - \frac{2D_k}{k}\right) \\ &= 2\left(\frac{D_k}{k} - \frac{D_{k+1}}{k+1}\right) \qquad (3.33)\end{aligned}$$

As noted previously there are two possible cases, i.e.,

$$D_{k+1} = \begin{cases} D_k \\ D_k + 1 \end{cases}$$

1. For $D_{k+1} = D_k$, the change in peak value is:

$$\Delta \rho_{max}(k) = 2D_k\left[\frac{1}{k} - \frac{1}{k+1}\right] = \frac{2D_k}{k(k+1)} \qquad (3.34)$$

Using the relationship

$$2D_k = k\left(1 - \rho_{max}(k)\right)$$

the change can be expressed as follows:

$$\Delta \rho_{max}(k) = \frac{1}{k+1}\left(1 - \rho_{max}(k)\right) \qquad (3.35)$$

This corresponds to an increase in the positive peak value.

2. For $D_{k+1} = D_k + 1$ the change in peak value is:

$$\Delta \rho_{max}(k) = 2\left[\frac{D_k}{k} - \frac{D_k + 1}{k+1}\right]$$

$$= \frac{2D_k}{k(k+1)} - \frac{2}{k+1} \qquad (3.36)$$

Making the same substitution as above:

$$\Delta \rho_{max}(k) = \frac{1}{k+1}\left(-1 - \rho_{max}(k)\right) \qquad (3.37)$$

This corresponds to a decrease in the positive peak value. Since $\rho_{max}(k) > 0$ the decrease in magnitude of the positive peak value is larger than the possible increase for a given value of k and therefore the bound exhibits relatively large negative steps and relatively small positive steps as seen in Figs. 3.2 through 3.6.

In order to examine the negative peak value of the partial period correlation function let D_k' be the number of 1's in the k-tuple of maximum weight. The equations have the same form as above with ρ_{min} substituted for ρ_{max} and D_k' substituted for D_k. Then for $D_{k+1}' = D_k'$ the change in peak value is:

$$\Delta \rho_{min}(k) = \frac{1}{k+1}\left(1 - \rho_{min}(k)\right) \qquad (3.38)$$

For this case this results in a decrease in the magnitude of the negative peak value.

For $D_{k+1}' = D_k' + 1$ the change in peak value is:

$$\Delta \rho_{min}(k) = \frac{1}{k+1}\left(-1 - \rho_{min}(k)\right) \qquad (3.39)$$

This corresponds to an increase in the magnitude of the negative peak value. It should be noted that $\rho_{min}(k) < 0$ and thus the possible decrease in magnitude is larger than the possible increase in magnitude and the bound on the negative peak values exhibits the same general shape (in magnitude) as the bound on positive peak values.

For the computer calculation of the limiting values of the partial period correlation function one sequence was chosen as a reference sequence and the bounds were computed for the cross-correlation between that sequence and all other sequences of the same length. In Table 3.3 the experimental bounds are shown for the cross-correlation between the 7-stage maximals and the reference sequence [7, 3, 0], for values of k which lead to high rate, medium rate, and low rate codes. It can be seen in Table 3.3 that the peak values of the correlation functions can vary over a considerable range. For example, for $k = 50$, the example considered previously, the peak positive value of partial period correlation for the selected

Sequences \ k	14	20	25	45	50	55	100	127
[7, 3, 2, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.600 / -0.680	0.467 / -0.467	0.440 / -0.440	0.418 / -0.455	0.280 / -0.280	0.118 / -0.134
[7, 5, 4, 3, 2, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.680 / -0.680	0.511 / -0.511	0.440 / -0.440	0.455 / -0.455	0.280 / -0.300	0.118 / -0.134
[7, 6, 4, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.700 / -0.900	0.760 / -0.760	0.467 / -0.511	0.440 / -0.480	0.418 / -0.455	0.280 / -0.280	0.118 / -0.134
[7, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.700	0.680 / -0.600	0.556 / -0.511	0.480 / -0.480	0.455 / -0.418	0.280 / -0.280	0.118 / -0.134
[7, 5, 3, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.680 / -0.680	0.556 / -0.467	0.480 / -0.480	0.455 / -0.455	0.280 / -0.280	0.118 / -0.134
[7, 5, 4, 3, 0] ; [7, 3, 0]	0.857 / -1.0	0.700 / -0.800	0.760 / -0.760	0.556 / -0.511	0.520 / -0.520	0.491 / -0.491	0.360 / -0.440	0.181 / -0.323
[7, 6, 5, 4, 2, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.760 / -0.760	0.511 / -0.600	0.440 / -0.560	0.455 / -0.564	0.280 / -0.380	0.181 / -0.323
[7, 6, 5, 2, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.760 / -0.600	0.467 / -0.467	0.440 / -0.440	0.418 / -0.418	0.300 / -0.380	0.181 / -0.323
[7, 4, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.840 / -0.680	0.511 / -0.556	0.520 / -0.520	0.455 / -0.455	0.320 / -0.260	0.150 / -0.165
[7, 6, 5, 4, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.900	0.840 / -0.920	0.556 / -0.689	0.560 / -0.720	0.491 / -0.600	0.320 / -0.460	0.181 / -0.323
[7, 6, 5, 4, 3, 2, 0] ; [7, 3, 0]	0.857 / -1.0	0.700 / -0.800	0.680 / -0.760	0.467 / -0.644	0.480 / -0.600	0.455 / -0.527	0.360 / -0.460	0.181 / -0.323
[7, 6, 3, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.700	0.760 / -0.600	0.600 / -0.511	0.520 / -0.440	0.491 / -0.455	0.300 / -0.360	0.181 / -0.323
[7, 6, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.760 / -0.680	0.511 / -0.511	0.480 / -0.480	0.418 / -0.455	0.260 / -0.300	0.118 / -0.134
[7, 6, 4, 2, 0] ; [7, 3, 0]	0.857 / -1.0	0.700 / -0.900	0.680 / -0.680	0.467 / -0.511	0.440 / -0.440	0.455 / -0.418	0.260 / -0.280	0.118 / -0.134
[7, 4, 3, 2, 0] ; [7, 3, 0]	0.857 / -1.0	0.700 / -0.800	0.680 / -0.760	0.511 / -0.556	0.440 / -0.520	0.418 / -0.491	0.260 / -0.280	0.118 / -0.134
[7, 6, 5, 3, 2, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.900 / -0.800	0.840 / -0.680	0.467 / -0.511	0.480 / -0.480	0.455 / -0.455	0.260 / -0.300	0.118 / -0.134
[7, 5, 2, 1, 0] ; [7, 3, 0]	0.857 / -1.0	0.800 / -0.800	0.860 / -0.680	0.511 / -0.511	0.480 / -0.480	0.418 / -0.418	0.260 / -0.300	0.118 / -0.134

Table 3.3. Actual cross-correlation bounds (7-stage maximals)

sequences ranges from 0.440 to 0.560.

The bounds which have been discussed in this chapter are shown in Table 3.4 and the values of partial period correlation calculated using these bounds are shown in Table 3.5. Comparing these bounds with the experimental values it can be seen that for small values of k (high rate codes) the Hamming bound provides best results. The asymptotic bounds are not appropriate for this range of values of k. For medium values of k (45, 50, 55) the Elias bound provides the best bound, however it is high for $k = L$. For k close to L (low rate codes) the binomial values are the best bounds. (It should be noted that the asymptotic Plotkin bound is equal to the binomial bound.) The upper bound on the partial period correlation function provided by the asymptotic V-G bound gives some idea of the range of possible values. The asymptotic bounds are plotted in Fig. 3.7 along with the actual bounds for two cases. The actual bounds shown in Fig. 3.7 were chosen to exhibit the approximate range of peak values found using the computer.

3.8 Summary

The study of coding theory bounds provides a general feeling for the way in which the partial period correlation changes as the integration time is decreased. It can be seen from the comparison of the theoretical bounds (Table 3.5) that the appropriate bound depends

Average Distance (Plotkin)	$d_0 \leq k\left(\frac{2^{2n-1}}{2^{2n}-1}\right)$	$\rho_{max} \geq -\frac{1}{2^{2n}-1}$	Not a function of k Use for low rates
Plotkin Bound	$2d_0 - \log_2 d_0 - 2 \leq k - 2n$	Table look-up	
Asymptotic (d_0 large)	$2d_0 \leq k - 2n$	$\rho_{max} \geq \frac{2n}{k}$	
Hamming Bound	$\sum_{i=0}^{e} \binom{k}{i} \leq 2^{k-2n}$; $d_0 = 2e+1$	Table look-up	High rate
Approximate	$\log_2 \binom{k}{e} \leq k - 2n$; $d_0 = 2e+1$	Table look-up	
Asymptotic (k large)	$\frac{2n}{k} \leq 1 - H\left(\frac{e}{k}\right)$ or $d_0 \leq 2k\,\delta\left(\frac{2n}{k}\right)$	$\rho_{max} \geq 1 - 4\delta\left(\frac{2n}{k}\right)$	
Elias Bound (Asymptotic)	$d_0 \leq 2k\,\delta\left(\frac{2n}{k}\right)\left\{1 - \delta\left(\frac{2n}{k}\right)\right\} + \epsilon;\ \epsilon > 0$	$\rho_{max} \geq 1 - 4\delta\left(\frac{2n}{k}\right)\left\{1 - \delta\left(\frac{2n}{k}\right)\right\} - \frac{2\epsilon}{k}$	Medium rate
Griesmer Bound	$\sum_{i=0}^{2n-1} \left[\frac{d_0 - 1}{2^i}\right] \leq k - 2n$ where $[x]$ = greatest integer $\leq x$	Table look-up	
Binomial	$d_0 \leq \frac{k}{2} - n$ $d_{max} \geq \frac{k}{2} + n$	$\rho_{max} \geq \frac{2n}{k}$ $\rho_{min} \leq -\frac{2n}{k}$	Same as asymptotic Plotkin Bound
Varsharmov-Gilbert Bound	$\sum_{i=0}^{d_0-2} \binom{k-1}{i} \leq 2^{r_{max}}$	Table look-up	For codes with max.-min. distance
Asymptotic (k large)	$d_0 > k\,\delta\left(\frac{2n}{k}\right)$	$\rho_{max} \leq 1 - 2\delta\left(\frac{2n}{k}\right)$	

Table 3.4. Theoretical cross-correlation bounds

k	14	20	25	45	50	55	100	127
Rate $R = \frac{2n}{k}$	1.000	0.700	0.560	0.311	0.280	0.255	0.140	0.110
Check Digits $r = k - 2n$	0	6	11	31	36	41	86	113
$\delta\left(\frac{2n}{k}\right)$ (From Fig. 3.1)	0	0.053	0.091	0.184	0.199	0.212	0.283	0.307
Average Distance (Plotkin)	$.61 \times 10^{-4}$	$.61 \times 10^{-4}$	$.61 \times 10^{-4}$	$.61 \times 10^{-4}$	$.61 \times 10^{-4}$	$.61 \times 10^{-4}$	$.61 \times 10^{-4}$	$.61 \times 10^{-4}$
Plotkin Bound	0.858	0.500	0.360	0.178	0.160	0.145	0.070	0.055
Asymptotic (d_0 large)				0.311	0.280	0.255	0.140	0.110
Hamming Bound	0.857	0.700	0.600	0.155	0.080			
Asymptotic (k large)				0.264	0.204	0.152	-0.132	-0.228
Elias Bound (Asymptotic)	1.000	0.799	0.669	0.400	0.361	0.332	0.187	0.149
Griesmer Bound	1.000	0.600	0.360	0.243	0.200	0.164	0.080	0.055
Binomial (pos. peak)	1.000	0.700	0.560	0.311	0.280	0.255	0.140	0.110
(neg. peak)	-1.000	-0.700	-0.560	-0.311	-0.280	-0.255	-0.140	-0.110
Varsharmov-Gilbert Bound	0.735	0.700	0.680	0.510	0.480			
Asymptotic (k large)	1.000	0.894	0.818	0.632	0.602	0.576	0.434	0.386

Table 3.5. Theoretical cross-correlation bounds for various rates (n = 7)

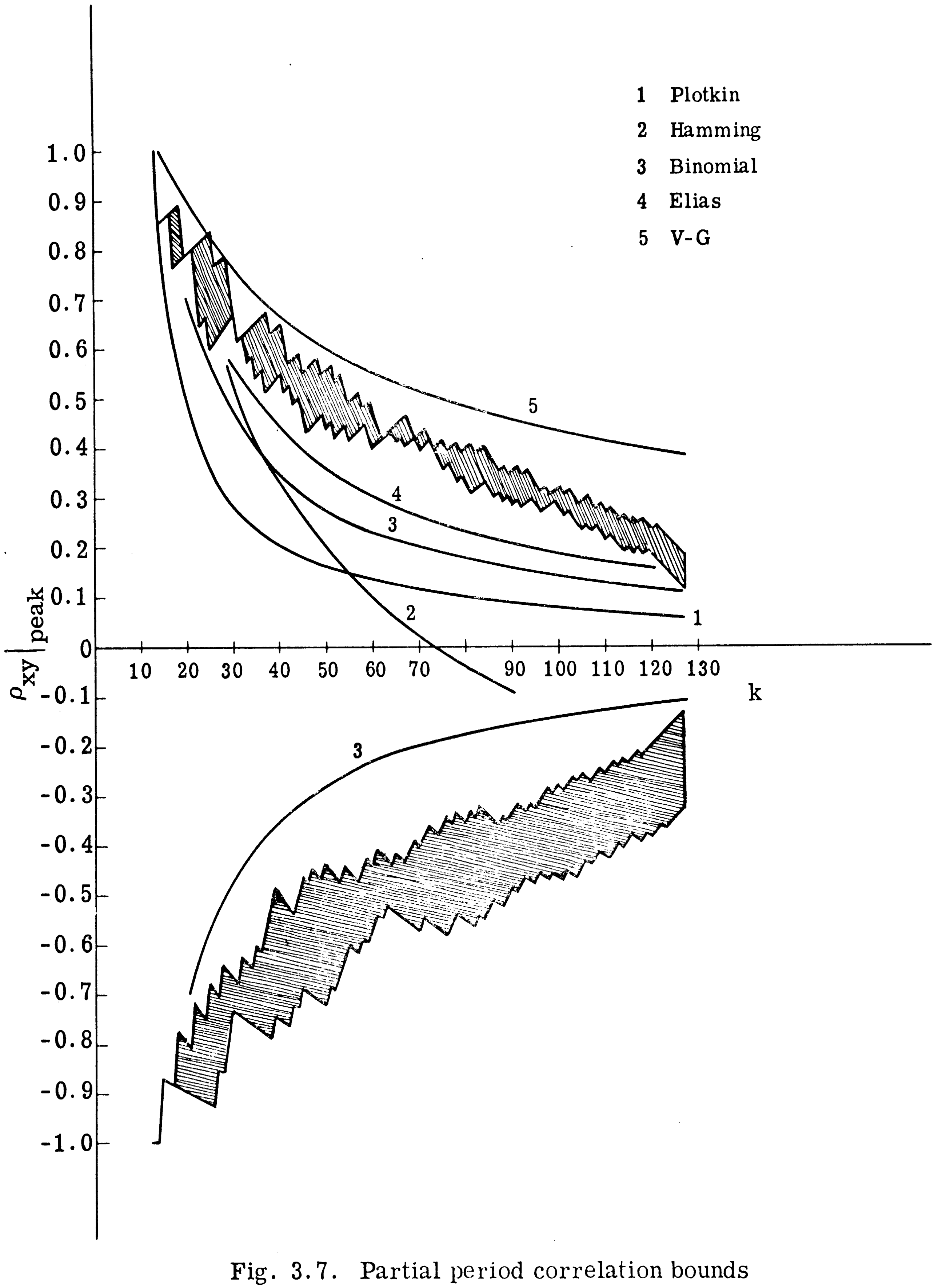

Fig. 3.7. Partial period correlation bounds

on the value of k. In general the Hamming bound and the Binomial model are appropriate for values of k close to 2n. The asymptotic Elias bound is somewhat better for values of k leading to medium rate codes and the Binomial model is again applicable at values of k close to L.

CHAPTER IV

MOMENTS OF THE DISTRIBUTION OF CORRELATION VALUES

The relationship between the partial period correlation problem and the study of weight distributions for linear codes has been established. The k-tuples which are operated on to calculate the partial period correlation values form a linear code. In the previous chapter it is shown that the minimum distance between these code vectors (the code vector of minimum weight) leads to the maximum value of partial period correlation. If the weight of all possible code vectors is known, it is possible to transform this information into complete knowledge concerning the distribution of correlation values. The weight structure of codes has been the subject of much study in the area of error correcting codes since the weight structure is related to the error correcting capability of the code. Since we are more concerned with the correlation values for this study, the purpose of this chapter is to study the distribution of possible correlation values when the correlation is carried out over a portion of the period of the sequences. The goal here is to determine how this distribution depends on the particular sequences which are being correlated.

This study is an extension of the study by Lindholm, Ref. 19, in which the pseudo-randomness properties of sub-sequences, or segments, taken from linear maximal binary sequences were studied. The purpose of Lindholm's study was to find methods of choosing a linear maximal sequence in such a way that these sub-sequences had good pseudo-randomness properties. This was accomplished by examining the moments of the distribution of weights of the sub-sequences (k-tuples). The results of this paper show that, while the first two moments of the weight distribution are independent of the particular sequence used, the third moment is dependent on the sequence. This provides a method of finding those sequences for which the weight distribution is skewed and which, therefore, may be eliminated from consideration as sequences which provide sub-sequences with good pseudo-randomness properties. It should be noted at this point that this study can be considered to be a study of the partial period autocorrelation case. In the autocorrelation case the k-tuples which are generated by mod-2 addition of the sequence and a shifted version of itself are, because of the shift-and-add property of PR sequences, k-tuples from the same sequence. That is, in Fig. 2.1 if $\{b_i\}$ is equal to $\{a_i\}$ then the sequence $\{c_i(\tau)\}$ is equal to the same sequence, $\{a_i\}$, at some phase shift which is determined by the characteristic polynomial. This is, of course, only true for non-zero values of phase shift, τ.

For the cross-correlation problem studied here the weight distribution, or correlation distribution, is the distribution of the weights of k-tuples taken from the family of non-maximal sequences discussed previously (Section 2.2). The moments of this distribution are calculated and it is shown that the same general properties exist for these non-maximal sequences as found by Lindholm for the maximal sequences.

4.1 Theoretical Calculation of Moments

The family of non-maximal sequences that are generated, during the cross-correlation calculation, by the mod-2 addition of two linear maximal sequences, can be labeled as follows: (These are assumed to be $\{0, 1\}$ sequences.)

$$\begin{aligned} \{a_i\} &= \{a_0, a_1, a_2, \ldots, a_{L-1}\} \\ \{b_i\} &= \{b_0, b_1, b_2, \ldots, b_{L-1}\} \\ \{c_i(\tau)\} &= \{a_{i+\tau} \oplus b_i\} \qquad i = 0, 1, 2, \ldots, L-1 \\ &\qquad \text{for } \tau = 0, 1, 2, \ldots, L-1 \end{aligned} \tag{4.1}$$

From the properties of the family of non-maximal sequences given previously it can be shown that:

$$\sum_{i=0}^{L-1} a_i + \sum_{i=0}^{L-1} b_i + \sum_{\tau=0}^{L-1} \sum_{i=0}^{L-1} c_i(\tau) = 2^{2n-1} \tag{4.2}$$

where $L = 2^n - 1$ is the period of the maximal sequences, $\{a_i\}$ and $\{b_i\}$.

Since $\{a_i\}$ and $\{b_i\}$ are linear maximal sequences, it is known that

$$\sum_{i=0}^{L-1} a_i = \sum_{i=0}^{L-1} b_i = 2^{n-1}$$

Therefore

$$\sum_{\tau=0}^{L-1} \sum_{i=0}^{L-1} c_i(\tau) = 2^{2n-1} - 2^n \tag{4.3}$$

The weight of a sub-sequence of k digits is given by

$$W_s = \sum_{i=0}^{k-1} d_{i+s} \tag{4.4}$$

$$s = \text{starting point}$$

This is the equation for the weight of k digits out of a general sequence $\{d_i\}$. The weight distribution is the distribution of the quantity W_s for all possible starting points. The pth moments for this distribution can be defined in the usual manner, i.e.,

$$\overline{W^p} = \frac{1}{L_T} \sum_{s=0}^{L_T-1} W_s^{\,p} \tag{4.5}$$

The pth moment is the average over all possible starting points, that is, all possible weights, and thus L_T is the total number of starting points for unique k-tuples. For the case under consideration here we are concerned with a family of sequences, rather than a single sequence, and some care must be taken in applying the equation given above. Since we are interested in k-tuples which are possible from the family of sequences then any k-tuple which starts in a sequence must be taken entirely from that sequence. The member of the family of sequences from which the k-tuple is taken is determined by the starting point and, for the family of sequences defined by Eq. 4.1 above, there must be a unique assignment of starting points in the sequence $\{d_i\}$ to particular sequences from the family. This assignment is as follows:

$$\{d_{i+s}\} = \{d_s, d_{s+1}, \ldots, d_{s+k-1}\} = \begin{cases} \{a_{i+s}\} & \text{for} \quad 0 \leq s \leq L-1 \\ \{b_{i+s}\} & \text{for} \quad L \leq s \leq 2L-1 \\ \{c_{i+s}(0)\} & \text{for} \quad 2L \leq s \leq 3L-1 \\ \{c_{i+s}(1)\} & \text{for} \quad 3L \leq s \leq 4L-1 \\ \vdots & \\ \{c_{i+s}(L-1)\} & \text{for} \quad (L+1)L \leq s \leq L^2+2L-1 \end{cases} \tag{4.6}$$

Since all of the sequences in the family are periodic sequences with period L the subscript may be reduced mod-L in all of the equations concerning the weight distribution, however the total number of starting points must be used for the normalization factor in Eq. 4.5.

The moments of the weight distribution can be calculated, using Eq. 4.5, and the results used to examine the shape of the correlation distribution. In studying higher order moments it is convenient to make use of the family of $\{1, -1\}$ sequences which are equivalent to the family of sequences defined above. As noted previously, the $\{0, 1\}$ sequences can be transformed into $\{1, -1\}$ sequences by making use of the transformation

$$\begin{aligned} x_i &= 1 - 2\, a_i \\ y_i &= 1 - 2\, b_i \\ z_i(\tau) &= 1 - 2\, c_i(\tau) \\ \{z_i(\tau)\} &= \{x_{i+\tau} \cdot y_i\} \end{aligned} \qquad (4.7)$$

Using the $\{1, -1\}$ sequences, the sum over k digits is equal to the difference between the number of 1's and the number of -1's in the k-tuple. This corresponds to the number of agreements minus the number of disagreements in the original two sequences for the starting point and phase shift which generated the k-tuple and consequently the quantity which is calculated by this sum is the unnormalized correlation. Therefore the distribution making use of $\{1, -1\}$ sequences is a distribution of correlation values. The correlation values are calculated using

$$C_s = \sum_{i=0}^{k-1} v_{i+s} \tag{4.8}$$

where

$$v_{i+s} = 1 - 2\,d_{i+s}\ .$$

It can readily be seen that

$$C_s = k - 2\,W_s \tag{4.9}$$

The assignment of starting points in the general sequence $\{v_i\}$ to the sequences $\{x_i\}$, $\{y_i\}$, and $\{z_i(\tau)\}$ is the same as the assignment shown above in Eq. 4.6 for the $\{0,1\}$ sequences. As noted previously, equivalent information about the sequences is contained in either of these distributions, and the relationship between the pth moments for these distributions is shown in Eq. 4.10.

$$\begin{aligned}\overline{C^p} &= \frac{1}{L_T} \sum_{s=0}^{L_T-1} C_s^{\,p} \\ &= \frac{1}{L_T} \sum_{s=0}^{L_T-1} (k - 2\,W_s)^p \\ &= \sum_{\ell=0}^{p} \binom{p}{\ell} k^{p-\ell} (-2)^{\ell}\, \overline{W^{\ell}}\end{aligned} \tag{4.10}$$

The primary advantage involved in using the $\{1,-1\}$ sequences is that the well known shift-and-add property of linear maximal $\{0,1\}$

sequences becomes a shift-and-multiply property for the transformed sequences. This allows some simplification of the moment calculation. For this reason the correlation distribution is used for the calculation of moments.

Before proceeding with the calculation of the moments it should be noted that the correlation distribution, as defined above, will include k-tuples from the original two maximal sequences. It is necessary to include k-tuples from the original two sequences in order to consider the vectors involved to be elements of an Abelian group (and hence a linear code). For the cross-correlation between linear maximal sequences, however, it may at times be convenient to consider only those k-tuples which are a result of multiplication, bit by bit, of the original two maximal sequences. [That is, k-tuples from the sequences $\{z_i(\tau)\}$.] This leads to a modified correlation distribution which does not contain as many correlation values. The moments of the correlation distribution are calculated for both cases. For the entire family of sequences the number of starting points for unique k-tuples, and hence the total number of possible correlation values is $L_T = (L+2)L$, i.e., $L_T = 2^{2n}-1$, while for the case in which only the sequences $\{z_i(\tau)\}$ are considered, L_T will be equal to L^2, i.e., $L_T = (2^n-1)^2$.

The pth moment of the correlation distribution for the entire family of sequences becomes

$$\overline{C^p} = \frac{1}{L_T} \sum_{s=0}^{L_T-1} C_s^{\ p}$$

$$= \frac{1}{L_T} \sum_{s=0}^{L_T-1} \left(\sum_{i=0}^{k-1} v_{i+s} \right)^p$$

$$= \frac{1}{L_T} \left\{ \sum_{s=0}^{L-1} \left(\sum_{i=0}^{k-1} x_{i+s} \right)^p + \sum_{s=0}^{L-1} \left(\sum_{i=0}^{k-1} y_{i+s} \right)^p \right.$$

$$\left. + \sum_{s=0}^{L-1} \sum_{\tau=0}^{L-1} \left(\sum_{i=0}^{k-1} z_{i+s}(\tau) \right)^p \right\} \qquad (4.11)$$

This expansion of the equation for the pth moment into the sum of the moments for the individual sequences of the family is justified by the fact that the member of the family from which the k-tuple is taken is determined by the starting point. The first two terms in the expanded version of Eq. 4.11 are just the pth moments of the two original sequences. Lindholm's results may be applied directly for these two terms. The last term involves the non-maximal sequences and must be considered separately. The analysis is, of course, very much like that for the maximal sequence terms.

4.1.1 Moments for k-tuples from Maximal Sequences (Lindholm, Ref. 19). Before proceeding to the analysis of the last term in Eq. 4.11 above, it would be appropriate to review Lindholm's results which are applicable to the first two terms. In the following discussion the notation is essentially the same as that used by Lindholm in Ref. 19.

Lindholm has shown that the pth moment of the correlation distribution for a linear maximal sequence is given by

$$\overline{C_x{}^p} = \frac{1}{L} \sum_{s=0}^{L-1} \left(\sum_{i=0}^{k-1} x_{i+s} \right)^p$$

$$\overline{C_x{}^p} = F(0,p,k) + \frac{1}{L} \sum_{j=1}^{p} F(j,p,k) \left\{ -\binom{k}{j} + (L+1)\, B_{xj} \right\} \qquad (4.12)$$

In this equation B_{xj} is the number of polynomials with j terms, of degree less than k, which are divisible by $f(X)$, the characteristic polynomial of the sequence $\{x_i\}$. The coefficients, $F(j,p,k)$, arise in the expansion of the pth order summation into a series of jth order summations. That is, the expression

$$\left(\sum_{i=0}^{k-1} x_{i+s} \right)^p = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \cdots \sum_{\ell=0}^{k-1} x_{i+s}\, x_{j+s} \cdots x_{\ell+s} \qquad (4.13)$$

can be expanded into a series of jth order summations over terms in which the subscripts are all different by considering the number of ways in which $p-j$ subscripts will be the same in the product on the right hand side of Eq. 4.13. With the $\{1,-1\}$ sequences the shift-and-multiply property is given by

$$x_{i+s}\,x_{j+s} = \begin{cases} 1 & i = j \\ x_{m+s} & i \neq j \end{cases} \tag{4.14}$$

In this equation the value of m, the phase of the product sequence for $i \neq j$, is a function of the characteristic polynomial of the sequence $\{x_i\}$ and, therefore, depends on the particular sequence used. The calculation of the coefficients in the expansion of the pth order summation of Eq. 4.13 into a series of jth order summations is, therefore, accomplished by counting the number of ways that $p-j$ subscripts are equal, in pairs, so that the product on the right hand side of the equation becomes a product of j terms. It should be noted that $p-j$ must be even since the product of an odd number of terms with equal subscripts will just be equal to the term. For example, for $p=3$ we have

$$\begin{aligned} \left(\sum_{i=0}^{k-1} x_{i+s}\right)^3 &= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\sum_{\ell=0}^{k-1} x_{i+s}\,x_{j+s}\,x_{\ell+s} \\ &= (3k-2)\sum_{i=0}^{k-1} x_{i+s} + 3!\sum_{i=0}^{k-3}\sum_{j=i+1}^{k-2}\sum_{\ell=j+1}^{k-1} x_{i+s}\,x_{j+s}\,x_{\ell+3} \end{aligned} \tag{4.15}$$

in which the coefficient $F(3,3,k) = 3!$ is the number of ways in which a set of three different subscripts can occur, and $F(1,3,k) = 3k-2$ is equal to the number of ways in which two of the subscripts will be identical. (There are three ways to choose the pairs of subscripts which will be equal, k values of the subscript, and, in the 3k possibilities, the set for which all three subscripts are equal will occur three times. Thus the coefficient becomes 3k-2.) The values of the coefficients, $F(j, p, k)$, for other values of p can be calculated and are listed by Lindholm for values of $p \leq 5$. Additional comments on the calculation are given in Appendix A.

As noted previously the term B_{xj} is the number of j-term polynomials of degree less than k which are divisible by the characteristic polynomial, $f(X)$, of the sequence $\{x_i\}$. That is, we are looking for the number of polynomials of the form

$$g(X) = \sum_{i=0}^{j-1} X^{e_i} \tag{4.16}$$

for which the largest value of e_i is $\leq k-1$, and such that the polynomial is divisible by the characteristic polynomial. If we assume that the e_i's are chosen such that $e_k > e_\ell$ if $k > \ell$ then the term X^{e_0} can be factored out and the polynomial $g(X)$ becomes

$$g(X) = X^{e_0}\left(X^{e_{j-1} - e_0} + X^{e_{j-2} - e_0} + \ldots + 1\right) \tag{4.17}$$

If $g(X)$ is divisible by the characteristic polynomial $f(X)$ then the polynomial enclosed in parentheses on the right hand side of Eq. 4.17 must be divisible by $f(X)$. Therefore the problem of calculating B_{xj} can be reduced to finding the number of j-term polynomials of the form

$$g_1(X) = X^{d_i} + X^{c_i} + \ldots + X^{a_i} + 1 \qquad (4.18)$$

which are divisible by $f(X)$. If the number of such polynomials is B^*_{xj} then, as shown by Lindholm, B_{xj} can be found using the relation:

$$B_{xj} = k\, B^*_{xj} - \sum_{i=1}^{B^*_{xj}} d_i \qquad (4.19)$$

The problem of determining the value of B^*_{xj} can be solved in a variety of ways. The method described by Lindholm in Ref. 19 makes use of the coset relationships. Another technique for finding the polynomials makes use of the generating matrix described previously. If the columns of the generating matrix are labeled from 0 for the left-most column to $L-1$ for the column on the extreme right then the polynomials divisible by the characteristic polynomial of the sequence can be found by determining the subsets of columns for which the vector sum, mod-2, is equal to zero. That is, if the polynomial divisible by the characteristic polynomial is given as in Eq. 4.18 above, then the sum of the j columns labeled $0, a_i, \ldots, d_i$ will be equal to zero, mod-2. This leads to a simple computer search for polynomials. [Since, for polynomials of the form (4.18), the left-most column is always included in the sum it is only necessary to find sets of $j-1$ columns for which the sum is equal to the 0th column.]

For example, for the third moment the number of trinomials is needed. Since the generating matrix for a linear maximal binary sequence contains all possible non-zero columns, the sum of any two columns must be equal to a third column in the matrix. Therefore it can be seen that the number of polynomials of the form $X^d + X^c + 1$ for $d \leq L$ must be equal to $(L-1)/2$. The total number of trinomials which are divisible by the characteristic polynomial is found by using Eq. 4.19.

Using these results, it can be shown that the correlation distribution for a linear maximal sequence and its reverse sequence are identical. While this conclusion may be obvious, the derivation points out an interesting property of the term B_{xj}. It has been shown that the value of B_{xj} is found by determining the number of j-term polynomials of the form

$$X^d + X^c + \ldots + X^a + 1 \qquad (4.20)$$

that are divisible by the characteristic polynomial of a linear maximal sequence. For each polynomial of this form there are $k-(d+1)$ polynomials of the form given in Eq. 4.17 of degree $\leq k-1$ which are divisible by the characteristic polynomial. Therefore the total number of j-term polynomials divisible by the characteristic polynomial depends on the degree of the polynomials of the form (4.20) (as shown in Eq. 4.19). The number of j-term polynomials of degree $\leq k-1$ which are divisible by the characteristic polynomial of an n-stage shift-register generator is equal to the number of j-term polynomials which are divisible by the characteristic polynomial for the reverse sequence. That is, if the characteristic polynomial of an n-stage shift-register generator is

$$f(X) = X^n + X^m + \ldots + X^q + 1 \tag{4.21}$$

then the characteristic polynomial for the n-stage generator which generates the reverse sequence is

$$f^*(X) = X^n f\left(\frac{1}{X}\right) = X^n + X^{n-q} + \ldots X^{n-m} + 1 \tag{4.22}$$

Then, if $f(X) | g_1(X)$, $f^*(X) | g_1^*(X)$ where $g_1^*(X) = X^{d_i} g_1\left(\frac{1}{X}\right)$. Therefore the degree of the polynomials divisible by $f^*(X)$ will be identical to the degree of the polynomials divisible by $f(X)$ and, as seen from Eq. 4.19, B_{xj} will be identical for the reverse sequences. Since B_{xj} is the same for a sequence and its reverse it can be seen from Eq. 4.12 that the moments of the k-tuples for a sequence and its reverse are identical. (This obvious fact can also be seen by considering the individual k-tuples. By definition the k-tuples taken from a sequence are the reverses of the k-tuples taken from the reverse sequence and the operation of summing over the k digits leads to identical correlation values.) Therefore the partial period auto-correlation distribution for a sequence and its reverse are identical.

4.1.2 Moments of Partial Period Cross-Correlation Distribution. The first two terms in the expansion of the pth moment of the correlation distribution, Eq. 4.11, can be expressed in the form shown in Eq. 4.12. The third term in the expansion of the pth moment must be analyzed further here. This term involves the family of non-maximal sequences and, proceeding as before, we have

$$
\begin{aligned}
L_T \overline{C_z^{\,p}} &= \sum_{s=0}^{L-1} \sum_{\tau=0}^{L-1} \left(\sum_{i=0}^{L-1} z_{i+s}(\tau) \right)^p \\
&= \sum_{s=0}^{L-1} \sum_{\tau=0}^{L-1} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \cdots \sum_{m=0}^{k-1} z_{i+s}(\tau)\, z_{j+s}(\tau) \ldots z_{m+s}(\tau) \right)
\end{aligned} \tag{4.23}
$$

As noted previously $z_{i+s}(\tau) = x_{i+s+\tau}\, y_{i+s}$ and the product of sequence terms above becomes

$$
\begin{aligned}
z_{i+s}(\tau)\, z_{j+s}(\tau) \ldots z_{m+s}(\tau) &= \left(x_{i+s+\tau}\, x_{j+s+\tau} \ldots \right. \\
&\quad \left. x_{m+s+\tau} \right) \left(y_{i+s}\, y_{j+s} \ldots y_{m+s} \right)
\end{aligned} \tag{4.24}
$$

The shift-and-multiply property of $\{1, -1\}$ linear maximal sequences states that the product of an even number of terms with the same subscript will be equal to 1 . If the subscripts are not equal the resulting sequence will be a shifted version of the same sequence where the shift is determined by the characteristic polynomial of the sequence involved. Therefore the shift involved in multiplying terms from the sequence $\{x_i\}$ will not be the same as the shift involved in multiplying terms from the sequence $\{y_i\}$ even though the subscripts in the two products may be the same. However it can be seen that for both maximal sequences and, therefore, for the sequence $\{z_{i+s}(\tau)\}$ the product of an even number of terms with the same subscript is equal to 1. Then, as before, the summation of the product of p-terms where all subscripts range from 0 to $k-1$ can be expanded to sums of products of j terms over subscripts which are different $(j \leq p)$. Therefore for $p=3$ the term involving the sequences $\{z_i(\tau)\}$ reduces to

$$\left(\sum_{i=0}^{k-1} z_{i+s}(\tau)\right)^3 = (3k-2)\sum_{m=0}^{k-1} z_{m+s}(\tau) + 3!\sum_{i=0}^{k-1}\sum_{j=i+1}^{k-2}\sum_{m=j+1}^{k-1} z_{i+s}(\tau)\, z_{j+s}(\tau)\, z_{m+s}(\tau) \tag{4.25}$$

This equation has the same form as the equation given previously (Eq. 4.15) for the linear maximal sequence alone. As before the value of the triple sum depends on the number of 3 term polynomials which are divisible by the characteristic polynomials of the sequences. However in this case there are three possibilities, i.e., B_{x3}, the number of trinomials divisible by the characteristic polynomial for the sequence $\{x_i\}$, B_{y3}, the number of trinomials divisible by the characteristic polynomial for the sequence $\{y_i\}$, and B_{xy3}, the number of trinomials divisible by both characteristic polynomials. (B_{xy3} is therefore the number of trinomials divisible by the characteristic polynomial for the equivalent 2n-stage generator.) For those subscripts which do not result in a trinomial relationship which is divisible by the characteristic polynomial of either of the maximal generators the product of the terms will result in one of the sequences of the family of non-maximal sequences at some phase shift which is dependent upon the characteristic polynomial of the equivalent 2n-stage register. That is, for subscripts $i < j < m$ such that no trinomial relationship exists for either of the maximal sequences the product becomes

$$z_{i+s}(\tau)\, z_{j+s}(\tau)\, z_{m+s}(\tau) = \left(x_{i+s+\tau}\, x_{j+s+\tau}\, x_{j+s+\tau}\, x_{m+s+\tau}\right) \left(y_{i+s}\, y_{j+s}\, y_{m+s}\right)$$

$$\begin{aligned} x_{i+s+\tau}\, x_{j+s+\tau}\, x_{m+s+\tau} &= x_{m'+s+\tau} \\ y_{i+s}\, y_{j+s}\, y_{m+s} &= y_{\ell'+s} \\ x_{m'+s+\tau}\, y_{\ell'+s} &= z_{r+s}(\tau') \end{aligned} \tag{4.26}$$

where $\tau' = m' - \ell' + \tau$

For the triple sum of Eq. 4.25 there are $\binom{k}{3}$ total terms, of which B_{xy3} terms lead to a trinomial relationship for the sequence $\{z_i(\tau)\}$ and the product is equal to 1, $B_{x3} - B_{xy3}$ lead to trinomial relationships for the sequence $\{x_i\}$ alone and the product results in a term of the form $y_{\ell+s}$, $B_{y3} - B_{xy3}$ terms lead to a trinomial relationship for the sequence $\{y_i\}$ alone and the product leads to a term of the form x_{m+s}. Therefore the triple sum reduces to

$$\sum_{i=0}^{k-3} \sum_{j=i+1}^{k-2} \sum_{m=j+1}^{k-1} z_{i+s}(\tau)\, z_{j+s}(\tau)\, z_{m+s}(\tau)$$

$$= B_{xy3} + \left(B_{x3} - B_{xy3}\right) y_{\ell+s} + \left(B_{y3} - B_{xy3}\right) x_{m+s}$$

$$+ \left\{\binom{k}{3} - B_{x3} - B_{y3} + B_{xy3}\right\} z_{r+s}(\tau') \qquad (4.27)$$

This quantity will then be summed over all values of s and τ in order to determine the third moment. When the maximal sequences are summed over all values of s the result will be equal to -1. When the non-maximal sequences are summed over all values of s and τ the result will be equal to 1. (This can be seen by applying Eqs. 4.2 and 4.3 to the transformed sequences.)

The third moment of the correlation distribution for the entire family of sequences is therefore given by

$$\overline{C^3} = \frac{1}{L_T}\left\{\sum_{s=0}^{L-1}\left(\sum_{i=0}^{k-1} x_{i+s}\right)^3 + \sum_{s=0}^{L-1}\left(\sum_{i=0}^{k-1} y_{i+s}\right)^3 + \sum_{s=0}^{L-1}\sum_{\tau=0}^{L-1}\left(\sum_{i=0}^{k-1} z_{i+s}(\tau)\right)^3\right\}$$

$$= \frac{1}{L_T}\left\{(3k-2)\,k(-1) + 3!\left[\binom{k}{3} - B_{x3}\right](-1) + 3!\;B_{x3}L + (3k-2)\,k(-1)\right.$$

$$+ \;3!\left[\binom{k}{3} - B_{y3}\right](-1) + 3!\;B_{y3}L + (3k-2)\,(k)$$

$$+ \;3!\left[\binom{k}{3} - \left(B_{x3} + B_{y3} - B_{xy3}\right)\right] - \left(B_{y3} - B_{xy3}\right)L - \left(B_{x3} - B_{xy3}\right)L$$

$$\left. + \;L^2\,B_{xy3}\right\}.$$

$$\overline{C^3} = -\frac{k^3}{L(L+2)} + 3!\,\frac{(L+1)^2}{L(L+2)}\,B_{xy3} \tag{4.28}$$

In general, for the pth moment, the term involving the non-maximal sequences $\{z_i(\tau)\}$ reduces to

$$L_T\,\overline{C_z^{\,P}} = F(0,p,k)\,L^2 + \sum_{j=1}^{p} F(j,p,k)\;,$$

$$\left\{\binom{k}{j} - (L+1)\left(B_{xj} + B_{yj}\right) + (L+1)^2\,B_{xyj}\right\} \tag{4.29}$$

The coefficients $F(j,p,k)$ are the same coefficients which appear in the pth moments for the linear maximal sequences as shown in Eq. 4.12. B_{xj} and B_{yj} are the number of polynomials with j terms, with degree less than k which are divisible by the characteristic polynomials of the sequences $\{x_i\}$ and $\{y_i\}$ respectively. B_{xyj} is the number of polynomials with j terms which are divisible by both characteristic polynomials. It should be noted here that, for values of k in the range of interest, i.e., $2n \leq k \leq L$, B_{xj}, B_{yj}, and B_{xyj} will be equal to zero for $j \leq 2$. The characteristic polynomials for the two linear maximal sequences are both factors of the two term polynomial $X^L + 1$ and neither characteristic polynomial can divide a two term polynomial, $X^j + 1$ for $j < L$.

Using these results along with the results of Lindholm for the maximal sequences alone it can be seen that the pth moment for the entire family of sequences becomes

$$\overline{C^P} = \frac{1}{L_T}\left\{L\,F(0,p,k) + \sum_{j=1}^{p} F(j,p,k)\left[-\binom{k}{j} + (L+1)\,B_{xj}\right] + L\,F(0,p,k)\right.$$

$$+ \sum_{j=1}^{p} F(j,p,k)\left[-\binom{k}{j} + (L+1)\,B_{yj}\right] + F(0,p,k)\,L^2$$

$$\left. + \sum_{j=1}^{p} F(j,p,k)\left[\binom{k}{j} - (L+1)\left(B_{xj} + B_{yj}\right) + (L+1)^2\,B_{xyj}\right]\right\}$$

$$= \frac{1}{L_T}\left\{F(0,p,k)\,L(L+2) + \sum_{j=1}^{p} F(j,p,k)\left[-\binom{k}{j} + (L+1)^2\,B_{xyj}\right]\right\} \qquad (4.30)$$

As noted previously $L_T = L(L+2)$ and therefore this equation can be reduced to

$$\overline{C^P} = F(0,p,k) + \frac{1}{L_T} \sum_{j=1}^{p} F(j,p,k) \left\{ -\binom{k}{j} + (L_T + 1)\, B_{xyj} \right\} \qquad (4.31)$$

Comparing this equation with Eq. 4.12 it can be seen that the pth moment for the entire family of sequences has exactly the same form as the pth moment for the maximal sequences.

The pth moment of the correlation distribution for k-tuples taken from the non-maximal sequences $\{z_i(\tau)\}$ can be found directly from Eq. 4.29. As noted previously L_T for the non-maximal sequences alone is equal to L^2 and hence the pth moment for this distribution becomes

$$\overline{C_z^{\,P}} = F(0,p,k) + \frac{1}{L^2} \sum_{j=1}^{p} F(j,p,k) \left\{ \binom{k}{j} - (L+1)\left(B_{xj} + B_{yj}\right) + (L+1)^2 B_{xyj} \right\} \qquad (4.32)$$

Thus the only additional quantity needed to calculate the moments of the correlation distribution for the cross-correlation case is B_{xyj}. $F(j,p,k)$, B_{xj}, and B_{yj} are calculated as before for the distribution of correlation values from the maximal sequence alone (Lindholm, Ref. 19). As noted previously, B_{xyj} is the number of polynomials with j-terms which are divisible by both characteristic polynomials. These polynomials can be found by using the generating matrix for the equivalent 2n-stage shift-register generator as before and B_{xyj} found by application of Eq. 4.19.

As noted previously, the values of the coefficients F(j, p, k) are tabulated by Lindholm for $p \leq 5$. In Appendix A the calculation of these coefficients is discussed alone with some comments on the number of j-term polynomials of degree less than L.

In the preceding discussion we have been concerned with calculating the moments of the correlation distribution. For comparison purposes it is sometimes useful to have the central moments of the distribution. In order to find the central moments of the correlation distribution it is necessary to subtract the mean value from each value of correlation. That is,

$$\overline{C_c^{\,P}} = \frac{1}{L_T} \sum_{s=0}^{L_T-1} \left(C_s - \overline{C^1}\right)^p$$

$$= \sum_{\ell=0}^{p} (-1)^{\ell} \binom{p}{\ell} \overline{C^{p-\ell}} \left(\overline{C^1}\right)^p \tag{4.33}$$

The first five moments for the correlation distribution for the case in which the entire family of sequences is considered are shown in Table 4.1. The case in which just the non-maximal sequences are considered is shown in Table 4.2. The central moments are calculated using Eqs. 4.34 and are also shown in the tables.

MOMENTS (ENTIRE FAMILY)	CENTRAL MOMENTS (ENTIRE FAMILY)
$\overline{C} = -\frac{k}{L(L+2)}$	
$\overline{C^2} = k\left[1 - \frac{k-1}{L(L+2)}\right]$	$\overline{C_c^{\,2}} = k\,\frac{(L+1)^2}{L(L+2)}\left(1 - \frac{k}{L(L+2)}\right)$
$\overline{C^3} = \frac{1}{L(L+2)}\left\{-k^3 + 3!\,(L+1)^2 B_{xy3}\right\}$	$\overline{C_c^{\,3}} = \frac{1}{L^3(L+2)^3}\left\{-k^3(L+1)^2(L^2+2L+2) + 3k^2 L(L+2)(L+1)^2\right\} + 3!\,\frac{(L+1)^2}{L(L+2)} B_{xy3}$ for $L >> 1$ $\overline{C_c^{\,3}} \approx -\frac{k^2}{L^2}(k-3) + 3!\,B_{xy3}$
$\overline{C^4} = k(3k-2) - \frac{k(k-1)^2(k+2)}{L(L+2)} + 4!\,\frac{(L+1)^2}{L(L+2)} B_{xy4}$	$\overline{C_c^{\,4}} = k\,\frac{(L+1)^2}{L(L+2)}\left\{3k-2-\frac{k^3}{L(L+2)}\left(\frac{L^4+4L^3+7L^2+6L+3}{L^2(L+2)^2}\right) + \frac{6k^2}{L^2(L+2)^2}\right\}$ $+ 4!\,\frac{(L+1)^2}{L(L+2)}\left[B_{xy4} + \frac{k}{L(L+2)} B_{xy3}\right]$ for $L >> 1$ $\overline{C^4} \approx k\left\{3k-2-\frac{k^3}{L^2}\right\} + 4!\left\{B_{xy4} + \frac{k}{L^2} B_{xy3}\right\}$
$\overline{C^5} = \frac{1}{L(L+2)}\left\{-k^5 + 5!\,(L+1)^2\left[B_{xy5} + \left(\frac{k-2}{2}\right) B_{xy3}\right]\right\}$	$\overline{C_c^{\,5}} = k^2\,\frac{(L+1)^2}{L(L+2)}\left\{5\,\frac{3k-2}{L(L+2)} - \frac{k^3}{L(L+2)}\left(\frac{L^2+2L+2}{L(L+2)}\right)\left(\frac{L^4+4L^3+6L^2+4L+2}{L^2(L+2)^2}\right) + \frac{10k^2}{L^3(L+2)^3}\right\}$ $+ 5!\,\frac{(L+1)^2}{L(L+2)}\left[B_{xy5} + \frac{k}{L(L+2)} B_{xy4} + \left(k-2+\frac{k^2}{L^2(L+2)^2}\right)\frac{B_{xy3}}{2}\right]$ for $L >> 1$ $\overline{C_c^{\,5}} \approx 5\,\frac{k^2}{L^2}(3k-2) - \frac{k^5}{L^2} + 5!\left\{B_{xy5} + \frac{k}{L^2} B_{xy4} + (k-2)\,\frac{B_{xy3}}{2}\right\}$

Table 4.1. Moments of the correlation distribution (entire family)

MOMENTS ($\{z_i(\tau)\}$ ONLY)	CENTRAL MOMENTS ($\{z_i(\tau)\}$ ONLY)
$\overline{C_z} = \frac{k}{L^2}$	
$\overline{C_z^2} = k\left[1 + \frac{k-1}{L^2}\right]$	$\overline{C_{cz}^2} = k\left(\frac{L^2-1}{L^2}\right)\left(1 + \frac{k}{L^2}\right)$
$\overline{C_z^3} = \frac{1}{L^2}\{k^3 - 3![(L+1)(B_{x3} + B_{y3}) - (L+1)^2 B_{xy3}]\}$	$\overline{C_{cz}^3} = \frac{1}{L^6}[k^3(L^2-1)(L^2-2) - 3k^2L^2(L^2-1)] - \frac{3!}{L^2}[(L+1)(B_{x3} + B_{y3}) - (L+1)^2 B_{xy3}]$ for $L >> 1$ $\overline{C_{cz}^3} \approx \frac{k^2}{L^2}(k-3) - 3!\left[\frac{B_{x3} + B_{y3}}{L} - B_{xy3}\right]$
$\overline{C_z^4} = k(3k-2) + \frac{k(k-1)^2(k+2)}{L^2} - 4!\left[\frac{L+1}{L^2}(B_{x4} + B_{y4}) - \frac{(L+1)^2}{L^2}B_{xy4}\right]$	$\overline{C_{cz}^4} = k\frac{L^2-1}{L^2}\left\{3k-2 + \frac{k^3}{L^2}\left[\frac{L^4+3L^2+3}{L^4}\right] + \frac{6k^2}{L^4}\right\} - 4!\frac{L+1}{L^2}[(B_{x4} + B_{y4})$ $- \frac{k}{L^2}(B_{x3} + B_{y3})] + 4!\frac{(L+1)^2}{L^2}[B_{xy4} - \frac{k}{L^2}B_{xy3}]$ for $L >> 1$ $\overline{C_{cz}^4} = k(3k-2) + \frac{k^4}{L^2} - 4!\left[\frac{B_{x4} + B_{y4}}{L} - \frac{k}{L^2}\frac{B_{x3} + B_{y3}}{L}\right] + 4![B_{xy4} - \frac{k}{L^2}B_{xy3}]$
$\overline{C_z^5} = \frac{1}{L^2}\left\{k^5 - 5!(L+1)\left[(B_{x5} + B_{y5}) + \frac{(k-2)}{2}(B_{x3} + B_{y3})\right]\right.$ $\left. + 5!(L+1)^2\left[B_{xy5} - \frac{k-2}{2}B_{xy3}\right]\right\}$	$\overline{C_{cz}^5} = k^2\frac{L^2-1}{L^2}\left\{-5\frac{3k-2}{L^2} + \frac{k^3}{L^2}\left(\frac{L^2-2}{L^2}\right)\left(\frac{L^4-2L+2}{L^4}\right) - 10\frac{k^2}{L^6}\right\}$ $- 5!\left(\frac{L+1}{L^2}\right)\left\{(B_{x5} + B_{y5}) - \frac{k}{L^2}(B_{x4} + B_{x5}) + \left(k-2 + \frac{k^2}{L^4}\right)\left(\frac{B_{x3} + B_{y3}}{2}\right)\right\}$ $+ 5!\frac{(L+1)^2}{L^2}\left\{B_{xy5} - \frac{k}{L^2}B_{xy4} + \left(k-2 + \frac{k^2}{L^4}\right)\frac{B_{xy3}}{2}\right\}$ for $L >> 1$ $\overline{C_{cz}^5} = -5\frac{k^2}{L^2}(3k-2) + \frac{k^5}{L^2} - 5!\left\{\frac{B_{x5} + B_{y5}}{L} - \frac{k}{L^2}\frac{B_{x4} + B_{y4}}{L} + (k-2)\frac{B_{x3} + B_{y3}}{2L}\right\}$ $+ 5!\left\{B_{xy5} - \frac{k}{L^2}B_{xy4} + (k-2)\frac{B_{xy3}}{2}\right\}$

Table 4.2. Moments of the correlation distribution $\left(\{z_i(\tau)\}\text{ only}\right)$

$$\overline{C_c^{\,2}} = \overline{C^2} - \left(\overline{C^1}\right)^2 = \sigma^2$$

$$\overline{C_c^{\,3}} = \overline{C^3} - 3\left(\overline{C^1}\right)\overline{C^2} + 2\left(\overline{C^1}\right)^3$$

$$\overline{C_c^{\,4}} = \overline{C^4} - 4\left(\overline{C^1}\right)\overline{C^3} + 6\left(\overline{C^1}\right)^2\overline{C^2} - 3\left(\overline{C^1}\right)^4 \qquad (4.34)$$

$$\overline{C_c^{\,5}} = \overline{C^5} - 5\left(\overline{C^1}\right)\overline{C^4} + 10\left(\overline{C^1}\right)^2\overline{C^3} - 10\left(\overline{C^1}\right)^3\overline{C^2} + 4\left(\overline{C^1}\right)^5$$

4.2 Comparison with the Binomial Distribution

In this chapter a correlation distribution has been considered in which the correlation values are the result of summing k-tuples which have been generated by multiplying, bit by bit, k-tuples taken from linear maximal binary sequences. Since these sequences are called pseudo-random sequences it would be of interest to compare the moments for this distribution with the moments of a correlation distribution in which the correlation values are generated by summing k-tuples taken from a $p = .5$ Bernoulli sequence. The distribution of the weights of k-tuples taken from such a sequence is a binomial distribution and, therefore, the correlation distribution is a binomial distribution, i.e., there are $\binom{k}{i}$ vectors which lead to a correlation equal to $k - 2i$. The moments of this distribution are calculated using the equation

$$\overline{C_r^{\,p}} = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}(k-2i)^p \qquad (4.35)$$

The moments of the binomial distribution are well known and are given below.

$$\overline{C_r^{\,1}} = 0$$

$$\mu_2 = \sigma^2 = k$$

$$\mu_3 = 0$$

$$\mu_4 = 3k^2 - 2k$$

$$\mu_5 = 0 \qquad (4.36)$$

The odd moments for this distribution will always be equal to zero and the second and fourth moment are shown above. Since the mean value is equal to zero, the higher order moments are central moments.

From Tables 4.1 and 4.2 it can be seen that the moments of the partial period correlation distribution are somewhat different than the moments given above for the binomial distribution. For both cases, i.e., entire family and $\{z_i(\tau)\}$ alone, the mean and variance approach the binomial distribution value as L increases. For the higher moments the terms involving B_{xj}, B_{yj} and B_{xyj} could lead to significant differences between the moments of the correlation distribution and the moments of the binomial distribution. These differences are dependent on the particular sequences involved and can best be illustrated by an example.

For the example of the moment calculation two cross-correlation distributions are considered. For the first case the two sequences are those generated by the SSRG with feedback equations [7, 3, 0] and [7, 5, 4, 3, 2, 1, 0] . These sequences correspond to a "preferred pair" (Gold, Ref. 6) and therefore have the best full period cross-correlation. For the second case the two sequences are [7, 3, 0] and [7, 6, 5, 4, 0] . The correlation distribution for $k = 50$ is used and from Table 3.3 it can be seen that the case 2 sequences have the largest cross-correlation peaks for this value of k . The first 4 moments are calculated and therefore it is necessary to determine the number of 3- and 4-term polynomials, of degree less than 50, divisible by the characteristic polynomials for the sequences given above. The characteristic polynomial for the sequence [7, 3, 0] is $X^7 + X^4 + 1$ (using the definition of Birdsall and Ristenbatt, Ref. 4; also Ref. 8) and is denoted (7, 4, 0).[1] Thus the characteristic polynomials for [7, 5, 4, 3, 2, 1, 0] and [7, 6, 5, 4, 0] become (7, 6, 5, 4, 3, 2, 0) and (7, 3, 2, 1, 0) respectively. The trinomials of degree less than 50 divisible by the three characteristic polynomials are given in Table 4.3. The 4-term polynomials of degree less than 50 divisible by the characteristic polynomials are shown in Tables 4.4, 4.5 and 4.6.

Given the degree of the polynomials (Tables 4.3 through 4.6)

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[1]This definition of the characteristic polynomial differs from the definition of Golomb (Ref. 2), however the results will be the same as long as a consistent definition is used for all characteristic polynomials.

(7, 4, 0)	(7, 6, 5, 4, 3, 2, 0)	(7, 3, 2, 1, 0)
(7, 4, 0)	(13, 10, 0)	(19, 13, 0)
(14, 8, 0)	(19, 18, 0)	(21, 5, 0)
(28, 16, 0)	(25, 17, 0)	(28, 11, 0)
(31, 30, 0)	(26, 20, 0)	(36, 29, 0)
(34, 23, 0)	(38, 36, 0)	(37, 33, 0)
(37, 27, 0)	(41, 11, 0)	(38, 26, 0)
(40, 19, 0)	(44, 37, 0)	(39, 9, 0)
(44, 35, 0)	(47, 24, 0)	(41, 40, 0)
(49, 13, 0)	(49, 21, 0)	(42, 10, 0)
		(47, 2, 0)
		(48, 23, 0)

Table 4.3. Trinomials of degree less than 50 divisible by (7, 4, 0), (7, 6, 5, 4, 3, 2, 0), and (7, 3, 2, 1, 0)

(10, 4, 3, 0)	(11, 8, 7, 0)	(14, 5, 1, 0)	(14, 11, 4, 0)	(15, 14, 12, 0)
(17, 10, 8, 0)	(18, 12, 7, 0)	(20, 8, 6, 0)	(21, 15, 4, 0)	(21, 18, 8, 0)
(21, 12, 11, 0)	(22, 17, 2, 0)	(22, 16, 14, 0)	(23, 22, 6, 0)	(24, 12, 3, 0)
(25, 21, 16, 0)	(26, 2, 1, 0)	(26, 15, 6, 0)	(27, 26, 3, 0)	(27, 25, 9, 0)
(28, 10, 2, 0)	(28, 22, 8, 0)	(28, 13, 9, 0)	(28, 19, 12, 0)	(28, 23, 20, 0)
(29, 10, 5, 0)	(29, 19, 6, 0)	(30, 19, 3, 0)	(30, 25, 17, 0)	(30, 28, 24, 0)
(31, 18, 2, 0)	(31, 24, 16, 0)	(31, 27, 23, 0)	(32, 30, 1, 0)	(32, 11, 6, 0)
(32, 20, 7, 0)	(33, 18, 1, 0)	(33, 23, 3, 0)	(33, 11, 9, 0)	(33, 31, 26, 0)
(34, 17, 9, 0)	(34, 29, 15, 0)	(34, 20, 16, 0)	(34, 26, 19, 0)	(34, 30, 27, 0)
(35, 23, 4, 0)	(35, 34, 7, 0)	(35, 33, 10, 0)	(35, 32, 16, 0)	(36, 15, 3, 0)
(36, 27, 6, 0)	(36, 21, 10, 0)	(36, 24, 14, 0)	(36, 18, 17, 0)	(36, 31, 22, 0)
(37, 21, 13, 0)	(37, 33, 19, 0)	(37, 24, 20, 0)	(37, 20, 23, 0)	(37, 34, 31, 0)
(38, 16, 1, 0)	(38, 37, 4, 0)	(38, 22, 5, 0)	(38, 27, 7, 0)	(38, 15, 13, 0)
(38, 35, 30, 0)	(39, 22, 1, 0)	(38, 16, 5, 0)	(39, 19, 9, 0)	(39, 37, 11, 0)
(39, 13, 12, 0)	(39, 38, 14, 0)	(39, 26, 17, 0)	(40, 31, 3, 0)	(40, 13, 5, 0)
(40, 25, 11, 0)	(40, 16, 12, 0)	(40, 22, 15, 0)	(40, 26, 23, 0)	(40, 33, 27, 0)
(41, 12, 2, 0)	(41, 30, 4, 0)	(41, 6, 5, 0)	(41, 31, 7, 0)	(41, 19, 10, 0)
(41, 38, 23, 0)	(41, 37, 35, 0)	(42, 30, 8, 0)	(42, 25, 10, 0)	(42, 31, 14, 0)
(42, 36, 16, 0)	(42, 24, 22, 0)	(43, 11, 2, 0)	(43, 35, 13, 0)	(43, 30, 15, 0)
(43, 36, 19, 0)	(43, 41, 21, 0)	(43, 29, 27, 0)	(44, 24, 1, 0)	(44, 34, 4, 0)
(44, 36, 5, 0)	(44, 23, 7, 0)	(44, 29, 21, 0)	(44, 41, 27, 0)	(44, 32, 28, 0)
(44, 38, 31, 0)	(44, 42, 39, 0)	(45, 9, 3, 0)	(45, 44, 8, 0)	(45, 23, 11, 0)
(45, 35, 14, 0)	(45, 29, 18, 0)	(45, 32, 22, 0)	(45, 26, 25, 0)	(45, 39, 30, 0)
(46, 3, 1, 0)	(46, 27, 2, 0)	(46, 29, 11, 0)	(46, 44, 12, 0)	(46, 42, 13, 0)
(46, 23, 18, 0)	(46, 32, 19, 0)	(47, 26, 4, 0)	(47, 43, 5, 0)	(47, 21, 6, 0)
(47, 42, 9, 0)	(47, 27, 10, 0)	(47, 32, 12, 0)	(47, 20, 18, 0)	(47, 44, 19, 0)
(47, 46, 28, 0)	(47, 40, 35, 0)	(48, 24, 6, 0)	(48, 39, 7, 0)	(48, 37, 8, 0)
(48, 18, 13, 0)	(48, 27, 14, 0)	(48, 42, 23, 0)	(49, 19, 5, 0)	(49, 10, 6, 0)
(49, 16, 9, 0)	(49, 20, 17, 0)	(49, 27, 21, 0)	(49, 41, 29, 0)	(49, 47, 36, 0)
(49, 44, 43, 0)				

Table 4.4. Quartics of degree less than 50 divisible by (7, 4, 0)

(8, 2, 1, 0)	(9, 8, 3, 0)	(15, 7, 3, 0)	(16, 4, 2, 0)	(16, 10, 3, 0)
(18, 16, 6, 0)	(19, 15, 5, 0)	(20, 18, 1, 0)	(21, 19, 8, 0)	(22, 18, 9, 0)
(22, 17, 12, 0)	(23, 14, 12, 0)	(23, 20, 13, 0)	(24, 17, 6, 0)	(25, 14, 4, 0)
(25, 20, 7, 0)	(26, 19, 1, 0)	(26, 21, 2, 0)	(26, 17, 7, 0)	(26, 23, 10, 0)
(27, 18, 2, 0)	(27, 6, 4, 0)	(27, 12, 5, 0)	(27, 20, 8, 0)	(28, 26, 3, 0)
(28, 17, 15, 0)	(28, 23, 16, 0)	(29, 6, 1, 0)	(29, 28, 13, 0)	(29, 20, 16, 0)
(30, 14, 6, 0)	(30, 8, 7, 0)	(30, 15, 9, 0)	(30, 27, 25, 0)	(31, 3, 1, 0)
(31, 9, 2, 0)	(31, 17, 5, 0)	(31, 28, 19, 0)	(31, 27, 22, 0)	(32, 16, 1, 0)
(32, 8, 4, 0)	(32, 20, 6, 0)	(32, 31, 10, 0)	(32, 29, 18, 0)	(33, 17, 8, 0)
(33, 16, 11, 0)	(33, 30, 26, 0)	(34, 22, 1, 0)	(34, 27, 3, 0)	(34, 20, 9, 0)
(35, 23, 8, 0)	(35, 15, 11, 0)	(35, 27, 13, 0)	(36, 32, 12, 0)	(36, 30, 13, 0)
(36, 35, 25, 0)	(37, 14, 3, 0)	(37, 36, 19, 0)	(38, 30, 10, 0)	(38, 28, 11, 0)
(38, 35, 17, 0)	(38, 37, 18, 0)	(38, 33, 23, 0)	(39, 11, 3, 0)	(39, 5, 4, 0)
(39, 12, 6, 0)	(39, 35, 7, 0)	(39, 33, 10, 0)	(39, 36, 20, 0)	(39, 24, 22, 0)
(39, 30, 23, 0)	(39, 38, 26, 0)	(40, 36, 2, 0)	(40, 24, 3, 0)	(40, 29, 5, 0)
(40, 22, 11, 0)	(41, 14, 1, 0)	(41, 24, 21, 0)	(41, 36, 28, 0)	(41, 30, 29, 0)
(41, 37, 31, 0)	(42, 30, 3, 0)	(42, 21, 5, 0)	(42, 37, 6, 0)	(42, 9, 7, 0)
(42, 15, 8, 0)	(42, 23, 11, 0)	(42, 38, 16, 0)	(42, 34, 25, 0)	(42, 33, 28, 0)
(43, 10, 2, 0)	(43, 4, 3, 0)	(43, 11, 5, 0)	(43, 34, 6, 0)	(43, 32, 9, 0)
(43, 35, 19, 0)	(43, 23, 21, 0)	(43, 29, 22, 0)	(43, 37, 25, 0)	(44, 39, 1, 0)
(44, 31, 11, 0)	(44, 29, 12, 0)	(44, 43, 17, 0)	(44, 36, 18, 0)	(44, 38, 19, 0)
(44, 34, 24, 0)	(44, 40, 27, 0)	(45, 36, 1, 0)	(45, 40, 8, 0)	(45, 24, 9, 0)
(45, 16, 12, 0)	(45, 28, 14, 0)	(45, 39, 18, 0)	(45, 44, 20, 0)	(45, 37, 26, 0)
(46, 15, 6, 0)	(46, 37, 8, 0)	(46, 14, 9, 0)	(46, 36, 21, 0)	(46, 38, 24, 0)
(46, 40, 26, 0)	(47, 30, 4, 0)	(47, 23, 5, 0)	(47, 25, 6, 0)	(47, 32, 7, 0)
(47, 21, 11, 0)	(47, 27, 14, 0)	(47, 37, 34, 0)	(47, 43, 42, 0)	(48, 17, 1, 0)
(48, 33, 2, 0)	(48, 5, 3, 0)	(48, 11, 4, 0)	(48, 19, 7, 0)	(48, 34, 12, 0)
(48, 46, 13, 0)	(48, 30, 21, 0)	(48, 29, 24, 0)	(48, 43, 39, 0)	(49, 27, 1, 0)
(49, 20, 2, 0)	(49, 22, 3, 0)	(49, 29, 4, 0)	(49, 18, 8, 0)	(49, 24, 11, 0)
(49, 34, 31, 0)	(49, 46, 38, 0)	(49, 40, 39, 0)	(49, 47, 41, 0)	

Table 4.5. Quartics of degree less than 50 divisible by (7, 6, 5, 4, 3, 2, 0)

(8, 7, 4, 0)	(12, 10, 3, 0)	(12, 11, 7, 0)	(15, 5, 2, 0)	(15, 14, 12, 0)
(16, 14, 8, 0)	(17, 8, 2, 0)	(22, 11, 9, 0)	(23, 3, 1, 0)	(23, 16, 6, 0)
(23, 18, 9, 0)	(24, 20, 6, 0)	(24, 22, 14, 0)	(24, 21, 18, 0)	(25, 15, 3, 0)
(25, 13, 6, 0)	(25, 14, 10, 0)	(25, 18, 17, 0)	(26, 21, 10, 0)	(26, 22, 17, 0)
(27, 9, 1, 0)	(27, 18, 3, 0)	(27, 8, 5, 0)	(27, 17, 15, 0)	(28, 27, 6, 0)
(29, 19, 8, 0)	(29, 20, 15, 0)	(30, 19, 2, 0)	(30, 10, 4, 0)	(30, 29, 17, 0)
(30, 27, 20, 0)	(30, 28, 24, 0)	(31, 26, 2, 0)	(31, 22, 8, 0)	(31, 24, 16, 0)
(31, 23, 20, 0)	(32, 18, 1, 0)	(32, 9, 3, 0)	(32, 28, 16, 0)	(32, 26, 19, 0)
(32, 27, 23, 0)	(32, 31, 30, 0)	(33, 30, 1, 0)	(33, 20, 9, 0)	(33, 21, 16, 0)
(33, 31, 18, 0)	(34, 26, 1, 0)	(34, 16, 4, 0)	(34, 14, 7, 0)	(34, 15, 11, 0)
(34, 19, 18, 0)	(35, 9, 2, 0)	(35, 10, 6, 0)	(35, 14, 13, 0)	(36, 12, 1, 0)
(36, 13, 8, 0)	(36, 23, 10, 0)	(36, 35, 16, 0)	(37, 16, 5, 0)	(37, 17, 12, 0)
(37, 27, 14, 0)	(38, 10, 5, 0)	(38, 20, 7, 0)	(38, 32, 13, 0)	(39, 25, 4, 0)
(39, 30, 14, 0)	(39, 37, 20, 0)	(39, 28, 22, 0)	(40, 11, 2, 0)	(40, 29, 3, 0)
(40, 37, 4, 0)	(40, 34, 5, 0)	(40, 24, 13, 0)	(40, 25, 20, 0)	(40, 35, 22, 0)
(41, 10, 1, 0)	(41, 28, 2, 0)	(41, 36, 3, 0)	(41, 33, 4, 0)	(41, 23, 12, 0)
(41, 24, 19, 0)	(41, 34, 21, 0)	(42, 40, 1, 0)	(42, 26, 5, 0)	(42, 31, 15, 0)
(42, 38, 21, 0)	(42, 29, 23, 0)	(43, 28, 1, 0)	(43, 10, 2, 0)	(43, 19, 4, 0)
(43, 9, 6, 0)	(43, 29, 7, 0)	(43, 38, 15, 0)	(43, 18, 16, 0)	(43, 31, 21, 0)
(43, 33, 24, 0)	(44, 41, 7, 0)	(44, 33, 8, 0)	(44, 23, 11, 0)	(44, 21, 14, 0)
(44, 22, 18, 0)	(44, 26, 25, 0)	(45, 19, 3, 0)	(45, 37, 7, 0)	(45, 40, 8, 0)
(45, 26, 9, 0)	(45, 17, 11, 0)	(45, 36, 24, 0)	(45, 34, 27, 0)	(45, 35, 31, 0)
(45, 29, 38, 0)	(46, 6, 2, 0)	(46, 10, 9, 0)	(46, 32, 12, 0)	(46, 36, 18, 0)
(46, 45, 21, 0)	(46, 41, 27, 0)	(46, 43, 35, 0)	(46, 42, 39, 0)	(47, 23, 7, 0)
(47, 41, 11, 0)	(47, 44, 12, 0)	(47, 30, 13, 0)	(47, 21, 15, 0)	(47, 40, 28, 0)
(47, 38, 31, 0)	(47, 39, 35, 0)	(47, 43, 42, 0)	(48, 7, 2, 0)	(48, 17, 4, 0)
(48, 29, 10, 0)	(48, 40, 12, 0)	(48, 39, 18, 0)	(48, 44, 28, 0)	(48, 42, 36, 0)
(49, 47, 4, 0)	(49, 32, 5, 0)	(49, 14, 6, 0)	(49, 23, 8, 0)	(49, 13, 10, 0)
(49, 33, 11, 0)	(49, 42, 19, 0)	(49, 22, 20, 0)	(49, 35, 25, 0)	(49, 37, 28, 0)

Table 4.6. Quartics of degree less than 50 divisible by (7, 3, 2, 1, 0)

the constants necessary for the moment calculation can be found using Eq. 4.19. These results are shown in Fig. 4.1. The first 4 moments can them be calculated using the equations of Tables 4.1 and 4.2 and the results are shown in Fig. 4.2.

From Eq. 4.36 the odd moments of the binomial correlation distribution are zero. For $k = 50$ the second moment is equal to 50 and the fourth moment is equal to 7200. From the results above it can be seen that the first four moments for Case 1, [7, 3, 0] and [7, 5, 4, 3, 2, 1, 0], are reasonably close to the moments of the binomial distribution, however the fourth moment of Case 2, [7, 3, 0] and [7, 6, 5, 4, 0], differs by about 20 percent. The correlation distributions for these two cases are shown in Figs. 4.3 and 4.4. It can be seen that the distribution for [7, 3, 0] correlated with [7, 6, 5, 4, 0] has a considerably wider range than the Case 1 distribution.

4.3 Summary

In this chapter it is shown that the results of Lindholm (Ref. 19) for the moments of the distribution of weights of k-tuples from a linear maximal binary sequence can be extended to moments of the distribution of correlation values for the partial period cross-correlation between linear maximal binary sequences of the same length. For the cross-correlation distribution the k-tuples used to calculate the correlation values are taken from a family of non-maximal

	$\{x_i\} \Rightarrow [7, 3, 0]$ $\{y_i\} \Rightarrow [7, 5, 4, 3, 2, 1, 0]$	$\{x_i\} \Rightarrow [7, 3, 0]$ $\{y_i\} \Rightarrow [7, 6, 5, 4, 0]$
B_{x3}	166	166
B_{y3}	148	154
B_{xy3}	0	0
B_{x4}	1812	1812
B_{y4}	1833	1798
B_{xy4}	0	94

Fig. 4.1. Number of polynomials divisible by the characteristic polynomials (7, 4, 0), (7, 6, 5, 4, 3, 2, 0) and (7, 3, 2, 1, 0) for k = 50

	$\{x_i\} \Rightarrow [7, 3, 0]$ $\{y_i\} \Rightarrow [7, 5, 4, 3, 2, 1, 0]$		$\{x_i\} \Rightarrow [7, 3, 0]$ $\{y_i\} \Rightarrow [7, 6, 5, 4, 0]$	
	Entire Family	$\{z_i(\tau)\}$	Entire Family	$\{z_i(\tau)\}$
$\overline{C}$	-3.05×10^{-3}	3.10×10^{-3}	-3.05×10^{-3}	3.10×10^{-3}
$\overline{C_c^2}$	49.85	50.15	49.85	50.15
$\overline{C_c^3}$	-7.17	-7.67	-7.17	-7.95
$\overline{C_c^4}$	7019	7093	8795	8904

Fig. 4.2. Moments of example correlation distributions for k = 50

C	$\{z_i(\tau)\}$	$\{x_i\}$	$\{y_i\}$	Entire Family
22	16			16
20	27			27
18	60			60
16	132			132
14	258			258
12	445			445
10	786	11	4	801
8	983	6	9	998
6	1219	8	7	1234
4	1495	11	16	1522
2	1705	8	15	1728
0	1878	15	21	1914
- 2	1718	21	23	1762
- 4	1554	25	7	1586
- 6	1219	8	7	1234
- 8	957	7	6	970
-10	673	7	6	686
-12	422		5	427
-14	322		1	323
-16	146			146
-18	76			76
-20	25			25
-22	13			13

Fig. 4.3. Correlation distribution for $k = 50$
$\{x_i\} = [7, 3, 0]$, $\{y_i\} = [7, 5, 4, 3, 2, 1, 0]$

C	$\{z_i(\tau)\}$	$\{x_i\}$	$\{y_i\}$	Entire Family
28	2			2
26	6			6
24	12			12
22	32			32
20	62			62
18	92			92
16	100			100
14	189			189
12	379			379
10	592	11	1	604
8	996	6	12	1014
6	1320	8	12	1340
4	1581	11	14	1606
2	1839	8	21	1868
0	1830	15	14	1859
- 2	1756	21	6	1783
- 4	1575	25	10	1610
- 6	1255	8	17	1280
- 8	912	7	14	933
-10	681	7	6	694
-12	451			451
-14	217			217
-16	107			107
-18	49			49
-20	44			44
-22	22			22
-24	5			5
-26	6			6
-28	5			5
-30	6			6
-32	2			2
-34	3			3
-36	1			1

Fig. 4.4. Correlation distribution for $k = 50$
$\{x_i\} = [7, 3, 0]$, $\{y_i\} = [7, 6, 5, 4, 0]$

sequences. The mean and variance of the correlation distribution are constant for all pairs of sequences of the same length. The pth moment $(p \geq 3)$ of the distribution depends on the number of p-term (or less) polynomials divisible by the characteristic polynomials of the two maximal sequence generators and the equivalent non-maximal sequence generator.

The third moment is the lowest order moment in which there is any possibility of finding differences in the moment structure, however it may be necessary, as in the example, to go to higher moments to find differences. The number of polynomials divisible by the characteristic polynomial increases very rapidly for higher order moments and thus the computer search technique described in this chapter is very useful in the calculation of higher order moments.

The calculation of the moments of the partial period correlation distribution for a given pair of sequences provides one method of comparing that pair of sequences with others of the same length or with truly random sequences $(p = .5$ Bernoulli sequences).

CHAPTER V

FREQUENCY DOMAIN ANALYSIS

In the preceding discussion of correlation the linear maximal sequences have been assumed to be sequences of digits taken from either the set $\{0, 1\}$ or the set $\{1, -1\}$. These sequences may be represented by binary waveforms in which the time axis has been divided into equal intervals and the level of the waveform during each interval determined by the corresponding digit of the sequence. The correlation of two sequences can be found by multiplying two sequence waveforms and integrating the resulting waveform over the appropriate time interval. For periodic sequences, the waveforms are periodic and it can be shown that the correlation function can be expressed in terms of the Fourier coefficients of the two waveforms. (See, for example Stein and Jones, Ref. 32.) For real aperiodic waveforms the correlation function can be expressed in terms of the frequency spectra of the two waveforms.

In this chapter we first consider the Fourier series of linear maximal binary sequences. For the characteristic phase position of the linear maximal binary sequence, defined in Chapter II, the calculation of the Fourier series coefficients is greatly simplified by making use of the cyclotomic cosets. It has been found that the Fourier series coefficients for all maximal sequences of the same length are related in a well specified manner. The correlation function for linear maximal sequences is given in terms of the Fourier coefficients for both full period correlation and partial period correlation. Although the equation for the partial period case becomes somewhat cumbersome, it does provide another technique for studying the effect of correlating over a time interval less than a full period of the maximal sequences.

5.1 Fourier Series of Pseudo-Random Sequences (see Golomb, Ref. 2)

A periodic function, $x(t)$, of period T_0, which satisfies the Dirichlet conditions may be represented by a Fourier series as follows: (see for example Guillemin, Ref. 33)

$$x(t) = \sum_{r=-\infty}^{\infty} \alpha_r e^{jr\omega_0 t} ; \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$\alpha_r = \frac{1}{T_0} \int_a^{a+T_0} x(t)\, e^{-jr\omega_0 t}\, dt \tag{5.1}$$

In this section the case in which $x(t)$ is a linear maximal binary sequence is considered.

The waveform corresponding to the $\{1, -1\}$ sequence is the sum of pulses of width t_c, the clock period. Let $v(t)$ be such a pulse, i.e.,

$$v(t) = \begin{cases} 1 & 0 \leq t < t_c \\ 0 & \text{elsewhere} \end{cases} \tag{5.2}$$

Then the sequence waveform x(t) becomes:

$$x(t) = \sum_{m=-\infty}^{\infty} x_m \, v(t - mt_c)$$

$$x_m = \text{mth digit of the } \{1, -1\} \text{ sequence} \tag{5.3}$$

This sequence has transitions at integral multiples of the clock period, t_c, and has period $T_0 = Lt_c$. ($L = 2^n - 1$ the length of the linear maximal binary sequence generated by an n-stage shift-register generator.) One period of the sequence waveform can be described by:

$$x(t) = \sum_{m=0}^{L-1} x_m \, v(t - mt_c) \tag{5.4}$$

The Fourier coefficients are then found as follows:

$$\alpha_r = \frac{1}{Lt_c} \int_0^{Lt_c} \left(\sum_{m=0}^{L-1} x_m \, v(t - mt_c) \right) e^{-jr\omega_0 t} \, dt$$

$$= \frac{1}{Lt_c} \sum_{m=0}^{L-1} x_m \left\{ \int_0^{Lt_c} v(t - mt_c) \, e^{-jr\omega_0 t} \, dt \right\} \tag{5.5}$$

Since $v(t - mt_c)$ is equal to 0 except in the range $mt_c \leq t < (m+1)t_c$

and equal to 1 in that range the integral in Eq. 5.5 can be evaluated as follows:

$$\int_0^{Lt_c} v(t - mt_c)\, e^{-jr\omega_o t}\, dt = \int_{mt_c}^{(m+1)t_c} e^{-jr\omega_0 t}\, dt$$

$$= t_c\, e^{-jr(2m+1)\frac{\pi}{L}} \left(\frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \right) \quad (5.6)$$

and α_r becomes

$$\alpha_r = \frac{1}{L} \sum_{m=0}^{L-1} x_m\, e^{-jr(2m+1)\frac{\pi}{L}}\, \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}}$$

$$= \left(e^{-jr\frac{\pi}{L}}\, \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \right) \frac{1}{L} \sum_{m=0}^{L-1} x_m\, e^{-jrm\frac{2\pi}{L}} \quad (5.7)$$

In the above equation $x_m = +1$ or -1 depending on the value of the mth digit of the linear maximal sequence. Therefore the value of the summation depends on the particular sequence used and any differences in the Fourier coefficients for linear maximal sequences of the same length must be contained in this term. The summation can be modified as shown below

$$\frac{1}{L}\sum_{m=0}^{L-1} x_m\, e^{-jrm\frac{2\pi}{L}} = \frac{1}{L}\left\{\sum_{m=0}^{L-1}(x_m+1)\, e^{-jrm\frac{2\pi}{L}} - \sum_{m=0}^{L-1} e^{-jrm\frac{2\pi}{L}}\right\}$$

$$= -\frac{1}{L}\,\frac{e^{-jr\pi}}{e^{-jr\frac{\pi}{L}}}\,\frac{\sin r\pi}{\sin r\frac{\pi}{L}} + \frac{1}{L}\sum_{m=0}^{L-1}(x_m+1)\, e^{-jrm\frac{2\pi}{L}} \tag{5.8}$$

Then the Fourier coefficients become:

$$\alpha_r = -\frac{1}{L}\, e^{-jr\pi}\,\frac{\sin r\pi}{r\frac{\pi}{L}} + \frac{1}{L}\, e^{-jr\frac{\pi}{L}}\,\frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}}\sum_{m=0}^{L-1}(x_m+1)\, e^{-jrm\frac{2\pi}{L}} \tag{5.9}$$

The first term on the right hand side of Eq. 5.9 is zero for all values of r except $r = 0$. For $r = 0$ the coefficient is:

$$\alpha_0 = -\frac{1}{L}(L) + \frac{1}{L}\sum_{m=0}^{L-1}(x_m+1) \tag{5.10}$$

As noted previously for linear maximal sequences, $x_m = 1$ for $(L-1)/2$ values of m and $x_m = -1$ for $(L+1)/2$ values of m. Therefore:

$$\alpha_0 = -1 + \frac{2}{L}\left(\frac{L-1}{2}\right) = -\frac{1}{L} \tag{5.11}$$

and the equation for the Fourier coefficients becomes

$$\alpha_r = \begin{cases} -\frac{1}{L} & r = 0 \\ \frac{1}{L} e^{-jr\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \sum_{m=0}^{L-1} (x_m+1) e^{-jrm\frac{2\pi}{L}} \; ; & r \neq 0 \end{cases} \tag{5.12}$$

5.1.1 Fourier Coefficients for the Characteristic Phase of a Linear Maximal Sequence. In Section 2.3 the characteristic phase position for linear maximal binary sequences was defined. It was noted that for the characteristic phase position the sequence has the property of being constant for all positions in a given coset. That is, the digits in the sequence will have value +1 (or -1) for all positions corresponding to m an element of a coset with assigned value +1 (or -1). Therefore, if it is assumed that the sequence is in the characteristic phase position, the calculation of the Fourier coefficients reduces to a sum over values of m in a selected set of cosets. The summation of Eq. 5.12 becomes

$$\sum_{m=0}^{L-1} (x_m+1) e^{-jrm\frac{2\pi}{L}} = 2\left[\sum_{\ell=0}^{n_1-1} \left(e^{-jr\frac{2\pi}{L}m_1}\right)^{2^\ell} + \sum_{\ell=0}^{n_2-1} \left(e^{-jr\frac{2\pi}{L}m_2}\right)^{2^\ell} + \dots\right] \tag{5.13}$$

where the m_i are the coset leaders for those cosets which have assigned value +1 and n_i is the number of elements in the ith coset. For proper cosets $n_i = n$ and for improper cosets n_i is equal to n or some factor of n.

The calculation is further reduced by noting that the value of this sum is the same for all values of r which are elements of a coset. The structure of the cyclotomic cosets is discussed in Section 2.3 where it is noted that these cosets are formed by multiplying integers in the range 0 to L - 1 by the elements of the subgroup $(1, 2, 4, \ldots, 2^{n-1})$. Thus for a given value of r the general term in the summation becomes:

$$\sum_{\ell=0}^{n_i-1}\left(e^{-jr\frac{2\pi}{L}m_i}\right)^{2^\ell} = e^{-jr\frac{2\pi}{L}m_i} + e^{-jr\frac{2\pi}{L}2m_i} + e^{-jr\frac{2\pi}{L}4m_i} + \ldots + e^{-jr\frac{2\pi}{L}2^{n_i-1}m_i}$$

$$= \sum_{\ell=0}^{n_i-1} e^{-jr\theta m_\ell} \tag{5.14}$$

where $\theta = 2\pi/L$ and the m_ℓ are the elements of the coset with coset leader m_i. (It should be noted that the elements of the coset are formed by multiplying the coset leader by the elements of the subgroup defined above and reducing the result mod-L. This is also true for the angles since the exponential term is periodic with period L.) If r is one of the elements of the subgroup and m_ℓ one of the elements of a coset, the product $r\,m_\ell$ is an element of the same coset (by definition), and the sum of Eq. 5.14 is, therefore, the same for any r in the subgroup. If r is an element of one of the cyclotomic cosets, r is of the form

$$r = r_i 2^s \tag{5.15}$$

where r_i is the coset leader and the product rm_ℓ, for values of m_ℓ in the coset with coset leader m_i, is an element of the coset containing the term $r_i m_i$. Thus each of the sums on the right hand side of Eq. 5.13 has a constant value for all values of r in a coset and the phase angle contribution to the Fourier coefficient from each of these terms is the same for all values of r which are elements of a coset.

As an example of the use of this technique for the calculation of Fourier series, consider the maximal sequences generated by the four-stage shift registers (i.e., $n = 4$). The coset assignments for these sequences are given in Fig. 2.1. The cosets and the corresponding values of the sums are shown below:

Cosets	$\sum_{\ell=0}^{n_i-1}\left(e^{-j\frac{2\pi}{L}m_i}\right)^{2^\ell}$		
{0}	e^{-j0}	$= 1 + j0$	$= e^{j0^o}$
{1, 2, 4, 8}	$\sum_{\ell=0}^{3}\left(e^{-jr\frac{2\pi}{15}}\right)^{2^\ell}$	$= 0.500 - j\,1.936$	$= 2e^{-j75.5^o}$
{7, 14, 13, 11}	$\sum_{\ell=0}^{3}\left(e^{-jr\frac{2\pi}{15}\cdot 7}\right)^{2^\ell}$	$= 0.500 + j1.936$	$= 2e^{j75.5^o}$
{5, 10}	$\sum_{\ell=0}^{1}\left(e^{-jr\frac{2\pi}{15}\cdot 5}\right)^{2^\ell}$	$= -1.0 + j0$	$= 1e^{j180^o}$
{3, 6, 12, 9}	$\sum_{\ell=0}^{3}\left(e^{-jr\frac{2\pi}{15}\cdot 3}\right)^{2^\ell}$	$= -1.0 + j0$	$= 1e^{j180^o}$

(5.16)

From Fig. 2.1 it can be seen that the coset leaders for the cosets with assigned value +1 for the sequence [4, 1, 0] are 0, 7, and 5. Therefore, for this sequence, the sum shown in Eq. 5.13 becomes:

$$\sum_{m=0}^{15} (x_m+1)\, e^{-jrm\frac{2\pi}{15}} = 2\left[e^{-jr\frac{2\pi}{15}\cdot 0} + \sum_{\ell=0}^{3}\left(e^{-jr\frac{2\pi}{15}\cdot 7}\right)^{2^\ell} + \sum_{\ell=0}^{1}\left(e^{-jr\frac{2\pi}{L}\cdot 5}\right)^{2^\ell}\right] \tag{5.17}$$

The evaluation of this sum is necessary for the calculation of the Fourier coefficients and the results of this evaluation are given in Table 5.1 for r between 0 and 15. The values of rm_i are given in this figure and it should be noted that the product rm_i has been reduced mod-L. Since, for each sum, the product $rm_i\, 2^\ell$ will have value equal to each element of a coset at least once as ℓ takes on all values in its range, the coset leader may be used in place of the actual value of rm_i. For this reason the coset leaders are shown in Table 5.1. The value of the sum has been calculated and using this value the Fourier coefficients are found using Eq. 5.12. It should be noted that the value of the Fourier coefficient for $r = 0$ is always $-1/L$. The magnitude of the sum is constant for $r \neq 0$ mod-L and the phase angles for the sum are periodic in r with period L. It can be seen from Table 5.1 that the phase angle contribution from the sum is constant for r an element of a coset.

As noted previously, the Fourier coefficients are calculated by multiplying the sum by a term which is not dependent on the

r	r · 0	r · 7	r · 5	$\sum_{m=0}^{15} (x_m+1)\, e^{-jr\frac{2m\pi}{15}}$	$\lvert\alpha_r\rvert$	ϕ_r
0	0	0	0		0.0667	180^o
1	0	7	5	$4\, e^{j75.5}$	0.2647	63.5^o
2	0	$14 \equiv 7$	$10 \equiv 5$	$4\, e^{j75.5}$	0.2589	51.5^o
3	0	$6 \equiv 3$	0	$4\, e^{j0^o}$	0.2495	-36^o
4	0	$13 \equiv 7$	5	$4\, e^{j75.5}$	0.2366	27.5^o
5	0	5	$10 \equiv 5$	$4\, e^{j180^o}$	0.2205	120^o
6	0	$12 \equiv 3$	0	$4\, e^{j0^o}$	0.2018	-72^o
7	0	$4 \equiv 1$	5	$4\, e^{-j75.5}$	0.1809	-159.5^o
8	0	$11 \equiv 7$	$10 \equiv 5$	$4\, e^{j75.5}$	0.1583	-20.5^o
9	0	3	0	$4\, e^{j0^o}$	0.1346	-108^o
10	0	$10 \equiv 4$	5	$4\, e^{j180^o}$	0.1103	60^o
11	0	$2 \equiv 1$	$10 \equiv 5$	$4\, e^{-j75.5^o}$	0.0860	152.5^o
12	0	$9 \equiv 3$	0	$4\, e^{j0^o}$	0.0624	-144^o
13	0	1	5	$4\, e^{-j75.5^o}$	0.0398	128.5^o
14	0	$8 \equiv 1$	$10 \equiv 5$	$4\, e^{-j75.5^o}$	0.0189	116.5^o
15	0	0	0	$14\, e^{j0^o}$	0.0	---

Table 5.1. Fourier coefficients for the sequence generated by [4, 1, 0]

particular sequence used. Therefore the information concerning the particular sequence used is all contained in the phase angle of the sum. For the characteristic phase position, the Fourier series can be completely specified by listing the phase angles for each coset. That is,

$$\alpha_r = \left(e^{-jr\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \right) a_r \tag{5.18}$$

where

$$a_r = \frac{1}{L} \sum_{m=0}^{L-1} (x_m)\, e^{-jr\frac{2m\pi}{L}}$$

Then a_r has the following properties:

$$a_r = |a_r|\, e^{j\theta_r}$$

$$|a_r| = \begin{cases} \dfrac{\sqrt{L+1}}{L} & r \neq 0 \mod L \\ 1/L & r = 0 \mod L \end{cases} \tag{5.19}$$

θ_r is dependent on the particular sequence used.

$\theta_r = -\theta_{L-r} = \theta_{L+r}$

θ_r is constant for r an element of a coset.

The Fourier coefficients found above are calculated for the characteristic phase position of the sequence. The magnitude of the

Fourier coefficients for any periodic waveform is not dependent on the phase of the waveform. Therefore the modification to the results given above, for the characteristic phase position, to include the Fourier coefficients for other phases will take the form of a change in the phase angle. If the sequence is shifted by s_x digits the Fourier coefficients for the shifted version of the sequence can be found from the Fourier coefficients given above by multiplying the the coefficients by a term of the form $e^{-js_x\left(r\frac{2\pi}{L}\right)}$. It should be noted at this point that the exponential term, $e^{-jr\frac{\pi}{L}}$, in Eq. 5.18 is a result of the definition of the basic pulse. If the basic pulse is defined to be symmetric about the origin, this term is eliminated.

5.1.2 Comparison of Fourier Series Coefficients for Linear Maximal Sequences of Equal Period. In comparing the Fourier series coefficients for linear maximal sequences of the same period, it is only necessary to consider the phase angle of the term a_r, i.e., θ_r. It was shown in the previous section that these phase angles for linear maximal binary sequences in the characteristic phase position are calculated by making use of the coset assignments. In Section 2.3 it was shown that the coset assignments for the set of maximal sequences of a given length were cyclic permutations of the assignment for the original sequence. Using these facts it can be shown that, for all sequences of the same length, the phase angles of a_r which are associated with values of r in a coset are cyclic permutations of the phase angles for the original sequence. These phase angle assignments will be cyclic permutations in the same sense that the coset assignments discussed in Section 2.3 are cyclic permutations, i.e., when the coset assignments of +1 and -1 are complete cyclic permutations, the phase angle "assignment" is also, however when the coset assignments are split into groups the same relationship will apply to the phase angles.

This property of the phase angles of the coefficients a_r will be shown by considering the case which was discussed in Section 2.3, i.e , the case in which the length of the sequence was prime and the cosets could be arranged in such a way that each coset was a fixed integer, m, times the previous coset. For this case each of the cosets will have n terms and the general term in the summation for a sequence in which the ith coset has assigned value +1 will be

$$\sum_{\ell=0}^{n-1} \left(e^{-jr\frac{2\pi}{L}m_i}\right)^{2^\ell} = \sum_{\ell=0}^{n-1} \left(e^{-jr\frac{2\pi}{L}m^{i-1}}\right)^{2^\ell} \qquad (5.20)$$

where m_i is the coset leader of the ith coset. For the sequence which is generated by sampling the original sequence by a factor m, it has been shown in Section 2.3 that the +1 assigned to the ith coset for the original sequence is now assigned to the (i-1)st coset and the general term becomes

$$\sum_{\ell=0}^{n-1}\left(e^{-jr\frac{2\pi}{L}m_{i-1}}\right)^{2^{\ell}} = \sum_{\ell=0}^{n-1}\left(e^{-jr\frac{2\pi}{L}m^{i-2}}\right)^{2^{\ell}}$$

$$= \sum_{\ell=0}^{n-1}\left(e^{-j\frac{r}{m}\frac{2\pi}{L}m^{i-1}}\right)^{2^{\ell}} \qquad (5.21)$$

It can be seen by examining Eq. 5.20 and 5.21 that the results will be equivalent when the r of Eq. 5.21 is m times the r of Eq. 5.20 and therefore the phase angle which is assigned to the coset containing r for the first sequence will be assigned to the coset containing mr for the second sequence. The coset assignments of phase angles will permute in the same sense that the coset assignments of +1's and -1's permute. Those values of L which result in improper cosets will again result in a somewhat more complicated cyclic relationship of phase angles; however, as noted for the coset assignments, the cyclic relationship will hold in some sense. The phase angle assignment permutes in the opposite direction to the coset assignment as noted in the discussion of Eqs. 5.20 and 5.21. In Figs. 5.1 through 5.4 the phase angle assignments for the linear maximal sequences generated by shift-register generators of lengths 5, 6, 7, and 8 are shown. The phase angles given in these figures are the θ_r as defined by Eq. 5.19 and correspond to the characteristic phase position of the sequences. It should be noted

n = 5

N	S_1	S_2	S_3	S_4	S_5	S_6
C_0	180^o	180^o	180^o	180^o	180^o	180^o
C_1	-17.97^o	-128.36^o	149.43^o	17.97^o	128.36^o	-149.43^o
C_2	-149.43^o	-17.97^o	-128.36^o	149.43^o	17.97^o	128.36^o
C_3	128.36^o	-149.43^o	-17.97^o	-128.36^o	149.43^o	17.97^o
C_4	17.97^o	128.36^o	-149.43^o	-17.97^o	-128.36^o	149.43^o
C_5	149.43^o	17.97^o	128.36^o	-149.43^o	-17.97^o	-128.36^o
C_6	-128.36^o	149.43^o	17.97^o	128.36^o	-149.43^o	-17.97^o

Fig. 5.1. Fourier coefficient phase angle, n = 5

n = 6						
N	S_1	S_2	S_3	S_4	S_5	S_6
C_0	180°	180°	180°	180°	180°	180°
C_1	71.85°	-121.20°	-54.45°	-71.85°	121.20°	54.45°
C_2	54.45°	71.85°	-121.20°	-54.45°	-71.85°	121.20°
C_3	121.20°	54.45°	71.85°	-121.20°	-54.45°	-71.85°
C_4	-71.85°	121.20°	54.45°	71.85°	-121.20°	-54.45°
C_5	-54.45°	-71.85°	121.20°	54.45°	71.85°	-121.20°
C_6	-121.20°	-54.45°	-71.85°	121.20°	54.45°	71.85°
C_7	138.55°	-138.55°	138.55°	-138.55°	138.55°	-138.55°
C_8	-138.55°	138.55°	-138.55°	138.55°	-138.55°	138.55°
C_9	-41.45°	41.45°	-41.45°	41.45°	-41.45°	41.45°
C_{10}	41.45°	-41.45°	41.45°	-41.45°	41.45°	-41.45°
C_{11}	0°	0°	0°	0°	0°	0°
C_{12}	0°	0°	0°	0°	0°	0°

Fig. 5.2. Fourier coefficient phase angle, n = 6

n = 7

N	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}
C_0	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°
C_1	42.14°	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09
C_2	11.09°	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75
C_3	-171.75°	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93
C_4	92.93°	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39
C_5	-99.39°	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23
C_6	-62.23°	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60
C_7	-140.60°	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04
C_8	-101.04°	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73
C_9	-144.73°	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14
C_{10}	-42.14°	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09
C_{11}	-11.09°	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93	171.75
C_{12}	171.75°	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39	-92.93
C_{13}	-92.93°	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23	99.39
C_{14}	99.39°	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60	62.23
C_{15}	62.23°	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04	140.60
C_{16}	140.60°	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73	101.04
C_{17}	101.04°	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14	144.73
C_{18}	144.73°	101.04	140.60	62.23	99.39	-92.93	171.75	-11.09	-42.14	-144.73	-101.04	-140.60	-62.23	-99.39	92.93	-171.75	11.09	42.14

Fig. 5.3. Fourier coefficient phase angle, n = 7

n = 8

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
C_0	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°
C_1	29.1°	-44.3°	-111.5°	-130.2°	158.7°	96.8°	-162.7°	-8.7°	-29.1°	44.3°	111.5°	130.2°	-158.7°	-96.8°	162.7°	8.7°
C_2	-8.7°	29.1°	-44.3°	-111.5°	-130.2°	158.7°	96.8°	-162.7°	8.7°	-29.1°	44.3°	111.5°	130.2°	-158.7°	-96.8°	162.7°
C_3	-162.7°	-8.7°	29.1°	-44.3°	-111.5°	-130.2°	158.7°	96.8°	162.7°	8.7°	-29.1°	44.3°	111.5°	130.2°	-158.7°	-96.8°
C_4	96.8°	-162.7°	-8.7°	29.1°	-44.3°	-111.5°	-130.2°	158.7°	-96.8°	162.7°	8.7°	-29.1°	44.3°	111.5°	130.2°	-158.7°
C_5	158.7°	96.8°	-162.7°	-8.7°	29.1°	-44.3°	-111.5°	-130.2°	-158.7°	-96.8°	162.7°	8.7°	-29.1°	44.3°	111.5°	130.2°
C_6	-130.2°	158.7°	96.8°	-162.7°	-8.7°	29.1°	-44.3°	-111.5°	130.2°	-158.7°	-96.8°	162.7°	8.7°	-29.1°	44.3°	111.5°
C_7	-111.5°	-130.2°	158.7°	96.8°	-162.7°	-8.7°	29.1°	-44.3°	111.5°	130.2°	-158.7°	-96.8°	162.7°	8.7°	-29.1°	44.3°
C_8	-44.3°	-111.5°	-130.2°	158.7°	96.8°	-162.7°	-8.7°	29.1°	44.3°	111.5°	130.2°	-158.7°	-96.8°	162.7°	8.7°	-29.1°
C_9	-29.1°	44.3°	111.5°	130.2°	-158.7°	-96.8°	162.7°	8.7°	29.1°	-44.3°	-111.5°	-130.2°	158.7°	96.8°	-162.7°	-8.7°
C_{10}	8.7°	-29.1°	44.3°	111.5°	130.2°	-158.7°	-96.8°	162.7°	-8.7°	29.1°	-44.3°	-111.5°	-130.2°	158.7°	96.8°	-162.7°
C_{11}	162.7°	8.7°	-29.1°	44.3°	111.5°	130.2°	-158.7°	-96.8°	-162.7°	-8.7°	29.1°	-44.3°	-111.5°	-130.2°	158.7°	96.8°
C_{12}	-96.8°	162.7°	8.7°	-29.1°	44.3°	111.5°	130.2°	-158.7°	96.8°	-162.7°	-8.7°	29.1°	-44.3°	-111.5°	-130.2°	158.7°
C_{13}	-158.7°	-96.8°	162.7°	8.7°	-29.1°	44.3°	111.5°	130.2°	158.7°	96.8°	-162.7°	-8.7°	29.1°	-44.3°	-111.5°	-130.2°
C_{14}	130.2°	-158.7°	-96.8°	162.7°	8.7°	-29.1°	44.3°	111.5°	-130.2°	158.7°	96.8°	-162.7°	-8.7°	29.1°	-44.3°	-111.5°
C_{15}	111.5°	130.2°	-158.7°	-96.8°	162.7°	8.7°	-29.1°	44.3°	-111.5°	-130.2°	158.7°	96.8°	-162.7°	-8.7°	29.1°	-44.3°
C_{16}	44.3°	111.5°	130.2°	-158.7°	-96.8°	162.7°	8.7°	-29.1°	-44.3°	-111.5°	-130.2°	158.7°	96.8°	-162.7°	-8.7°	29.1°
C_{17}	138.3°	80.7°	-29.4°	-92.1°	-138.3°	-80.7°	29.4°	92.1°	-138.3°	-80.7°	29.4°	92.1°	138.3°	80.7°	-29.4°	-92.1°
C_{18}	92.1°	138.3°	80.7°	-29.4°	-92.1°	-138.3°	-80.7°	29.4°	-92.1°	-138.3°	-80.7°	29.4°	92.1°	138.3°	80.7°	-29.4°
C_{19}	29.4°	92.1°	138.3°	80.7°	-29.4°	-92.1°	-138.3°	-80.7°	-29.4°	-92.1°	-138.3°	-80.7°	29.4°	92.1°	138.3°	80.7°
C_{20}	-80.7°	29.4°	92.1°	138.3°	80.7°	-29.4°	-92.1°	-138.3°	80.7°	-29.4°	-92.1°	-138.3°	-80.7°	29.4°	92.1°	138.3°
C_{21}	-138.3°	-80.7°	29.4°	92.1°	138.3°	80.7°	-29.4°	-92.1°	138.3°	80.7°	-29.4°	-92.1°	-138.3°	-80.7°	29.4°	92.1°
C_{22}	-92.1°	-138.3°	-80.7°	29.4°	92.1°	138.3°	80.7°	-29.4°	92.1°	138.3°	80.7°	-29.4°	-92.1°	-138.3°	-80.7°	29.4°
C_{23}	-29.4°	-92.1°	-138.3°	-80.7°	29.4°	92.1°	138.3°	80.7°	29.4°	92.1°	138.3°	80.7°	-29.4°	-92.1°	-138.3°	-80.7°
C_{24}	80.7°	-29.4°	-92.1°	-138.3°	-80.7°	29.4°	92.1°	138.3°	-80.7°	29.4°	92.1°	138.3°	80.7°	-29.4°	-92.1°	-138.3°
C_{25}	-33.3°	-135.3°	33.3°	135.3°	-33.3°	-135.3°	33.3°	135.3°	33.3°	135.3°	-33.3°	-135.3°	33.3°	135.3°	-33.3°	-135.3°
C_{26}	135.3°	-33.3°	-135.3°	33.3°	135.3°	-33.3°	-135.3°	33.3°	-135.3°	33.3°	135.3°	-33.3°	-135.3°	33.3°	135.3°	-33.3°
C_{27}	33.3°	135.3°	-33.3°	-135.3°	33.3°	135.3°	-33.3°	-135.3°	-33.3°	-135.3°	33.3°	135.3°	-33.3°	-135.3°	33.3°	135.3°
C_{28}	-135.3°	33.3°	135.3°	-33.3°	-135.3°	33.3°	135.3°	-33.3°	135.3°	-33.3°	-135.3°	33.3°	135.3°	-33.3°	-135.3°	33.3°
C_{29}	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°
C_{30}	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°
C_{31}	-29.0°	29.0°	-29.0°	29.0°	-29.0°	29.0°	-29.0°	29.0°	29.0°	-29.0°	29.0°	-29.0°	29.0°	-29.0°	29.0°	-29.0°
C_{32}	29.0°	-29.0°	29.0°	-29.0°	29.0°	-29.0°	29.0°	-29.0°	-29.0°	29.0°	-29.0°	29.0°	-29.0°	29.0°	-29.0°	29.0°
C_{33}	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°
C_{34}	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°	180°

Fig. 5.4. Fourier coefficient phase angle, n = 8

that the phase angle for $r = 0$ is always 180^{0}.

Thus it has been shown that the calculation of the Fourier series coefficients for a linear maximal binary sequence in the characteristic phase position is greatly simplified by making use of the cyclotomic cosets. It is necessary to calculate one sequence dependent angle for each coset. (This can be further reduced by making use of the property $\theta_r = -\theta_{L-r}$.) Using these angles, the Fourier coefficients for any phase position can be calculated by a simple adjustment of each phase angle. The magnitude of the Fourier coefficients does not depend on the particular sequence or phase. If the calculation of the sequence dependent angles has been made for one sequence and the coset assignments, as defined in Chapter II, are known, the sequence dependent angles for all sequences of the same length can be found as a permutation of the angles for the original sequence.

5.1.3 Fourier Coefficients for a Shortened Sequence. In the Fourier series calculation of Section 5.1.1 the results were dependent on the fact that the full period of the sequence was used. In order to examine how these results are modified when a portion of the sequence is considered, the time waveform can be multiplied by a periodic window, i.e.,

Let $x(t)$ be a $\{1, -1\}$ sequence waveform, as before, with Fourier coefficients given by Eq. 5.7. Let

$$x_s(t) = w(t)\, x(t) \tag{5.22}$$

where

$$w(t) = \begin{cases} 1 & 0 \leq t + \ell L t_c < k t_c\,; \quad \ell = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{elsewhere} \end{cases}$$

Then $x_s(t)$ is periodic with period Lt_c and contains the first k digits of the sequence $x(t)$. The window, $w(t)$, is also periodic with period Lt_c and can be represented as a Fourier series.

$$w(t) = \sum_{r=-\infty}^{\infty} w_r\, e^{jr\omega_0 t} \qquad \omega_0 = \frac{2\pi}{Lt_c}$$

$$w_r = \frac{1}{Lt_c} \int_0^{Lt_c} w(t)\, e^{-jr\omega_0 t}\, dt$$

$$= \frac{1}{Lt_c} \int_0^{kt_c} e^{-jr\omega_0 t}\, dt$$

$$w_r = \frac{k}{L}\, e^{-jrk\frac{\pi}{L}}\, \frac{\sin rk\frac{\pi}{L}}{rk\frac{\pi}{L}} \tag{5.23}$$

Then Eq. 5.22 becomes

$$x_s(t) = \sum_{r=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} \alpha_m\, w_{r-m} \right) e^{jr\omega_0 t} \tag{5.24}$$

Using Eq. 5.18 and Eq. 5.23 the coefficient of this Fourier series may be reduced as follows:

$$f_r = \sum_{m=-\infty}^{\infty} \alpha_r \, w_{r-m}$$

$$= \sum_{m=-\infty}^{\infty} \left(e^{-jm\frac{\pi}{L}} \frac{\sin m\frac{\pi}{L}}{m\frac{\pi}{L}} a_m \right) \left(\frac{k}{L} e^{-j(r-m)k\frac{\pi}{L}} \frac{\sin(r-m)k\frac{\pi}{L}}{(r-m)k\frac{\pi}{L}} \right)$$

$$= \frac{k}{L} e^{-jrk\frac{\pi}{L}} \sum_{m=-\infty}^{\infty} e^{j(k-1)m\frac{\pi}{L}} \frac{\sin m\frac{\pi}{L} \sin(r-m)k\frac{\pi}{L}}{m\frac{\pi}{L}(r-m)k\frac{\pi}{L}} a_m$$

$$= \frac{k}{L} e^{-jrk\frac{\pi}{L}} \sum_{\ell=-\infty}^{\infty} \sum_{m=0}^{L-1} e^{j(k-1)(m+\ell L)\frac{\pi}{L}} \left(\frac{\sin(m+\ell L)\frac{\pi}{L} \sin(r-m-\ell L)k\frac{\pi}{L}}{(m+\ell L)\frac{\pi}{L}(r-m-\ell L)k\frac{\pi}{L}} a_{m+\ell L} \right) \tag{5.25}$$

From the definition of a_m and properties of sine and exponential functions:

$$\begin{aligned} a_{m+\ell L} &= a_m \\ \sin(m+\ell L)\frac{\pi}{L} &= (-1)^{\ell} \sin m\frac{\pi}{L} \\ \sin(r-m-\ell L)k\frac{\pi}{L} &= (-1)^{\ell k} \sin(r-m)k\frac{\pi}{L} \\ e^{j(k-1)(m+\ell L)\frac{\pi}{L}} &= (-1)^{(k-1)\ell} e^{j(k-1)m\frac{\pi}{L}} \end{aligned} \tag{5.26}$$

and the coefficient becomes:

$$f_r = \frac{k}{L} e^{-jrk\frac{\pi}{L}} \sum_{m=0}^{L-1} a_m e^{j(k-1)m\frac{\pi}{L}} \sin m\frac{\pi}{L} \sin(r-m)k\frac{\pi}{L}$$

$$\left(\sum_{\ell=-\infty}^{\infty} \frac{(-1)^{2\ell k}}{\left(m\frac{\pi}{L} + \ell\pi\right)\left[(r-m)k\frac{\pi}{L} - \ell k\pi\right]} \right)$$

The sum over ℓ may be evaluated, using series no. 827 of Jolley, Ref. 35, i.e.,

$$\sum_{\ell=-\infty}^{\infty} \frac{a-\theta}{(\theta-\ell)(a-\ell)} = \pi(\cot \pi\theta - \cot \pi a)$$

Therefore, let $\theta = -\frac{m}{L}$ and $a = \frac{r-m}{L}$, and the sum becomes:

$$\frac{1}{k\pi^2} \sum_{\ell=-\infty}^{\infty} \frac{1}{\left(\frac{m}{L} + \ell\right)\left(\frac{r-m}{L} - \ell\right)} = \frac{1}{k} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \left(\frac{1}{\sin m\frac{\pi}{L} \sin(r-m)\frac{\pi}{L}} \right) \tag{5.27}$$

The Fourier coefficient for the shortened periodic linear maximal binary sequence becomes:

$$f_r = \frac{1}{L} e^{-jrk\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \sum_{m=0}^{L-1} a_m e^{j(k-1)m\frac{\pi}{L}} \frac{\sin(r-m)k\frac{\pi}{L}}{\sin(r-m)\frac{\pi}{L}} \tag{5.28}$$

For $k = L$, f_r reduces to α_r as required. For $k < L$,

f_r is a weighted sum of the full period coefficients. The weights of the full period coefficients are determined by the term $\left(\frac{\sin(r-m)k\frac{\pi}{L}}{\sin(r-m)\frac{\pi}{L}}\right)$ which has maximum value for $r = m$, mod-L. Therefore the coefficient a_r has the maximum weight. As the value of k is reduced the weight of the adjacent terms increases and their effect on the value of f_r becomes more pronounced.

5.1.4 Frequency Spectra of Linear Maximal Binary Sequences.

For a periodic waveform which may be represented by a Fourier series the frequency spectrum may be found directly from the Fourier series. (See for example Ref. 32.)

$$g(t) = \sum_{n=-\infty}^{\infty} g_n \, e^{jn\omega_0 t} \; ; \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$G(f) = \sum_{n=-\infty}^{\infty} g_n \, \delta(f - nf_0) \tag{5.29}$$

The Fourier series for linear maximal binary sequences has been found in Section 5.1.1 and thus for the sequence $x(t)$, as defined in Eq. 5.3, with Fourier coefficients given by Eq. 5.18 the frequency spectrum is:

$$X(f) = \sum_{r=-\infty}^{\infty} a_r \, \delta(f - rf_0)$$

$$= \sum_{r=-\infty}^{\infty} e^{-jr\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} a_r \, \delta(f - rf_0) \qquad (5.30)$$

$$a_r = |a_r| \, e^{j\theta_r}$$

Thus, the frequency spectrum of a periodic linear maximal binary sequence, with period T_0, is a line spectrum which has components at frequencies which are linear multiples of the reciprocal of the period of the sequence. The envelope of the lines has the form $\text{sinc}\left(r\frac{\pi}{L}\right)$ and the phase angle of the components is the phase angle of the Fourier series coefficient.

The frequency spectrum of one period, or less, of the linear maximal sequences can be found quite easily by multiplying the sequence waveform, x(t), by a window of length k digits and recalling that multiplication in the time domain corresponds to convolution in the frequency domain. That is, let

$$w(t) = \begin{cases} 1 & 0 < t < kt_c \\ 0 & \text{elsewhere} \end{cases} \qquad (5.31)$$

Then

$$W(f) = \int_{-\infty}^{\infty} w(t)\, e^{-j\omega t}\, dt$$

$$= kt_c\, e^{-jkt_c \pi f} \left(\frac{\sin kt_c \pi f}{kt_c \pi f} \right) \qquad (5.32)$$

The waveform consisting of the first k digits of the sequence then is defined as

$$x_p(t) = w(t)\, x(t) \qquad (5.33)$$

and the frequency spectrum of this portion of the sequence becomes

$$\mathbf{X}_p(f) = W(f) * X(f)$$

$$= \int_{-\infty}^{\infty} W(f-\lambda)\, \mathbf{X}(\lambda)\, d\lambda$$

$$= kt_c \int_{-\infty}^{\infty} e^{-jkt_c \pi (f-\lambda)} \frac{\sin kt_c \pi (f-\lambda)}{kt_c \pi (f-\lambda)} \sum_{r=-\infty}^{\infty} \alpha_r\, \delta(\lambda - rf_0)\, d\lambda$$

$$= kt_c \sum_{r=-\infty}^{\infty} \alpha_r\, e^{-jkt_c \pi (f - rf_0)} \frac{\sin kt_c \pi (f - rf_0)}{kt_c \pi (f - rf_0)} \qquad (5.34)$$

where

$$\alpha_r = e^{-jr\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}}\, a_r$$

It can be seen from Eq. 5.34 that the spectrum of k digits of the linear maximal sequence is an infinite sum of (sinc x) functions centered at integral multiples of the fundamental frequency of the periodic waveform, i.e., the impulse functions of Eq. 5.30, and having amplitude proportional to the area of the impulses in the spectrum of the periodic waveform.

For $k = L$ the term $\operatorname{sinc} kt_c\pi(f - rf_0)$ has a zero at values of frequency equal to integral multiples of one over the sequence period except for $f = rf_0$. Thus the Fourier spectrum for one period of the sequence is proportional to the spectrum of the periodic sequence at integral multiples of f_0, and is continuous for frequencies between these points. For $k < L$ the spectrum consists of terms of the form shown above centered about each multiple of f_0; however, the zeros of the $\operatorname{sinc} kt_c\pi(f - rf_0)$ terms do not occur at integral values of f_0 and therefore the value of the spectrum at integral multiples of f_0 will consist of the sum of the value for $k = L$ plus contributions due to the adjacent values of α_r. The contribution from the adjacent terms is dependent on the values of k and becomes larger as the value of k is reduced.

5.2 Fourier Series Representation of Cross-Correlation Function

The cross-correlation function of x(t) and y(t) is defined to be: (see for example Davenport and Root, Ref. 34)

$$\rho_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau)\, y^*(t)\, dt \tag{5.35}$$

For real, periodic waveforms $x(t)$ and $y(t)$ the above equation can be written: (see for example Stein and Jones, Ref. 32)

$$\rho_{xy}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t+\tau)\, y(t)\, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, y(t-\tau)\, dt \qquad (5.36)$$

T_0 = period of $x(t)$ and $y(t)$.

The integral, of course, may be taken over any interval of length T_0, the period of the waveforms. Since $x(t)$ and $y(t)$ are periodic, with period T_0, they may be represented by Fourier series if the Dirichlet conditions are satisfied. (This will be the case for all waveforms of interest here.) Therefore we have

$$x(t) = \sum_{r=-\infty}^{\infty} \alpha_r e^{jr\omega_0 t} \; ; \qquad \omega_0 = \frac{2\pi}{T_0}$$

where

$$\alpha_r = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-jr\omega_0 t}\, dt$$

and

$$y(t) = \sum_{r=-\infty}^{\infty} \beta_r e^{jr\omega_0 t} \; ; \qquad \omega_0 = \frac{2\pi}{T_0}$$

where

$$\beta_r = \frac{1}{T_0} \int_0^{T_0} y(t)\, e^{-jr\omega_0 t}\, dt \tag{5.37}$$

Then the cross-correlation may be expressed as:

$$\rho_{xy}(\tau) = \frac{1}{T_0} \int_0^{T_0} \left(\sum_{r=-\infty}^{\infty} \alpha_r\, e^{jr\omega_0 (t+\tau)} \right) y(t)\, dt$$

$$= \sum_{r=-\infty}^{\infty} \alpha_r\, e^{jr\omega_0 \tau} \left(\frac{1}{T_0} \int_0^{T_0} y(t)\, e^{jr\omega_0 t}\, dt \right)$$

$$\rho_{xy}(\tau) = \sum_{r=-\infty}^{\infty} \alpha_r\, \beta_r^{*}\, e^{jr\omega_0 \tau} \tag{5.38}$$

β_r^{*} is the complex conjugate of the Fourier coefficient defined above for $y(t)$.

It should be noted that Eq. 5.38 for the cross-correlation is true for any pair of periodic waveforms, with equal periods, that satisfy the necessary conditions for representation in the form of a Fourier series. For the problem of interest here $x(t)$ and $y(t)$ are linear maximal pseudo-random binary sequence waveforms.

5.2.1 Full Period Cross-Correlation Function. In order to consider the full period cross-correlation between linear maximal sequences let $x(t)$ and $y(t)$ be $\{1, -1\}$ sequence waveforms, in the characteristic phase position as defined in Section 5.1. (The basic pulse in position 0 has transitions at $t = 0$ and $t = t_c$.) For this case it has been shown (see Eq. 5.7) that the Fourier coefficients are given by:

For $x(t)$

$$\alpha_r = e^{-jr\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \frac{1}{L} \sum_{m=0}^{L-1} x_m \, e^{-jr\frac{2m\pi}{L}}$$

For $y(t)$ (5.39)

$$\beta_r = e^{-jr\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \frac{1}{L} \sum_{m=0}^{L-1} y_m \, e^{-jr\frac{2m\pi}{L}}$$

In these equations x_m and y_m are, respectively, equal to +1 or -1 corresponding to the mth pulse of $x(t)$ and $y(t)$. Let

$$a_r = \frac{1}{L} \sum_{m=0}^{L-1} x_m \, e^{-jr\frac{2m\pi}{L}}$$

(5.40)

and $$b_r = \frac{1}{L} \sum_{m=0}^{L-1} y_m \, e^{-jr\frac{2m\pi}{L}}$$

Then the full period cross-correlation between these two sequences becomes:

$$\rho_{xy}(\tau) = \sum_{r=-\infty}^{\infty} \frac{\sin^2 r\frac{\pi}{L}}{\left(r\frac{\pi}{L}\right)^2} a_r b_r^* e^{jr\omega_0\tau} \tag{5.41}$$

The τ in Eq. 5.41 refers to a shift in the waveform $x(t)$. Since the waveforms of interest here are digital sequences it is only necessary to consider phase shifts of an integral number of clock periods. (The cross-correlation for values of τ between the points for which the shift is an integral number of clock periods can be found by joining the values at these points by straight lines.) Therefore let $\tau = s_x t_c$ where s_x is the number of pulses the sequence $x(t)$ is shifted to the left. That is, instead of the pulse starting at $t = 0$ being the pulse corresponding to the zero pulse in the characteristic sequence, the pulse starting at $t = 0$ will be the pulse labeled s_x in the characteristic sequence. Then the cross-correlation becomes:

$$\rho_{xy}(s_x t_c) = \sum_{r=-\infty}^{\infty} \frac{\sin^2 r\frac{\pi}{L}}{\left(r\frac{\pi}{L}\right)^2} a_r b_r^* e^{jrs_x\frac{2\pi}{L}} \tag{5.42}$$

This may be rewritten as follows:

$$\rho_{xy}(s_x t_c) = \sum_{\ell=-\infty}^{\infty} \sum_{r=0}^{L-1} \frac{\sin^2 (r+\ell L)\frac{\pi}{L}}{\left[(r+\ell L)\frac{\pi}{L}\right]^2} a_{r+\ell L} b^*_{r+\ell L} e^{j(r+\ell L)s_x\frac{2\pi}{L}} \tag{5.43}$$

Using the definition of a_r and b_r and the properties of the exponential and sine functions, as before in Eq. 5.26, the cross-correlation can be rewritten as follows:

$$\rho_{xy}(s_x t_c) = \sum_{r=0}^{L-1} \sin^2 r\frac{\pi}{L}\, a_r\, b_r{}^{*}\, e^{jr s_x \frac{2\pi}{L}} \left(\sum_{\ell=-\infty}^{\infty} \frac{1}{\left(r\frac{\pi}{L} + \ell\pi\right)^2} \right) \tag{5.44}$$

The final summation of Eq. 5.44 can be further reduced as follows: (series no. 822 of Jolley, Ref. 35)

$$\sum_{\ell=-\infty}^{\infty} \frac{1}{\left(r\frac{\pi}{L} + \ell\pi\right)^2} = \operatorname{cosec}^2 \frac{r\pi}{L} \tag{5.45}$$

and the full period cross-correlation becomes

$$\rho_{xy}(s_x t_c) = \sum_{r=0}^{L-1} a_r\, b_r{}^{*}\, e^{jr s_x \frac{2\pi}{L}} \tag{5.46}$$

The coefficients a_r and b_r are just the sequence dependent part of the Fourier coefficients for the sequence waveforms, x(t) and y(t). These coefficients have the following properties:

$$a_r = |a_r| e^{j\theta_{a_r}} \qquad b_r^* = |b_r| e^{-j\theta_{b_r}}$$

$$|a_r| = |b_r^*| = \begin{cases} \frac{1}{L} & r = 0 \\ \frac{\sqrt{L+1}}{L} & r = 1, 2, \ldots, L-1 \end{cases} \tag{5.47}$$

θ_{a_r} and θ_{b_r} are constant for r an element of a coset

$$\theta_{a_{L-r}} = -\theta_{a_r} \qquad \theta_{b_{L-r}} = -\theta_{b_r}$$

$$\theta_{a_0} = \theta_{b_0} = 180^{\circ}$$

By making use of these properties of the coefficients the cross-correlation can be reduced to:

$$\rho_{xy}(s_x t_c) = \frac{1}{L^2} + \frac{L+1}{L^2} \sum_{r=1}^{L-1} e^{j\left(\theta_{a_r} - \theta_{b_r} + r\, s_x \frac{2\pi}{L}\right)} \tag{5.48}$$

Since the phase angles θ_{a_r} and θ_{b_r} are constant for r an element of a coset, this equation can be further reduced to a set of summations over r in cosets from which it can be seen that the cross-correlation will be constant for s_x an element of a coset. This fact was noted by Gold, Ref. 6, and may be useful in reducing the amount of computation necessary to calculate the cross-correlation function using Eq. 5.48.

It is interesting to note the ease with which Eq. 5.48 can be used to show the full period autocorrelation properties. In this case the sequence waveforms x(t) and y(t) will be identical and therefore θ_{a_r} and θ_{b_r} will have the same value and the equation will reduce to:

$$\rho_x(s_x t_c) = \frac{1}{L^2} + \frac{L+1}{L^2} \sum_{r=1}^{L-1} e^{jr s_x \frac{2\pi}{L}} \tag{5.49}$$

$$= \begin{cases} 1 & s_x = 0 \qquad \text{mod } L \\ -\frac{1}{L} & s_x = 1, 2, \ldots, L-1 \ ; \ \text{mod } L \end{cases}$$

<u>5.2.2 Partial Period Cross-Correlation Function.</u> The primary purpose of this work is the study of the properties of the partial period, or short time, correlation function. In the previous section the full period correlation has been described in terms of the Fourier series coefficients for the periodic linear maximal binary sequence. The same general technique can be used to describe the partial period correlation function.

For the partial period analysis let

$$z(t, \tau) = x(t+\tau)\, y(t) = \sum_{r=-\infty}^{\infty} \gamma_r\, e^{jr\omega_0 t} \tag{5.50}$$

Since $z(t, \tau)$ is the product of periodic signals it will be periodic itself and can be represented by the Fourier series as shown above. As before

$$\begin{aligned} x(t+\tau) &= \sum_{r=-\infty}^{\infty} \alpha_r e^{jr\omega_0(t+\tau)} \\ y(t) &= \sum_{r=-\infty}^{\infty} \beta_r e^{jr\omega_0 t} \end{aligned} \tag{5.51}$$

where α_r and β_r are defined as before. Thus

$$\gamma_r = \sum_{m=-\infty}^{\infty} \alpha_m e^{jm\omega_0\tau} \beta_{r-m} \tag{5.52}$$

The correlation over a portion of the period may be defined as

$$\rho_{xy}(k, \tau) = \frac{1}{kt_c} \int_0^{kt_c} x(t+\tau)\, y(t)\, dt \tag{5.53}$$

$$= \frac{1}{kt_c} \int_0^{kt_c} \left(\sum_{r=-\infty}^{\infty} \gamma_r e^{jr\omega_0 t} \right) dt$$

$$= \sum_{r=-\infty}^{\infty} \gamma_r \left(\frac{1}{kt_c} \int_0^{kt_c} e^{jr\omega_0 t}\, dt \right)$$

Therefore

$$\rho_{xy}(k, \tau) = \sum_{r=-\infty}^{\infty} \gamma_r \, e^{jrk\frac{\pi}{L}} \left(\frac{\sin rk\frac{\pi}{L}}{rk\frac{\pi}{L}} \right) \tag{5.54}$$

For $k = L$, $\operatorname{sinc}\left(rk\frac{\pi}{L}\right)$ is equal to zero except for $r = 0$ and the equation for the cross-correlation reduces to Eq. 5.38 above.

For $k < L$ let $\tau = s_x t_c$ and, using Eqs. 5.39 and 5.52 it can be seen that

$$\gamma_r = \sum_{m=-\infty}^{\infty} \alpha_m \beta_{r-m} \, e^{jms_x \frac{2\pi}{L}}$$

$$= \sum_{m=-\infty}^{\infty} e^{-jr\frac{\pi}{L}} \frac{\sin m\frac{\pi}{L}}{m\frac{\pi}{L}} \, \frac{\sin(r-m)\frac{\pi}{L}}{(r-m)\frac{\pi}{L}} \, a_m b_{r-m} \, e^{jms_x \frac{2\pi}{L}}$$

Using the same techniques as before the coefficient reduces to

$$\gamma_r = e^{-jr\frac{\pi}{L}} \frac{\sin r\frac{\pi}{L}}{r\frac{\pi}{L}} \sum_{m=0}^{L-1} a_m b_{r-m} \, e^{jms_x \frac{2\pi}{L}} \tag{5.55}$$

and the partial period cross-correlation is

$$\rho_{xy}(k, s_x) = \sum_{r=-\infty}^{\infty} e^{jr(k-1)\frac{\pi}{L}} \frac{\sin rk\frac{\pi}{L} \, \sin r\frac{\pi}{L}}{k\left(r\frac{\pi}{L}\right)^2} \sum_{m=0}^{L-1} a_m b_{r-m} \, e^{jms_x \frac{2\pi}{L}} \tag{5.56}$$

This equation can be reduced as before to give the result

$$\rho_{xy}(k, s_x) = \sum_{r=0}^{L-1} \left(\sum_{m=0}^{L-1} a_m \, b_{r-m} \, e^{jms_x \frac{2\pi}{L}} \right) e^{jr(k-1)\frac{\pi}{L}} \, \frac{\sin r k \frac{\pi}{L}}{k \sin r \frac{\pi}{L}} \tag{5.57}$$

For this equation the integral is taken over k digits of the sequence waveform $z(t, \tau)$ and the value of s_x is the phase shift in the sequence waveform $x(t)$. Thus the equation as it stands accounts for all possible starting points in the sequence $x(t)$ and the starting point in the sequence $y(t)$ can be taken into account by adjusting the phase angle of the term b_{r-m}. The coefficients a_m and b_{r-m} in Eq. 5.57 are

$$a_m = |a_m| \, e^{j\theta_{a_m}}$$

$$b_m = |b_m| \, e^{j\left(\theta_{b_m} + m \, s_y \frac{2\pi}{L}\right)} \tag{5.58}$$

θ_{a_m} and θ_{b_m} are the phase angles for the Fourier coefficients of the waveforms in the characteristic phase position, and the quantity s_y is the number of digits the starting point in the waveform $y(t)$ differs from the characteristic phase starting point. Therefore the partial period cross-correlation becomes

$$\rho_{xy}(k, s_x, s_y) = \sum_{r=0}^{L-1} \left(\sum_{m=0}^{L-1} |a_m| \, |b_{r-m}| \, e^{j\left(\theta_{a_m} + \theta_{b_{r-m}}\right)} e^{j\left[(r-m)\, s_y \frac{2\pi}{L} + m s_x \frac{2\pi}{L}\right]} \right) e^{jr(k-1)\frac{\pi}{L}} \frac{\sin r k \frac{\pi}{L}}{k \sin r \frac{\pi}{L}} \tag{5.59}$$

$$2n \leq k \leq L$$

$$0 \leq s_x \leq L-1$$

$$0 \leq s_y \leq L-1$$

Thus, as seen from Eq. 5.59, the partial period cross-correlation, expressed in terms of the Fourier series coefficients, is very complex. It does, however, provide another method of studying the partial period correlation function.

5.3 Summary

In this chapter linear maximal binary sequences and their correlation functions have been considered in the frequency domain. It is shown that the calculation of the Fourier series coefficients is greatly simplified by making use of the cyclotomic cosets. The cyclotomic cosets, and the coset assignments of Chapter II, are used to show that the Fourier series coefficients for all maximal sequences of the same length are related in a well specified manner. The frequency spectra for periodic linear maximal sequences, and for segments of the sequences, are given in terms of the Fourier series coefficients. The correlation functions, for both the full period and the partial period correlation, are expressed in terms of the Fourier series coefficients. The results, although complex, provide another technique for examining the structure of the correlation function and how it changes as the correlation time is reduced.

CHAPTER VI

CORRELATION AMBIGUITY

A digital system, using PR techniques where correlation over the full period of the sequences is impractical, is forced to operate with the more complex partial period correlation function. The full period correlation between PR sequences is a function of the relative delay between the sequences while the partial period correlation for the same value of delay is also a function of the starting points in the sequences (Eqs. 2.4 and 2.11). It is this ambiguity in the partial period correlation for a given relative delay which leads to the more complex correlation function.

In this chapter two matrices, called the "correlation ambiguity matrix" and the "difference matrix", are introduced as techniques for displaying this ambiguity. The correlation ambiguity matrix is an array of the unnormalized correlation values for a given pair of sequences and fixed k. The difference matrix is an array of elements which, for the given pair of sequences and fixed k, are proportional to the difference between the elements of the correlation ambiguity matrix and the full period correlation values.

These matrices provide a display which may be used to compare the partial period correlation, for fixed k, with the full period correlation function and, thus, show graphically how the correlation function is modified as the correlation interval is reduced.

6.1 Correlation Ambiguity Matrix

The partial period correlation for linear maximal binary sequence waveforms is a function of the number of digits, k, over which the correlation is taken and the starting points, s_x and s_y, in the sequences, i.e.,

$$\rho_{xy}(k, s_x, s_y) = \frac{1}{kt_c} \int_0^{kt_c} x(t + s_x t_c)\, y(t + s_y t_c)\, dt \qquad (6.1)$$

(For this equation to be valid, it is assumed that the transitions of the two sequence waveforms are lined up, and that the correlation is taken over an integral number of digits. Equation 6.1 is then equivalent to Eq. 2.11 for $s = s_y$ and $\tau = s_x - s_y$.) For fixed k the set of all correlation values can be written as a matrix of elements of the form $\rho_{s_x s_y}$. The correlation values are periodic, with period L, in both s_x and s_y and, therefore, all possible correlation values are contained in an $L \times L$ matrix. The correlation function, for non-integer values of s_x and s_y, is a surface which has the values given in the matrix for integer values of s_x and s_y. This matrix contains all of the essential information about the partial period correlation function. It provides a simple way of showing how the partial period correlation differs from the full period correlation, and is called the "correlation ambiguity matrix."

Some examples of correlation ambiguity matrices are shown in Figs. 6.1 through 6.9 (additional examples are given in Appendix B). It should be noted that in these figures the elements of the matrices are the unnormalized correlation values, i.e., $C_{s_x s_y} = k\,\rho_{s_x s_y}$. The correlation function surface for non-integer values of s_x and s_y is piecewise planar surface. That is the elements of the correlation ambiguity matrix are points on the correlation surface which may be joined by straight lines in the s_x direction, the s_y direction, and along diagonals of constant $(\tau = s_x - s_y)$. The intersecting lines connecting these points define planes which make up the surface desired. The zero contour of the correlation function surface defined by the correlation ambiguity matrix is shown in these figures and the positive portion of the surface is shaded.

The full period cross-correlation does not depend on the starting point since the entire period of the sequence is involved in the integration. Thus in the figures for which $k = L$, i.e., Figs. 6.1 and 6.5, the correlation values are constant on diagonals corresponding to constant values of τ. This matrix is the limiting case for the partial period correlation ambiguity matrices for the given pair of sequences and the average value of the partial period correlation values along the diagonal must be proportional to the full period value (as noted previously in Eq. 2.12). The correlation ambiguity matrices

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1
1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1
2	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9
3	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1
4	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7
5	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9
6	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1
7	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1
8	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7
9	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7
10	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7
11	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9
12	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7
13	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1
14	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1
15	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1
16	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9
17	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7
18	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1
19	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7	7
20	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7	7
21	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1	7
22	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9	-1
23	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1	-9
24	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7	-1
25	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1	7
26	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1	-1
27	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1	-1
28	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1	-1
29	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9	-1
30	-1	-1	-1	-1	7	-1	-9	-1	7	7	7	-1	7	-9	-1	-1	-1	7	-9	7	7	7	-1	-1	-9	7	-1	-9	-1	-1	-9

Fig. 6.1. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, k = 31

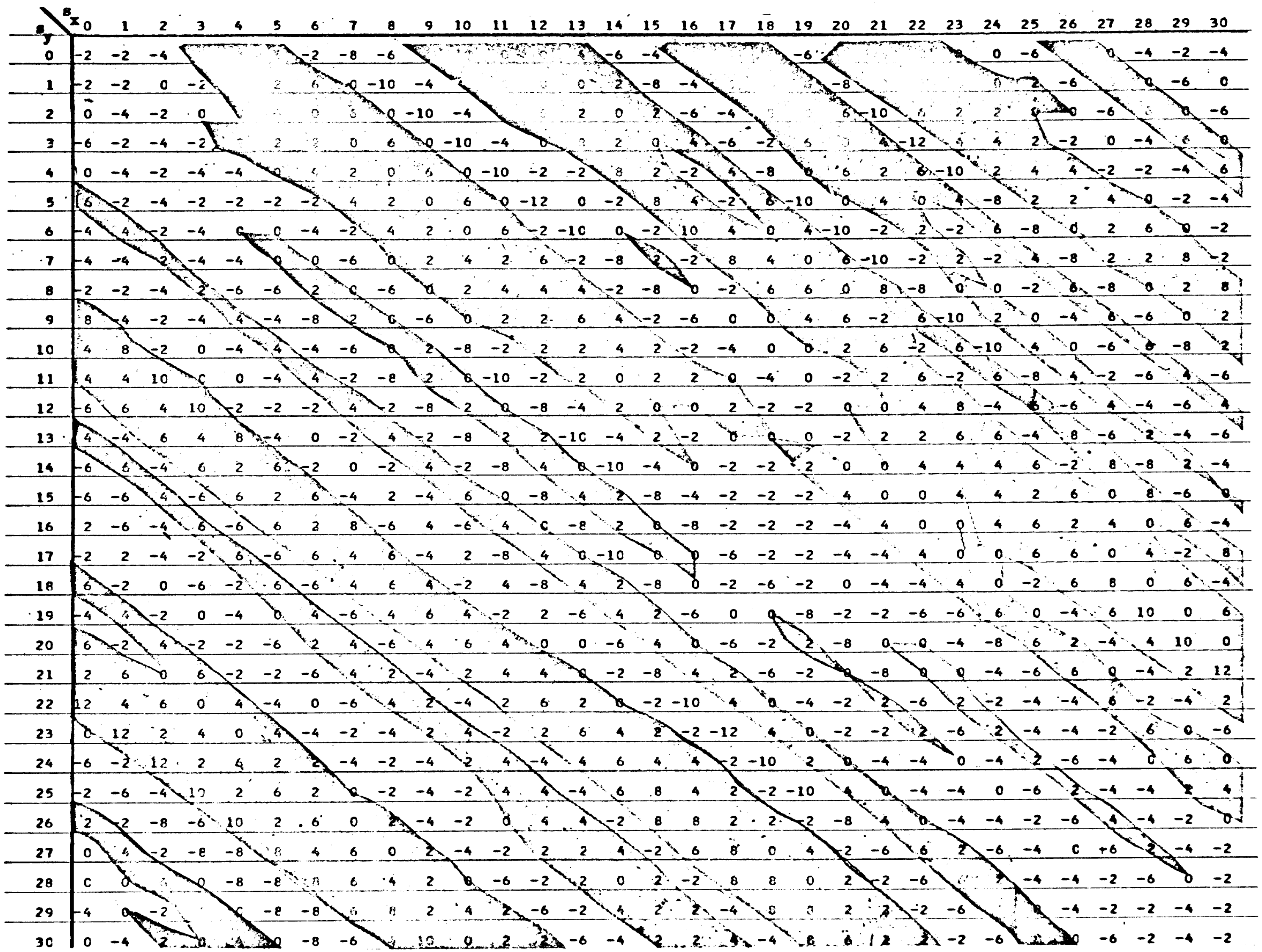

Fig. 6.2. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 20$

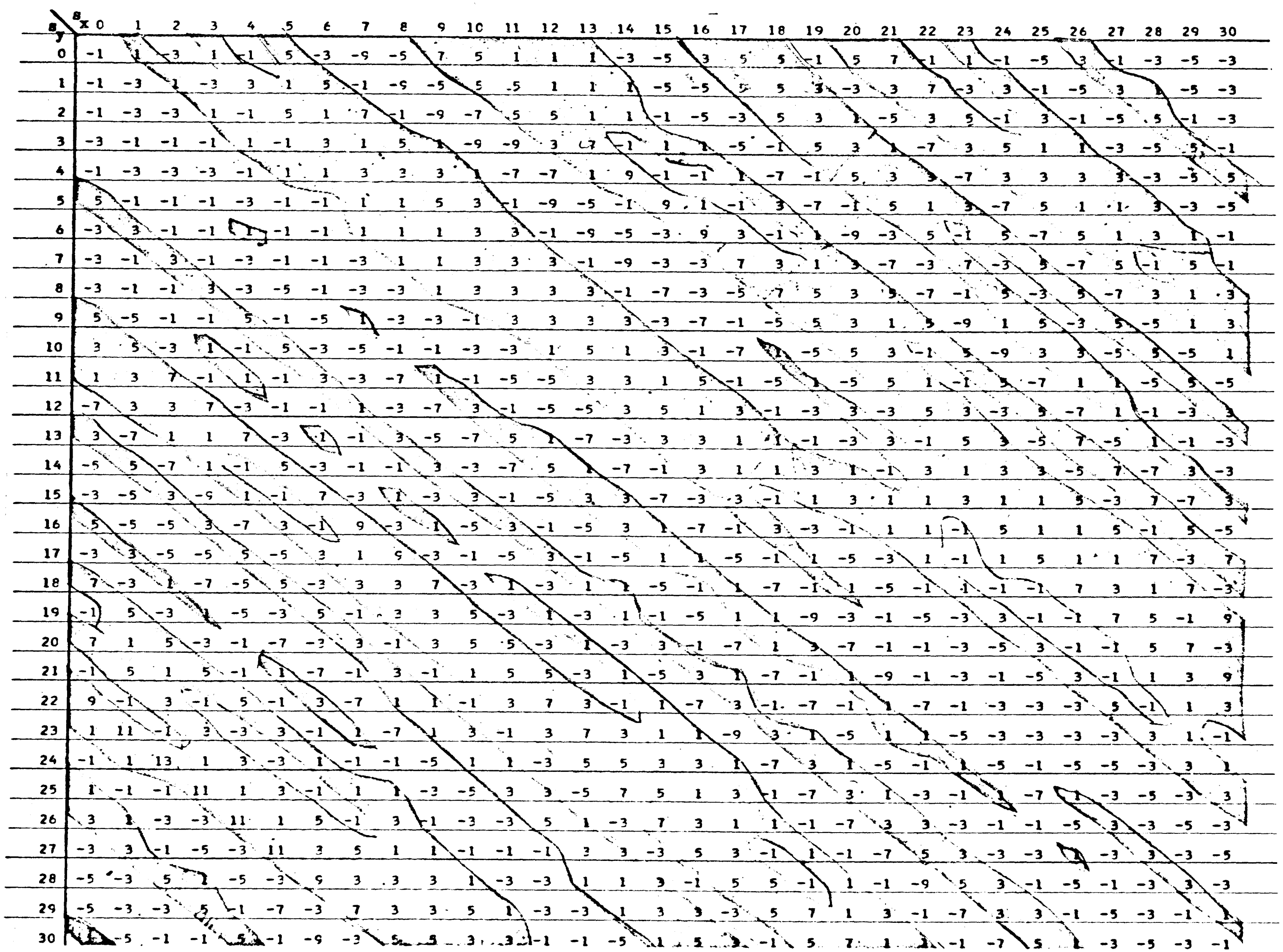

Fig. 6.3. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 15$

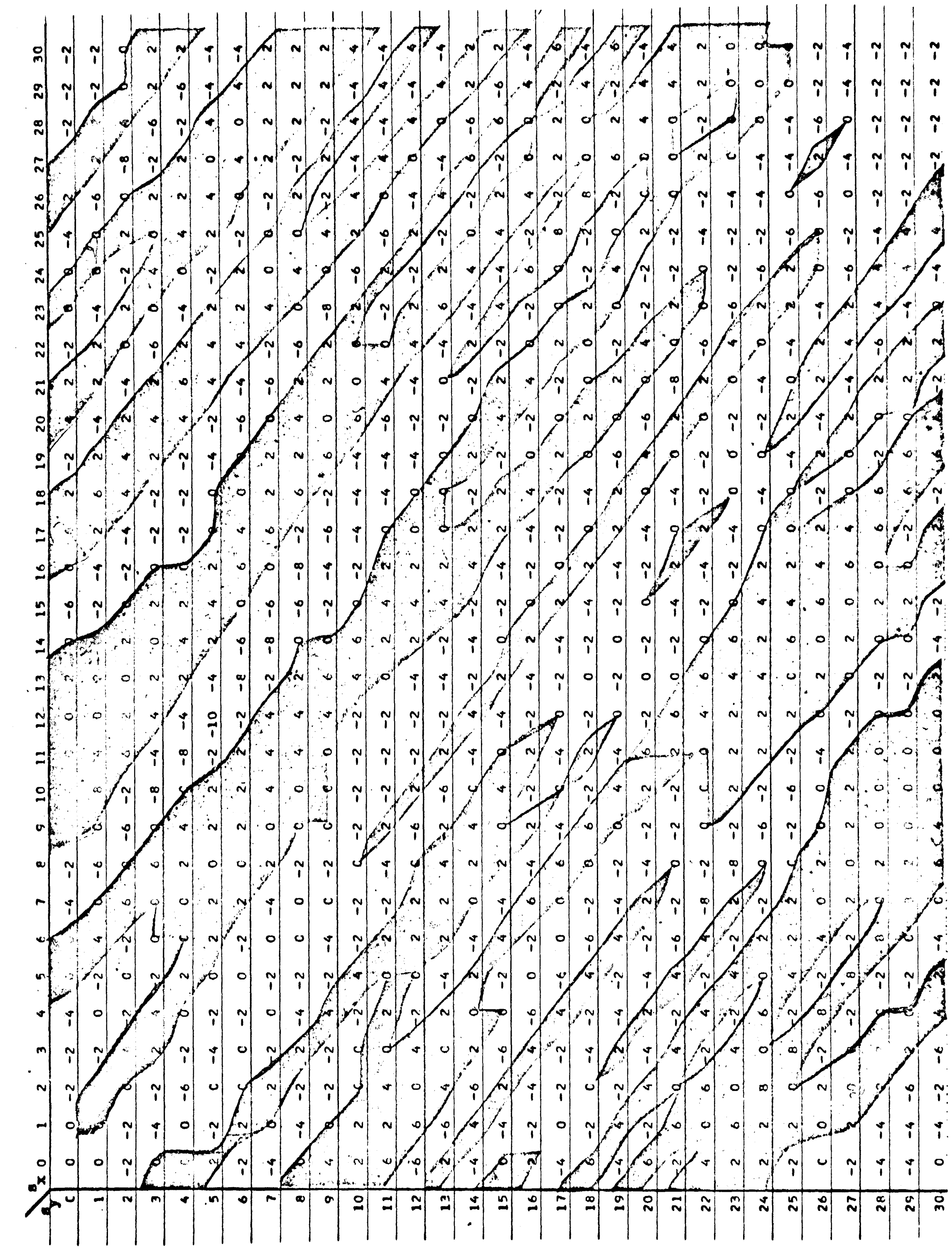

Fig. 6.4. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 10$

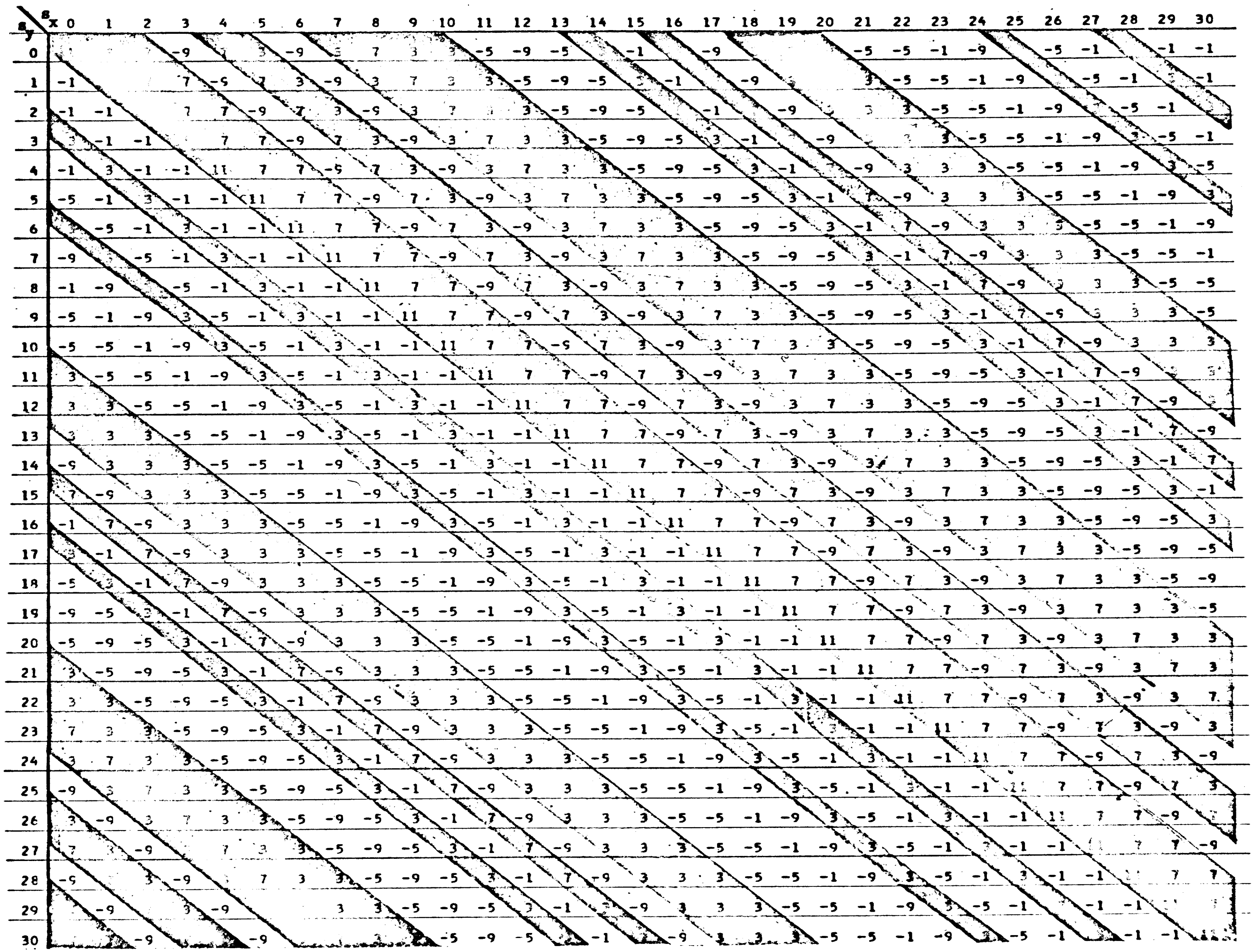

Fig. 6.5. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 31$

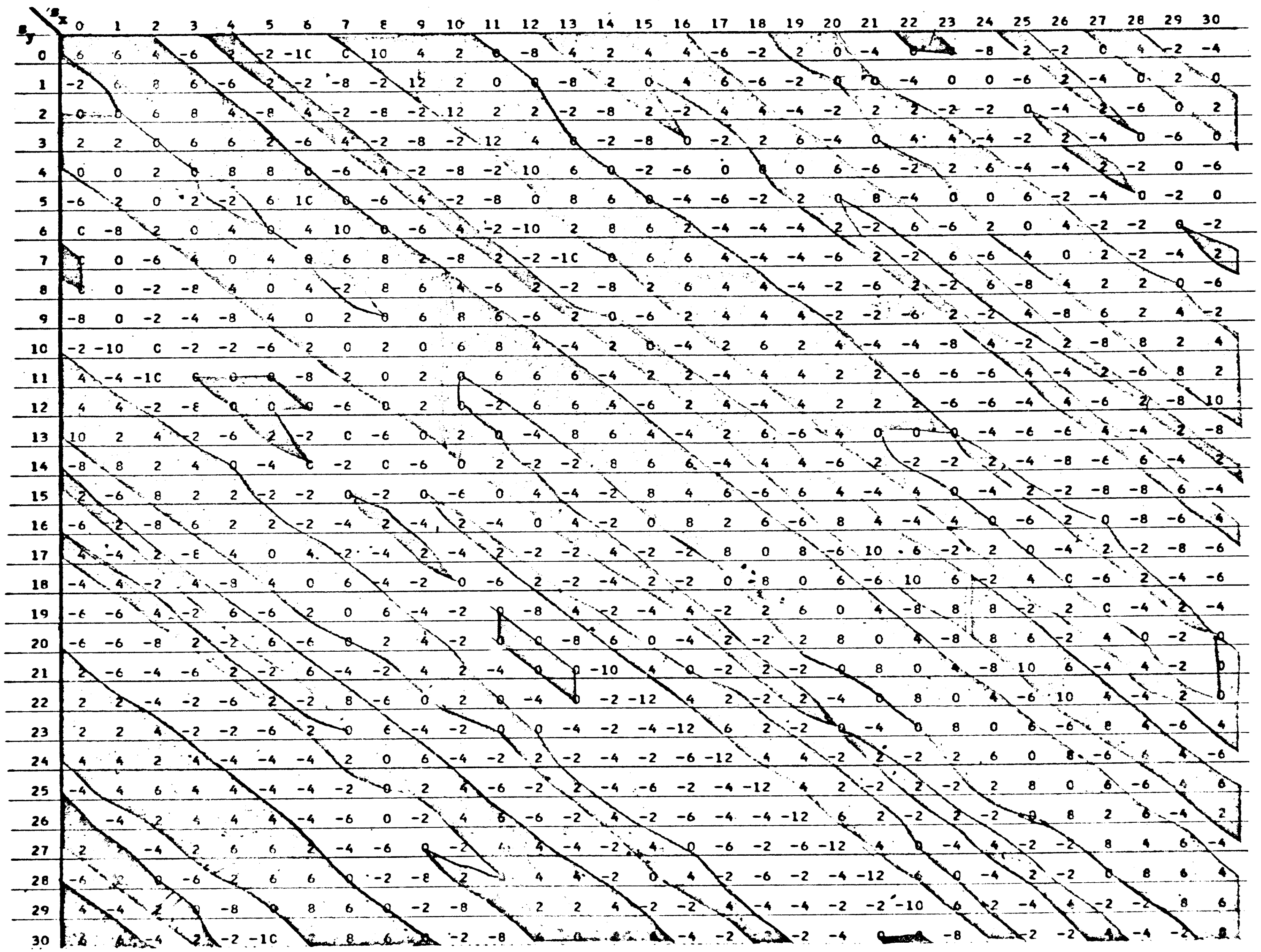

Fig. 6.6. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 20$

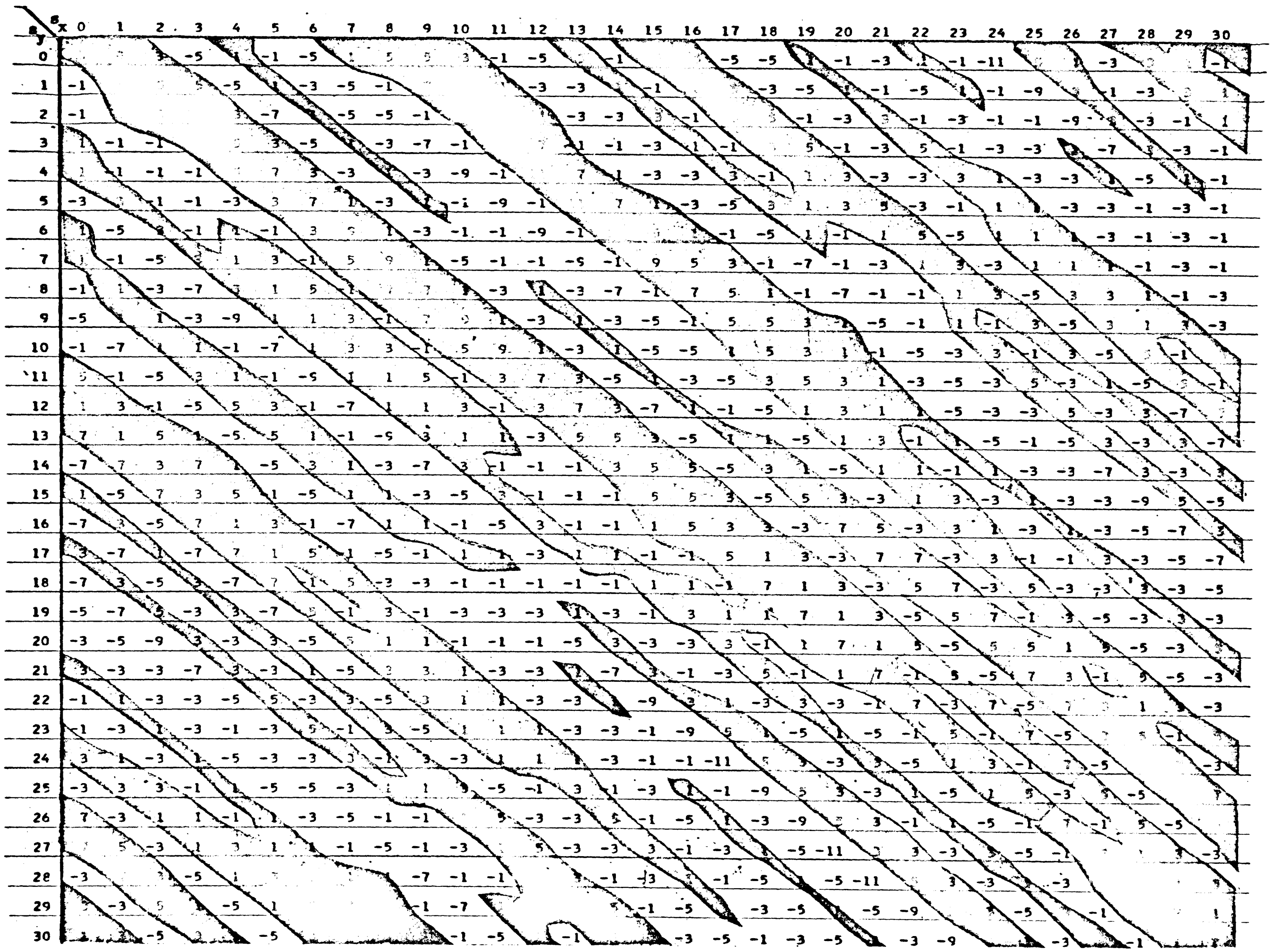

Fig. 6.7. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 15$

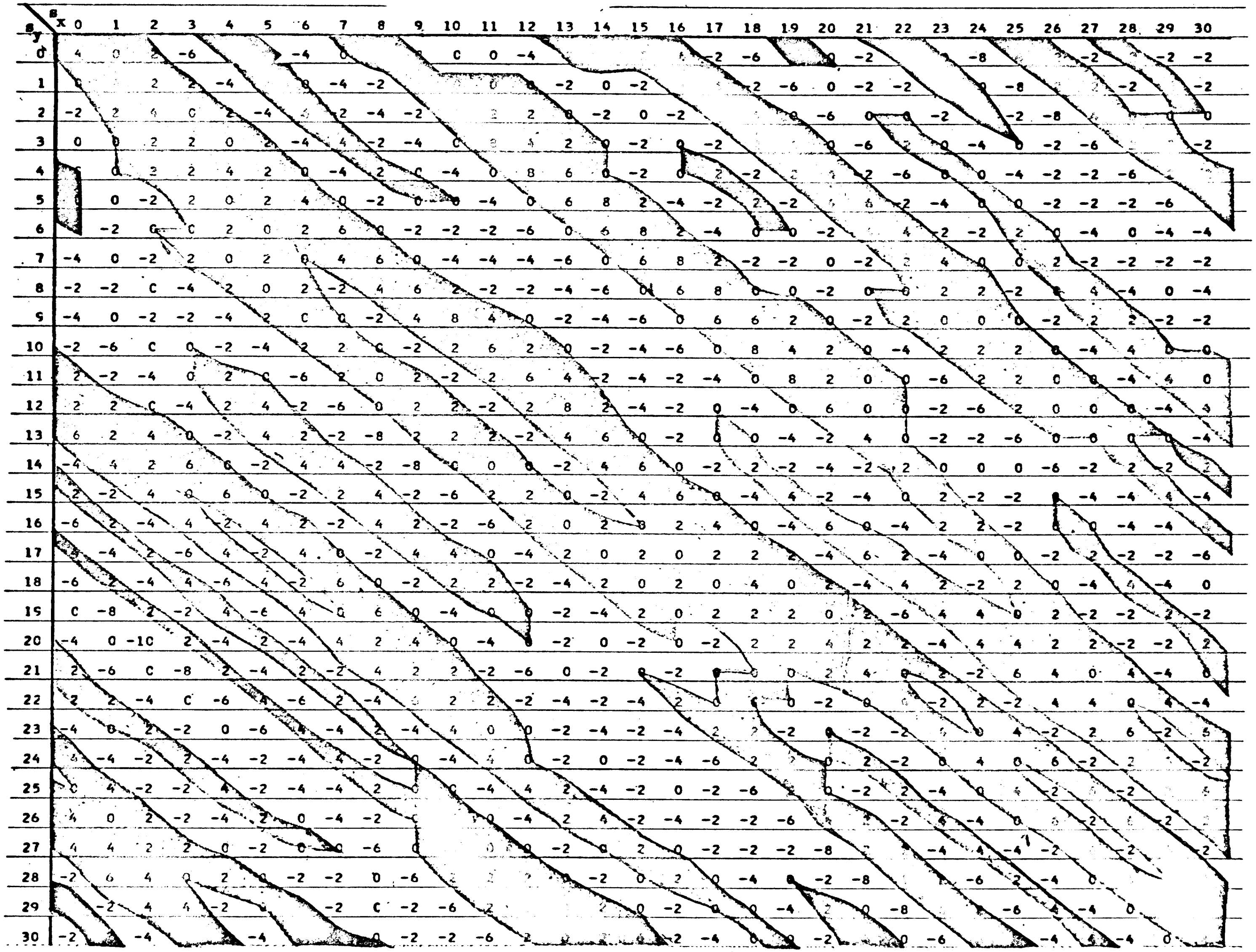

Fig. 6.8. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 10$

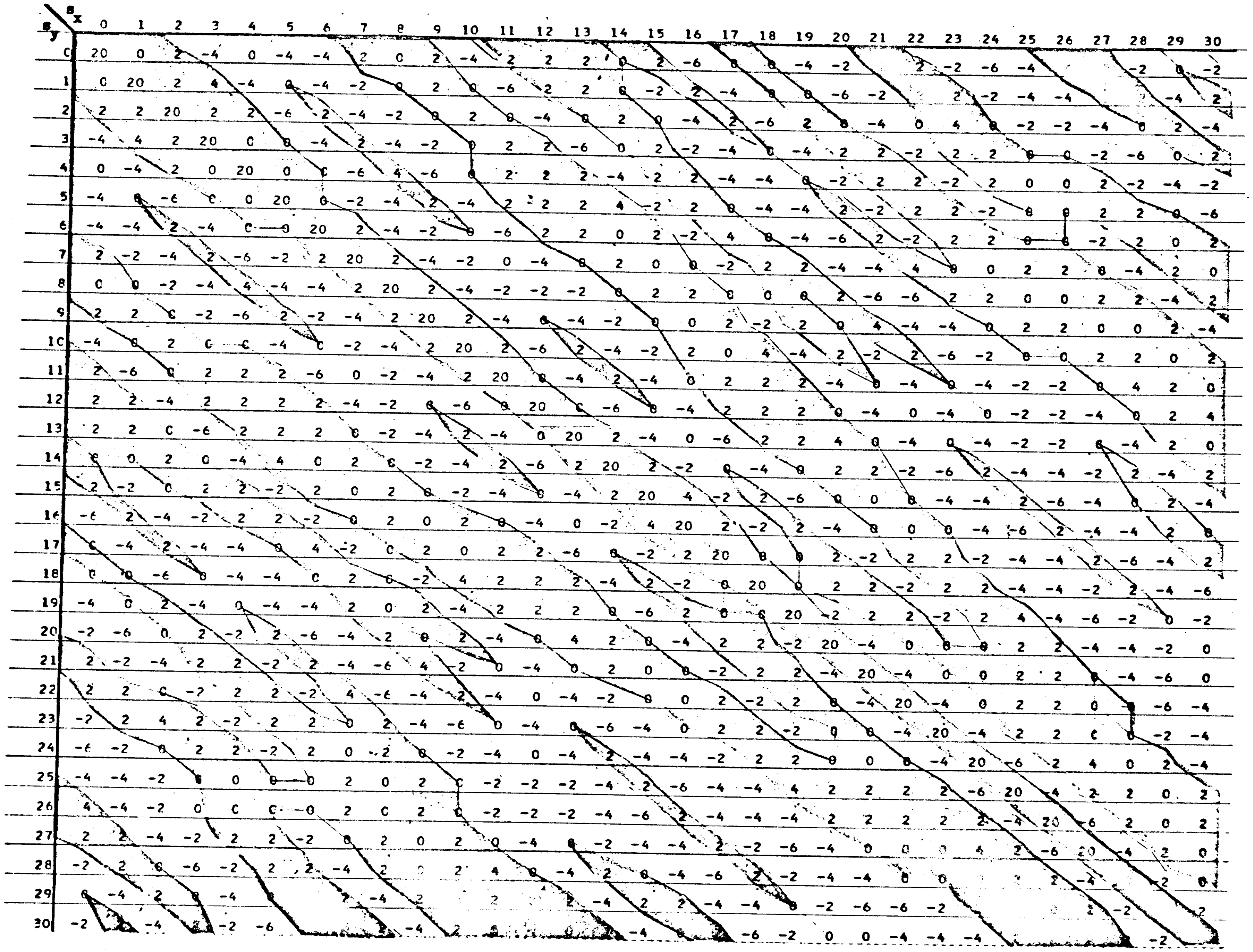

Fig. 6.9. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 2, 0]$, $k = 20$

for several values of k are shown and it can be seen that the peaks and valleys of these diagrams follow the diagonal characteristic of the full period correlation ambiguity matrix.

For example, the full period correlation ambiguity matrix for the correlation between the sequences [5, 2, 0] and [5, 4, 2, 1, 0], Fig. 6.1, shows relatively deep valleys for diagonals corresponding to $\tau = 0$, 7, 14, 19, 25, and 28. Referring to Figs. 6.2 through 6.4, the correlation ambiguity matrices with k = 20, 15, and 10, for the same pair of sequences, it can be seen that the deepest valleys of these correlation functions occur for the same values of τ. The ridges in the correlation function with the largest positive values occur for values of τ leading to largest full period correlation values. While this effect is less pronounced for the lower values of k, it is still visible in the matrix for k = 10 (Fig. 6.4).

A correlation ambiguity matrix for the autocorrelation case is given for comparison purposes (Fig. 6.9). It is seen that the matrix for this case has value k along the principal diagonal ($\tau = 0$) and is symmetric about this diagonal. The diagonal peaks and valleys are still present although not as pronounced as in the cross-correlation case for the same value of k. This is, of course, due to the fact that the full period correlation ambiguity diagram for the autocorrelation case has value -1 for all $s_x \neq s_y$. The average value along any diagonal except $s_x = s_y$ is, therefore, proportional to -1 and thus the peak values are not expected to be as large as in the cross-correlation case.

The correlation ambiguity matrix for correlation over L - k digits can be found quite easily from the correlation ambiguity matrix for correlation over k digits. That is, let $C_{s_x s_y}(m)$ be the correlation ambiguity element for correlation over m digits of the sequences. Then

$$C_{s_x s_y}(L) = \sum_{i=0}^{L-1} x_{i+s_x} y_{i+s_y}$$

$$= \sum_{i=0}^{k-1} x_{i+s_x} y_{i+s_y} + \sum_{i=k}^{L-1} x_{i+s_x} y_{i+s_y}$$

$$= C_{s_x s_y}(k) + C_{s_x+k\, s_y+k}(L - k) \tag{6.2}$$

Therefore the correlation ambiguity matrix for correlation over L - k digits can be derived directly using the matrix for k digit correlation and the full period correlation values.

6.2 Difference Matrix

The correlation ambiguity matrix, defined in the previous section, is a matrix of unnormalized partial period correlation values in which the elements, $C_{s_x s_y}(k)$, can be found using

$$C_{s_x s_y}(k) = \sum_{i=0}^{k-1} x_{i+s_x} y_{i+s_y} \tag{6.3}$$

The correlation ambiguity diagram for $k = L$ is constant along diagonals, i.e., constant for a given value of $\tau = s_x - s_y$, and it has been shown (see Eq. 2.12) that the mean value for the partial period cross-correlation for a given value of τ is equal to the full period cross-correlation for that value of τ. Thus the average value of the elements of a diagonal in the correlation ambiguity matrix is proportional to the full period correlation value. In order to show this let $s_y = s$ and $s_x = s + \tau$ and average over s, i.e.,

$$\frac{1}{L}\sum_{s=0}^{L-1} C_{s_x s_y}(k) = \frac{1}{L}\sum_{s=0}^{L-1}\left(\sum_{i=0}^{k-1} x_{i+s+\tau}\, y_{i+s}\right)$$

$$= \frac{k}{L}\left(\sum_{i=0}^{L-1} x_{i+s_x}\, y_{i+s_y}\right) = \frac{k}{L}\, C_{s_x s_y}(L) \qquad (6.4)$$

Another matrix of interest then is a matrix in which the elements are porportional to the difference between the partial period correlation value and the mean value along the diagonal. This matrix is called the "difference matrix", and some examples of this matrix are shown in Figs. 6.10 through 6.12. The elements of these matrices are found using the equation

$$V_{s_x s_y}(k) = L\left[C_{s_x s_y}(k) - \frac{k}{L}\, C_{s_x s_y}(L)\right] \qquad (6.5)$$

The difference matrix, like the correlation ambiguity matrix,

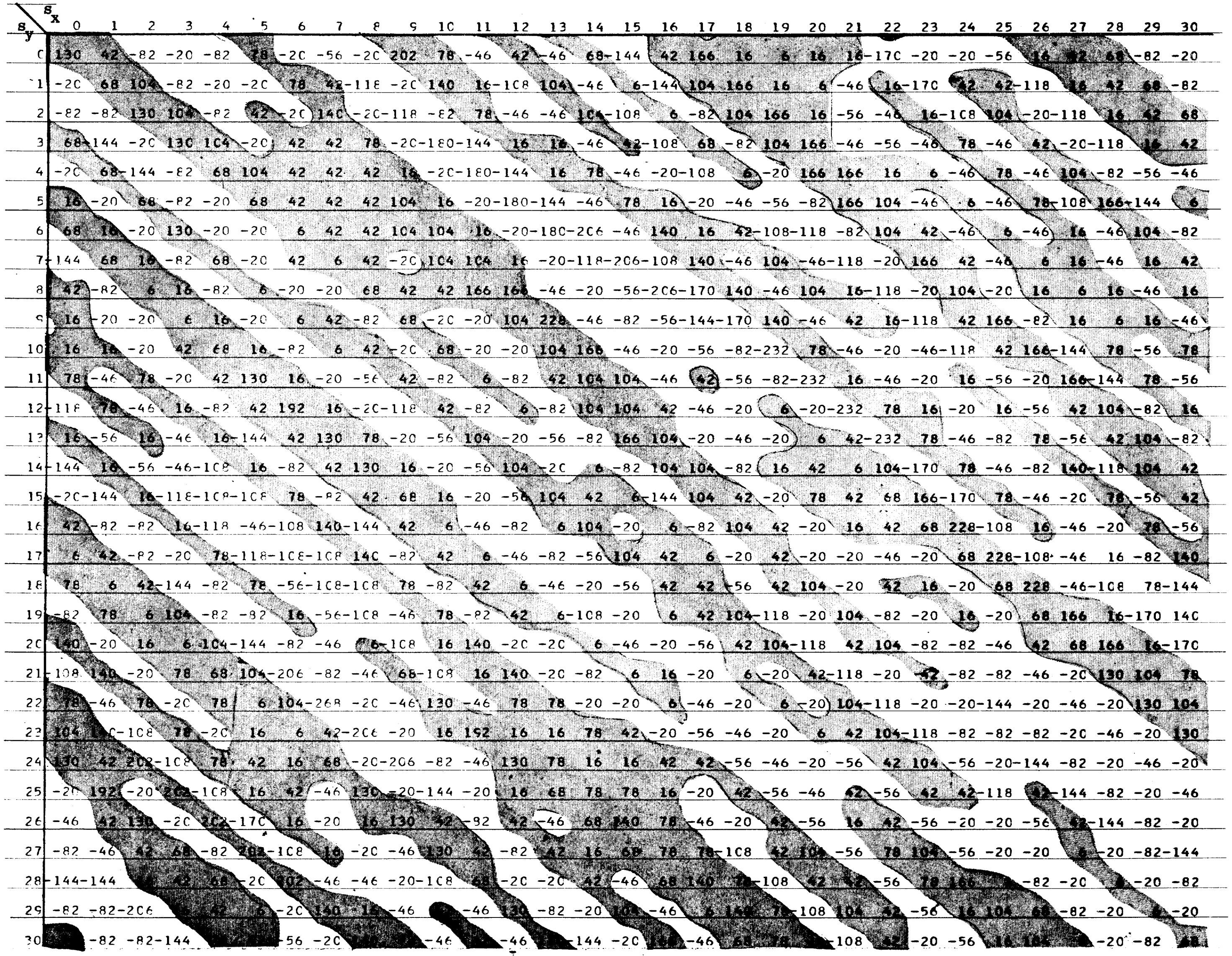

Fig. 6.10. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 11$

$s_y \backslash s_x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	118	-42	-104	82	82	46	-42	-68	-166	-16	170	-140	20	-16	-6	-104	144	82	46	-6	108	-16	-140	20	20	-6	46	20	56	-42	-104
1	-42	118	20	-42	82	82	46	20	-130	-104	-78	108	-140	20	-78	-68	-104	206	82	46	-68	108	-16	-140	20	82	-6	-16	20	-6	20
2	20	-104	118	20	20	144	20	46	20	-130	-104	-78	46	-78	20	-78	-6	-104	268	20	46	-130	46	-78	-78	20	20	-6	46	20	-6
3	-6	-42	-104	118	82	82	82	20	46	20	-130	-104	-140	108	-78	20	-16	-6	-42	206	20	-16	-192	-16	-16	-78	-42	20	56	46	20
4	20	56	-42	-104	56	20	144	82	20	46	20	-130	-42	-202	108	-78	-42	-16	-68	20	206	82	46	-130	-78	-16	-16	-42	-42	56	46
5	46	-42	56	-42	-42	118	-42	144	82	20	46	20	-192	20	-202	108	-16	-42	46	-130	20	144	20	-16	-68	-78	-78	-16	20	-42	56
6	56	-16	-42	56	20	20	56	-42	144	82	20	46	-42	-130	20	-202	170	-16	20	-16	-130	-42	82	-42	46	-68	-140	-78	46	20	-42
7	-104	56	-78	-104	56	20	20	-6	20	82	144	82	46	-42	-68	82	-202	108	-16	20	46	-130	-42	82	-42	-16	-68	-78	-78	108	-42
8	-42	-42	56	-78	-166	-6	82	20	-6	20	82	144	144	-16	-42	-68	20	-202	46	46	20	108	-68	20	20	-42	46	-68	-140	-78	108
9	108	-104	-42	56	-16	-104	-68	82	20	-6	20	82	82	206	-16	-42	-6	20	-140	-16	46	-42	46	-130	82	20	-104	46	-6	-140	-78
10	-16	108	-42	20	56	-16	-104	-6	20	82	-68	-42	82	82	144	-78	-42	56	20	-140	-78	46	-42	46	-130	144	20	-166	46	-68	-78
11	-16	-16	170	20	20	56	-16	-42	-68	82	20	-130	-42	82	20	82	-78	20	56	20	-202	-78	46	-42	46	-68	144	-42	-166	-16	-6
12	-6	46	-16	170	-42	-42	118	-16	-42	-68	82	20	-68	-104	82	20	20	-78	-42	118	20	-140	-16	108	-104	46	-6	144	-104	-166	-16
13	-16	56	46	-16	108	-104	20	118	-16	-42	-68	82	82	-130	-104	82	-42	20	-140	20	118	82	-78	46	46	-104	108	-6	82	-104	-166
14	-166	46	56	46	-78	46	-42	20	118	-16	-42	-68	144	20	-130	-104	20	-42	-42	-78	20	180	144	-16	-16	46	-42	108	-68	82	-104
15	-166	-166	-16	-6	46	-78	46	-104	82	56	46	20	-68	144	82	-68	-104	-42	-42	-42	-16	20	180	144	-16	-78	46	20	108	-6	20
16	82	-166	-104	46	-6	46	-78	108	-166	144	-6	-16	20	-68	82	20	-68	-42	-42	-42	-104	-16	20	180	144	46	-78	-16	20	46	56
17	118	82	-104	-42	46	-6	46	-16	46	-104	82	-68	-16	20	-130	20	20	-6	-42	-42	-104	-104	-16	20	180	206	46	-140	-16	-42	108
18	46	118	20	-166	-42	46	-6	-16	46	-16	-42	144	-68	-16	82	-68	20	-42	-6	-42	20	-104	-104	-16	20	118	206	108	-140	46	-104
19	-104	-16	118	20	-104	20	-16	-6	-16	46	-16	-42	82	-6	-16	82	-6	20	20	-68	-42	-42	-166	-166	46	20	56	206	170	-140	46
20	46	-42	-16	118	-42	-166	82	-16	-6	-16	46	-16	20	20	-6	-16	20	-6	-42	82	-68	20	20	-104	-228	46	82	56	144	170	-140
21	-78	46	20	46	118	-42	-166	144	-78	56	-78	-16	-16	20	-42	-68	-16	82	-6	-42	20	-68	20	20	-104	-166	46	20	56	82	232
22	232	-16	46	20	-16	56	20	-166	144	-78	56	-78	46	-78	20	-42	-130	-16	20	56	-42	82	-6	82	-42	-104	-104	46	-42	56	82
23	20	232	-78	-16	20	-16	56	-42	-104	82	-16	118	-78	46	-16	82	-42	-192	-16	20	118	-42	82	-6	82	-104	-104	-42	46	20	-6
24	-6	-42	232	-78	46	82	-78	56	-42	-104	82	-16	56	-16	46	-16	144	-42	-130	-78	20	56	-104	20	56	82	-166	-104	20	46	20
25	-42	-6	-104	170	-78	46	82	-140	118	-104	-42	144	-16	56	46	108	-16	82	-42	-130	-16	20	56	-104	20	-6	82	-104	-104	82	-16
26	-78	-42	-68	-166	170	-78	46	20	-78	56	-42	20	144	-16	118	108	108	-78	82	-42	-68	-16	20	56	-104	-42	-6	144	-104	-42	20
27	20	-16	-42	-68	-228	108	-16	46	20	-78	56	-42	82	82	-16	118	46	108	-140	144	-42	-6	46	82	-6	-104	20	-6	82	-104	-42
28	20	20	46	20	-68	-228	108	46	-16	82	-140	-6	-42	82	20	-78	118	108	108	-140	82	-42	-6	46	82	56	-104	-42	-6	20	-42
29	-104	20	-42	-16	20	-68	-228	46	108	-78	144	-78	-6	-42	144	82	-78	56	108	108	-78	82	-42	-6	46	20	56	-42	-42	56	-42
30	20	-104	82	20	-16	20	-68	-166	-16	108	-140	82	-78	-6	-104	82	82	-16	56	108	108	-78	82	-42	-6	108	82	-6	-42	-104	118

Fig. 6.11. Correlation difference matrix, $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 20$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-34	46	-16	-6	-78	-122	-130	-60	170	64	2	100	-68	224	2	144	-16	-6	-122	2	-60	-24	100	20	-68	2	38	20	64	-42	-104
1	-42	-34	108	46	-6	-78	-122	-68	-122	232	2	-60	100	-68	162	-60	144	46	-6	-122	-60	-60	-24	100	20	-6	2	-24	20	2	20
2	20	20	-34	108	-16	-68	-16	-122	-68	-122	232	2	2	38	-68	162	-122	144	-16	56	-122	2	2	38	38	20	56	2	-86	20	2
3	2	82	20	-34	46	-78	-6	-16	-122	-68	-122	232	64	-60	38	-68	100	-122	82	46	56	-60	64	64	-24	38	82	56	-60	-86	20
4	20	-60	82	20	28	108	-140	-6	-16	-122	-68	-122	170	126	-60	38	-6	100	-60	20	46	-6	-122	2	126	-24	-24	82	118	-60	-86
5	-86	82	-60	82	-42	-34	170	-140	-6	-16	-122	-68	-60	108	126	-60	-24	-6	38	2	20	108	56	-60	-60	126	38	-24	20	118	-60
6	-60	-148	82	-60	144	20	-96	170	-140	-6	-16	-122	-130	2	108	126	2	-24	56	-24	2	-42	46	-6	2	-60	64	38	38	20	118
7	180	-60	-86	144	-60	144	20	-34	108	-78	-68	-78	-122	-130	-60	46	126	64	-24	56	-86	2	-42	46	-6	64	-60	2	38	-24	82
8	20	180	-122	-148	144	-60	144	-42	28	46	-16	-6	-78	-122	-68	2	46	64	64	-24	118	-86	2	-42	46	-68	64	2	2	100	-86
9	-148	20	118	-184	-142	144	-60	82	20	-34	108	46	-6	-78	-60	-6	2	-16	64	64	38	118	-86	2	-42	-16	-68	126	2	64	38
10	38	-210	20	118	-122	-86	82	-60	82	20	-34	108	-16	56	-78	-60	56	2	46	2	64	-24	56	-148	64	-42	-78	-68	188	2	64
11	64	-24	-210	20	180	-60	-148	82	-60	82	20	-34	46	46	56	-78	2	56	64	-16	2	2	-86	-6	-86	64	-104	-78	-6	188	2
12	64	64	38	-148	20	180	-60	-86	20	2	20	-42	-34	46	-16	-6	-78	64	56	64	-78	2	2	-86	-6	-24	64	-166	-78	-68	250
13	250	2	64	38	-86	82	118	-60	-86	20	2	20	-104	28	46	-16	56	-78	126	-6	64	-140	-60	-60	-24	-6	-86	64	-104	-78	-68
14	-68	188	2	64	100	-24	20	118	-60	-86	20	2	-42	-42	28	46	46	56	-16	64	-6	2	-202	-122	2	-24	-68	-86	126	-104	-78
15	-78	-6	188	2	2	38	38	20	118	-60	-86	20	64	-104	-42	28	-16	46	-6	46	64	56	64	-140	-184	2	38	-68	-148	126	-104
16	-166	-78	-68	126	2	2	38	-24	82	56	2	-24	20	64	-42	20	28	-78	46	-6	108	64	56	64	-140	-246	2	100	-68	-86	64
17	64	-104	-78	-68	64	-60	64	38	-24	82	56	2	38	-42	64	-42	-42	28	-140	108	-6	170	126	118	2	-140	-184	2	38	-68	-86
18	-24	64	-42	-16	-68	64	-60	126	-24	38	20	-6	2	38	-104	2	-42	20	28	-140	46	-6	170	126	118	64	-140	-246	2	-24	-6
19	-6	-86	64	-42	46	-6	2	-60	126	-24	38	20	-68	64	38	-104	64	-42	82	-34	-140	-16	-68	108	188	118	2	-140	-184	2	-24
20	-86	-6	-148	2	-42	46	-6	-60	2	64	38	100	20	-68	126	100	-104	2	-42	82	28	-140	-16	-68	108	126	118	64	-140	-122	-60
21	2	-86	56	-86	2	-42	46	56	-122	64	2	-24	100	20	-130	64	100	-42	2	-42	20	28	-140	-16	-68	170	126	56	64	-202	-60
22	2	2	-24	118	-86	2	-42	108	-6	-60	2	-60	-24	100	-42	-192	64	162	-42	2	-104	20	28	-140	-16	-6	170	64	56	2	-140
23	-78	2	64	38	118	-86	2	20	46	56	-122	-60	-60	-24	38	-104	-192	126	162	-42	-60	-104	20	28	-140	46	-6	108	64	-6	64
24	64	-16	2	64	-24	56	-24	2	20	46	56	-122	2	-122	-24	38	-166	-192	64	224	-42	2	-42	82	-34	-140	108	-6	46	64	-6
25	56	64	46	64	64	-24	56	38	-60	82	-16	-6	-122	2	-184	-86	38	-104	-192	64	162	-42	2	-42	82	28	-140	46	-6	-16	126
26	64	56	2	-16	64	64	-24	-6	100	-122	144	46	-6	-122	64	-122	-86	-24	-104	-192	126	162	-42	2	-42	20	28	-78	46	56	-78
27	-78	2	56	2	46	126	2	-24	-6	100	-122	144	-16	56	-122	64	-60	-86	38	-166	-192	64	100	-104	64	-42	-42	28	-16	46	56
28	-6	-78	-60	-6	2	46	126	-60	38	-68	162	-60	144	-16	118	-60	64	-122	-86	38	-104	-192	64	100	-104	2	-42	20	28	46	-16
29	-16	56	-78	-60	-68	-60	108	126	-60	38	-68	162	2	82	-16	118	-122	64	-184	-24	38	-42	-130	126	38	-104	64	-42	-42	28	46
30	46	46	56	-78	-122	-130	2	108	126	-60	38	-68	224	-60	82	-16	56	-122	2	-122	-24	100	20	-68	64	38	-42	64	-104	-42	[illegible]

Fig. 6.12. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 20$

is periodic in s_x and s_y, with period L, and thus all possible element values are contained in an $L \times L$ matrix. As before the approximate zero contour for the surface defined by this matrix has been shown and the positive portions of the surface shaded in order to show how this matrix compares with the correlation ambiguity matrix. It can be seen that, while the general diagonal characteristic is still present, the correspondence between peaks and valleys and the full period correlation values has been eliminated. (The difference matrix for $k = L$ is obviously the zero matrix.) The sum of the elements along a diagonal (constant $\tau = s_x - s_y$) is equal to zero and, thus, continuous ridges, or valleys, for constant τ are not possible. The positive regions in the diagrams correspond to portions of the correlation ambiguity matrix which have values greater than $\frac{k}{L}$ times the full period value. It should be noted that the difference between the normalized partial period correlation and the normalized full period correlation can be found from the elements of this matrix by dividing the elements by the quantity kL.

Using Eqs. 6.2 and 6.5, it can be shown that the elements of the difference matrix for k and $L - k$ are related in a very simple fashion, i.e.,

$$V_{s_x+k\ s_y+k}(L-k) = L\left[C_{s_x+k\ s_y+k}(L-k) - \frac{(L-k)}{L} C_{s_x s_y}(L)\right]$$

$$= L\left[C_{s_x s_y}(L) - C_{s_x\ s_y}(k) - \frac{(L-k)}{L} C_{s_x s_y}(L)\right]$$

$$= L\left[\frac{k}{L} C_{s_x s_y}(L) - C_{s_x\ s_y}(k)\right]$$

$$= -V_{s_x\ s_y}(k) \qquad (6.6)$$

Thus the elements of the difference matrix, for correlation over $L - k$ digits, are the negative of the elements of the matrix for correlation over k digits, displaced by k digits along the diagonal. An example of this is given by Figs. 6.10 and 6.11. The difference matrix for $k = 20$ (Fig. 6.11) has elements which are the negative of the elements in the difference matrix for $k = 11$ displaced by k $\left(\text{i.e.,}\ V_{s_x, s_y}(20) = -V_{20+s_x, 20+s_y}(11)\right)$. It is only necessary to compute the matrices for $k < (L+1)/2$ and from these matrices the remaining matrices can be found using Eq. 6.6 above.

6.3 Matrix Utilization

The two matrices, introduced in this chapter, provide tools which may be used in the study of the partial period correlation function or in the design and analysis of PR systems using partial period correlation. The difference matrix is more useful for the study of the partial period correlation function since it is a direct view of the way the correlation function is modified as the correlation interval is decreased. The correlation ambiguity matrix represents the actual correlation function a system must use if it is operating in a partial period environment. Therefore the correlation ambiguity matrix provides a graphic display of the partial period correlation function which is necessary in the design and analysis of PR systems using partial period correlation.

Before considering the utilization of these matrices, a few comments on the matrices themselves are in order. As noted previously, for $k = L$ the correlation ambiguity matrix is constant along diagonals of constant delay, and the difference matrix is the zero matrix. As the value of k is decreased from L, the elements of the correlation ambiguity matrix change slowly. For $k = L-1$ the values of the elements can differ from the full period values by, at most ± 1. For each subsequent decrease of 1 in the value of k, the element values can change by at most ± 1. Thus the changes in the correlation function are relatively slow as k is decreased and the diagonal characteristic of the full period correlation matrix can not change rapidly as a function of k. The expected value of the elements along any diagonal is proportional to the full period correlation. The difference matrix shows a larger variation for small values of k; however, it also changes slowly as a function of k. As noted previously, the sum of the elements along any diagonal of the difference matrix must be equal to zero and therefore, along any diagonal there must be positive and negative elements, and continuous ridges, or valleys, for constant τ are not possible.

The correlation ambiguity matrix contains all the essential information to analyze partial period correlation problems. For digital communications systems which obtain synchronization by correlating over a portion of the period of the maximal sequence, it is necessary to know the amplitude of the subpeaks for the particular sequence. For correlation over the entire period the value of the correlation function for non-zero phase shift is known to be uniformly small; however, for the partial period case this is no longer true and setting the synchronization threshold without considering the subpeaks could lead to a high probability of false synchronization. The autocorrelation ambiguity matrix provides not only the size of the subpeaks but the location as well. The structure of the entire partial period autocorrelation function is given by the matrix. If the communications system must operate in an environment which contains signals modulated by sequences of the same length, the cross-correlation becomes important. For example, in the synchronization situation above, it is necessary to consider whether the correlation between the reference sequence and another sequence of the same length will lead to a high probability of false synchronization. Thus the size of the peaks of the cross-correlation function become important and can be found readily using the correlation ambiguity matrix. This situation would also arise in the recognition of addresses for a RADA system, i.e., if partial period correlation is necessary and sequences of the same length are employed to address different users, it is necessary to know the size of the cross-correlation peaks in order to determine the probability of the wrong user responding to a particular sequence. In a code multiplexing situation, the interference between the signals is given by the cross-correlation function. In all of these cases, where partial period correlation is necessary, the correlation ambiguity matrix provides a graphic means of viewing the problem.

One obvious disadvantage of matrices of the type introduced in this chapter is the size. Since the matrix, as presented in this chapter, is an LxL matrix, it is impractical to consider this particular representation for sequences generated by more than 6 or 7 stage generators. Other displays could be used to extend the useful range of sequence lengths, e.g., computer controlled CRT displays for the entire matrix, or portions of the matrix. Some results for the longer sequences can be deduced from the results of this chapter. The peaks and valleys of the partial period ambiguity matrix will, in general, occur for the same values of relative delay as the peaks and valleys of the full period correlation. (The full period correlation functions for sequences generated by up to 13 stage generators have been tabulated by Gold, Ref. 36.) Since, as noted by Gold, Ref. 6, the full period correlation values are constant for τ an element of a coset, the full period correlation function can be tabulated with much less effort than the partial period correlation. The search for extreme values in the partial period correlation function can be limited to those values of τ corresponding to extreme values in the full period correlation.

CHAPTER VII

SUMMARY

7.1 Conclusions

One of the properties of pseudo-random sequences which makes them desirable for use in digital systems is the correlation property. The full period autocorrelation and cross-correlation properties have been studied extensively; however, it is not always possible to correlate over a full period of the sequences. In order to determine system performance in this situation, it is necessary to consider the partial period correlation. While the full period correlation properties are well documented, very little is known about the partial period correlation properties. This study is an effort to fill this gap in the body of technical knowledge concerning PR sequences. Specifically, this is a study of the properties of the partial period correlation function for linear maximal binary sequences of the same length. The partial period autocorrelation has been the subject of some study during the past few years and is obviously included here as the case in which the two sequences are identical.

In this study the partial period correlation between linear maximal binary sequences of the same length has been considered from a number of viewpoints. It has been shown that the problem is directly related to the study of the weight distribution of linear codes and the standard coding theory bounds have been applied to the partial period correlation. The results of this application of coding theory bounds show that a general idea of the behavior of the partial period correlation peak values can be achieved with this approach. Since the appropriate coding theory bound depends on the rate of the code, the appropriate correlation bound depends on the number of digits, k, in the correlation interval. The peak correlation values for a pair of sequences can be compared with the bounds to determine how close the values are to the theoretical limits on the partial period correlation. The bounds establish a range of possible correlation peaks as a function of k.

The moments of the distribution of correlation values for all possible starting points have been examined in detail. The mean and variance of this distribution depend only on k, the number of digits in the correlation, and L, the length of the sequences, but not on the particular sequences. The calculation of the pth moment $(p \geq 3)$ for correlation over k digits depends on finding the number of p-term polynomials of degree less than k which are divisible by the characteristic polynomials of the individual maximal sequences. This computation becomes somewhat cumbersome; however, it does provide a means of comparing the distributions of correlation values for various pairs of sequences of the same length. It also provides a way of comparing the distribution with a binomial distribution of correlation values, which would be the distribution for a $p = 0.5$ Bernoulli sequence.

The correlation function for periodic waveforms can be described in terms of the Fourier series coefficients. Thus the next major area of investigation here was concerned with the Fourier series calculation for linear maximal binary sequences and the frequency domain description of the cross-correlation function. The calculation of the Fourier series coefficients, for linear maximal sequences in the characteristic phase position, is found to be greatly simplified by making use of the cyclotomic cosets. It is also found that the Fourier series calculations for the set of maximal sequences of a given length, in the characteristic phase position, lead to coefficients with phase angles that are related in a well defined manner. That is, the cyclic coset assignment noted by Golomb carries over into the assignment of phase angles to the Fourier coefficients. The correlation function, both full period and partial period, has been expressed in terms of the Fourier coefficients. The results, although complex, provide another technique for examining the structure of the correlation function and how it is modified as the correlation interval is reduced.

The correlation ambiguity matrix has been introduced as a technique for displaying the partial period correlation function. This display is useful for comparing the partial period correlation function with the full period correlation and provides information which is necessary for the design and analysis of systems which must use partial period correlation. It shows graphically that the peaks and valleys of the partial period correlation occur for the same values of relative delay, τ, as the peaks and valleys of the full period correlation function. The diagonal characteristic of the full period correlation function carries over to the partial period case although it becomes less pronounced for small values of k. The difference matrix, in which the elements are calculated by finding the difference between the elements of the correlation ambiguity matrix and the mean value along the diagonal, has also been introduced. This matrix shows graphically how the correlation function is modified as the correlation interval is reduced. The correlation ambiguity matrix shows the complexity of the partial period correlation function and thus indicates the problems involved for a system which is forced to operate in an environment in which full period correlation is impossible.

The purpose of this study was to examine the properties of the partial period correlation function for linear maximal sequences of the same length. The goal in this research has been to find techniques for specifying the partial period cross-correlation properties as a function of the particular sequences involved.

The moment calculation, Fourier series analysis, and correlation ambiguity matrix description are all sequence dependent techniques which can be used in the study of the partial period correlation function. These three techniques provide different ways of viewing the effect of partial period correlation. The correlation ambiguity matrix presents the entire correlation function; however, for long sequences, it becomes too large to be practical in the form described here. The Fourier series approach requires the calculation of the phase angles for the Fourier coefficients. The techniques described here reduce the number of calculations necessary by using the cyclotomic cosets and the coset assignments. The moment calculation depends on the calculation of polynomials divisible by the characteristic polynomial. For extremely long sequences and moderate values of k the moment calculation provides the only practical technique, of the three, for examining the correlation function.

While no convenient general method was found by which the correlation properties can be specified, these techniques provide methods of examining the correlation properties in specific cases. The results of this study should form a foundation on which further studies of the partial period correlation can be built.

7.2 Future Work

Future studies of the partial period correlation properties could go in a number of directions. The moment structure of the distribution of correlation values for sequences of the same length has been specified; however, the results could be extended to include the correlation of sequences with different periods. These results would be of interest in the analysis of systems operating in an environment containing signals modulated by sequences of different lengths. The correlation ambiguity matrix study could be extended by a search for more efficient display techniques. As noted previously the correlation ambiguity matrix, as described in Chapter VI, is impractical for long sequences; however, more efficient display techniques could extend the usefulness of this approach.

APPENDIX A

CALCULATION OF $F(j, p, k)$ AND N_j FOR MOMENT EQUATIONS

In this appendix the calculations necessary to find the coefficients, $F(j, p, k)$, for the moment equations of Chapter IV are reviewed, along with the calculation of the number, N_j, of j-term polynomials, of degree less than L, which are divisible by the characteristic polynomials of maximal sequences. While N_j is not directly involved in the moment equations, it is a bound on the number of polynomials, B_{xj}, for values of k less than L.

A.1 Calculation of $F(j, p, k)$

The coefficients used in the moment equations of Chapter IV are given by Lindholm for $p \leq 5$, however some comments on the calculation of these coefficients seem appropriate. As noted previously the coefficients $F(j, p, k)$ arise in the expansion of the pth order summation, where each subscript takes on each of the k possible values as shown in Eq. 4.13, into a series of jth order summations over terms in which the subscripts are all different. This expansion is accomplished by considering the number of ways in which the product of $p - j$ terms in the summation will be equal to 1. The coefficient $F(j, p, k)$ is just the number of ways in which $p - j$

elements out of the p elements in the product are equal to 1 times the number of permutations of the remaining subscripts. In the following discussion of the coefficients it is assumed that $k \geq p$. If this is not the case then it will be necessary to eliminate some of the terms.

The coefficient $F(p, p, k)$ is the number of permutations of a set of p different subscripts in Eq. 4.13. The number of such permutations for $k \geq p$ is $p!$ and hence the value of the coefficient $F(p, p, k)$ is always $p!$. Note that if $p \geq k$ it is not possible to choose p subscripts which are different and therefore this term will be equal to zero for $p \geq k$.

The next coefficient is $F(p-2, p, k)$ and in order to find the value of this coefficient it is necessary to count the number of ways in which either two or three of the subscripts are identical, leading to a sum over the product of $p-2$ terms with different subscripts. Note that if three subscripts are identical the product is the term with the same subscript. The coefficient of interest here is the coefficient for the summation over the product of $p-2$ terms in which the subscripts are different, i.e.,

$$\sum_{i_{p-2}=0}^{k-(p-2)} \; \sum_{i_{p-3}=i_{p-2}+1}^{k-(p-3)} \cdots \sum_{i_2=i_3+1}^{k-2} \; \sum_{i_1=i_2+1}^{k-1} v_{i_{p-2}+s} \, v_{i_{p-3}+s} \cdots v_{i_1+s} \tag{A.1}$$

The number of terms in this summation is $\binom{k}{p-2}$ and the simplest way to find the coefficient for summations of this type is to count the total number of ways in which 2 or 3 subscripts will be identical and then factor out $\binom{k}{p-2}$. For two subscripts identical there are k ways to choose the subscripts, $\binom{p}{2}$ positions for the two subscripts, $\binom{k-1}{p-2}$ ways to choose the remaining subscripts and $(p-2)!$ permutations of the remaining subscripts. For three subscripts the same we again have k choices of subscript, $\binom{p}{3}$ positions for the subscripts, $\binom{k-1}{p-3}$ ways to choose the remaining subscripts and $(p-3)!$ permutations of the remaining subscripts. The total number of ways in which the product of p terms over all possible subscripts will reduce to the product of p-2 terms with different subscripts is:

$$\binom{k}{p-2} F(p-2, p, k) = k\binom{p}{2}\binom{k-1}{p-2}(p-2)! + k\binom{p}{3}\binom{k-1}{p-3}(p-3)!$$

$$= \frac{p!}{2!}\,\frac{k!}{(p-2)!\,[k-1-(p-2)]\,!} + \frac{p!}{3!}\,\frac{k!}{(p-3)!\,[k-1-(p-3)]\,!} \qquad \text{A.2)}$$

Therefore the coefficient becomes

$$F(p-2, p, k) = \frac{p!}{2!}(k-p+2) + \frac{p!}{3!}(p-2)$$

$$= \frac{p!}{3!}(3k-2p+4) \qquad \text{(A.3)}$$

For the coefficient $F(p-4, p, k)$ it is necessary to determine the number of ways in which the product of four terms is equal to 1. This will be possible if all four subscripts are identical, the subscripts are identical in pairs, five subscripts are identical, or there are two identical subscripts plus three identical subscripts. Proceeding as above it can be shown that the coefficient reduces to

$$F(p-4, p, k) = \frac{p!}{5!} \left[15 k^2 + (70 - 20p) k + (5p - 9)(p - 4)\right] \quad (A.4)$$

Continuing in this fashion the coefficients can be found for higher order moments however the equations above are adequate for calculating the moments for $p \leq 5$ and should be sufficient for most purposes.

A.2 Calculation of N_j

It was noted in Chapter IV that the number of polynomials of the form $X^d + X^c + 1$ for $d \leq L$ will be equal to $(L-1)/2$ for all maximal sequences. Thus for the maximal sequences the number of three term polynomials of degree less than L increases linearly with L, and the number becomes quite large for long sequences. Since it is necessary to know the number of j term polynomials in order to calculate the moments, the number of such polynomials of degree less than L will provide some information on the amount of work necessary to calculate the higher order moments. Let N_j be the number of polynomials of degree less than L containing j terms and having the form $X^d + X^c + \ldots + X^a + 1$ from which B_{xj} can be calculated. Thus, from before, $N_3 = (L-1)/2$. As noted previously the generating matrix for the linear maximal sequence contains all possible non-zero n-tuple columns. The search for polynomials of the form given above therefore becomes a search for subsets of j columns for which the vector sum is equal to zero. For the polynomial of the form given above the first column of the generating matrix is always an element of the subset. For $j = 4$ there are $\binom{L-1}{2}$ ways to choose the next two columns to be added to the first column and each set of three columns will either be equal to zero or be equal to some fourth column. Since there are $(L-1)/2$ ways to choose trinomials in this way, there will be

$$\binom{L-1}{2} - \left(\frac{L-1}{2}\right) \tag{A.5}$$

ways in which the columns can be chosen to give a fourth column. Since each four term polynomial will appear in three different ways using this technique the number of four term polynomials is

$$N_4 = \frac{1}{3}\left[\left(\frac{L-1}{2}\right) - \left(\frac{L-1}{2}\right)\right] = \frac{(L-1)(L-3)}{3!} \tag{A.6}$$

Thus the number of four term polynomials of degree less than L for linear maximal sequences increases as the square of the length of the sequence and becomes quite large very rapidly. For example the characteristic polynomial for a 7 stage maximal sequence will divide 2604 four term polynomials of the form $X^d + X^c + X^b + 1$ with $d < L$.

For N_5 the calculation becomes much more complex. In this case the subsets of five columns of the generating matrix which add to zero must be found. As above the first column is always used and in this case the next three columns may be chosen in $\binom{L-1}{3}$ ways. In this case the sum of the four columns chosen is either zero, equal to one of the four columns chosen (if a trinomial relationship exists for three of the columns), or equal to one of the other columns of the matrix. N_5 is the number of times the last case occurs. The number of non-zero columns is given by

$$\binom{L-1}{3} - N_4 \qquad \text{(A.7)}$$

We must determine how many of this set of non-zero sums exhibit a trinomial relationship. The $'N_3$ trinomials which contain zero will occur $L-3$ times (since each trinomial will occur with each of the other columns in the set of all possible subsets of four columns) and each trinomial relationship which does not contain zero will occur once. Since B_3 is the total number of trinomial relationships of degree less than L the number of such columns, i.e., relationships not containing zero, will be $B_3 - N_3$. In calculating the polynomials in this way each polynomial will be repeated four times and thus the number of 5 term polynomials of degree less than L which are divisible by the characteristic polynomial of a linear maximal sequence will be

$$N_5 = \frac{1}{4}\left[\binom{L-1}{3} - N_4 - (L-3)\,N_3 - (B_3 - N_3)\right]$$

$$= \frac{1}{4}\left[\frac{(L-1)(L-2)(L-3)}{3!} - \frac{(L-1)(L-3)}{3!} - \frac{(L-4)(L-1)}{2} - B_3\right]$$

$$= \frac{(L-1)(L^2-9L+21)}{4!} - \frac{B_3}{4} \qquad \text{(A.8)}$$

Again the calculations can be made for the higher order polynomials however the results above should be adequate for practical purposes. It should be noted that these results are for the number of polynomials divisible by the maximal sequence and not the number of polynomials divisible by both maximal sequences. No general statements of this type can be made about the number of polynomials divisible by the characteristic polynomial for the equivalent 2n-stage generator.

APPENDIX B

EXAMPLE MATRICES

In this appendix additional examples of the correlation ambiguity matrix and the difference matrix are given. Although the matrices in this appendix are presented without the shading used for the examples in Chapter VI, the diagonal structure of the matrices is still apparent. For all the example matrices, both in Chapter VI and this appendix, the reference sequence, $\{x_i\}$, is the sequence generated by the 5-stage simple shift register generator with feedback connections [5, 2, 0] .

B.1 Correlation Ambiguity Matrices (Figs. B.1 through B.13)

From the results of Gold (Refs. 6 and 36) it can be shown that the full period cross-correlation between the reference sequence and the sequence [5, 4, 2, 1, 0] has peak values which are equal to or less than the peak values for any other pair of sequences of the same length. The correlation ambiguity matrices for this pair of sequences are given for several values of k . (Including the examples of Chapter VI the values of k are 31, 30, 25, 20, 15, 11, and 10.) The tabulation of full period cross-correlation functions (Ref. 36) shows that the correlation between the reference sequence and the sequence [5, 3, 0] has the maximum peak values. Therefore the correlation ambiguity matrices for this pair of sequences are given for the same values of k. The correlation ambiguity matrices for the cross-correlation between the reference sequence and each of the other 5-stage maximal sequences, i.e., [5, 4, 3, 2, 0], [5, 3, 2, 1, 0] and [5, 4, 3, 1, 0], are given for k = 31 and k = 20. The final example correlation ambiguity matrix is the autocorrelation case for k = 15.

B.2 Correlation Difference Matrix (Figs. B.14 through B.27)

The correlation difference matrices are included for the same values of k (except k = 31) as the correlation ambiguity matrices. For k = L the difference matrix is the zero matrix and, therefore, is not included. Two difference matrices for the autocorrelation case are given (Figs. B.26 and B.27) and it can be seen that these matrices are symmetrical about the principal diagonal which has value zero.

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-8	-2	0	0	0	8	-2	-8	-2	8	6	6	-2	8	-10	-2	0	0	8	-10	6	6	6	-2	0	-8	6	-2	-8	-2	0
1	0	-10	0	0	0	0	6	0	-10	0	6	6	6	0	6	-10	0	0	0	6	-10	6	6	6	0	0	-10	6	0	-10	0
2	0	-2	-8	0	0	0	-2	8	-2	-8	-2	6	6	8	-2	6	-8	0	0	-2	6	-10	6	6	8	0	-2	-10	8	-2	-8
3	-8	-2	0	-8	0	0	-2	0	6	0	-10	-2	6	8	6	-2	8	-8	0	-2	-2	6	-10	6	8	8	-2	-2	-8	6	0
4	-2	-8	-2	-2	-10	-2	0	-2	0	6	0	-8	0	6	8	8	-2	6	-10	0	0	0	8	-8	6	6	8	0	-2	-8	6
5	8	-2	-8	0	0	-8	-2	0	-2	0	6	-2	-10	0	6	6	8	0	8	-10	-2	-2	-2	6	-8	8	6	6	0	-2	-8
6	-8	6	0	-8	0	0	-10	0	-2	0	-2	6	-2	-8	-2	6	8	8	0	6	-10	-2	-2	-2	8	-8	6	6	8	-2	0
7	-2	-8	6	-2	-10	-2	0	-10	0	-2	0	0	8	-2	-8	0	6	6	6	0	8	-8	0	0	-2	6	-8	8	6	8	-2
8	-2	0	-10	6	-2	-10	0	-2	-8	-2	0	0	0	6	0	-8	-2	6	6	8	0	8	-8	0	-2	-2	8	-8	6	8	6
9	8	-2	0	-8	8	0	-10	0	-2	-8	-2	-2	-2	0	6	-2	-8	0	8	6	6	-2	6	-10	0	0	-2	6	-8	6	8
10	8	6	0	0	-8	8	-2	-8	-2	0	-10	-2	-2	0	-2	6	0	-8	0	6	6	6	-2	6	-8	0	-2	-2	8	-10	8
11	8	6	8	0	0	-8	6	0	-10	0	-2	-10	-2	0	-2	-2	8	0	-8	-2	6	6	6	-2	8	-8	-2	-2	0	6	-8
12	-10	8	6	6	-2	-2	-8	6	0	-10	0	0	-8	-2	0	0	-2	6	-2	-8	0	8	8	8	-2	6	-8	0	-2	0	6
13	6	-8	6	6	6	-2	0	-10	8	-2	-8	0	0	-10	0	0	-2	-2	6	0	-8	0	8	8	6	-2	8	-8	-2	0	-2
14	-2	8	-10	6	6	6	0	-2	-8	6	0	-8	0	-2	-8	0	-2	-2	-2	8	0	-8	0	8	6	6	0	8	-10	0	-2
15	-2	0	6	-10	6	6	8	-2	0	-10	8	0	-8	-2	0	-8	-2	-2	-2	0	8	0	-8	0	6	6	8	0	6	-8	-2
16	0	-2	0	8	-8	8	6	8	-2	0	-10	6	-2	-8	-2	-2	-8	0	0	-2	-2	6	-2	-10	0	8	6	6	0	6	-8
17	-8	-2	0	0	8	-8	6	8	6	0	-2	-10	6	0	-10	-2	0	-8	0	-2	-2	-2	6	-2	-8	0	6	6	8	-2	8
18	6	-8	-2	-2	-2	6	-8	6	8	6	0	0	-8	6	0	-8	-2	-2	-10	0	0	0	0	8	-2	-10	0	8	6	8	-2
19	0	6	-8	0	0	0	6	-8	6	8	6	-2	-2	-8	6	-2	-8	0	0	-10	-2	-2	-2	-2	8	0	-10	-2	8	6	8
20	6	0	6	-10	-2	-2	0	6	-8	6	8	8	0	-2	-8	8	-2	-10	-2	0	-8	0	0	0	-2	6	0	-8	-2	8	6
21	8	6	0	8	-8	0	-2	0	6	-8	6	6	6	0	-2	-10	8	0	-8	-2	-2	-10	-2	-2	0	0	6	-2	-8	-2	8
22	6	8	6	-2	6	-10	0	-2	0	6	-8	8	8	6	0	0	-10	6	-2	-8	0	0	-8	0	-2	-2	0	8	-2	-8	-2
23	-2	8	6	6	-2	6	-8	-2	0	-2	8	-8	8	6	8	0	-2	-10	6	0	-8	0	0	-8	-2	-2	0	0	6	0	-10
24	-8	-2	8	8	8	0	6	-8	-2	0	-2	6	-10	8	6	6	0	0	-8	6	-2	-10	-2	-2	-8	0	-2	-2	0	6	0
25	-2	-8	-2	6	6	6	0	6	-8	-2	0	0	8	-10	8	8	6	-2	-2	-8	8	0	-8	0	-2	-10	0	0	-2	0	6
26	6	0	-10	-2	6	6	8	-2	8	-10	0	0	0	6	-8	8	6	6	-2	0	-8	8	0	-8	-2	-2	-8	0	-2	0	-2
27	-2	8	-2	-10	-2	6	8	6	0	6	-8	0	0	-2	8	-8	6	6	6	0	0	-8	8	0	-10	-2	0	-8	-2	0	-2
28	0	-2	8	0	-8	0	6	8	6	0	6	-10	-2	0	-2	6	-8	8	8	6	-2	-2	-10	6	0	-8	-2	-2	-8	-2	0
29	-2	0	-2	6	-2	-10	0	6	8	6	0	8	-8	-2	0	0	6	-10	6	8	8	0	0	-8	6	-2	-8	0	-2	-8	-2
30	0	-2	0	0	8	0	-10	0	6	8	6	-2	6	-8	-2	-2	0	8	-8	6	6	6	-2	-2	-8	8	-2	-10	0	-2	-8

Fig. B.1. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 30$

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-3	-3	-1	3	-3	9	-1	-11	-1	3	9	3	-1	7	-9	-3	3	1	7	-7	5	5	3	-3	1	-5	7	-1	-7	-3	-3
1	-1	-5	-3	1	3	-1	9	-1	-11	1	3	9	1	1	5	-9	-3	3	1	7	-9	3	5	3	-1	1	-7	5	-1	-9	-1
2	1	-3	-5	-1	1	5	-1	9	-1	-9	1	3	7	3	-1	5	-9	-3	3	1	5	-11	3	5	5	-1	-1	-9	5	-3	-7
3	-7	1	-1	-5	1	1	3	1	7	-1	-11	-1	3	7	3	-3	7	-7	-1	1	1	5	-13	1	5	7	-1	-1	-7	5	-3
4	-3	-7	-1	-1	-7	1	3	1	3	7	1	-9	-1	3	7	5	-5	5	-9	1	1	1	7	-11	1	3	7	-1	-3	-7	5
5	5	-3	-5	-1	1	-7	-1	5	-1	3	5	-1	-9	-1	3	5	7	-3	7	-11	1	1	-1	5	-11	3	3	7	1	-3	-7
6	-7	5	-1	-5	1	1	-9	1	3	-1	1	3	-1	-9	-1	1	7	9	-1	5	-11	1	-1	-3	5	-9	3	3	9	1	-3
7	-5	-5	5	-3	-5	-1	1	-9	1	1	-1	1	5	-3	-7	-1	1	7	9	-1	7	-9	1	-1	-5	5	-7	5	3	11	-1
8	-3	-3	-5	3	-3	-7	-1	1	-9	-1	1	-1	3	3	-1	-7	-1	1	7	9	1	9	-9	1	-3	-5	7	-5	5	5	9
9	9	-3	-1	-5	5	-3	-9	1	-1	-9	-3	-1	-1	3	3	-3	-5	1	3	5	9	1	7	-11	1	-1	-5	7	-3	5	5
10	7	7	-3	1	-5	7	-3	-9	1	1	-9	-3	-3	1	1	3	-3	-5	1	3	3	7	1	7	-9	1	-3	-7	7	-5	7
11	7	7	9	-3	3	-5	5	-1	-11	1	-1	-11	-3	-3	1	-1	5	-1	-3	-1	3	3	5	-1	7	-7	1	-3	-5	7	-5
12	-7	9	7	7	-3	1	-5	5	-1	-13	1	-1	-9	-5	-1	1	-1	5	-1	-3	1	5	3	5	-3	7	-5	3	-3	-3	5
13	5	-7	7	7	5	-3	3	-7	7	-1	-11	3	-1	-9	-5	1	-1	-3	3	1	-3	1	7	5	5	-5	7	-5	1	-3	-3
14	-3	5	-9	7	5	5	-1	1	-5	7	1	-9	3	-1	-9	-3	-1	-3	-5	5	1	-3	3	9	5	3	-5	7	-7	1	-3
15	-5	-1	5	-11	7	3	5	-1	1	-7	7	1	-7	1	1	-9	-3	-1	-3	-5	7	3	-3	3	7	5	5	-3	7	-5	-1
16	-1	-5	1	5	-9	7	1	7	-3	1	-9	5	1	-7	1	-1	-7	-1	1	-5	-5	7	1	-5	3	9	5	5	-1	7	-5
17	-5	-1	-3	1	7	-9	5	3	5	-3	-1	-11	5	1	-7	-1	1	-5	1	-1	-5	-5	5	-1	-5	5	9	5	7	-1	7
18	7	-5	-3	-3	-1	7	-7	3	5	5	-1	1	-11	5	1	-5	-3	-1	-7	3	-1	-5	-3	7	-1	-7	5	9	3	7	-1
19	1	5	-5	-1	-3	1	7	-7	3	7	5	-1	-1	-5	3	1	-5	-3	-1	-7	1	-3	-5	-3	9	-1	-9	3	9	1	9
20	9	1	3	-5	-3	-3	3	5	-5	3	9	7	-1	-1	-9	5	-1	-7	-5	1	-7	1	-1	-3	-3	7	-1	-9	1	9	1
21	3	7	1	5	-5	-1	-3	3	5	-3	3	9	5	1	-3	-9	5	-1	-7	-5	-1	-9	1	-1	-1	-3	5	-3	-9	-1	11
22	9	5	7	-1	5	-7	-1	-3	3	3	-3	3	11	3	3	-3	-9	5	-1	-7	-3	1	-9	1	-3	-1	-1	7	-3	-7	-3
23	-5	11	5	5	-1	3	-7	-1	-3	1	3	-3	5	9	5	3	-3	-9	5	-1	-5	-1	1	-9	-1	-3	1	1	7	-1	-9
24	-7	-7	11	7	5	1	3	-7	-1	-1	1	3	-5	7	7	5	3	-3	-9	5	-3	-7	-1	1	-7	-1	-5	-1	1	5	1
25	-1	-5	-7	9	7	3	1	3	-7	-3	-1	1	5	-7	9	7	5	3	-3	-9	7	-1	-7	-1	-1	-7	1	-3	-1	3	3
26	3	-1	-7	-7	7	7	5	-1	5	-7	-1	1	1	5	-7	11	5	3	1	-1	-9	7	1	-5	-1	-3	-7	1	-5	-1	3
27	1	5	-1	-9	-7	5	7	5	-1	3	-7	-1	3	-1	7	-7	11	5	3	1	1	-7	7	1	-7	-1	-1	-5	1	-3	-3
28	-1	-1	5	1	-9	-5	5	7	5	1	3	-7	-3	5	-3	7	-7	11	5	3	-1	-1	-7	7	3	-7	-3	-3	-5	-1	-1
29	-1	-1	-3	5	-1	-9	-3	3	9	5	3	5	-7	-3	5	-1	5	-9	9	7	3	-1	1	-5	7	1	-7	-3	-5	-5	-1
30	-1	-1	1	-3	7	-1	-11	-1	1	9	3	1	5	-7	-3	3	1	7	-7	7	7	3	-3	-1	-5	9	1	-7	-1	-5	-5

Fig. B.2. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 25$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	1	1	-3	-1	-3	5	-1	-5	-1	9	5	1	1	1	-1	-5	1	5	3	-3	3	3	-3	-1	-1	-5	3	1	-1	-3	-1
1	-1	-1	3	-3	-1	-1	5	1	-7	-1	7	3	-1	3	1	-3	-5	3	5	3	-3	1	3	-3	1	1	-7	3	1	-1	-3
2	-3	-3	1	3	-3	1	-1	7	-1	-7	-3	5	1	1	3	-1	-3	-3	3	5	3	-5	1	3	-1	3	-1	-7	3	1	-1
3	-1	-5	-1	1	3	-1	1	1	5	-1	-9	-5	3	3	1	1	-1	-1	-3	3	5	1	-5	1	5	1	1	-1	-7	3	1
4	-1	-1	-5	-3	-1	3	1	1	1	3	-1	-9	-5	3	5	1	-1	-1	-3	-1	5	5	3	-3	1	5	1	3	-3	-5	1
5	3	-1	-1	-3	-1	-1	1	1	1	3	1	1	-9	-5	1	5	3	-1	1	-5	-3	5	3	1	-3	1	5	-1	5	-5	-3
6	-1	3	-1	1	-1	-1	-3	1	1	3	3	3	-1	-9	-7	1	7	3	1	-1	-7	-3	3	1	1	-3	1	3	1	3	-3
7	-5	-1	3	-3	-1	-1	1	-3	1	-1	3	3	3	-1	-7	-7	-1	7	1	3	1	-7	-1	5	1	1	-3	3	1	3	1
8	1	-3	-3	3	-3	-3	-1	-1	-1	1	1	5	5	1	-1	-5	-7	-3	7	1	3	3	-7	-1	3	-1	3	-3	3	1	3
9	3	-1	-1	-3	3	-1	-3	1	-3	-1	-1	-1	3	7	1	-3	-5	-5	-3	7	1	1	3	-7	1	5	-3	3	-3	3	1
10	3	3	-1	1	-1	3	-3	-3	1	-1	-1	-1	-1	3	5	1	-1	-5	-3	-5	5	1	-1	1	-7	1	5	-5	5	-5	5
11	5	1	5	-1	1	1	3	-1	-5	1	-3	-3	-3	1	3	3	1	1	-5	-3	-5	3	1	-1	3	-5	-1	5	-5	5	-5
12	-7	5	1	3	-3	1	3	3	-1	-7	1	-3	-3	-3	3	3	1	1	-1	-3	-1	-5	5	3	-1	3	-5	1	3	-3	3
13	3	-5	3	1	3	-5	1	1	5	-1	-5	3	-1	-5	-5	5	3	-1	1	-1	-3	1	-5	5	1	-3	5	-5	1	3	-3
14	-5	3	-5	1	-1	3	-3	1	1	3	-1	5	3	-1	-3	-3	3	3	-3	3	1	-3	3	-3	5	1	-3	7	-7	3	1
15	-1	-5	3	-7	-1	-1	5	-3	1	-1	3	-1	-5	3	1	-3	-5	3	1	-1	5	1	-1	5	-3	5	1	-1	5	-5	1
16	1	-3	-3	3	-7	1	-1	7	-5	1	-3	1	-3	-3	3	-1	-3	-3	3	1	-1	3	1	-1	7	-1	3	1	-1	5	-5
17	-3	1	-3	-1	5	-7	-1	-1	7	-3	1	-3	1	-3	-5	3	1	-3	-1	1	-1	-1	1	-1	-1	7	-1	1	3	-3	7
18	5	-3	1	-5	-3	5	-5	-1	-1	5	-1	1	-3	1	-1	-5	1	1	-5	1	3	-1	1	3	-1	-1	7	1	-1	5	-5
19	-3	5	-3	3	-3	-3	3	-5	-1	1	5	-3	1	-3	-1	-1	-3	1	3	-7	-1	3	-3	-1	3	-1	-1	5	3	-3	7
20	7	-1	3	-3	3	-5	-3	1	-3	-1	3	7	-1	-1	-3	1	-1	-5	1	3	-7	1	3	-3	-3	1	1	-1	5	3	-3
21	-1	7	-1	5	-1	3	-7	-3	1	-1	-1	3	7	-1	-3	-3	3	-1	-3	-1	1	-7	-1	1	-3	-3	1	-1	1	3	5
22	5	1	5	-1	5	-3	3	-9	-1	1	1	1	5	5	-1	-1	-3	1	-1	-3	-1	3	-7	-1	-1	-5	-1	1	-1	1	3
23	3	7	-1	5	-1	3	-3	1	-7	-1	3	3	3	3	5	1	-1	-5	1	-1	-3	1	3	-7	-3	-3	-3	-1	1	-1	1
24	1	1	5	-1	5	1	3	-1	-1	-7	-3	1	1	5	3	3	1	1	-5	1	-1	-5	1	3	-5	-1	-5	-3	-1	1	-1
25	-1	3	-1	9	-1	3	1	1	1	-1	-5	-1	3	-1	5	5	3	-1	1	-5	1	1	-5	1	1	-7	1	-5	-3	-1	1
26	1	1	1	-1	9	-3	3	-1	3	1	1	-3	1	1	-1	7	5	1	-1	1	-5	3	1	-5	-1	-1	-5	1	-3	-3	-1
27	-3	1	1	-1	-5	9	-1	3	-1	1	1	1	-3	1	3	-1	5	5	-1	1	3	-5	5	3	-5	-1	-1	-3	-1	-3	-5
28	-5	-5	3	1	-1	-1	9	1	1	-1	-1	-1	-1	-1	1	1	-1	7	5	-1	3	1	-5	5	5	-3	-3	-1	-3	-1	-3
29	-5	-3	-7	3	1	-3	-1	7	3	1	1	1	1	-3	-1	3	1	-3	7	5	-1	3	1	-5	3	3	-1	-3	-1	-3	-1
30	1	-3	-3	-5	5	1	-5	-1	7	5	1	1	1	1	-5	-1	5	1	-1	5	3	-1	1	-1	-5	3	3	-3	-1	-3	-1

Fig. B.3. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 11$

ε_y \ ε_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	10	8	6	-10	6	2	-8	2	8	2	4	-4	-8	-6	4	0	6	-10	2	4	4	-4	-4	0	-10	2	-4	0	2	0	-2
1	0	10	8	8	-8	8	2	-8	2	8	2	2	-6	-8	-6	2	0	8	-8	2	2	2	-6	-6	0	-8	2	-6	0	2	0
2	-2	0	10	6	6	-10	8	2	-8	2	8	4	4	-6	-8	-4	2	-2	6	-8	4	4	4	-4	-6	-2	-8	4	-6	0	2
3	2	0	-2	10	6	6	-8	6	4	-10	4	8	4	2	-4	-8	-6	2	-2	8	-8	4	4	4	-6	-6	0	-8	2	-4	-2
4	0	2	0	0	12	8	6	-8	6	4	-10	2	6	4	2	-6	-8	-4	4	-2	6	-10	2	2	4	-4	-6	-2	-8	2	-4
5	-6	0	2	-2	-2	10	8	6	-8	6	4	-8	4	6	4	4	-6	-10	-6	4	0	8	-8	4	2	2	-4	-4	-2	-8	2
6	4	-6	0	4	0	0	10	8	6	-8	6	2	-10	4	6	2	4	-4	-8	-6	2	-2	6	-10	4	4	2	-6	-4	-2	-8
7	-8	2	-4	0	4	0	-2	12	6	8	-10	6	2	-8	2	6	4	4	-4	-10	-6	2	-2	6	-8	4	2	2	-4	-6	0
8	-2	-8	2	-6	-2	2	0	-2	12	6	8	-8	8	2	-8	4	6	2	2	-4	-8	-4	4	0	6	-10	4	4	2	-4	-6
9	-6	0	-10	2	-6	-2	4	-2	0	10	8	8	-8	6	4	-8	2	6	2	4	-4	-8	-4	4	-2	6	-8	4	2	4	-6
10	-4	-6	0	-8	4	-4	-2	4	-2	0	10	6	6	-8	6	2	-8	4	8	2	2	-6	-10	-6	4	0	6	-10	4	2	4
11	4	-6	-4	0	-8	4	-6	0	2	0	-2	10	6	8	-10	6	4	-8	4	6	2	2	-6	-10	-4	4	-2	6	-8	2	4
12	4	2	-4	-4	0	-8	2	-4	-2	4	-2	-2	10	8	6	-10	8	4	-8	2	6	2	2	-6	-8	-4	2	-2	8	-10	4
13	4	2	4	-4	-4	0	-10	4	-6	0	2	-2	-2	12	6	6	-8	8	4	-10	2	6	2	2	-4	-8	-6	2	0	6	-8
14	-8	2	4	4	-4	-4	-2	-8	2	-4	-2	2	-2	0	10	6	8	-8	8	2	-10	2	6	2	4	-4	-10	-6	4	-2	8
15	6	-8	2	2	2	-6	-4	-2	-8	2	-4	0	4	-2	0	12	6	6	-10	8	4	-8	4	8	2	2	-4	-8	-6	4	-2
16	-2	8	-10	2	2	2	-4	-6	0	-10	4	-4	0	2	0	0	10	6	6	-8	8	4	-8	4	6	2	4	-4	-10	-4	2
17	2	0	6	-10	2	2	4	-6	-4	-2	-8	4	-4	-2	4	0	-2	10	6	8	-8	8	4	-8	2	6	4	4	-6	-8	-6
18	-4	2	0	8	-8	4	2	4	-6	-4	-2	-10	2	-4	-2	2	0	0	12	6	6	-10	6	2	-8	4	6	2	4	-6	-8
19	-8	-6	4	0	8	-8	2	4	2	-4	-6	-2	-10	4	-6	-2	4	0	0	10	6	6	-10	6	4	-8	2	6	4	2	-4
20	-6	-8	-6	2	-2	6	-8	2	4	2	-4	-4	0	-10	4	-4	-2	2	-2	0	12	8	8	-8	6	2	-8	4	6	4	2
21	4	-6	-8	-4	4	0	6	-8	2	4	2	-6	-6	0	-10	2	-4	0	4	-2	-2	10	6	6	-8	8	2	-10	4	6	4
22	4	2	-4	-8	-4	4	-2	8	-10	4	2	2	-6	-4	-2	-10	4	-4	0	2	-2	-2	10	6	8	-8	6	2	-8	2	8
23	8	2	4	-4	-8	-4	2	0	6	-8	2	2	2	-4	-6	-2	-8	4	-4	-2	2	-2	-2	10	8	8	-10	6	4	-10	4
24	2	8	2	2	-6	-10	-4	2	0	6	-8	4	4	2	-4	-4	-2	-10	2	-4	0	4	0	0	10	6	8	-8	6	4	-10
25	-8	2	8	4	4	-4	-10	-4	2	0	6	-10	2	4	2	-6	-4	0	-8	2	-6	-2	2	-2	0	12	6	6	-8	6	4
26	2	-8	2	6	2	2	-4	-10	-4	2	0	8	-8	2	4	4	-6	-6	-2	-8	4	-4	0	4	-2	-2	12	8	6	-8	6
27	8	2	-8	4	8	4	2	-4	-10	-4	2	-2	6	-8	2	2	4	-4	-4	-2	-10	2	-6	-2	4	0	-2	10	8	6	-8
28	-10	8	2	-10	2	6	4	2	-4	-10	-4	4	0	6	-8	4	2	2	-6	-4	0	-8	4	-4	-2	2	0	0	10	8	6
29	6	-8	6	2	-10	2	8	2	4	-6	-8	-4	4	-2	8	-8	2	2	2	-4	-4	0	-8	4	-6	-2	4	0	-2	12	6
30	6	8	-10	6	2	-10	4	6	4	2	-4	-8	-4	2	0	8	-10	2	2	4	-4	-4	0	-8	2	-6	0	4	-2	0	10

Fig. B.4. Correlation ambiguity matrix; $\{x_i\} \Rrightarrow [5, 2, 0]$, $\{y_i\} \Rrightarrow [5, 3, 0]$, $k = 30$

$\theta_y \backslash \theta_x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	9	9	3	-9	5	-3	-9	1	7	3	1	-1	-9	-1	3	1	7	-7	-1	1	5	-3	-1	1	-7	3	-5	-1	5	1	-3
1	-1	7	9	5	-9	7	-3	-9	1	9	3	1	-3	-7	-3	3	1	7	-7	-1	-1	3	-3	-1	3	-7	1	-7	-1	3	3
2	1	1	7	7	5	-11	7	-3	-9	-1	9	3	3	-5	-5	-3	3	1	7	-7	1	1	3	-3	-3	3	-5	3	-7	1	1
3	1	1	-1	7	5	5	-9	5	-1	-9	1	11	3	3	-5	-3	-5	1	-1	9	-7	1	3	5	-3	-5	3	-5	1	-7	1
4	3	-1	1	1	7	7	5	-9	5	1	-9	1	9	5	1	-5	-3	-5	1	-1	7	-9	1	3	7	-3	-7	1	-5	-1	-5
5	-5	3	-3	1	-1	7	9	3	-7	5	3	-7	1	9	5	3	-7	-5	-7	3	-1	7	-7	3	3	5	-3	-7	-1	-5	-1
6	1	-7	3	-1	1	1	7	9	3	-5	5	3	-9	3	7	5	3	-7	-5	-7	1	-3	7	-7	5	3	3	-5	-7	-3	-3
7	-3	1	-5	3	1	1	-1	9	7	3	-7	3	3	-9	3	5	7	5	-5	-7	-7	1	-5	5	-7	7	3	3	-3	-7	-3
8	-3	-3	[illegible]	-5	1	1	3	-3	11	7	9	-5	3	3	-9	5	3	5	3	-3	-7	-7	3	-3	5	-9	7	3	1	-3	-7
9	-7	-3	-5	-1	-7	1	3	1	-1	11	9	7	-5	3	3	-7	3	1	3	5	-3	-7	-5	5	-3	3	-9	7	1	1	-3
10	-3	-7	-1	-5	1	-7	-1	5	-1	-1	9	7	7	-5	3	1	-5	5	3	1	5	-3	-9	-7	5	-1	3	-9	9	1	1
11	3	-5	-7	1	-5	3	-7	-1	5	1	-1	9	5	9	-7	3	1	-5	5	3	-1	3	-3	-9	-5	5	-3	1	-9	7	3
12	3	3	-3	-7	3	-5	1	-5	-3	5	-1	-3	9	5	9	-9	5	3	-3	3	3	-1	1	-5	-9	-3	5	-3	3	-9	7
13	7	3	5	-3	-5	3	-7	3	-7	-3	3	-3	-3	9	5	7	-7	7	5	-5	3	3	-3	-1	-5	-7	-3	5	-1	3	-9
14	-7	5	3	7	-3	-3	3	-7	3	-5	-3	3	-5	-1	7	5	7	-7	7	5	-7	1	3	-3	1	-5	-9	-5	5	-3	5
15	5	-7	3	3	5	-3	-1	1	-5	3	-3	-1	3	-5	-1	9	3	5	-9	9	5	-7	3	5	-3	-1	-5	-9	-7	5	-3
16	-5	7	-7	1	3	3	-3	-1	1	-7	3	-3	1	1	-3	-1	9	3	5	-9	11	7	-7	3	3	-3	1	-3	-9	-5	3
17	1	-3	7	-9	1	1	3	-3	-1	-1	-7	3	-1	-1	3	-3	-1	9	3	5	-7	13	7	-7	1	3	-1	3	-3	-7	-7
18	-7	1	-1	7	-7	1	-1	5	-5	-1	-3	-9	3	-1	-1	1	-1	1	11	1	5	-7	11	5	-7	3	3	-1	5	-3	-7
19	-7	-7	3	-1	9	-7	-1	1	3	-5	-3	-5	-9	3	-1	-3	3	1	3	9	1	5	-9	9	5	-5	3	3	1	5	-3
20	-5	-5	-7	1	-1	7	-7	-1	1	1	-5	-3	-3	-11	5	-1	-3	3	1	3	11	3	5	-9	7	5	-3	5	3	3	3
21	5	-7	-5	-5	1	1	7	-7	-1	3	1	-5	-5	-1	-13	5	-1	-3	3	1	1	9	3	5	-7	7	3	-5	5	[illegible]	5
22	7	3	-7	-3	-5	3	1	7	-7	1	3	1	-7	-3	-3	-13	5	-1	-3	3	-1	-1	9	3	7	-7	5	1	-5	3	3
23	5	5	3	-5	-3	-3	3	1	7	-5	1	3	-1	-5	-5	-3	-13	5	-1	-3	1	-3	-1	9	5	7	-9	3	1	-7	5
24	3	7	5	1	-5	-5	-3	3	1	5	-5	1	5	-3	-3	-5	-3	-13	5	-1	-1	3	-3	-1	7	5	9	-7	3	3	-9
25	-9	3	9	5	3	-5	-7	-1	1	1	3	-7	1	5	-3	-5	-3	-1	-11	3	-1	-1	1	-5	-1	9	5	9	-5	3	3
26	3	-9	1	9	3	3	-3	-9	1	1	3	5	-7	1	5	-1	-7	-5	-3	-9	3	-1	1	3	-5	-3	9	5	7	-5	3
27	3	3	-7	1	11	3	1	-1	-11	1	-1	1	5	-7	1	3	1	-5	-3	-5	-9	3	-3	-1	3	-3	-3	9	7	7	-5
28	-7	5	3	-9	1	9	3	1	-1	-13	1	-1	3	3	-5	1	3	1	-5	-3	-3	-7	3	-3	-3	3	-1	-1	9	9	5
29	3	-5	5	[illegible]	-9	-1	9	3	1	-3	-13	1	1	1	5	-5	1	3	1	-5	-1	-1	-7	3	-5	-3	5	[illegible]	[illegible]	[illegible]	7
30	7	3	[illegible]	[illegible]	-1	-9	1	7	[illegible]	1	-1	-11	1	1	1	7	-7	-1	1	[illegible]	-5	-1	1	-5	3	-7	[illegible]	5	[illegible]	[illegible]	[illegible]

Fig. B.5. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 25$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	5	1	1	-5	5	1	-5	-1	3	1	1	1	-3	1	-1	3	5	-3	-5	1	-1	-1	5	-1	-9	3	3	-3	3	-3	-1
1	1	5	1	3	-3	5	-1	-5	-1	5	1	1	1	-3	-1	-1	5	5	-1	-7	-1	-1	-3	3	-1	-9	3	1	-1	1	-1
2	-1	3	3	1	3	-5	5	-3	-3	-1	7	3	3	-1	-3	1	-1	3	5	-1	-7	1	-1	-3	1	-3	-7	3	1	-1	1
3	1	1	1	3	1	1	-5	3	-1	-3	1	9	5	1	-1	-1	1	-3	3	5	-1	-5	1	-1	-5	-1	-1	-7	3	1	-1
4	-1	-1	3	1	3	3	1	-3	1	-1	-5	-1	7	7	1	-3	-1	3	-3	3	5	-3	-5	1	1	-3	-3	-1	-7	3	1
5	-1	-1	-1	1	-1	3	5	1	-3	-1	-1	-5	-1	7	9	1	-5	-1	1	-1	5	5	-1	-3	1	1	-3	-1	-3	-5	1
6	1	-3	1	-1	1	1	3	7	-1	-3	-3	-3	-7	1	7	7	1	-3	-1	1	-1	3	5	-1	-1	3	-1	-3	-1	-3	-5
7	-3	1	-3	3	1	1	-1	3	7	1	-3	-3	-3	-7	-1	7	9	1	-1	-3	-1	-1	1	3	-1	-1	3	-3	-1	-3	-1
8	-1	-1	-1	-3	3	-1	1	-3	5	7	3	-1	-1	-5	-7	1	7	7	1	-1	-3	1	-1	1	1	-3	1	3	-3	-1	-3
9	-5	-1	-1	-3	-5	3	1	1	-3	3	7	3	-1	-1	-3	-7	-1	7	5	3	1	-3	3	1	1	1	-3	3	1	-1	-3
10	-1	-5	-1	1	-1	-5	1	1	1	-1	3	7	3	-1	-3	-3	-5	-1	9	3	1	1	-5	1	1	1	1	-5	5	-1	1
11	3	-1	-5	1	3	-1	-7	1	1	3	-1	3	7	3	-3	-3	-1	-5	1	7	1	1	-1	-7	1	1	1	-1	-3	3	1
12	3	3	-1	-3	3	3	-3	-7	1	3	3	-1	3	7	1	-3	-1	-1	-3	-1	5	1	-1	-3	-7	1	1	-1	1	-5	5
13	5	1	5	-1	-3	5	3	-1	-9	1	1	1	-3	5	7	-1	-3	1	-1	-3	-1	3	1	-1	-1	-5	-1	1	-1	1	-5
14	-3	5	1	7	1	-3	3	3	-1	-7	1	1	1	-3	3	7	1	-3	3	-3	-5	-1	1	-1	-1	-1	-5	-3	3	-3	3
15	1	-3	5	-1	5	1	-1	3	3	-3	-7	1	1	1	-1	3	5	1	-5	5	-1	-5	1	3	-1	-1	-1	-3	-5	5	-5
16	-5	3	-5	5	-1	3	1	-3	5	3	-1	-5	3	-1	1	1	3	3	1	-5	5	1	-5	1	1	-3	1	-1	-3	-5	5
17	3	-5	3	-7	3	-1	5	1	-3	3	3	-1	-5	3	1	1	-1	3	1	3	-3	5	3	-3	1	1	-3	3	-3	-1	-7
18	-7	1	-3	3	-7	5	-1	7	-1	-3	1	1	-3	-3	3	-1	1	1	3	1	3	-5	5	3	-1	3	-1	-3	3	-3	-1
19	-1	-9	3	-3	3	-5	5	1	5	-1	-5	-1	-1	-1	-3	1	-1	3	1	3	1	1	-5	5	5	1	1	-1	-3	3	-3
20	-5	-1	-9	1	-5	3	-3	5	1	3	-1	-5	-1	-1	1	-3	-1	-1	1	3	5	1	3	-3	5	5	1	3	-3	-1	1
21	3	-5	-1	-7	3	-5	1	-3	5	3	3	-1	-5	-1	-3	1	-1	-1	1	-1	1	5	-1	1	-3	5	5	-1	5	-5	1
22	1	1	-3	-1	-7	5	-5	3	-5	5	1	1	-3	-3	-1	-5	1	1	-1	1	-1	-1	5	-1	3	-1	3	5	-1	5	-5
23	-5	-1	3	-3	-1	-5	5	-3	1	-5	3	-1	-1	-1	-3	-3	-5	3	1	-1	1	-3	-1	5	1	5	-3	3	5	-1	5
24	5	-3	-3	3	-3	-3	-5	3	-1	1	-3	5	1	-3	-1	-1	-3	-7	3	1	-1	3	-3	-1	3	-1	7	-3	3	5	-1
25	-1	3	-1	-3	3	-1	-3	-3	1	-1	-1	-5	3	3	-3	-3	-1	-1	-7	3	1	-3	3	-3	1	5	-3	7	-3	3	5
26	5	1	1	-1	-3	1	-1	-5	-1	1	1	1	-3	1	3	-1	-3	-3	-1	-7	3	3	-3	3	-5	-1	7	-3	7	-3	3
27	5	5	1	3	1	-3	-1	-1	-5	1	1	1	1	-3	-1	3	1	-3	-1	-3	-9	3	1	-5	3	-5	-1	5	-1	5	-1
28	-3	5	5	-1	1	1	-1	-1	-1	-7	1	1	1	1	-1	-1	1	1	-5	1	-1	-9	5	3	-5	3	-5	1	3	1	3
29	1	-3	5	3	-3	1	3	-1	-1	-3	-7	1	1	1	3	-1	-3	1	-1	-3	3	-1	-7	7	3	-5	3	-3	-1	5	-1
30	-1	3	-5	5	3	-5	1	1	1	-1	-1	-5	3	-1	1	5	-1	-5	1	-1	-3	5	-1	-7	5	1	-3	3	-3	-1	5

Fig. B.6. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 11$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1
1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1
2	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7
3	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1
4	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7
5	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7
6	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1
7	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1
8	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7
9	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7
10	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9
11	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7
12	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9
13	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1
14	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1
15	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1
16	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7
17	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7
18	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1
19	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7
20	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9	-9
21	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1	-9
22	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7	-1
23	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1	7
24	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9	-1
25	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1	-9
26	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1	-1
27	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1	-1
28	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1	-1
29	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9	-1
30	-1	-1	-1	-1	-9	-1	7	-1	-9	-9	7	-1	7	7	-1	-1	-1	-9	7	-9	7	7	-1	-1	7	7	-1	7	-1	-1	-9

Fig. B.7. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 3, 2, 0]$, k = 31

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-2	-2	0	-2	2	-2	2	8	-2	-8	-6	4	-4	0	6	-4	0	-6	-6	10	-8	4	4	0	0	6	2	-4	4	-2	4
1	4	-4	-2	0	0	4	-4	2	8	-2	-8	-6	2	-2	0	6	-2	0	-4	-8	10	-10	2	2	2	0	4	2	-2	4	-2
2	-4	4	-6	-4	0	0	4	-6	4	6	0	-6	-6	2	0	2	6	-4	0	-4	-6	10	-10	2	2	0	0	6	2	0	2
3	0	-4	2	-6	-4	0	0	2	-4	2	8	2	-6	-6	4	2	2	4	-4	0	-2	-6	10	-10	2	0	0	2	6	4	-2
4	0	0	-2	4	-8	-4	0	2	0	-2	0	6	2	-6	-8	2	2	4	4	-4	-2	-2	-6	10	-10	4	0	-2	2	4	6
5	6	2	0	-2	2	-10	-2	0	2	0	-2	0	8	0	-6	-8	0	2	2	6	-4	0	0	-4	8	-10	6	0	-4	2	4
6	4	8	2	0	-4	0	-8	-2	0	2	0	-2	2	6	0	-6	-10	0	0	4	6	-2	2	2	-6	8	-8	6	-2	-4	2
7	2	2	8	2	2	-2	-2	-8	-2	0	2	0	-4	4	6	0	-4	-10	2	-2	4	4	-4	0	4	-6	6	-8	8	-2	-4
8	-2	2	4	10	2	2	-2	0	-10	0	-2	0	0	-4	2	4	0	-2	-10	2	-4	4	4	-4	0	6	-6	4	-8	6	0
9	0	0	2	4	8	0	4	-2	0	-10	0	-2	2	-2	-4	2	2	0	-4	-8	2	-2	6	6	-6	0	8	-6	2	-8	6
10	6	2	0	2	2	6	2	4	-2	0	-10	0	0	0	-2	-4	0	2	-2	-2	-8	4	0	8	4	-6	2	8	-8	2	-8
11	-8	8	2	0	0	0	8	2	4	-2	0	-10	2	-2	0	-2	-6	0	0	0	-2	-6	6	2	6	4	-4	2	6	-8	2
12	2	-6	8	2	-2	-2	2	8	2	4	-2	0	-8	0	-2	0	-4	-6	-2	2	0	0	-4	8	0	6	6	-4	0	6	-8
13	-8	0	-6	8	4	0	-4	2	8	2	4	-2	-2	-6	0	-2	2	-4	-4	-4	2	-2	-2	-6	10	0	4	6	-2	0	6
14	4	-8	-2	-8	8	4	0	-6	4	6	4	6	-2	-2	-4	2	-2	0	-4	-4	-2	2	-2	-2	-6	8	0	6	6	0	-2
15	0	4	-6	0	-8	8	4	2	-8	6	4	2	6	-2	-4	-6	2	0	0	-4	-6	-2	2	-2	-2	-4	8	-2	6	4	2
16	2	-2	4	-6	2	-6	6	4	2	-8	6	4	0	8	-2	-4	-4	2	2	-2	-4	-8	-4	0	0	-2	-6	8	0	6	4
17	2	2	-4	2	-6	2	-6	4	6	0	-6	8	4	0	10	0	-4	-6	2	2	0	-4	-8	-4	0	-2	-2	-4	8	2	4
18	6	2	4	-2	2	-6	2	-4	2	8	-2	-8	8	4	-2	8	0	-2	-6	2	0	0	-4	-8	-4	2	-2	-4	-4	6	4
19	4	8	2	4	-4	0	-4	2	-4	2	8	-2	-6	6	4	-2	6	0	-4	-4	2	2	2	-2	-10	-4	4	-2	-6	-4	6
20	6	2	8	2	6	-2	-2	-4	2	-4	2	8	-4	-4	6	4	0	6	2	-6	-4	0	0	0	0	-10	-6	4	0	-6	-4
21	-6	6	0	6	2	6	-2	-4	-2	0	-2	4	8	-4	-2	8	4	-2	6	2	-4	-4	0	0	0	-2	-10	-4	4	2	-8
22	-10	-6	4	-2	6	2	6	-4	-2	-4	2	0	4	8	-2	0	8	2	-2	6	4	-4	-4	0	0	-2	-2	-8	-4	6	0
23	-2	-10	-8	2	-2	6	2	4	-2	-4	-2	4	0	4	10	0	0	6	2	-2	8	4	-4	-4	0	-2	-2	0	-8	-2	4
24	6	-2	-8	-6	2	-2	6	4	2	0	-6	-4	4	0	2	8	0	2	6	2	-4	8	4	-4	-4	2	-2	-4	0	-10	0
25	0	4	-2	-8	-4	4	-4	6	4	2	0	-6	-6	6	0	2	10	0	4	4	2	-6	6	2	-2	-4	0	-2	-2	0	-10
26	-8	0	6	0	-8	-4	4	-2	4	6	0	-2	-6	-6	4	-2	2	12	0	4	2	2	-6	6	2	0	-4	-2	-2	-4	2
27	2	-6	0	6	-2	-10	-2	4	-2	4	6	0	0	-8	-6	4	-4	2	10	2	4	4	4	-4	4	2	2	-4	-4	-2	-4
28	-4	0	-6	0	8	0	-12	-2	4	-2	4	6	-2	2	-8	-6	6	-4	4	8	2	2	2	2	-2	4	0	2	-2	-4	-2
29	0	-4	2	-4	0	8	0	-10	-4	6	-4	2	6	-2	0	-10	-6	8	-4	4	6	2	2	2	2	0	4	-2	2	-4	-2
30	-2	-2	-4	2	-2	2	6	0	-10	-4	6	-4	0	8	-2	0	-8	-6	10	-6	4	4	0	0	4	2	-2	4	0	2	-4

Fig. B.8. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 3, 2, 0]$, $k = 20$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9
1	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9
2	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7
3	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9
4	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1
5	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7
6	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7
7	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9
8	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1
9	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1
10	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1
11	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7
12	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1
13	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7
14	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1
15	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9
16	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7
17	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1
18	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7
19	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1
20	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1	-1
21	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1	-1
22	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7	-1
23	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7	7
24	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1	7
25	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1	-1
26	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7	-1
27	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1	7
28	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1	-1
29	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9	-1
30	-1	-1	7	-1	-1	7	7	-1	-1	-1	-1	7	-1	7	-9	-1	7	-1	7	-1	-1	-1	-9	7	7	-1	-9	7	-9	-9	-9

Fig. B.9. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 2, 1, 0]$, $k = 31$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-10	-2	0	2	-2	2	6	4	-2	0	-2	0	4	-4	6	-4	-4	2	2	10	0	0	4	-4	8	2	-2	-8	4	-6	0
1	0	-12	-2	0	4	0	0	6	4	-2	0	-2	-2	6	-4	6	-2	-4	4	0	10	-2	-2	2	-2	8	0	-2	-6	4	-6
2	-6	-2	-12	-2	2	6	-2	0	6	4	-2	0	-4	0	6	-4	8	-2	-2	2	0	8	-4	-4	4	-2	6	0	0	-6	4
3	4	-8	-2	-12	0	4	4	-2	0	6	4	-2	-2	-2	0	6	-2	8	0	-4	2	-2	6	-6	-2	4	-4	6	2	0	-6
4	-6	2	-8	-2	-10	2	2	4	-2	0	6	4	-4	0	-2	0	8	-2	10	-2	-4	0	-4	4	-4	-2	2	-4	8	2	0
5	2	-6	4	-6	-2	-10	2	4	2	0	-2	4	4	-4	-2	-4	0	10	-2	10	-4	-4	0	-4	4	-2	-2	0	-4	6	4
6	4	4	-6	4	-8	-4	-8	2	4	2	0	-2	6	2	-4	-2	-6	0	8	0	10	-2	-2	2	-6	4	0	-2	-2	-4	6
7	6	2	4	-6	6	-6	-6	-8	2	4	2	0	-4	8	2	-4	0	-6	2	6	0	8	-4	-4	4	-6	2	0	0	-2	-4
8	-4	4	2	4	-4	8	-8	-6	-8	2	4	2	-2	-2	8	2	-2	0	-4	0	6	-2	6	-6	-2	4	-8	2	2	0	-2
9	0	-4	6	4	4	-4	8	-6	-8	-6	0	2	2	-2	-4	6	2	0	0	-4	-2	6	-2	6	-6	0	4	-10	2	0	2
10	0	0	-6	4	4	4	-4	6	-4	-10	-4	2	2	2	0	-2	6	0	0	0	-2	-2	6	-2	6	-8	0	6	-10	4	-2
11	-4	0	-2	-8	4	4	4	-6	8	-6	-8	-2	2	2	4	2	-2	4	0	0	2	-2	-2	6	-2	4	-8	2	6	-8	2
12	2	-2	0	-2	-10	2	6	4	-6	8	-6	-8	0	0	2	4	0	-2	2	2	0	4	0	0	4	-2	6	-8	0	6	-8
13	-6	2	0	2	-2	-10	2	8	2	-4	6	-8	-8	0	-2	0	4	2	-2	2	0	0	4	0	0	6	-2	4	-8	-2	8
14	8	-4	2	0	0	-4	-8	2	8	2	-4	6	-6	-10	0	-2	-2	4	0	0	2	2	2	6	-2	0	8	-2	2	-8	-2
15	0	8	-2	4	0	0	-4	-6	0	10	0	-6	6	-6	-12	-2	-2	0	4	0	-2	2	2	2	6	0	0	6	-2	0	-6
16	-6	2	8	-2	2	-2	2	-4	-6	0	10	0	-4	4	-6	-12	-4	-2	-2	6	0	0	4	4	0	6	2	0	4	-2	0
17	0	-8	2	8	0	4	-4	2	-4	-6	0	10	-2	-2	4	-6	-10	-4	0	-4	6	-2	-2	2	6	0	4	2	2	4	-2
18	0	0	-6	4	8	0	4	-2	0	-2	-8	-2	10	-2	-4	2	-6	-8	-4	0	-6	6	-2	-2	2	8	0	2	2	0	6
19	6	2	0	-6	2	6	2	4	-2	0	-2	-8	0	8	-2	-4	0	-6	-10	-2	0	-4	8	0	-4	2	10	0	0	2	0
20	2	6	4	2	-6	2	6	4	2	0	-2	-4	-8	0	6	-4	-4	2	-6	-10	-4	0	-4	8	0	-2	2	8	0	-2	4
21	2	2	4	2	2	-6	2	4	6	0	2	0	-4	-8	2	8	-4	-6	2	-6	-8	-4	0	-4	8	-2	-2	4	8	2	-4
22	-6	2	0	2	2	2	-6	0	6	4	2	4	0	-4	-6	4	8	-6	-6	2	-4	-8	-4	0	-4	6	-2	0	4	10	0
23	-2	-6	0	-2	2	2	2	-8	2	4	6	4	4	0	-2	-4	4	6	-6	-6	4	-4	-8	-4	0	-6	6	0	0	6	8
24	8	0	-6	0	-4	0	4	2	-8	2	4	6	6	2	0	-2	-6	4	4	-4	-6	6	-2	-6	-6	0	-4	6	-2	0	6
25	6	6	0	-6	2	-2	-2	4	2	-8	2	4	4	8	2	0	0	-6	6	2	-4	-8	4	-4	-4	-6	-2	-4	8	-2	0
26	2	6	8	2	-6	2	-2	0	2	4	-10	0	4	4	6	0	0	2	-6	6	0	-4	-8	4	-4	-2	-6	-4	-4	6	0
27	-2	2	4	6	2	-6	2	-4	2	0	6	-8	0	4	6	8	0	-2	2	-6	8	0	-4	-8	4	-6	-2	-4	-4	-2	4
28	4	0	2	4	4	0	-4	2	-4	2	0	6	-6	-2	4	6	6	0	-4	4	-6	10	2	-2	-10	4	-4	-2	-6	-4	-2
29	0	4	2	4	4	4	0	-2	0	-2	0	-2	6	-6	-4	2	6	8	0	-4	2	-6	10	2	-2	-8	4	-6	-2	-8	-2
30	-4	0	2	0	4	4	4	-2	0	-2	0	2	-2	6	-4	-2	2	4	8	0	-2	2	-6	10	2	-4	-8	6	-6	0	-10

Fig. B.10. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 2, 1, 0]$, $k = 20$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7
1	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7
2	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1
3	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7
4	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1
5	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1
6	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1
7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7
8	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1
9	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1
10	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7
11	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1
12	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7
13	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1
14	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9
15	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7
16	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1
17	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1
18	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1
19	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1
20	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7	7
21	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9	7
22	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1	-9
23	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1	-1
24	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7	-1
25	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9	7
26	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1	-9
27	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9	-1
28	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9	-9
29	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9	-9
30	-9	-9	-1	-9	7	-1	-1	-9	7	7	-1	-1	-1	-1	7	-9	-1	7	-1	7	-1	-1	7	-1	-1	-1	7	-1	7	7	-9

Fig. B.11. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 3, 1, 0]$, $k = 31$

$s_y \backslash s_x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-6	-6	-8	2	-10	6	2	0	-6	4	6	0	0	0	-2	0	-4	2	6	-6	8	0	-4	0	0	6	-2	4	0	6	0
1	0	-8	-6	-8	4	-8	4	2	0	-6	4	6	-2	2	0	-2	2	-4	4	4	-6	6	-2	-6	2	0	4	-2	6	0	6
2	8	0	-6	-4	-8	4	-8	6	0	2	-8	2	6	-2	0	-2	-2	4	-4	4	2	-6	6	-2	-6	4	0	2	-2	4	2
3	4	8	2	-4	-4	-8	4	-6	4	2	0	-10	2	6	-4	-2	-2	0	4	-4	2	2	-6	6	-2	-4	4	-2	2	-4	6
4	6	6	8	2	-6	-6	-6	4	-6	4	2	0	-8	0	6	-4	-4	-2	-2	6	-4	4	4	-4	4	-2	-2	4	-4	2	-4
5	-4	4	6	8	4	-4	-8	-6	4	-6	4	2	-2	-6	0	6	-2	-4	0	-4	6	-6	2	2	-2	4	-4	-2	6	-4	2
6	0	-4	2	4	8	4	-4	-10	-4	2	-4	6	2	-2	-4	2	6	-4	-4	0	-2	6	-6	2	2	-4	4	-2	-2	8	-6
7	-6	2	-4	2	2	6	6	-4	-10	-4	2	-4	8	0	-2	-4	0	6	-6	-2	0	0	8	-4	0	2	-2	4	-4	-2	8
8	8	-4	2	-4	0	0	8	6	-4	-10	-4	2	-2	6	0	-2	-6	0	4	-4	-2	2	2	10	-6	0	4	-2	2	-4	-2
9	-2	6	-4	2	-2	2	-2	8	6	-4	-10	-4	0	0	6	0	0	-6	2	2	-4	-4	0	0	12	-6	-2	4	0	2	-4
10	-4	0	6	-4	0	-4	4	-2	8	6	-4	-10	-2	-2	0	6	-2	0	-8	4	2	-2	-2	2	-2	12	-4	-2	2	0	2
11	2	-2	0	6	-6	-2	-2	4	-2	8	6	-4	-8	-4	-2	0	4	-2	-2	-6	4	4	0	0	0	-2	14	-4	-4	2	0
12	2	2	0	2	6	-6	-2	0	2	0	6	4	-4	-8	-6	-4	0	6	-2	-2	-8	4	4	0	0	2	-2	12	-4	-6	4
13	4	4	2	0	0	4	-4	-2	0	2	0	6	6	-6	-8	-6	-6	0	4	0	-2	-6	6	6	-2	0	4	-2	10	-4	-6
14	-4	4	6	4	0	0	4	-2	-4	2	0	-2	6	6	-8	-10	-6	-4	0	4	-2	-2	-6	6	6	0	0	2	-2	8	-2
15	-4	-4	2	4	4	0	0	2	0	-6	4	2	-2	6	8	-6	-10	-8	-4	0	6	-2	-2	-6	6	4	0	2	2	0	6
16	8	-4	-2	4	4	4	0	2	0	2	-8	2	2	-2	4	6	-6	-8	-8	-4	-2	6	-2	-2	-6	8	4	-2	2	0	2
17	2	6	-4	-2	6	6	2	0	2	0	2	-8	0	4	-2	4	8	-6	-6	-10	-4	-4	4	-4	0	-6	6	4	0	2	0
18	-2	2	4	-6	-2	6	6	0	2	0	2	4	-8	0	6	0	4	6	-6	-6	-8	-4	-4	4	-4	-2	-6	8	4	2	0
19	-2	-2	0	2	-6	-2	6	4	2	0	2	4	4	-8	2	8	0	2	6	-6	-4	-8	-4	-4	4	-6	-2	-4	8	6	0
20	0	0	-2	0	0	-8	0	6	4	2	0	2	6	2	-8	2	6	0	0	8	-6	-2	-6	-2	-6	4	-4	-2	-6	8	6
21	4	0	-2	-4	0	0	-8	-2	8	2	4	2	2	6	4	-6	2	4	0	0	10	-6	-2	-6	-2	-8	4	-2	-2	-4	6
22	6	2	0	-2	-2	2	-2	-8	-2	8	2	4	0	4	6	4	-4	2	6	-2	0	8	-8	-4	-4	-2	-10	4	0	-2	-4
23	-2	6	4	2	-2	-2	2	0	-10	0	6	0	4	0	2	4	4	-2	2	6	-4	0	8	-8	-4	-2	-2	-12	4	-2	0
24	0	-4	6	4	4	0	-4	2	0	-10	0	6	-2	6	0	2	6	4	0	0	6	-6	-2	6	-6	-4	-4	-2	-10	4	-2
25	-2	2	-4	6	2	2	2	-4	2	0	-10	0	8	-4	6	0	0	6	2	2	0	8	-4	0	4	-6	-2	-4	-4	-10	4
26	2	-2	0	-6	6	2	2	0	-2	0	2	-8	0	8	-2	8	0	-2	6	2	4	0	8	-4	0	2	-6	0	-4	-2	-12
27	-10	2	0	2	-6	6	2	4	-2	0	-2	0	-8	0	6	-4	8	2	-2	6	0	4	0	8	-4	2	2	-8	0	-6	0
28	2	-10	4	2	2	-6	6	4	2	0	-2	-4	0	-8	-2	4	-4	10	2	-2	4	0	4	0	8	-2	2	0	-8	-2	-4
29	-6	2	-12	2	2	2	-6	4	6	0	2	0	-4	0	-6	0	4	-6	10	2	0	4	0	4	0	6	-2	4	0	-6	-4
30	-4	-8	2	-12	4	4	0	-6	4	6	0	2	-2	-2	0	-6	2	4	-4	8	2	-2	2	-2	6	0	4	-2	6	0	-6

Fig. B.12. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 3, 1, 0]$, k = 20

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	15	-3	1	-3	3	1	-3	-1	-1	3	-3	5	1	1	1	3	-5	-3	-3	-5	-3	3	-1	-3	-5	-1	3	3	1	3	-3
1	-3	15	-1	3	-3	3	-1	-3	-3	1	3	-5	3	3	-1	1	5	-5	-1	-3	-5	-3	1	-1	-3	-3	-3	1	3	1	3
2	1	-1	15	-1	1	-5	3	-3	-3	-3	3	3	-5	3	3	1	1	3	-5	1	-1	-3	-3	3	-3	-3	-3	-3	-1	5	-1
3	-3	3	-1	15	-3	-1	-5	1	-3	-3	-1	3	3	-5	3	5	1	-1	3	-3	3	1	-3	-1	1	-3	-3	-3	-5	1	3
4	3	-3	1	-3	15	-3	1	-5	3	-5	-3	1	5	1	-3	3	3	1	-3	3	-3	3	3	-3	-1	-1	-1	-1	-3	-5	1
5	1	3	-5	-1	-3	15	-1	1	-3	1	-5	-1	3	3	3	-3	1	3	-1	-3	3	-3	5	3	-3	-3	1	1	-1	-3	-5
6	-3	-1	3	-5	1	-1	15	1	1	-3	-1	-5	-1	3	3	1	-3	3	3	-3	-5	1	-3	3	5	-3	-3	1	3	-3	-1
7	-1	-3	-3	1	-5	1	1	15	3	-1	-3	1	-3	-3	5	3	-1	-3	1	3	-3	-5	3	-3	3	3	-1	-1	1	3	-3
8	-1	-3	-3	-3	3	-3	1	3	15	3	-3	-3	1	-3	-3	3	3	1	-3	-1	1	-5	-5	1	-1	3	3	-1	1	-1	5
9	3	1	-3	-3	-5	1	-3	-1	3	15	5	-3	-3	1	-3	-1	3	1	1	-1	1	3	-5	-3	-1	-1	3	3	-3	3	-3
10	-3	3	3	-1	-3	-5	-1	-3	-3	5	15	3	-5	-1	-1	-3	1	3	3	1	-1	1	1	-5	-3	1	-3	1	3	-3	3
11	5	-5	3	3	1	-1	-5	1	-3	-3	3	15	3	-5	-1	-3	-3	3	3	1	-1	-3	1	-1	-3	-3	1	-3	3	1	-1
12	1	3	-5	3	5	3	-1	-3	1	-3	-5	3	15	3	-5	-3	-3	-1	3	1	-1	-3	-3	-1	1	-3	-3	1	-1	1	3
13	1	3	3	-5	1	3	3	-3	-3	1	-1	-5	3	15	3	-3	-3	-5	-1	5	3	1	-3	-1	-3	1	-3	-3	-1	1	-1
14	1	-1	3	3	-3	3	3	5	-3	-3	-1	-1	-5	3	15	1	-3	-1	-5	-3	3	1	1	-5	1	-3	1	-3	-1	-3	3
15	3	1	1	5	3	-3	1	3	3	-1	-3	-3	-3	-3	1	15	3	-3	1	-5	-3	3	-1	1	-5	3	-5	-1	-3	-1	-3
16	-5	5	1	1	3	1	-3	-1	3	3	1	-3	-3	-3	-3	3	15	1	-3	3	-3	-1	3	1	-1	-5	3	-5	-3	-1	-3
17	-3	-5	3	-1	1	3	3	-3	1	1	3	3	-1	-5	-1	-3	1	15	-1	-3	3	-3	1	3	1	-3	-3	5	-5	-3	-1
18	-3	-1	-5	3	-3	-1	3	1	-3	1	3	3	3	-1	-5	1	-3	-1	15	1	-1	5	-3	3	1	1	-3	-3	3	-3	-5
19	-5	-3	1	-3	3	-3	-3	3	-1	-1	1	1	1	5	-3	-5	3	-3	1	15	1	-1	3	-3	3	3	-1	-5	-3	3	-3
20	-3	-5	-1	3	-3	3	-5	-3	1	1	-1	-1	-1	3	3	-3	-3	3	-1	1	15	1	-3	3	-3	5	1	-3	-5	-3	3
21	3	-3	-3	1	3	-3	1	-5	-5	3	1	-3	-3	1	1	3	-1	-3	5	-1	1	15	-1	-3	3	-1	3	-1	-3	-5	-3
22	-1	1	-3	-3	3	5	-3	3	-5	-5	1	1	-3	-3	1	-1	3	1	-3	3	-3	-1	15	-3	-1	3	-1	3	1	-5	-3
23	-3	-1	3	-1	-3	3	3	-3	1	-3	-5	-1	-1	-1	-5	1	1	3	3	-3	3	-3	-3	15	-3	1	1	-3	3	1	-5
24	-5	-3	-3	1	-1	-3	5	3	-1	-1	-3	-3	1	-3	1	-5	-1	1	1	3	-3	3	-1	-3	15	-5	3	3	-3	3	1
25	-1	-3	-3	-3	-1	-3	-3	3	3	-1	1	-3	-3	1	-3	3	-5	-3	1	3	5	-1	3	1	-5	15	-5	3	1	-1	1
26	3	-3	-3	-3	-1	1	-3	-1	3	3	-3	1	-3	-3	1	-5	3	-3	-3	-1	1	3	-1	1	3	-5	15	-5	5	-1	1
27	3	1	-3	-3	-1	1	1	-1	-1	3	1	-3	1	-3	-3	-1	-5	5	-3	-5	-3	-1	3	-3	3	3	-5	15	-3	3	1
28	1	3	-1	-5	-3	-1	3	1	1	-3	3	3	-1	-1	-1	-3	-3	-5	3	-3	-5	-3	1	3	-3	1	5	-3	15	-3	3
29	3	1	5	1	-5	-3	-3	3	-1	3	-3	1	1	1	-3	-1	-1	-3	-3	3	-3	-5	-5	1	3	-1	-1	3	-3	15	-3
30	-3	3	-1	3	1	-5	-1	-3	5	-3	3	-1	3	-1	3	-3	-3	-1	-5	-3	3	-3	-3	-5	1	1	1	1	3	-3	15

Fig. B.13. Correlation ambiguity matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 2, 0]$, $k = 15$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	22	-32	30	30	30	38	-32	22	-32	38	-24	-24	-32	38	-40	-32	30	30	38	-40	-24	-24	-24	-32	30	22	-24	-32	22	-32	30
1	30	-40	30	30	30	30	-24	30	-40	30	-24	-24	-24	30	-24	-40	30	30	30	-24	-40	-24	-24	-24	30	30	-40	-24	30	-40	30
2	30	-32	22	30	30	30	-32	38	-32	22	-32	-24	-24	38	-32	-24	22	30	30	-32	-24	-40	-24	-24	38	30	-32	-40	38	-32	22
3	22	-32	30	22	30	30	-32	30	-24	30	-40	-32	-24	38	-24	-32	38	22	30	-32	-32	-24	-40	-24	38	38	-32	-32	22	-24	30
4	-32	22	-32	-32	-40	-32	30	-32	30	-24	30	22	30	-24	38	38	-32	-24	-40	30	30	30	38	22	-24	-24	38	30	-32	22	-24
5	38	-32	22	30	30	22	-32	30	-32	30	-24	-32	-40	30	-24	-24	38	30	38	-40	-32	-32	-32	-24	22	38	-24	-24	30	-32	22
6	22	-24	30	22	30	30	-40	30	-32	30	-32	-24	-32	22	-32	-24	38	38	30	-24	-40	-32	-32	-32	38	22	-24	-24	38	-32	30
7	-32	22	-24	-32	-40	-32	30	-40	30	-32	30	30	38	-32	22	30	-24	-24	-24	30	38	22	30	30	-32	-24	22	38	-24	38	-32
8	-32	30	-40	-24	-32	-40	30	-32	22	-32	30	30	30	-24	30	22	-32	-24	-24	38	30	38	22	30	-32	-32	38	22	-24	38	-24
9	38	-32	30	22	38	30	-40	30	-32	22	-32	-32	-32	30	-24	-32	22	30	38	-24	-24	-32	-24	-40	30	30	-32	-24	22	-24	38
10	38	-24	30	30	22	38	-32	22	-32	30	-40	-32	-32	30	-32	-24	30	22	30	-24	-24	-24	-32	-24	22	30	-32	-32	38	-40	38
11	38	-24	38	30	30	22	-24	30	-40	30	-32	-40	-32	30	-32	-32	38	30	22	-32	-24	-24	-24	-32	38	22	-32	-32	30	-24	22
12	-40	38	-24	-24	-32	-32	22	-24	30	-40	30	30	22	-32	30	30	-32	-24	-32	22	30	38	38	38	-32	-24	22	30	-32	30	-24
13	-24	22	-24	-24	-24	-32	30	-40	38	-32	22	30	30	-40	30	30	-32	-32	-24	30	22	30	38	38	-24	-32	38	22	-32	30	-32
14	-32	38	-40	-24	-24	-24	30	-32	22	-24	30	22	30	-32	22	30	-32	-32	-32	38	30	22	30	38	-24	-24	30	38	-40	30	-32
15	-32	30	-24	-40	-24	-24	38	-32	30	-40	38	30	22	-32	30	22	-32	-32	-32	30	38	30	22	30	-24	-24	38	30	-24	22	-32
16	30	-32	30	38	22	38	-24	38	-32	30	-40	-24	-32	22	-32	-32	22	30	30	-32	-32	-24	-32	-40	30	38	-24	-24	30	-24	22
17	22	-32	30	30	38	22	-24	38	-24	30	-32	-40	-24	30	-40	-32	30	22	30	-32	-32	-32	-24	-32	22	30	-24	-24	38	-32	38
18	-24	22	-32	-32	-32	-24	22	-24	38	-24	30	30	22	-24	30	22	-32	-32	-40	30	30	30	30	38	-32	-40	30	38	-24	38	-32
19	30	-24	22	30	30	30	-24	22	-24	38	-24	-32	-32	22	-24	-32	22	30	30	-40	-32	-32	-32	-32	38	30	-40	-32	38	-24	38
20	-24	30	-24	-40	-32	-32	30	-24	22	-24	38	38	30	-32	22	38	-32	-40	-32	30	22	30	30	30	-32	-24	30	22	-32	38	-24
21	38	-24	30	38	22	30	-32	30	-24	22	-24	-24	-24	30	-32	-40	38	30	22	-32	-32	-40	-32	-32	30	30	-24	-32	22	-32	38
22	-24	38	-24	-32	-24	-40	30	-32	30	-24	22	38	38	-24	30	30	-40	-24	-32	22	30	30	22	30	-32	-32	30	38	-32	22	-32
23	-32	38	-24	-24	-32	-24	22	-32	30	-32	38	22	38	-24	38	30	-32	-40	-24	30	22	30	30	22	-32	-32	30	30	-24	30	-40
24	22	-32	38	38	38	30	-24	22	-32	30	-32	-24	-40	38	-24	-24	30	30	22	-24	-32	-40	-32	-32	22	30	-32	-32	30	-24	30
25	-32	22	-32	-24	-24	-24	30	-24	22	-32	30	30	38	-40	38	38	-24	-32	-32	22	38	30	22	30	-32	-40	30	30	-32	30	-24
26	-24	30	-40	-32	-24	-24	38	-32	38	-40	30	30	30	-24	22	38	-24	-24	-32	30	22	38	30	22	-32	-32	22	30	-32	30	-32
27	-32	38	-32	-40	-32	-24	38	-24	30	-24	22	30	30	-32	38	22	-24	-24	-24	30	30	22	38	30	-40	-32	30	22	-32	30	-32
28	30	-32	38	30	22	30	-24	38	-24	30	-24	-40	-32	30	-32	-24	22	38	38	-24	-32	-32	-40	-24	30	22	-32	-32	22	-32	30
29	-32	30	-32	-24	-32	-40	30	-24	38	-24	30	38	22	-32	30	30	-24	-40	-24	38	38	30	30	22	-24	-32	22	30	-32	22	-32
30	30	-32	30	30	38	30	-40	30	-24	38	-24	-32	-24	22	-32	-32	30	38	22	-24	-24	-24	-32	-32	22	38	-32	-40	30	-32	22

Fig. B.14. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 30$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	132	-68	-6	118	-68	104	-6	-116	-6	-82	104	-82	-6	42	-54	-68	118	56	42	8	-20	-20	-82	-68	56	70	42	-6	8	-68	-68
1	-6	70	-68	56	118	-6	104	-6	-116	56	-82	104	-144	56	-20	-54	-68	118	56	42	-54	-82	-20	-82	-6	56	8	-20	-6	-54	-6
2	56	-68	70	-6	56	180	-6	104	-6	-54	56	-82	42	-82	-6	-20	-54	-68	118	56	-20	-116	-82	-20	-20	-6	-6	-54	-20	-68	8
3	8	56	-6	70	56	56	118	56	42	-6	-116	-6	-82	42	-82	-68	42	8	-6	56	56	-20	-178	-144	-20	42	-6	-6	8	-20	-68
4	-68	8	-6	-6	8	56	118	56	118	42	56	-54	-6	-82	42	-20	-130	-20	-54	56	56	56	42	-116	-144	-82	42	-6	-68	8	-20
5	-20	-68	70	-6	56	8	-6	180	-6	118	-20	-6	-54	-6	-82	-20	42	-68	42	-116	56	56	-6	-20	-116	-82	-82	42	56	-68	8
6	8	-20	-6	70	56	56	-54	56	118	-6	56	-82	-6	-54	-6	-144	42	104	-6	-20	-116	56	-6	-68	-20	-54	-82	-82	104	56	-68
7	-130	70	-20	-68	70	-6	56	-54	56	56	-6	56	-20	-68	8	-6	-144	42	104	-6	42	-54	56	-6	-130	-20	8	-20	-82	166	-6
8	-68	-68	70	-82	-68	8	-6	56	-54	-6	56	-6	118	-82	-6	8	-6	-144	42	104	56	104	-54	56	-68	-130	42	70	-20	-20	104
9	104	-68	-6	70	-20	-68	-54	56	-6	-54	-68	-6	-6	118	-82	-68	70	56	-82	-20	104	56	42	-116	56	-6	-130	42	132	-20	-20
10	42	42	-68	56	70	42	-68	-54	56	56	-54	-68	-68	56	56	-82	-68	70	56	-82	-82	42	56	42	-54	56	-68	-192	42	70	42
11	42	42	104	-68	118	70	-20	-6	-116	56	-6	-116	-68	-68	56	-6	-20	-6	132	-6	-82	-82	-20	-6	42	8	56	-68	-130	42	70
12	8	104	42	42	-68	56	70	-20	-6	-178	56	-6	-54	-130	-6	56	-6	-20	-6	132	56	-20	-82	-20	-68	42	70	118	-68	-68	-20
13	-20	8	42	42	-20	-68	118	8	42	-6	-116	118	-6	-54	-130	56	-6	-68	-82	56	132	56	42	-20	-20	-130	42	70	56	-68	-68
14	-68	-20	-54	42	-20	-20	-6	56	70	42	56	-54	118	-6	-54	-68	-6	-68	-130	-20	56	132	118	104	-20	-82	-130	42	8	56	-68
15	-130	-6	-20	-116	42	-82	-20	-6	56	8	42	56	8	56	56	-54	-68	-6	-68	-130	42	118	132	118	42	-20	-20	-68	42	70	-6
16	-6	-130	56	-20	-54	42	-144	42	-68	56	-54	-20	56	8	56	-6	8	-6	56	-130	-130	42	56	70	118	104	-20	-20	-6	42	70
17	70	-6	-68	56	42	-54	-20	-82	-20	-68	-6	-116	-20	56	8	-6	56	70	56	-6	-130	-130	-20	-6	70	180	104	-20	42	-6	42
18	42	70	-68	-68	-6	42	8	-82	-20	-20	-6	56	-116	-20	56	70	-68	-6	8	118	-6	-130	-68	42	-6	8	180	104	-82	42	-6
19	56	-20	70	-6	-68	56	42	8	-82	42	-20	-6	-6	-54	-82	56	70	-68	-6	8	56	-68	-130	-68	104	-6	-54	118	104	-144	104
20	104	56	-82	70	-68	-68	118	-20	70	-82	104	42	-6	-6	-54	-20	-6	8	-130	56	8	56	-6	-68	-68	42	-6	-54	56	104	-144
21	-82	42	56	-20	70	-6	-68	118	-20	132	-82	104	-20	56	-68	-54	-20	-6	8	-130	-6	-54	56	-6	-6	-68	-20	-68	-54	-6	166
22	104	-20	42	-6	-20	8	-6	-68	118	-82	132	-82	166	-82	118	-68	-54	-20	-6	8	-68	56	-54	56	-68	-6	-6	42	-68	8	-68
23	-130	166	-20	-20	-6	-82	8	-6	-68	56	-82	132	-20	104	-20	118	-68	-54	-20	-6	70	-6	56	-54	-6	-68	56	56	42	-6	-54
24	8	-192	166	42	-20	56	-82	8	-6	-6	56	-82	70	42	42	-20	118	-68	-54	-20	-68	8	-6	56	8	-6	-130	-6	56	-20	56
25	-6	70	-192	104	42	-82	56	-82	8	-68	-6	56	-20	8	104	42	-20	118	-68	-54	42	-6	8	-6	-6	8	56	-68	-6	118	-82
26	-82	-6	8	-192	42	42	-20	-6	-20	8	-6	56	56	-20	8	166	-20	-82	56	-6	-54	42	56	70	-6	-68	8	56	-130	-6	118
27	56	-20	-6	-54	-192	-20	42	-20	-6	-82	8	-6	118	-6	42	8	166	-20	-82	56	56	8	42	56	8	-6	-6	70	56	-68	-68
28	-6	-6	-20	56	-54	-130	-20	42	-20	56	-82	8	-68	180	-68	42	8	166	-20	-82	-6	-6	8	42	118	8	-68	-68	70	-6	-6
29	-6	-6	-68	-20	-6	-54	-68	-82	104	-20	118	-20	8	-68	180	-6	-20	-54	104	42	-82	-6	56	70	42	56	8	-68	-130	70	-6
30	-6	-6	56	-68	42	-6	-116	-6	-144	104	-82	56	-20	8	-68	118	56	42	8	42	42	-82	-68	-6	70	104	56	8	-6	-130	70

Fig. B.15. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 25$

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	104	46	-78	46	-16	50	-78	-144	-140	112	50	-74	46	-74	42	-140	108	170	50	104	50	112	-136	46	-16	-20	-12	-16	42	-140	-78
1	-16	42	46	-78	108	46	50	-16	-144	-140	50	50	-74	46	-74	-20	-140	170	170	-12	42	-12	112	-198	108	-16	-20	-12	46	-20	-78
2	-16	-78	42	46	-16	170	46	112	-16	-144	-202	50	50	-74	46	-136	-20	-78	170	108	-74	-20	-12	50	-136	108	-16	-20	50	-16	42
3	42	-16	-16	104	46	-16	108	46	50	46	-144	-264	-12	112	-136	46	-74	-20	-16	170	108	-74	-82	-12	50	-74	46	-78	-20	50	-16
4	-16	42	-78	-78	104	46	46	108	108	-12	46	-82	-202	-74	174	-136	-16	-74	-82	-16	170	108	-12	-82	-12	-12	-12	108	-78	-20	50
5	50	-16	104	-16	-78	104	-16	46	46	170	-12	-16	-144	-140	-136	174	-74	-16	-12	-82	-16	170	46	-12	-82	50	-74	-74	108	-78	-20
6	42	-12	-16	104	46	-16	104	46	46	46	108	-12	-16	-144	-140	-198	174	-12	-16	-74	-144	-78	170	-16	50	-82	50	-74	-12	46	-16
7	-78	104	-12	-16	42	-16	-16	42	46	46	108	108	-12	-16	-144	-78	-198	112	-12	46	-12	-82	-78	232	-78	50	-82	50	-136	50	-16
8	-78	-16	104	-12	-78	-20	-16	-78	42	46	108	108	108	-12	-16	-82	-78	-260	112	50	108	50	-82	-16	170	-78	50	-82	-12	-74	-12
9	50	-140	-16	104	50	-16	-20	46	-78	42	-16	108	108	108	-12	-78	-82	-16	-260	50	-12	46	50	-144	46	170	-78	50	-20	-74	-12
10	-12	50	-78	46	104	50	-78	-20	-16	-16	42	-78	46	170	46	-12	-16	-82	46	-260	50	-12	-16	50	-144	108	108	-140	50	-20	-74
11	-74	-12	112	-16	46	104	-12	-78	-82	46	-16	-20	-140	108	108	46	50	-16	-20	46	-260	50	-74	-16	50	-82	46	46	-140	50	-20
12	-82	-12	-12	112	-78	-16	104	-74	-78	-82	108	-16	-20	-140	108	170	46	-12	-16	42	108	-198	50	-12	-78	50	-82	46	-16	-78	-12
13	-12	-82	-74	-74	112	-78	46	104	-12	-140	-82	170	46	-82	-78	108	108	46	-74	-16	42	108	-136	50	-12	-140	112	-20	46	-16	-78
14	-140	50	-82	-74	-136	50	-78	-16	104	-12	-78	-82	170	46	-82	-16	108	46	46	-12	46	104	108	-74	-12	-12	-140	112	-82	108	-78
15	-78	-140	-12	-144	-74	-136	112	-78	46	42	-12	-16	-20	108	108	-82	-78	108	-16	46	-12	46	166	108	-74	-74	50	-78	112	-82	108
16	170	-140	-140	-12	-82	-12	-136	174	-78	46	-20	-12	-16	-20	108	46	-82	-16	108	-78	-16	-74	46	104	170	-74	-74	50	-16	50	-20
17	42	108	-140	-140	50	-20	-12	-74	174	-78	-16	-20	-12	-16	-20	46	46	-20	-16	46	-140	-78	-74	-16	166	170	-74	-74	112	-78	112
18	112	42	46	-202	-140	50	42	-12	-12	112	-78	46	42	-74	46	-20	-16	46	-82	-16	46	-140	-16	-74	-16	104	232	-12	-74	112	-78
19	-16	50	42	46	-140	-78	50	104	-12	-12	50	-78	46	42	-74	-16	-20	46	46	-144	-78	-16	-140	-78	-12	-16	104	232	50	-136	174
20	112	46	50	42	-16	-202	-78	-12	104	-12	50	50	-78	46	42	-12	-16	-82	46	108	-82	-16	-16	-78	-140	-12	-16	104	170	112	-198
21	-136	50	46	50	104	46	-202	-16	-12	104	-74	50	50	-78	46	-20	-12	46	-82	-16	46	-144	-16	-78	-16	-140	-12	-16	166	108	174
22	174	-136	-12	-16	50	104	108	-202	46	-74	104	-12	112	-12	-16	46	-82	-12	-16	-82	-16	46	-82	-16	-78	-78	-78	50	-16	166	108
23	46	236	-136	-12	-78	-12	104	46	-202	46	-12	104	-12	112	-12	46	46	-144	-12	46	-20	46	46	-20	-78	-78	-78	-78	-12	46	104
24	104	46	258	-74	-12	-78	-74	104	-16	-140	46	-74	42	50	50	-12	108	46	-82	-12	46	-20	-16	46	-20	-16	-140	-140	-78	-12	46
25	46	104	-16	236	-74	-12	-16	-74	166	-78	-140	108	-12	-20	112	50	-74	108	-16	-82	-12	46	42	-16	46	-82	46	-78	-140	-78	-12
26	-12	46	42	-78	236	-74	50	-16	-12	104	-78	-78	170	-74	42	112	-12	-74	46	-16	-82	-12	108	42	-16	-16	-20	108	-78	-140	-78
27	-78	-12	-16	-20	-78	236	-12	50	46	-74	104	-16	-16	108	-12	42	50	-12	-136	46	-16	-82	50	108	42	-78	46	42	108	-78	-140
28	-140	-78	50	46	-20	-78	174	-12	-12	108	-74	42	-78	46	46	-12	104	50	50	-136	46	-16	-144	50	108	104	-140	-16	42	108	-78
29	-140	-78	-78	50	-16	-82	-78	112	-12	-12	170	-74	42	-78	46	108	-12	42	50	112	-74	108	-16	-82	-12	108	104	-140	-78	104	46
30	46	-140	-16	-16	50	-16	-144	-78	50	50	-12	108	-136	104	-140	46	170	-12	104	50	112	-74	46	-16	-82	50	46	42	-140	-78	104

Fig. B.16. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 15$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	90	10	-52	-52	-114	116	10	-34	-52	178	54	-70	10	-8	90	-176	10	196	-8	28	54	-8	-132	10	10	-34	-8	72	28	-52	-52
1	10	90	72	-52	10	-52	54	10	-96	10	178	54	-70	72	-70	28	-114	72	196	-8	-34	-8	-8	-194	10	10	-96	-8	72	28	-52
2	-52	-52	90	134	-52	10	-52	116	10	-96	-52	116	-8	-70	72	-70	28	-114	134	134	-8	-34	-70	-8	-132	72	10	-158	54	10	90
3	90	-114	-52	152	134	-52	10	10	116	10	-158	-114	54	-8	-70	72	-70	28	-52	72	134	-8	-96	-70	54	-70	72	-52	-96	-8	72
4	10	90	-176	-52	90	72	10	10	72	54	10	-158	-114	-8	54	-8	10	-132	28	-52	134	196	-8	-34	-70	54	-8	72	-52	-96	-8
5	-8	-52	90	-114	-52	90	72	72	10	72	-8	-52	-220	-114	-8	54	-8	10	-70	-34	-52	134	134	-8	28	-8	54	-70	134	-114	-34
6	28	-8	10	90	-52	10	28	72	10	72	72	-8	-52	-158	-176	-70	116	54	10	-70	-96	-114	134	72	-8	28	-70	54	-70	134	-114
7	-114	90	-8	-52	90	-52	10	-34	72	10	134	134	54	-52	-158	-176	-70	116	-8	72	-70	-96	-52	134	10	-70	28	-8	-8	-8	72
8	10	-114	28	-8	-114	28	10	10	28	10	10	134	134	-8	10	-96	-238	-132	116	-8	134	-8	-96	10	134	10	-8	28	-8	-8	-8
9	54	10	-52	28	54	-52	-34	10	-52	90	10	10	134	196	-70	-52	-34	-176	-132	116	-70	72	-8	-158	10	134	-52	-8	28	-8	-8
10	-8	-8	10	10	28	54	-52	28	10	-52	28	-52	-52	134	196	-70	-52	-34	-114	-194	116	-70	10	-8	-96	72	134	-114	54	-34	54
11	116	-8	54	10	72	90	-8	-52	-34	72	-52	28	-52	10	72	134	-8	10	-34	-114	-256	54	-70	-52	-8	-96	10	134	-114	54	-34
12	-96	116	-70	54	-52	10	152	-8	10	-96	72	-52	28	-114	72	134	72	-70	10	-34	-52	-194	54	-8	-52	-8	-34	10	134	-114	54
13	-8	-96	54	-70	-8	-114	72	152	54	-52	-96	72	-52	-34	-52	134	72	10	-70	10	28	10	-194	116	-8	-52	54	-34	10	134	-114
14	-114	54	-96	-8	-70	-8	-114	10	152	54	10	-34	134	-52	-34	-52	134	72	-52	-8	10	28	72	-194	54	-70	-52	116	-96	72	72
15	10	-114	-8	-96	-70	-132	54	-114	72	90	54	10	-34	72	10	28	-114	72	72	-52	54	72	28	134	-194	54	-8	-52	116	-96	72
16	72	-52	-114	54	-96	-70	-132	116	-114	72	28	-8	-52	-34	72	10	28	-114	134	10	-52	54	10	28	196	-132	54	-70	10	54	-34
17	-34	10	-52	-52	54	-96	-70	-70	116	-114	10	-34	-70	-52	-34	72	10	28	-52	72	10	-52	-8	10	90	258	-132	-8	-8	-52	116
18	116	28	10	-114	-52	54	-96	-132	-70	116	-52	72	28	-70	-52	-34	72	10	-34	10	72	10	10	-8	-52	28	258	-70	-70	54	-114
19	-114	54	28	72	-114	-52	54	-34	-132	-70	54	-114	10	28	-70	-52	-34	72	72	-96	10	72	-52	10	54	10	28	196	-8	-132	116
20	116	-52	54	-34	72	-114	-52	-8	-34	-132	-8	116	-52	10	28	-70	-52	-34	10	134	-96	10	134	-52	-52	-8	10	90	134	54	-194
21	-132	116	10	54	28	134	-176	-52	-70	28	-132	-8	116	10	-52	-34	-8	10	-34	10	72	-158	10	72	-52	-52	-70	10	90	134	54
22	54	-70	116	-52	54	28	134	-238	-52	-70	90	-70	54	116	10	-52	-34	-8	-52	28	10	72	-96	10	10	-114	-52	-8	-52	152	72
23	72	116	-70	54	-52	54	28	72	-238	-52	-8	152	-8	54	116	10	-52	-34	-70	10	28	10	134	-96	-52	-52	-114	10	-70	10	90
24	152	72	178	-70	116	10	-8	28	10	-176	-52	-8	152	54	-8	54	72	10	-34	-70	-52	-34	10	72	-96	-52	-114	-114	10	-70	10
25	-52	152	10	178	-132	54	72	-8	90	-52	-176	-52	-8	90	116	54	-8	10	10	-34	-8	10	-34	72	72	-96	10	-114	-114	10	-70
26	-70	10	152	-52	178	-132	54	10	-8	90	10	-114	10	-8	90	116	54	-8	-52	72	-34	-8	72	-34	10	10	-96	72	-176	-52	-52
27	-52	-8	10	90	-52	178	-132	-8	10	-8	152	72	-52	10	-8	90	116	54	-70	10	72	-34	54	72	-96	-52	10	-34	10	-114	-114
28	-114	-114	-8	72	90	-52	178	-70	-8	10	-70	90	10	-52	10	-8	90	116	116	-132	10	72	-96	54	134	-34	-52	-52	28	-52	-52
29	-114	-114	-176	-8	10	28	10	178	-8	-70	10	-70	90	-52	10	72	-70	28	116	116	-70	72	72	-34	54	134	28	-52	-52	28	-52
30	10	-114	-52	-176	54	72	-34	10	116	54	-70	10	-70	152	-114	-52	134	-8	28	116	54	-132	72	10	-34	54	72	28	-52	-52	28

Fig. B.17. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 2, 1, 0]$, $k = 10$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-20	38	-24	-40	-24	-28	22	-28	38	-28	34	26	22	-36	34	30	-24	-40	-28	34	34	26	26	30	-40	-28	26	30	-28	30	-32
1	30	-20	38	38	22	38	-28	22	-28	38	-28	-28	-36	22	-36	-28	30	38	22	-28	-28	-28	-36	-36	30	22	-28	-36	30	-28	30
2	-32	30	-20	-24	-24	-40	38	-28	22	-28	38	34	34	-36	22	26	-28	-32	-24	22	34	34	34	26	-36	-32	22	34	-36	30	-28
3	-28	30	-32	-20	-24	-24	22	-24	34	-40	34	38	34	-28	26	22	-36	-28	-32	38	22	34	34	34	-36	-36	30	22	-28	26	-32
4	30	-28	30	30	42	38	-24	22	-24	34	-40	-28	-24	34	-28	-36	22	26	34	-32	-24	-40	-28	-28	34	26	-36	-32	22	-28	26
5	-36	30	-28	-32	-32	-20	38	-24	22	-24	34	22	34	-24	34	34	-36	-40	-36	34	30	38	22	34	-28	-28	26	26	-32	22	-28
6	34	-36	30	34	30	30	-20	38	-24	22	-24	-28	-40	34	-24	-28	34	26	22	-36	-28	-32	-24	-40	34	34	-28	-36	26	-32	22
7	22	-28	26	30	34	30	-32	42	-24	38	-40	-24	-28	22	-28	-24	34	34	26	-40	-36	-28	-32	-24	22	34	-28	-28	26	-36	30
8	-32	22	-28	-36	-32	-28	30	-32	42	-24	38	22	38	-28	22	34	-24	-28	-28	26	22	26	34	30	-24	-40	34	34	-28	26	-36
9	-36	30	-40	-28	-36	-32	34	-32	30	-20	38	38	22	-24	34	22	-28	-24	-28	34	26	22	26	34	-32	-24	22	34	-28	34	-36
10	26	-36	30	22	34	26	-32	34	-32	30	-20	-24	-24	22	-24	-28	22	34	38	-28	-28	-36	-40	-36	34	30	-24	-40	34	-28	34
11	34	-36	26	30	22	34	-36	30	-28	30	-32	-20	-24	38	-40	-24	34	22	34	-24	-28	-28	-36	-40	26	34	-32	-24	22	-28	34
12	34	-28	26	26	30	22	-28	26	-32	34	-32	-32	-20	38	-24	-40	38	34	22	-28	-24	-28	-28	-36	22	26	-28	-32	38	-40	34
13	34	-28	34	26	26	30	-40	34	-36	30	-28	-32	-32	42	-24	-24	22	38	34	-40	-28	-24	-28	-28	26	22	-36	-28	30	-24	22
14	22	-28	34	34	26	26	-32	22	-28	26	-32	-28	-32	30	-20	-24	38	22	38	-28	-40	-28	-24	-28	34	26	-40	-36	34	-32	38
15	-24	22	-28	-28	-28	-36	26	-32	22	-28	26	30	34	-32	30	42	-24	-24	-40	38	34	22	34	38	-28	-28	26	22	-36	34	-32
16	-32	38	-40	-28	-28	-28	26	-36	30	-40	34	26	30	-28	30	30	-20	-24	-24	22	38	34	22	34	-24	-28	34	26	-40	26	-28
17	-28	30	-24	-40	-28	-28	34	-36	26	-32	22	34	26	-32	34	30	-32	-20	-24	38	22	38	34	22	-28	-24	34	34	-36	22	-36
18	26	-28	30	38	22	34	-28	34	-36	26	-32	-40	-28	26	-32	-28	30	30	42	-24	-24	-40	-24	-28	22	34	-24	-28	34	-36	22
19	22	-36	34	30	38	22	-28	34	-28	26	-36	-22	-40	34	-36	-32	34	30	30	-20	-24	-24	-40	-24	34	22	-28	-24	34	-28	26
20	-36	22	-36	-28	-32	-24	22	-28	34	-28	26	26	30	-40	34	26	-32	-28	-32	30	42	38	38	22	-24	-28	22	34	-24	34	-28
21	34	-36	22	26	34	30	-24	22	-28	34	-28	-36	-36	30	-40	-28	26	30	34	-32	-32	-20	-24	-24	22	38	-28	-40	34	-24	34
22	34	-28	26	22	26	34	-32	38	-40	34	-28	-28	-36	26	-32	-40	34	26	30	-28	-32	-32	-20	-24	38	22	-24	-28	22	-28	38
23	38	-28	34	26	22	26	-28	30	-24	22	-28	-28	-28	26	-36	-32	22	34	26	-32	-28	-32	-32	-20	38	38	-40	-24	34	-40	34
24	-28	38	-28	-28	-36	-40	26	-28	30	-24	22	34	34	-28	26	26	-32	-40	-28	26	30	34	30	30	-20	-24	38	22	-24	34	-40
25	22	-28	38	34	34	26	-40	26	-28	30	-24	-40	-28	34	-28	-36	26	30	22	-28	-36	-32	-28	-32	30	42	-24	-24	22	-24	34
26	-28	22	-28	-24	-28	-28	26	-40	26	-28	30	38	22	-28	34	34	-36	-36	-32	22	34	26	30	34	-32	-32	42	38	-24	22	-24
27	38	-28	22	34	30	34	-28	26	-40	26	-28	-32	-24	22	-28	-28	34	26	26	-32	-40	-28	-36	-32	34	30	-32	-20	38	-24	22
28	-40	38	-28	-40	-28	-24	34	-28	26	-40	26	34	30	-24	22	34	-28	-28	-36	26	30	22	34	26	-32	-28	30	30	-20	38	-24
29	-24	22	-24	-28	-40	-28	38	-28	34	-36	22	26	34	-32	38	22	-28	-28	-28	26	26	30	22	34	-36	-32	34	30	-32	42	-24
30	-24	38	-40	-24	-28	-40	34	-24	34	-28	26	22	26	-28	30	38	-40	-28	-28	34	26	26	30	22	-28	-36	30	34	-32	30	-20

Fig. B.18. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 30$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	4	104	-82	-54	-20	-168	-54	-44	42	18	-44	94	-54	94	18	56	42	8	-106	-44	80	32	94	56	8	18	-30	-6	80	56	-68
1	-6	-58	104	-20	-54	42	-168	-54	-44	104	18	-44	32	8	32	18	56	42	8	-106	-106	18	32	94	118	8	-44	-92	-6	18	118
2	56	56	-58	42	-20	-116	42	-168	-54	-106	104	18	18	-30	70	32	18	56	42	8	-44	-44	18	32	32	118	70	18	-92	56	-44
3	-44	56	-6	-58	-20	-20	-54	-20	-106	-54	-44	166	18	18	-30	132	-30	-44	-6	104	8	-44	18	80	32	-30	118	70	-44	-92	56
4	118	-106	56	56	-58	42	-20	-54	-20	-44	-54	-44	104	80	-44	-30	132	-30	-44	-6	42	-54	-44	18	142	32	-92	56	70	-106	-30
5	-30	118	-168	56	-6	-58	104	-82	8	-20	18	8	-44	104	80	18	-92	70	-92	18	-6	42	8	18	18	80	32	-92	-6	70	-106
6	-44	-92	118	-106	56	56	-58	104	-82	70	-20	18	-54	18	42	80	18	-92	70	-92	-44	-68	42	8	80	18	18	-30	-92	-68	132
7	132	-44	-30	118	-44	56	-6	4	42	-82	8	-82	18	-54	18	-20	142	80	-30	8	-92	-44	-130	-20	8	142	18	18	32	-92	-68
8	-68	132	-106	-30	56	-44	118	-68	66	42	-20	70	-82	18	-54	80	-82	80	18	32	8	-92	18	-68	-20	-54	142	18	-44	32	-92
9	-92	-68	70	-106	-92	56	18	56	-6	66	104	42	70	-82	18	8	18	-144	18	80	32	8	-30	80	-68	-82	-54	142	-44	-44	32
10	32	-92	-6	70	-44	-92	-6	80	-6	-6	4	42	42	70	-82	-44	70	80	-82	-44	80	32	-54	-92	80	-6	-82	-54	204	-44	-44
11	18	-30	-92	56	70	18	-92	-6	80	56	-6	4	-20	104	8	-82	-44	70	80	-82	-106	18	32	-54	-30	80	-68	-144	-54	142	18
12	18	18	32	-92	118	70	-44	-30	-66	80	-6	-68	4	-20	104	-54	-20	18	132	18	-82	-106	-44	-30	-54	32	80	-68	-82	-54	142
13	142	18	80	32	-30	118	8	18	-92	-66	18	-68	-68	4	-20	42	8	42	80	70	18	-82	-168	-106	-30	8	32	80	-6	-82	-54
14	8	80	18	142	32	32	118	8	18	-30	-68	18	-130	-6	-58	-20	42	8	42	80	8	-44	-82	-168	-44	-30	-54	-30	80	-68	-20
15	-20	8	18	18	80	32	94	56	70	18	32	-6	18	-130	-6	4	-82	-20	-54	104	80	8	18	-20	-168	-106	-30	-54	-92	80	-68
16	-130	42	8	-44	18	18	32	94	56	8	18	32	56	-44	-68	-6	4	-82	-20	-54	166	142	8	18	-82	-168	-44	32	-54	-30	18
17	-44	-68	42	-54	-44	-44	18	32	94	-6	8	18	94	-6	18	-68	-6	4	-82	-20	8	228	142	8	-44	-82	-106	18	32	8	-92
18	-92	-44	-6	42	8	-44	-106	80	-30	94	-68	-54	18	94	-6	-44	-6	56	66	-144	-20	8	166	80	8	18	-82	-106	80	32	8
19	8	-92	18	-6	104	8	-106	-44	18	-30	32	-130	-54	18	94	-68	18	56	118	4	-144	-20	-54	104	80	70	18	-82	-44	80	32
20	-30	70	-92	-44	-6	42	8	-106	-44	-44	-30	32	-68	-116	80	94	-68	18	56	118	66	-82	-20	-54	42	80	132	80	-82	18	18
21	80	-92	70	-30	-44	56	42	8	-106	18	-44	-30	-30	-6	-178	80	94	-68	18	56	56	4	-82	-20	8	42	18	70	80	-144	80
22	142	18	-92	132	-30	18	56	42	8	-44	18	-44	-92	32	-68	-178	80	94	-68	18	-6	-6	4	-82	42	8	-20	-44	70	18	-82
23	-20	80	18	-30	132	32	18	56	42	70	-44	18	-106	-30	-30	-68	-178	80	94	-68	-44	-68	-6	4	-20	42	-54	-82	-44	8	80
24	18	42	80	-44	-30	70	32	18	56	-20	70	-44	80	-168	32	-30	-68	-178	80	94	-6	18	-68	-6	-58	-20	104	8	-82	18	-54
25	-54	18	104	80	18	-30	8	94	-44	56	-82	8	-44	80	-168	-30	32	-6	-116	18	94	-6	-44	-130	-6	4	-20	104	70	-82	18
26	18	-54	-44	104	18	18	32	-54	156	-44	118	-20	8	-44	80	-106	-92	-30	-68	-54	18	94	56	18	-130	-68	4	-20	42	70	-82
27	-92	18	8	-44	166	18	-44	94	-116	156	-106	56	-20	8	-44	18	-44	-30	32	-130	-54	18	32	-6	18	-68	-68	4	42	42	70
28	8	-20	18	-54	-44	104	18	-44	94	-178	156	-106	118	-82	70	-44	18	-44	-30	32	-68	8	18	32	-68	18	-6	-6	4	104	-20
29	-82	70	-20	-44	-54	-106	104	18	-44	32	-178	156	-44	56	-20	70	-44	18	-44	-30	94	-6	8	18	-30	-68	80	56	-6	66	42
30	42	-82	8	-20	-106	-54	-44	42	80	-44	94	-116	156	-44	56	42	8	-106	-44	18	-30	94	56	70	18	-92	-68	80	-6	-6	66

Fig. B.19. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 25$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-10	-12	-12	-20	-74	-76	-20	-14	50	110	48	44	-20	168	-76	170	50	-20	-200	-14	-76	-18	106	-16	-206	110	106	-78	48	46	-16
1	-16	-10	50	50	-20	-74	-138	-20	-76	112	110	-14	-18	42	106	-76	232	50	42	-200	-14	-76	-80	106	-16	-144	48	44	-78	48	46
2	-16	46	-10	50	-12	-82	-74	-200	-20	-76	174	110	-14	-18	42	168	-76	170	50	104	-138	48	-76	-18	44	-16	-144	48	-18	-16	-14
3	-14	-16	-16	-72	50	-12	-20	-74	-138	-82	-76	236	172	-76	44	42	106	-76	108	50	104	-138	110	-76	-18	-18	46	-82	48	-18	-16
4	46	-76	-16	-16	-10	112	-12	42	-74	-138	-144	-76	236	172	-76	-18	42	168	-76	46	-12	42	-138	48	-14	-18	-18	46	-20	-14	44
5	-18	108	-76	-16	-78	-72	112	-74	42	-74	-76	-144	-76	236	172	-14	-18	-20	168	-14	108	50	42	-76	-14	-14	-18	-18	-16	42	-76
6	-14	-80	108	-76	46	-16	-72	174	-74	42	-136	-76	-144	-76	236	110	-14	44	-20	106	-76	46	50	-20	-14	-14	-14	-18	44	-78	104
7	166	-76	-80	108	-14	108	-16	-10	174	-74	-20	-136	-76	-144	-76	174	110	48	44	-82	44	-138	46	-12	42	-14	-14	-14	44	-18	-16
8	-16	166	-138	-142	108	-14	170	-16	52	112	-74	42	-74	-138	-82	-76	112	110	-14	44	-82	44	-76	46	-12	-20	48	48	-14	44	-18
9	-80	46	166	-138	-204	46	-14	108	-16	52	174	-74	42	-74	-138	-20	-76	50	110	48	106	-20	44	-14	-16	-12	-20	48	-14	48	-18
10	44	-142	46	166	-76	-142	46	48	108	-16	-10	174	-74	42	-74	-200	-20	-14	50	48	-14	44	-20	-18	48	-16	-12	-20	110	-76	110
11	110	44	-80	108	166	-76	-204	46	-14	170	-16	-72	112	-12	-20	-74	-138	-20	48	50	48	-14	-18	-20	-18	110	-78	-74	-20	110	-76
12	-14	48	44	-80	170	228	-76	-142	46	-14	108	-16	-72	112	-12	-82	-74	-76	-20	-14	-12	-14	-14	-80	42	-18	110	-78	-12	-82	172
13	172	-14	110	106	-80	170	166	-76	-204	108	-14	46	-78	-10	50	-12	-20	-74	-14	-20	-14	-12	-76	-14	-80	104	-80	48	-78	-12	-82
14	-82	172	48	172	106	-80	108	166	-138	-142	108	-76	-16	-16	-72	50	50	-20	-12	-14	-20	-14	-74	-76	-14	-18	42	-142	48	-78	-12
15	-74	-20	172	48	110	44	-80	46	166	-138	-80	108	-76	-16	-16	-10	50	-12	-20	50	48	42	-14	-12	-138	-14	-18	42	-204	110	-140
16	-202	-12	-20	172	-14	48	44	-142	46	166	-76	-80	108	-76	-16	46	-10	-12	-12	42	112	110	42	48	-74	-138	-14	-18	-20	-142	48
17	48	-202	-74	-82	172	-14	110	44	-80	-16	166	-14	-18	46	-14	-16	-16	-10	-74	-12	42	112	172	42	48	-136	-76	48	-18	-20	-142
18	-142	48	-140	-12	-82	172	-76	110	-18	-18	-16	104	-76	44	-16	-14	46	-16	52	-74	-12	42	50	172	42	110	-198	-138	48	-18	-20
19	-20	-142	110	-78	-12	-82	110	-76	48	44	-18	-78	42	-14	-18	-16	48	46	46	52	-74	-12	-20	50	172	104	48	-260	-138	48	-18
20	-18	-20	-204	48	-78	-12	-20	110	-14	-14	44	44	-16	-20	48	-18	-78	48	-16	46	52	-74	50	-20	50	110	166	110	-260	-138	48
21	48	-18	42	-142	48	-78	-74	-20	48	48	-14	-18	-18	46	-82	48	44	-78	110	-16	46	52	-136	50	-20	112	48	104	110	-260	-138
22	-76	-14	-18	42	-80	110	-78	-12	-20	48	-14	-14	-18	-18	46	-144	48	106	-78	48	-78	-16	52	-198	112	-20	112	48	166	48	-198
23	-136	-138	-14	-18	104	-18	110	-16	-12	-20	-14	-14	-14	-18	-18	-16	-144	110	106	-140	-14	-140	-16	-10	-136	112	-20	112	110	104	110
24	48	-74	-138	-14	-80	42	-18	48	-16	-12	42	-14	-14	-14	-18	44	-16	-206	110	168	-78	48	-140	46	-72	-136	112	-20	50	172	42
25	42	48	-12	-76	-14	-80	-20	-18	-14	46	-12	-20	-76	48	-76	-18	106	-16	-144	110	168	-78	-14	-140	46	-10	-198	50	-20	50	172
26	172	42	-14	-74	-76	-14	-18	-20	44	-76	46	50	42	-138	110	-76	-80	106	-78	-144	110	168	-16	-14	-140	-16	52	-136	50	-20	50
27	112	110	42	-14	-12	-14	-14	44	-20	44	-138	46	50	42	-138	48	-76	-18	106	-140	-206	48	168	-78	48	-140	-16	52	-74	-12	42
28	42	112	48	-20	-14	-12	48	-14	106	-82	44	-76	108	-12	104	-138	-14	-76	-80	106	-140	-206	110	168	-78	-14	-78	46	52	-74	-12
29	-12	42	50	-14	-20	-14	50	48	48	44	-82	106	-14	46	50	104	-200	-14	-138	-80	106	-140	-144	110	168	-140	48	-16	46	52	-74
30	-74	-12	-20	-12	-14	-20	48	50	110	-14	44	-20	168	-76	108	50	42	-200	-76	-138	-80	106	-78	-144	110	106	-78	110	-16	46	52

Fig. B.20. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 15$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	34	-46	-46	-56	78	-2	-56	-64	16	-2	-2	86	6	86	-64	104	78	6	-188	-2	-64	24	210	-20	-180	60	148	-82	60	-82	-20
1	42	34	-46	16	6	78	-64	-56	-64	78	-2	-2	86	6	24	-64	166	78	68	-250	-64	-64	-38	148	-20	-180	60	86	-20	-2	-20
2	-20	104	-28	-46	16	-56	78	-126	6	-64	140	60	60	24	6	86	-64	104	78	68	-250	-2	-64	-38	86	-82	-118	60	86	-20	-2
3	-2	42	42	-28	-46	-46	-56	16	-64	6	-2	202	122	-2	24	68	86	-126	104	78	68	-188	-2	-64	-100	24	-20	-118	60	86	-20
4	-20	-64	104	42	-28	16	-46	6	-46	-64	-56	-64	140	184	-2	-38	68	148	-126	104	78	6	-188	-2	-2	-38	-38	-20	-118	60	86
5	24	-20	-64	42	-20	-28	78	-46	6	-108	-64	-56	-64	140	246	-2	-100	68	86	-64	166	78	68	-126	-2	-2	-38	24	-82	-56	-2
6	-2	-38	42	-64	42	42	-28	140	-108	6	-170	-126	-118	-2	-140	184	-2	-38	68	86	-64	104	78	68	-64	60	-64	-38	24	-82	-56
7	6	-2	-38	104	-2	42	-20	-28	140	-46	6	-170	-126	-118	-64	140	246	-2	24	6	24	-64	42	16	68	-64	60	-126	24	-38	-20
8	-20	68	-64	-38	104	-64	42	-82	34	140	16	68	-108	-188	-118	-2	140	184	-2	24	6	86	-64	42	-46	6	-2	60	-126	24	-38
9	-100	-20	68	-126	-100	104	-2	42	-82	-28	140	16	68	-108	-126	-118	-64	140	122	60	86	6	148	-2	42	-46	6	60	-2	-64	-38
10	24	-100	-20	130	-64	-100	42	-2	42	-20	-28	140	16	68	-170	-126	-56	-64	202	60	-2	86	-56	86	-2	42	-46	-56	122	-64	-2
11	60	24	-100	42	192	-64	-162	42	-2	104	-20	-28	140	16	6	-170	-64	-56	-2	140	-2	-2	24	-118	86	-2	42	-108	6	60	-2
12	60	60	24	-38	104	192	-126	-162	42	60	104	-20	-28	140	-46	6	-108	-64	6	-64	78	-2	-64	-38	-118	86	-2	-20	-46	-56	122
13	122	-2	122	24	-38	166	192	-64	-224	42	-2	42	-82	34	140	-108	6	-46	-64	6	-64	16	-2	-64	24	-56	24	-2	-20	-46	-56
14	6	122	-2	184	86	-38	104	192	-64	-162	42	-2	42	-82	-28	140	-46	6	16	-126	-56	-64	-46	-64	-64	24	-56	-38	60	-82	16
15	-46	6	122	-64	122	86	24	104	192	-126	-162	42	-2	42	-20	-28	78	-46	-56	78	-64	-56	-2	16	-64	-64	24	6	-100	122	-144
16	-144	16	-56	122	-64	60	86	-38	166	192	-64	-100	104	-64	42	42	-28	16	-46	-56	78	-2	-56	-2	-46	-126	-2	24	6	-100	122
17	60	-144	16	-118	60	-64	122	86	-38	104	192	-64	-100	104	-2	42	-20	-28	-46	16	6	78	60	6	-2	-46	-126	60	-38	68	-162
18	-162	-2	-82	16	-118	122	-64	184	24	-38	42	130	-126	-38	104	-64	42	42	-28	-46	16	-56	78	60	68	60	-108	-126	60	-38	68
19	68	-224	60	-82	16	-56	122	-2	122	24	-100	-20	68	-64	-38	42	-64	104	42	-28	-46	-46	-56	78	122	130	-2	-108	-126	60	-38
20	-100	68	-224	-2	-144	16	6	122	-2	60	24	-100	-20	68	-2	-38	-20	-64	42	104	34	-46	16	6	78	122	130	60	-170	-64	-2
21	60	-100	68	-162	60	-144	-46	6	122	60	60	24	-100	-20	6	-2	24	-20	-2	-20	42	34	-108	-46	6	78	122	68	122	-232	-2
22	-2	-2	-38	68	-162	122	-144	16	-56	122	-2	-2	-38	-38	-20	-56	-2	86	-20	-2	-20	-20	34	-108	16	68	16	122	68	122	-232
23	-232	-64	60	-38	68	-100	122	-82	-46	-56	60	-64	-64	24	-38	-82	-56	60	86	-20	-2	-82	-20	34	-46	78	6	16	122	68	122
24	122	-170	-126	60	-38	6	-100	60	-20	-46	6	122	-2	-126	24	24	-82	-118	60	86	-20	60	-82	-20	-28	-108	140	6	16	122	68
25	68	60	-108	-126	60	24	6	-38	-2	-20	-108	-56	60	60	-126	-38	24	-20	-118	60	86	-82	60	-82	42	34	-170	140	6	16	122
26	122	130	-2	-108	-126	-2	24	-56	24	-2	42	-46	6	-2	60	-64	-38	-38	-20	-118	60	148	-82	60	-144	-20	96	-170	140	6	16
27	78	122	130	60	-46	-126	-64	24	-56	86	-2	42	-46	6	-64	60	-2	-38	24	-82	-180	60	86	-144	60	-144	-20	34	-108	78	68
28	6	78	122	68	-2	-46	-64	-64	24	-118	86	-2	42	-46	68	-64	-2	-2	-100	86	-20	-180	122	148	-144	60	-144	42	-28	-46	16
29	-46	6	78	60	6	-2	16	-64	-64	-38	-118	86	-2	42	16	68	-126	-2	-64	-38	148	-20	-118	184	148	-144	60	-82	-20	34	-108
30	-108	16	-56	78	60	-56	-2	-46	-2	-64	24	-56	148	-64	42	78	68	-188	-2	-64	-38	210	-20	-118	122	86	-82	60	-82	-20	34

Fig. B.21. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 11$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	14	-70	-8	-96	54	32	-34	-30	-8	-30	-30	50	-34	112	-30	72	54	28	-216	32	-30	-12	236	10	-158	94	112	-52	32	-52	-52
1	10	14	-8	-8	-34	116	-30	-34	-52	54	-30	-30	50	28	50	-92	134	116	28	-216	-30	-92	-12	174	10	-158	32	112	-52	32	-52
2	-52	72	14	-70	-8	-34	116	-92	-34	-92	116	32	32	50	28	50	-92	134	54	90	-216	-30	-30	-12	112	-52	-158	94	50	10	-30
3	-30	10	72	-48	-70	-8	-34	54	-92	-34	-30	178	94	32	50	28	50	-92	72	116	90	-216	32	-30	-74	50	-52	-96	32	112	-52
4	10	-30	72	72	14	-8	-70	-34	-8	-30	-34	-30	178	156	-30	-12	90	112	-92	72	54	28	-216	-30	-30	-74	-12	-52	-96	32	112
5	50	10	-92	72	10	-48	54	-70	28	-70	-30	-34	-30	116	218	32	-74	28	112	-92	134	116	28	-154	-30	-30	-12	-12	-52	-96	32
6	32	-12	10	-30	72	10	-48	116	-70	28	-132	-92	-96	-30	116	218	32	-74	90	50	-92	134	54	28	-92	32	-30	-74	50	-114	-34
7	-34	-30	-12	72	-30	72	10	14	116	-70	-34	-194	-154	-96	-30	116	218	32	-12	28	50	-92	72	54	90	-30	32	-92	-12	-12	-52
8	-52	28	-30	-74	72	-30	72	-52	14	116	-8	28	-132	-154	-96	-30	116	218	-30	50	28	50	-30	72	-8	28	-30	94	-154	50	-74
9	-74	10	28	-92	-74	72	-30	10	-52	14	178	54	90	-132	-154	-96	-30	116	156	32	50	28	112	-30	10	-70	28	32	32	-92	-12
10	-12	-136	10	90	-92	-74	72	32	10	-52	-48	116	-8	90	-132	-154	-96	-30	178	94	32	50	-34	112	32	72	-70	-34	94	-30	-30
11	32	-12	-74	10	152	-30	-136	72	-30	72	-52	-48	116	54	28	-194	-92	-34	-30	178	32	-30	50	-96	112	32	10	-70	-34	94	-30
12	32	32	50	-74	72	214	-92	-136	10	32	72	-52	-48	178	-8	-34	-132	-30	-34	-30	116	-30	-30	-12	-96	112	-30	10	-70	-34	94
13	156	32	94	50	-12	134	152	-92	-198	72	32	72	-52	14	116	-70	28	-70	-30	-34	-92	54	-30	-92	-12	-96	50	-30	10	-70	-34
14	-34	54	32	156	50	-12	134	214	-92	-198	10	-30	10	-52	14	116	-70	28	-8	-92	-34	-92	-8	-30	-30	50	-96	-12	32	-52	-8
15	-8	28	94	-30	156	50	-12	72	214	-92	-136	72	32	10	-52	14	116	-70	-34	54	-92	-34	-30	-8	-92	-92	50	-34	-74	94	-114
16	176	-8	-34	94	-92	94	112	-12	134	152	-92	-136	72	-30	72	10	-48	54	-70	-34	116	-30	-34	32	-8	-92	-30	50	-34	-74	94
17	94	-114	-8	-96	94	-92	94	50	-12	134	214	-30	-74	72	-30	72	10	-48	-8	-8	-34	116	32	-34	-30	-70	-92	32	-12	28	-136
18	-136	32	-114	54	-96	94	-92	156	50	-12	72	152	-92	-74	72	-30	72	10	14	-70	-8	-34	54	32	28	32	-70	-154	94	-74	90
19	90	-198	32	-52	54	-96	94	-30	156	50	-74	10	90	-92	-74	72	-30	72	72	-48	-70	-8	-96	54	94	90	32	-132	-92	32	-12
20	-74	90	-260	32	-114	-8	-34	94	32	94	50	-74	10	28	-30	-12	10	-92	72	72	14	-8	-8	-34	54	94	152	32	-132	-92	32
21	32	-136	90	-198	32	-114	-8	28	94	32	32	-12	-136	10	28	-30	-12	10	-30	10	72	14	-70	-8	28	116	94	90	94	-194	-30
22	32	32	-74	90	-136	94	-176	-8	-34	156	32	32	-12	-74	-52	-34	32	50	10	-30	-52	10	14	-132	-8	28	54	54	50	94	-194
23	-194	-30	32	-12	90	-136	94	-114	-8	-34	94	-30	-30	-12	-74	-52	-34	32	112	-52	-30	-52	-52	14	-70	54	28	-8	156	28	156
24	94	-194	-92	32	-74	28	-74	94	-52	-70	-34	94	-30	-92	50	-12	-114	-96	32	112	10	32	-52	10	14	-70	116	28	-8	156	28
25	90	94	-132	-92	94	-12	-34	-74	32	10	-70	-34	94	32	-154	-12	50	-52	-96	32	50	-52	32	-114	10	14	-132	116	28	-8	156
26	94	90	32	-132	-154	32	50	-34	-12	-30	10	-70	-34	32	94	-92	-74	-12	-52	-96	94	112	-52	94	-114	10	76	-132	116	28	-8
27	54	94	152	32	-70	-92	-30	50	-96	50	-30	10	-70	28	-30	32	-30	-12	-12	-52	-158	32	112	-114	94	-114	-52	76	-132	116	28
28	28	116	94	90	32	-70	-92	-92	50	-96	112	32	72	-70	28	-30	32	-30	-74	50	-52	-158	94	112	-176	32	-114	10	14	-70	54
29	-8	28	54	94	28	-30	-8	-92	-30	-12	-96	112	32	10	-8	90	-92	-30	-30	-74	112	10	-158	156	112	-176	94	-114	10	14	-70
30	132	-8	-34	54	32	-34	32	-8	-30	-92	-12	-96	112	-30	72	54	28	-154	-30	-30	-12	174	10	-96	156	112	-114	94	-114	10	14

Fig. B.22. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 0]$, $k = 10$

$s_y \backslash s_x$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	118	-42	20	-42	82	118	82	108	-42	-68	-6	-16	-104	-140	46	-104	20	-166	-6	170	-68	-16	-16	20	20	46	-78	-104	-16	-42	144
1	144	56	-42	20	20	144	56	82	108	-42	-68	-6	-78	-42	-140	46	-42	20	-104	-68	170	-130	-78	-78	82	20	-16	-78	-42	-16	-42
2	104	144	-6	-104	20	20	144	-6	144	46	20	-6	-6	-78	20	-78	46	-104	20	-104	-6	170	-130	-78	-78	20	20	46	-78	20	-78
3	140	-104	82	-68	-104	20	20	82	56	82	108	82	-6	-6	-16	82	-78	-16	-104	20	-42	-6	170	-130	-78	-140	20	82	46	-16	-42
4	20	-140	-42	144	-68	-104	20	82	20	118	20	46	82	-6	-68	-78	82	-16	-16	-104	-42	-42	-6	170	-130	-16	-140	-42	82	-16	46
5	46	82	-140	-42	82	-130	-42	20	82	20	118	20	108	20	-6	-68	-140	82	-78	46	-104	20	20	56	108	-130	46	-140	-104	82	-16
6	-16	108	82	-140	-104	20	-68	-42	20	82	20	118	82	46	20	-6	-130	-140	20	-16	46	-42	82	82	-6	108	-68	46	-202	-104	82
7	82	-78	108	82	-78	-42	-42	-68	-42	20	82	20	56	144	46	20	56	-130	-78	-42	-16	-16	-104	20	144	-6	46	-68	108	-202	-104
8	-42	82	-16	170	82	-78	-42	20	-130	20	-42	20	20	56	82	-16	20	118	-130	-78	-104	-16	-16	-104	20	206	-6	-16	-68	46	-140
9	140	20	82	-16	108	20	-16	-42	20	-130	20	-42	82	-42	56	82	-78	20	56	-68	-78	-42	46	46	-166	20	268	-6	-78	-68	46
10	46	-78	20	82	-78	46	82	-16	-42	20	-130	20	20	20	-42	56	20	-78	-42	118	-68	-16	20	108	-16	-166	82	268	-68	-78	-68
11	-68	108	-78	20	20	-140	108	82	-16	-42	20	-130	82	-42	20	-42	-6	20	-140	20	118	-6	46	82	46	-16	-104	82	206	-68	-78
12	-78	-6	108	-78	-42	-42	-78	108	82	-16	-42	20	-68	20	-42	20	-104	-6	-42	-78	20	180	56	108	20	46	46	-104	20	206	-68
13	-68	-140	-6	108	-16	20	-104	-78	108	82	-16	-42	-42	-6	20	-42	82	-104	56	-104	-78	-42	118	-6	170	20	-16	46	-42	20	206
14	144	-68	-202	-68	108	-16	20	-166	-16	46	144	46	-42	-42	56	82	-42	20	-104	56	-42	-78	-42	118	-6	108	20	46	46	20	-42
15	20	144	-6	-140	-68	108	-16	82	-228	46	-16	82	46	-42	-104	-6	82	20	20	-104	-6	-42	-78	-42	118	56	108	-42	46	-16	82
16	82	-42	144	-6	-78	-6	46	-16	82	-228	46	-16	20	108	-42	-104	56	82	82	-42	-104	-68	-104	-140	20	118	-6	108	20	46	-16
17	-78	82	-104	82	-6	-78	-6	-16	46	20	-166	108	-16	20	170	20	-104	-6	82	82	20	-104	-68	-104	-140	-42	118	56	108	82	-16
18	46	-78	144	-42	82	-6	-78	56	-78	108	-42	-228	108	-16	-42	108	20	-42	-6	82	20	20	-104	-68	-104	-78	-42	56	56	46	144
19	144	108	-78	144	-104	20	56	-78	56	-78	108	-42	-166	46	-16	-42	46	20	-104	56	82	82	82	-42	-130	-104	-16	-42	-6	56	46
20	46	82	108	-78	206	-42	-42	56	-78	56	-78	108	-104	-104	46	-16	20	46	82	-166	56	20	20	20	20	-130	-166	-16	20	-6	56
21	-6	46	20	46	-78	206	-42	-104	118	-140	118	-16	108	-104	-42	108	-16	-42	46	82	-104	56	20	20	20	-42	-130	-104	-16	82	-68
22	130	-6	-16	-42	46	-78	206	-104	-42	56	-78	180	-16	108	-42	20	108	-78	-42	46	144	-104	56	20	20	-42	-42	-68	-104	46	20
23	-42	-130	-68	-78	-42	46	-78	144	-42	-104	118	-16	180	-16	170	20	20	46	-78	-42	108	144	-104	56	20	-42	-42	20	-68	-42	-16
24	46	-42	-68	-6	-78	-42	46	-16	82	20	-166	56	-16	180	-78	108	20	82	46	-78	-104	108	144	-104	56	82	-42	-104	20	-130	20
25	20	-16	-42	-68	56	-16	-104	46	-16	82	20	-166	-6	46	180	-78	170	20	144	-16	-78	-166	46	82	-42	56	20	-42	-42	20	-130
26	-68	20	46	20	-68	56	-16	-42	-16	46	20	-42	-166	-6	-16	118	-78	232	20	144	-78	-78	-166	46	82	20	56	-42	-42	-104	82
27	82	-6	20	46	-42	-130	118	-16	-42	-16	46	20	20	-228	-6	-16	56	-78	170	82	144	-16	-16	-104	-16	82	82	56	-104	-42	-104
28	104	20	-6	20	108	20	-192	118	-16	-42	-16	46	-42	82	-228	-6	46	56	-16	108	82	82	-78	-78	-42	-16	20	82	118	-104	-42
29	20	-104	82	56	20	108	20	-130	56	46	-104	-78	46	-42	20	-290	-6	108	56	-16	46	82	82	-78	-78	20	-16	-42	82	56	-42
30	-42	-42	-104	82	118	82	46	20	-130	56	46	-104	-140	108	-42	20	-228	-6	170	-6	-16	-16	20	20	-16	-78	-42	-16	20	82	56

Fig. B.23. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 3, 2, 0]$, $k = 20$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	130	-42	20	-78	-42	82	46	-16	-42	20	-42	20	-16	-104	46	56	-104	-78	82	170	20	20	144	56	108	-78	-42	-68	-16	-6	180
1	180	-192	-42	20	-16	20	20	46	-16	-42	20	-42	-42	46	-104	46	118	-104	-16	20	170	-42	-42	82	118	108	-140	-42	-6	-16	-6
2	-6	118	-192	-42	82	46	-42	20	46	-16	-42	20	-104	20	46	-104	108	118	-42	-78	20	108	-104	-104	144	118	46	-140	20	-6	-16
3	-16	-68	118	-192	20	144	-16	-42	20	46	-16	-42	-42	-42	20	46	-42	108	180	-104	-78	-42	46	-166	-42	144	56	46	-78	20	-6
4	-6	-78	-68	118	-130	82	82	-16	-42	20	46	-16	-104	20	-42	20	108	-42	170	118	-104	-140	-104	-16	-104	-42	82	56	108	-78	20
5	82	-6	-16	-6	118	-130	82	144	-78	20	-42	-16	-16	-104	-42	-104	20	170	-42	170	56	-104	-140	-104	-16	-42	-42	20	56	46	-16
6	-16	144	-6	-16	-68	56	-68	82	144	-78	20	-42	46	-78	-104	-42	-166	20	108	20	170	118	-42	-78	-166	-16	20	-42	-42	56	46
7	46	-78	144	-6	46	-6	-6	-68	82	144	-78	20	-104	108	-78	-104	20	-166	82	46	20	108	56	-104	-16	-166	-78	20	20	-42	56
8	56	-16	-78	144	56	108	-68	-6	-68	82	144	-78	-42	-42	108	-78	-42	20	-104	20	46	-42	46	-6	-42	-16	-228	-78	82	20	-42
9	20	56	46	-16	144	56	108	-6	-68	-6	20	82	-78	-42	-104	46	-78	20	20	-104	-42	46	-42	46	-6	20	-16	-250	-78	20	82
10	20	20	-6	-16	-16	144	56	46	56	-130	56	82	82	-78	20	-42	46	-140	20	20	-42	-42	46	-42	46	-68	20	46	-250	-16	-42
11	104	20	-42	-68	-16	-16	144	-6	108	-6	-68	118	82	82	-16	82	-42	-16	-140	20	82	-42	-42	46	-42	-16	-68	82	46	-228	-78
12	-78	-42	20	-42	-130	-78	46	144	-6	108	-6	-68	180	20	82	-16	20	-42	-78	-78	20	144	20	20	-16	-42	46	-68	20	46	-228
13	166	-78	20	82	-42	-130	-78	108	82	56	46	-68	-68	180	-42	20	-16	82	-42	-78	-140	20	144	20	20	46	-42	-16	-68	-42	108
14	108	-104	-78	20	20	-104	-68	-78	108	82	56	46	-6	-130	180	-42	-42	-16	20	20	-78	-78	82	206	-42	20	108	-42	-78	-68	-42
15	20	108	-42	-16	20	20	-104	-6	-140	170	20	-6	46	-6	-192	118	-42	20	-16	20	-42	-78	-78	82	206	20	20	46	-42	-140	-6
16	-6	82	108	-42	-78	-42	82	-104	-6	-140	170	20	56	-16	-6	-192	56	-42	-42	46	20	20	-16	-16	20	206	82	20	-16	-42	-140
17	140	-56	82	108	20	-16	-104	82	-104	-6	-140	170	-42	118	-16	-6	-130	56	20	-104	46	-42	-42	-78	46	20	144	82	82	-16	-42
18	20	-140	-6	144	108	20	-16	-42	20	-42	-68	-202	170	-42	56	-78	-6	-68	56	20	-166	46	-42	-42	-78	108	20	82	82	20	46
19	46	82	-140	-6	82	46	82	-16	-42	20	-42	-68	-140	108	-42	56	-140	-6	-130	118	20	-104	108	20	-104	-78	170	20	20	82	20
20	82	46	144	-78	-6	82	46	144	-78	20	-42	-104	-68	-140	46	-104	56	-78	-6	-130	56	20	-104	108	20	-42	-78	108	20	-42	144
21	82	82	-16	82	-78	-6	82	-16	206	-140	82	20	-104	-68	-78	108	-104	-6	-78	-6	-68	56	20	-104	108	-42	-42	-16	108	82	-104
22	166	82	20	-78	82	-78	-6	20	46	144	-78	144	20	-104	-6	-16	108	-166	-6	-78	56	-68	56	20	-104	46	-42	20	-16	170	20
23	-42	-166	20	-42	-78	82	-78	-68	82	-16	206	-16	144	20	-42	56	-16	46	-166	-6	-16	56	-68	56	20	-166	46	20	20	46	108
24	108	20	-166	20	-104	-140	144	-78	-68	82	-16	206	46	82	20	-42	-6	-16	-16	-104	-6	46	118	-6	-6	20	-104	46	-42	20	46
25	46	46	20	-166	82	-42	-202	144	-78	-68	82	-16	144	108	82	20	20	-6	46	-78	-104	-68	-16	56	56	-6	-42	-104	108	-42	20
26	82	46	108	82	-166	82	-42	-140	82	-16	-130	20	-16	144	46	20	20	82	-6	46	-140	-104	-68	-16	56	118	-6	-104	-104	46	20
27	-42	82	-16	46	82	-166	82	-104	-78	20	46	-68	20	-16	206	108	20	-42	82	-6	108	-140	-104	-68	-16	-6	118	56	-104	-42	-16
28	-16	20	82	-16	-16	20	-104	82	-104	-78	20	46	-6	-42	-16	206	46	20	-104	144	-6	170	-78	-42	-130	-16	56	118	-6	-104	-42
29	20	-16	82	144	-16	-16	20	-42	20	-42	-140	-42	46	-6	-104	-78	206	108	20	-104	82	-6	170	-78	-42	-68	-16	-6	118	-68	-42
30	104	20	-78	20	144	-16	-16	-42	20	-42	20	-78	-42	46	56	-42	-78	144	108	20	-42	82	-6	170	-78	-104	-68	46	-6	180	-130

Fig. B.24. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 3, 2, 1, 0]$, $k = 20$

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	-6	-6	-68	82	-130	46	82	20	-6	-16	46	20	20	20	-42	-140	56	82	46	-166	108	20	-104	-140	20	206	-42	-16	20	46	-140
1	140	-68	-6	-68	144	-68	-16	82	20	-6	-16	46	-42	82	20	-42	-78	56	144	-16	-166	46	-42	-166	-78	20	144	-42	46	20	46
2	108	-140	-6	56	-68	144	-68	46	20	82	-68	-78	46	-42	20	-42	-42	-16	56	144	-78	-166	46	-42	-166	-16	20	82	-42	-16	82
3	144	108	-78	56	56	-68	144	-6	-16	82	20	-130	-78	46	-104	-42	-42	20	-16	56	82	-78	-166	46	-42	-104	-16	-42	82	-104	46
4	46	206	108	-78	-6	-6	-6	144	-6	-16	82	20	-68	-140	46	-104	-104	-42	-42	46	56	144	-16	-104	-16	-42	-42	-16	-104	82	-104
5	104	-16	206	108	-16	56	-68	-6	144	-6	-16	82	-42	-6	-140	46	-42	-104	20	-104	46	-6	82	-78	-42	-16	-104	-42	46	-104	82
6	20	-104	-78	144	108	-16	56	-130	56	82	56	46	82	-42	56	-78	46	-104	-104	20	-42	46	-6	82	-78	-104	-16	-42	-42	108	-166
7	166	82	-104	-78	82	46	46	56	-130	56	82	56	108	20	-42	56	-140	46	-166	-42	20	20	108	56	20	-78	-42	-16	-104	-42	108
8	108	-104	82	-104	-140	20	108	46	56	-130	56	82	118	46	20	-42	-6	-140	-16	-104	-42	82	82	170	-6	20	-16	-42	-78	-104	-42
9	-42	46	-104	82	-42	-78	-42	108	46	56	-130	56	20	180	46	20	20	-6	-78	-78	-104	-104	20	20	232	-6	-42	-16	20	-78	-104
10	104	20	46	-104	20	-104	-16	-42	108	46	56	-130	118	-42	180	46	-42	20	-68	-16	-78	-42	-42	82	-42	232	56	-42	-78	20	-78
11	-78	-42	20	46	-166	-42	-42	-16	-42	108	46	56	-68	56	-42	180	-16	-42	-42	-6	-16	-16	20	20	20	-42	254	56	-104	-78	20
12	82	-78	20	82	46	-166	-42	20	-78	20	46	-16	56	-68	-6	-104	180	46	-42	-42	-68	-16	-16	20	20	82	-42	232	56	-166	-16
13	-16	144	-78	20	20	-16	-104	-42	20	-78	20	46	46	-6	-68	-6	-166	180	-16	20	-42	-6	46	46	-42	20	144	-42	170	56	-166
14	104	-16	206	-16	20	20	-16	-42	-104	82	-140	-42	46	46	-68	-130	-6	-104	180	-16	-42	-42	-6	46	46	20	20	82	-42	108	118
15	56	-104	-78	144	-16	20	20	-78	20	-166	144	-78	-42	46	108	-6	-130	-68	-104	180	46	-42	-42	-6	46	-16	20	82	82	20	46
16	108	56	-42	-16	144	-16	20	82	-140	82	-228	82	-78	-42	-16	46	-6	-68	-68	-104	118	46	-42	-42	-6	108	-16	-42	82	20	82
17	82	46	56	-42	46	206	-78	20	82	-140	82	-228	20	-16	-42	-16	108	-6	-6	-130	-104	56	-16	-104	20	-6	46	-16	20	82	20
18	-42	82	-16	-6	-42	46	206	-140	82	20	-78	144	-228	20	46	20	-16	46	-6	-6	-68	-104	56	-16	-104	-42	-6	108	-16	82	20
19	-42	-42	20	-78	-6	-42	46	144	-78	20	82	-16	144	-228	82	108	20	-78	46	-6	56	-68	-104	56	-16	-166	-42	56	108	46	20
20	20	20	-42	20	-140	-68	20	46	144	-78	20	82	46	82	-228	82	46	20	-140	108	-6	118	-6	-42	-6	-16	-104	-42	-6	108	46
21	-16	20	-42	-104	20	-140	-68	-42	108	82	-16	82	82	46	144	-166	82	-16	20	-140	170	-6	118	-6	-42	-68	-16	-42	-42	56	46
22	46	-78	20	-42	-42	82	-202	-68	-42	108	82	-16	20	144	46	144	-104	82	46	-42	-140	108	-68	56	56	-42	-130	-16	20	-42	56
23	118	46	-16	82	-42	-42	82	-140	-130	20	46	20	-16	20	82	-16	144	-42	82	46	-104	-140	108	-68	56	118	-42	-192	-16	-42	20
24	20	56	46	-16	144	20	-104	82	-140	-130	20	46	-42	46	20	82	46	144	20	20	46	-166	-202	46	-6	56	56	-42	-130	-16	-42
25	-42	82	56	46	-78	82	82	-104	82	-140	-130	20	108	-104	46	20	20	46	82	82	20	108	-104	-140	-16	-6	118	56	-104	-130	-16
26	-78	-42	20	-6	46	-78	82	20	-42	20	-78	-68	20	108	-42	108	20	-42	46	82	144	20	108	-104	-140	-78	-6	180	56	-42	-192
27	130	-78	20	82	-6	46	-78	144	-42	20	-42	-140	-68	20	46	-104	108	82	-42	46	20	144	20	108	-104	-78	-78	-68	180	-6	20
28	82	-130	-16	82	82	-6	46	-16	82	20	-42	-104	-140	-68	-42	-16	-104	170	82	-42	-16	20	144	20	108	-42	-78	-140	-68	118	56
29	-6	82	-192	-78	82	82	-6	-16	46	20	82	20	-104	-140	-6	20	-16	-166	170	82	20	-16	20	144	20	46	-42	-16	-140	-6	56
30	56	-68	82	-192	-16	144	20	-6	-16	46	20	82	-42	-42	-140	-6	82	-16	-104	108	82	-42	-78	-42	206	20	-16	-42	46	-140	-6

Fig. B.25. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 4, 3, 1, 0]$, $k = 20$

s_y \ s_x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0	20	82	-104	20	-104	-104	82	20	82	-104	82	82	82	20	82	-166	20	20	-104	-42	82	82	-42	-166	-104	144	82	-42	20	-42
1	20	0	82	144	-104	20	-104	-42	20	82	20	-166	82	82	20	-42	82	-104	20	20	-166	-42	82	82	-42	-104	-104	82	82	-104	82
2	82	82	0	82	82	-166	82	-104	-42	20	82	20	-104	20	82	20	-104	82	-166	82	20	-104	20	144	20	-42	-42	-104	20	82	-104
3	104	144	82	0	20	20	-104	82	-104	-42	20	82	82	-166	20	82	-42	-104	20	-104	82	82	-42	82	82	20	20	-42	-166	20	82
4	20	-104	82	20	0	20	20	-166	144	-166	20	82	82	82	-104	82	82	-104	-104	20	-42	82	82	-42	82	20	20	82	-42	-104	-42
5	104	20	-166	20	20	0	20	-42	-104	82	-104	82	82	82	144	-42	82	20	-104	-104	82	-42	82	82	-42	20	20	82	82	20	-166
6	104	-104	82	-104	20	20	0	82	-104	-42	20	-166	82	82	20	82	-42	144	20	-104	-166	82	-42	82	82	20	20	-42	82	20	82
7	82	-42	-104	82	-166	-42	82	0	82	-104	-42	20	-104	20	82	20	20	-42	82	82	-104	-104	144	20	20	82	82	20	-104	82	20
8	20	20	-42	-104	144	-104	-104	82	0	82	-104	-42	-42	-42	20	82	82	20	20	20	82	-166	-166	82	82	20	20	82	82	-104	82
9	82	82	20	-42	-166	82	-42	-104	82	0	82	-104	20	-104	-42	20	20	82	-42	82	20	144	-104	-104	20	82	82	20	20	82	-104
10	104	20	82	20	20	-104	20	-42	-104	82	0	82	-166	82	-104	-42	82	20	144	-104	82	-42	82	-166	-42	20	20	82	82	20	82
11	82	-166	20	82	82	82	-166	20	-42	-104	82	0	20	-104	82	-104	20	82	82	82	-104	20	-104	20	-104	-42	-42	20	144	82	20
12	82	82	-104	82	82	82	82	-104	-42	20	-166	20	0	20	-166	20	-104	82	82	82	20	-104	20	-104	20	-42	-42	-104	20	82	144
13	82	82	20	-166	82	82	82	20	-42	-104	82	-104	20	0	82	-104	20	-166	82	82	144	20	-104	20	-104	-42	-42	20	-104	82	20
14	20	20	82	20	-104	144	20	82	20	-42	-104	82	-166	82	0	82	-42	20	-104	20	82	82	-42	-166	82	-104	-104	-42	82	-104	82
15	82	-42	20	82	82	-42	82	20	82	20	-42	-104	20	-104	82	0	144	-42	82	-166	20	20	20	-104	-104	82	-166	-104	20	82	-104
16	166	82	-104	-42	82	82	-42	20	82	20	82	20	-104	20	-42	144	0	82	-42	82	-104	20	20	20	-104	-166	82	-104	-104	82	20
17	20	-104	82	-104	-104	20	144	-42	20	82	20	82	82	-166	20	-42	82	0	20	20	82	-42	82	82	-42	-104	-104	82	-166	-104	82
18	20	20	-166	20	-104	-104	20	82	20	-42	144	82	82	82	-104	82	-42	20	0	20	82	82	-42	82	82	-104	-104	-42	82	-104	-166
19	104	20	82	-104	20	-104	-104	82	20	82	-104	82	82	82	20	-166	82	20	20	0	-42	82	82	-42	82	144	-104	-166	-42	20	-42
20	-42	-166	20	82	-42	82	-166	-104	82	20	82	-104	20	144	82	20	-104	82	82	-42	0	-104	20	20	20	82	82	-104	-104	-42	20
21	82	-42	-104	82	82	-42	82	-104	-166	144	-42	20	-104	20	82	20	20	-42	82	82	-104	0	-104	20	20	82	82	20	-104	-166	20
22	82	82	20	-42	82	82	-42	144	-166	-104	82	-104	20	-104	-42	20	20	82	-42	82	20	-104	0	-104	20	82	82	20	20	-166	-104
23	-42	82	144	82	-42	82	82	20	82	-104	-166	20	-104	20	-166	-104	20	82	82	-42	20	20	-104	0	-104	82	82	20	20	-42	-104
24	166	-42	20	82	82	-42	82	20	82	20	-42	-104	20	-104	82	-104	-104	-42	82	82	20	20	20	-104	0	-166	82	144	20	82	-104
25	104	-104	-42	20	20	20	20	82	20	82	20	-42	-42	-42	-104	82	-166	-104	-104	144	82	82	82	82	-166	0	-104	82	82	20	82
26	144	-104	-42	20	20	20	20	82	20	82	20	-42	-42	-42	-104	-166	82	-104	-104	-104	82	82	82	82	82	-104	0	-166	82	20	82
27	82	82	-104	-42	82	82	-42	20	82	20	82	20	-104	20	-42	-104	-104	82	-42	-166	-104	20	20	20	144	82	-166	0	-104	82	20
28	-42	82	20	-166	-42	82	82	-104	82	20	82	144	20	-104	82	20	-104	-166	82	-42	-104	-104	20	20	20	82	82	-104	0	-42	20
29	20	-104	82	20	-104	20	20	82	-104	82	20	82	82	82	-104	82	82	-104	-104	20	-42	-166	-166	-42	82	20	20	82	-42	0	-42
30	-42	82	-104	82	-42	-166	82	20	82	-104	82	20	144	20	82	-104	20	82	-166	-42	20	20	-104	-104	-104	82	82	20	20	-42	0

Fig. B.26. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 2, 0]$, $k = 20$

y \ x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0	-20	104	-82	104	-20	-82	-82	-82	104	-20	104	-20	104	42	42	-20	-144	-82	-20	-82	166	-20	-82	-82	-82	166	-20	42	104	-82
1	-20	0	-20	166	-20	104	-82	-82	-82	-20	104	-20	104	-20	42	42	104	-20	-82	-144	-82	-82	104	-82	-82	-82	-82	104	42	-20	166
2	104	-20	0	-82	104	-20	166	-82	-82	-144	-20	104	-20	104	42	42	-20	104	-82	-20	-82	-82	-20	166	-82	-82	-82	-20	42	104	-82
3	-82	166	-82	0	-82	42	-20	104	-20	-82	-82	42	166	-82	104	104	42	-82	104	-82	-20	-20	-82	-20	104	-144	-20	-82	-20	42	104
4	104	-20	104	-82	0	-144	42	-82	166	-20	-20	-20	104	104	-82	166	104	-20	-82	104	-82	42	-20	-82	-82	42	-82	-20	-82	-20	42
5	-20	104	-20	42	-144	0	-82	42	-82	104	-20	-20	-20	104	166	-82	104	104	-82	-20	166	-82	104	42	-82	-82	42	-20	-82	-20	-82
6	-82	-82	166	-20	42	-82	0	-20	-20	-82	42	-82	-82	42	104	104	-82	166	104	-82	-20	104	-82	104	104	-20	-144	42	-20	-82	-20
7	-82	-82	-82	104	-82	42	-20	0	-20	-82	-82	42	-82	-82	104	104	42	-82	104	166	-20	-20	166	-20	104	104	-20	-82	-20	42	-144
8	-82	-82	-82	-20	166	-82	-20	-20	0	42	-82	-82	42	-82	-144	104	166	42	-20	42	104	-20	-82	104	-20	104	104	-82	-20	-82	104
9	104	-20	-144	-82	-20	104	-82	-82	42	0	104	-20	-20	-20	-82	-82	104	104	42	-20	42	166	-20	-82	42	-82	166	104	-82	-20	-82
10	-20	104	-20	-82	-20	-20	42	-82	-82	104	0	104	-20	-20	-82	-82	-20	104	166	-20	-82	42	104	-82	-82	42	-82	104	166	-144	42
11	104	-20	104	42	-20	-20	-82	42	-82	-20	104	0	104	-20	-82	-82	-20	-20	166	104	-82	-82	-20	42	-82	-82	42	-144	166	104	-82
12	-20	104	-20	166	104	-20	-82	-82	42	-20	-20	104	0	104	-82	-82	-20	-20	42	104	42	-82	-144	-82	42	-82	-82	-20	-82	104	166
13	104	-20	104	-82	104	104	42	-82	-82	-20	-20	-20	104	0	166	-82	-144	-20	-82	104	166	42	-20	-82	-82	42	-82	-20	-82	-20	42
14	42	42	42	104	-82	166	104	104	-144	-82	-82	-82	-82	166	0	104	-82	-82	-20	-82	104	104	42	-20	-20	-20	-20	-82	-20	-82	-20
15	42	42	42	104	166	-82	104	104	104	-82	-82	-82	-82	-82	104	0	166	-82	-20	-82	-144	104	42	-20	-20	-20	-20	-82	-20	-82	-20
16	-20	104	-20	42	104	104	-82	42	166	104	-20	-20	-20	-144	-82	166	0	104	-82	-20	-82	-82	104	42	-82	-82	42	-20	-82	-20	-82
17	-144	-20	104	-82	-20	104	166	-82	42	104	104	-20	-20	-20	-82	-82	104	0	42	-20	42	-82	-20	166	42	-82	-82	104	-82	-20	-82
18	-82	-82	-82	104	-82	-82	104	104	-20	42	166	166	42	-82	-20	-20	-82	42	0	42	-20	104	-82	-20	104	-20	-20	-82	104	-82	-20
19	-20	-144	-20	-82	104	-20	-82	166	42	-20	-20	104	104	104	-82	-82	-20	-20	42	0	42	-82	104	-82	42	166	-82	-20	-82	104	-82
20	-82	-82	-82	-20	-82	166	-20	-20	104	42	-82	-82	42	166	104	-144	-82	42	-20	42	0	-20	-82	104	-20	104	104	-82	-20	-82	104
21	166	-82	-82	-20	42	-82	104	-20	-20	166	42	-82	-82	42	104	104	-82	-82	104	-82	-20	0	-82	-144	104	-20	104	42	-20	-82	-20
22	-20	104	-20	-82	-20	104	-82	166	-82	-20	104	-20	-144	-20	42	42	104	-20	-82	104	-82	-82	0	-82	-82	166	-82	104	42	-20	-82
23	-82	-82	166	-20	-82	42	104	-20	104	-82	-82	42	-82	-82	-20	-20	42	166	-20	-82	104	-144	-82	0	-20	-20	104	-82	104	42	-20
24	-82	-82	-82	104	-82	-82	104	104	-20	42	-82	-82	42	-82	-20	-20	-82	42	104	42	-20	104	-82	-20	0	-20	-20	166	-144	166	-20
25	-82	-82	-82	-144	42	-82	-20	104	104	-82	42	-82	-82	42	-20	-20	-82	-82	-20	166	104	-20	166	-20	-20	0	-20	42	104	-82	104
26	166	-82	-82	-20	-82	42	-144	-20	104	166	-82	42	-82	-82	-20	-20	42	-82	-20	-82	104	104	-82	104	-20	-20	0	-82	104	42	-20
27	-20	104	-20	-82	-20	-20	42	-82	-82	104	104	-144	-20	-20	-82	-82	-20	104	-82	-20	-82	42	104	-82	166	42	-82	0	-82	104	42
28	42	42	42	-20	-82	-82	-20	-20	-20	-82	166	166	-82	-82	-20	-20	-82	-82	104	-82	-20	-20	42	104	-144	104	104	-82	0	-82	104
29	104	-20	104	42	-20	-20	-82	42	-82	-20	-144	104	104	-20	-82	-82	-20	-20	-82	104	-82	-82	-20	42	166	-82	42	104	-82	0	-82
30	-82	166	-82	104	42	-82	-20	-144	104	-82	42	-82	166	42	-20	-20	-82	-82	-20	-82	104	-20	-82	-20	-20	104	-20	42	104	-82	0

Fig. B.27. Correlation difference matrix; $\{x_i\} \Rightarrow [5, 2, 0]$, $\{y_i\} \Rightarrow [5, 2, 0]$, $k = 11$

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Cooley Electronics Laboratory
The University of Michigan
Ann Arbor, Michigan

2a. REPORT SECURITY CLASSIFICATION

2b. GROUP

3. REPORT TITLE

PARTIAL-PERIOD CORRELATION OF PSEUDO-RANDOM BINARY SEQUENCES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

C. E. L. Technical Report 202 March 1970

5. AUTHOR(S) (First name, middle initial, last name)

James L. Daws, Jr.

6. REPORT DATE

March 1970

7a. TOTAL NO. OF PAGES

7b. NO. OF REFS

8a. CONTRACT OR GRANT NO.

DAAB07-68-C-0138

b. PROJECT NO.

1 HO 21101 A04 01 02

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

01482-14-T TR202

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

ECOM-0138-14-T

10. DISTRIBUTION STATEMENT

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

U. S. Army Electronics Command
Fort Monmouth, New Jersey 07703
Attn: AMSEL-WL-S

13. ABSTRACT

The partial-period correlation between linear maximal sequences of the same length is studied from a numberof viewpoints. The purpose of this study of the partial-period correlation is to determine how this correlation function depends on the particular sequences used. It is shown that the study is directly related to the study of weight distributions of linear codes and the stndard coding theory bounds are applied to the correlation problem. The coding theory bounds are used to determine how the peak values of correlation behave as the correlation interval is decreased, and as a means of establishing a range of possible correlation peaks. Peak values of the partial-period correlation function for selected pairs of sequences are compared with the bounds and for the cases examined the peak values are not close to the bounds except at the end points. The moments of the distribution of correlation values are calculated. The moments provide a method for comparing the correlation distributions for pairs of sequences. The correlation function is calculated in terms of the Fourier series coefficients of the periodic sequences. The frequency domain description of the correlation function leads to extremely complex equations which provide another technique for examining the structure of the partial-period correlation function. The correlation ambiguity matrix is introduced as a technique for displaying the partial-period correlation functions. The difference matrix is also introduced as a possible tool in the study of partial-period correlation functions. This matrix is a direct method of displaying how the full period correlation function is modified as a correlation interval is decreased.

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Pseudo-random sequences						
Partial-period correlation						
Coding theory						
Correlation distribution						
Fourier series						