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Unified Solution of the Expected Maximum of a Discrete Time Random Walk and the Discrete Flux to a Spherical Trap

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Two random-walk related problems which have been studied independently in the past, the expected maximum of a random walker in one dimension and the flux to a spherical trap of particles undergoing discrete jumps in three dimensions, are shown to be closely related to each other and are studied using a unified approach as a solution to a Wiener-Hopf problem. For the flux problem, this work shows that a constant $c = 0.29795219$ which appeared in the context of the boundary extrapolation length, and was previously found only numerically, can be derived analytically. The same constant enters in higher-order corrections to the expected-maximum asymptotics. As a byproduct, we also prove a new universal result in the context of the flux problem which is an analogue of the Sparre Andersen theorem proved in the context of the random walker's maximum.

KEY WORDS: Random walk, adsorption to a trap, Wiener-Hopf, diffusion, Sparre Anderson theorem.

1. INTRODUCTION

Random walks arise in an astounding variety of problems in physics as well as mathematics, computer science, etc., and great progress has been made in solving many of their deep and subtle properties. Two seemingly unrelated problems, the expected maximum of a random walker in *one* dimension undergoing jumps drawn from a *uniform* distribution, and the flux to a trap of particles undergoing random walks in *three* dimensions, have been studied independently over the last several

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years, and their solutions seem to involve a similar numerical constant 0.29795219, a coincidence that has not been noticed before. The similarity of these constants suggests that these two problems may be intimately related. For the first problem, the constant was first computed numerically by evaluating a rather complicated double series expansion[1,2] and very recently, an exact closed form expression of the constant has been found[3] that is valid not just for the uniform jump distribution, but for any arbitrary continuous and symmetric jump distribution. For the second problem of flux to a spherical trap, the corresponding constant was computed only numerically[4]. Therefore, finding the relation between the two problems raises the possibility that the flux problem can be solved analytically for the first time. Indeed, this is what we accomplish in this paper.

The two problems we consider are:

Problem I. The asymptotic behavior of the expected maximum position of a discrete time random walker moving on a continuous line. The position x_n of the walker after n steps evolves for $n \geq 1$ via,

$$x_n = x_{n-1} + \xi_n \tag{1}$$

starting at $x_0 = 0$, where the step lengths ξ_n's are independent and identically distributed (i.i.d.) random variables with zero mean and each drawn from the same probability distribution, $\text{Prob}(\xi_n \leq x) = \int_{-\infty}^{x} f(y)\,dy$, $f(x)$ being a continuous and symmetric probability density normalized to unity. Let M_n denote the positive maximum of the random walk up to n steps (see Fig. 1),

$$M_n = \max(0, x_1, x_2, \ldots, x_n). \tag{2}$$

We are interested in the asymptotic large-n behavior of the expected maximum $E(M_n)$. This question arose some years ago in the context of a packing problem in two dimensions where n rectangles of variable sizes are packed in a semi-infinite strip of width one[1,2]. It was shown in ref. 2 that for the special case of the uniform jump distribution, $f(x) = 1/2$ for $-1 \leq x \leq 1$ and $f(x) = 0$ outside, for large n,

$$E[M_n] = \sqrt{\frac{2n}{3\pi}} - 0.29795219028\cdots + O(n^{-1/2}). \tag{3}$$

The leading $\sqrt{n}$ behavior is easy to understand and can be derived from the corresponding behavior of a continuous-time Brownian motion after a suitable rescaling[2]. However, the leading finite-size correction term turns out to be a non-trivial constant $-c$ with $c = 0.29795219028\ldots$ that was computed in ref. 2 by enumerating a somewhat intricate double series obtained after a lengthy calculation. Recently, it was shown [3] that for arbitrary continuous and symmetric jump

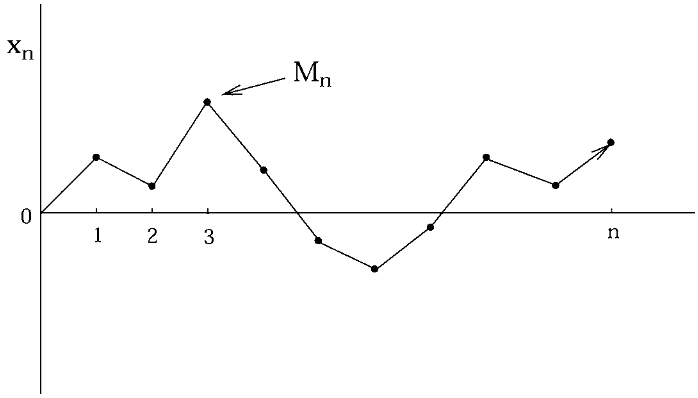

Fig. 1. A typical configuration of a random walker in one dimension up to n steps, starting at 0 at $n = 0$ with M_n denoting the maximum up to n steps.

distribution $f(x)$ with a finite second moment $\sigma^2 = \int_0^\infty x^2 f(x)dx$, the expected maximum has a similar asymptotic behavior as in the uniform case, namely,

$$E[M_n] = \sigma\sqrt{\frac{2n}{\pi}} - c + O(n^{-1/2}). \tag{4}$$

Moreover, an exact expression for the constant c was found[3]

$$c = -\frac{1}{\pi}\int_0^\infty \frac{dk}{k^2}\ln\left[\frac{1-\hat{f}(k)}{\sigma^2 k^2/2}\right], \tag{5}$$

where $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$ is the Fourier transform of $f(x)$. In particular, for the uniform distribution, $f(x) = 1/2$ for $-1 \le x \le 1$ and $f(x) = 0$ outside, one has $\hat{f}(k) = \sin(k)/k$ and one gets from Eq. (5) an exact expression,

$$c = -\frac{1}{\pi}\int_0^\infty \frac{dk}{k^2}\ln\left[\frac{6}{k^2}\left(1-\frac{\sin k}{k}\right)\right] = 0.29795219028\ldots \tag{6}$$

Problem II. The calculation of flux to a spherical trap in three dimensions. Consider first the classic Smoluchowski problem[5] where point particles are initially distributed uniformly with density ρ_0 outside a sphere of radius R in three dimensions. Each particle subsequently performs continuous-time Brownian motion with a diffusion constant D, independent of each other. One is interested in computing the flux of particles $\Phi(t)$ to the sphere at time t. This can be done by solving the diffusion equation outside the sphere with an absorbing boundary condition on the surface of the sphere and the result is well known[5–7]. One gets

$$\Phi(t) = 4\pi R D\rho_0 \left[1 + \frac{R}{\sqrt{\pi\, D\, t}}\right]. \tag{7}$$

valid for all $t > 0$. Also, as $t \to \infty$, the density profile outside the sphere becomes time independent and has a simple form

$$\rho(r) = \rho_0 \left(1 - \frac{R}{r}\right) \tag{8}$$

for all $r \geq R$. Far from the sphere the density remains unchanged from its initial value ρ_0 and as one approaches the surface of the sphere, the density vanishes.

An interesting issue, first studied in ref. 4, is how do the steady-state profile in Eq. (8) and correspondingly the expression of flux in Eq. (7) get modified when each of the point particles, instead of performing continuous-time Brownian diffusion, undergoes discrete 'Rayleigh flights', i.e. a particle jumps, at every discrete time step τ, a fixed step length l whose direction is chosen arbitrarily in the three-dimensional space (see Fig. 2). In ref. 4, it was shown that the expression for the flux, at late times $t = n\tau$ and for $0 < l \leq 2R$, now gets replaced by

$$\Phi(t) = 4\pi(R - c'\, l)D'\rho_0 \left[1 + \frac{R - c'\, l}{\sqrt{\pi\, D'\, t}} + O(t^{-3/2})\right], \tag{9}$$

where $D' = l^2/6\tau$ and c' is a constant whose numerical value was obtained by an iterative numerical solution of the density profile, with the result

$$c' \approx 0.29795219 \tag{10}$$

The density profile $\rho_n(r)$ after n steps also gets modified rather drastically, as found numerically in ref. 4. In particular, the steady-state density profile $\rho_\infty(r)$ in the discrete problem turns out to be quite different from its continuous-time counterpart. While very far away from the sphere the steady-state density profile behaves as

$$\rho_\infty(r \gg R) = \rho_0 \left(1 - \frac{R - c'l}{r}\right) \tag{11}$$

where c' is as in Eq. (10), the density actually tends to a nonzero constant as one approaches the surface of the sphere from outside

$$\rho_\infty(r \to R) = 0.408245\, \frac{\rho_0 l}{R} \tag{12}$$

where the constant 0.408245 was evaluated numerically in ref. 4. This is in stark contrast to the continuous-time Brownian case where the density vanishes on the surface of the sphere.

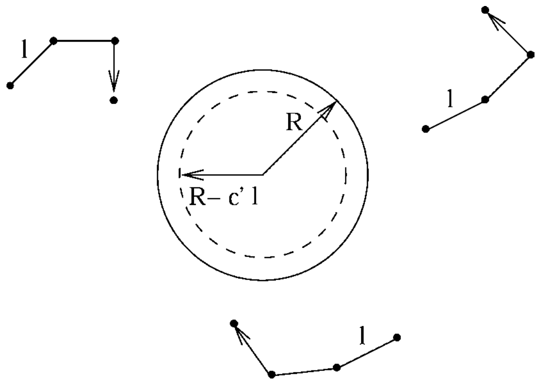

Fig. 2. Independent Rayleigh flights in three dimensions in presence of a spherical trap. The discreteness of the jumps shows up effectively in a renormalized sphere with a smaller radius where $c' l$ is the Milne extrapolation length.

The distance $c' l$ is the 'Milne extrapolation length'[(8–10)] and represents the distance inside the surface where the far steady-state solution in Eq. (11) extrapolates to $\rho_\infty = 0$. This steady-state density distribution implies the leading-order term of the flux given in Eq. (9). Thus at late times, the continuum formula for the flux in Eq. (7) still remains valid for the discrete jump case, but with an effectively smaller radius $R - c' l$ of the trap as in Eq. (9). Thus the effect on the flux due to the discrete nature of the jumps, at least at late times, is simply to renormalize the radius of the trap to a smaller value.

Comparing Eqs. (6) and (10) one finds, rather amazingly, that the two constants c and c', in these two *a priori* unrelated problems, are identical at least up to 8 decimal places! This raises an interesting question: are they equal? In this paper, we indeed prove that $c = c'$. In the process, we also provide exact solutions to many other features of the flux problem that were observed numerically in ref. 4. For example, we will calculate exactly the steady-state density profile and will prove that indeed it approaches to a constant on the surface of the sphere as in Eq. (12) and the constant 0.408245 is actually $1/\sqrt{6}$. Our method consists in showing that both of these problems can be cast into the same Wiener-Hopf type problem involving an integral equation over half-space, albeit with different initial conditions. We then obtain explicit solutions to this Wiener-Hopf problem with these two different initial conditions. The general solution turns out to be a product of two parts, one that explicitly depends on the initial condition and the other part which is a 'Green's function' that is independent of the initial condition. The constant c, given by the exact expression in Eq. (6), is part of this 'initial condition independent' Green's function and hence it appears in both problems.

2. MAXIMUM OF A RANDOM WALKER AS A WIENER-HOPF PROBLEM

We consider a discrete time random walker hopping on a continuous line. The position x_n of the walker evolves via Eq. (1), starting at $x_0 = 0$. The maximum M_n, defined in Eq. (2), is a random variable. Let $q_n(z) = \mathrm{Prob}(M_n \leq z)$ denote the cumulative distribution of the maximum and $q_n'(z) = dq_n/dz$ its probability density with $x \in [0, \infty[$. Then, a simple integration by parts gives

$$E[M_n] = \int_0^\infty z q_n'(z) dz = -\int_0^\infty z \frac{d}{dz}(1 - q_n(z))\, dz = \int_0^\infty (1 - q_n(z))\, dz, \tag{13}$$

where we have used the normalization condition, $q_n(\infty) = 1$. Using translational invariance and the fact that the jump distribution is symmetric, one can also interpret $q_n(z)$ as the probability that a random walker, starting initially at position $z > 0$, stays positive up to step n. Then it is easy to write down, using the Markov property of the evolution in Eq. (1), the following recurrence relation[3], valid for all $z \geq 0$,

$$q_n(z) = \int_0^\infty q_{n-1}(z') f(z - z')\, dz', \quad \text{starting with} \quad q_0(z) = 1. \tag{14}$$

The generating function $\tilde{q}(z, s) = \sum_{n=0}^{\infty} q_n(z) s^n$ then satisfies an inhomogeneous Wiener-Hopf integral equation [3]

$$\tilde{q}(z, s) = s \int_0^\infty \tilde{q}(z', s) f(z - z') dz' + q_0(z), \tag{15}$$

where the inhomogeneous term $q_0(z) = 1$ arises from the initial condition. We need to thus solve this integral equation to obtain the full probability distribution $q_n(z)$ of the maximum. The mean value $E[M_n]$ can then be computed from Eq. (13).

3. FLUX TO A TRAP AS A WIENER-HOPF PROBLEM

We consider a sphere of radius R in three dimensions. Outside the sphere point particles are initially distributed with uniform density ρ_0. Particles subsequently perform independent Rayleigh flights, i.e. at every time step τ, each particle, independently of others, jumps a fixed distance l in a direction chosen randomly. The object of interest is the flux at late times t to the sphere. For simplicity, we assume $\tau = 1$ (so that the jumps occur at integer steps) and also $l = 1$. Since the particles are independent, one can alternately think of a single particle whose probability density $\rho_n(\vec{r})$ at $\vec{r}$ after n steps evolves via the Markov equation [4]

$$\rho_n(\vec{r}) = \int W(\vec{r}\,|\,\vec{r}\,')\rho_{n-1}(\vec{r}\,')d\vec{r}\,' \tag{16}$$

where $W(\vec{r}|\vec{r}\,') = \delta(|\vec{r} - \vec{r}\,'| - 1)/4\pi$ is the jump probability density at each step from $\vec{r}\,'$ to $\vec{r}$ and the integral in Eq. (16) extends over the full three-dimensional space outside the sphere of radius R. The recursion relation in Eq. (16) starts with the initial condition, $\rho_0(\vec{r}) = \rho_0$ for $r > R$ and $\rho_0(\vec{r}) = 0$ for $r \leq R$. Since the initial condition is spherically symmetric, it is clear that from Eq. (16) that this symmetry will be maintained at all n, implying $\rho_n(\vec{r}) = \rho_n(r)$. Thanks to this spherical symmetry, the three-dimensional problem thus becomes a one-dimensional problem where one considers only the radial direction after integrating out the angular coordinates. Defining $P_n(r) \equiv 4\pi r^2 \rho_n(r)$ as the probability that the particle is in the shell $[r, r + dr]$ after n steps, it follows that $P_n(r)$ evolves via the recurrence equation

$$P_n(r) = \int w(r|r')P_{n-1}(r')dr' \quad \text{starting with} \quad P_0(r) = 4\pi r^2 \rho_0 \theta(r - R) \tag{17}$$

where $\theta(x)$ is the Heaviside theta function and $w(r|r')$ is the jump probability density from radius r' to r, which can be easily calculated by integrating the kernel $W(\vec{r}|\vec{r}')$ over the angular coordinates[4,11]

$$\begin{aligned} w(r|r')dr &= \frac{r\,dr}{2r'} \qquad \text{for} \quad |r' - 1| < r < r' + 1 \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{18}$$

To simplify, one introduces a new quantity $F_n(r) = P_n(r)/4\pi\rho_0 r$. The recursion relation for $F_n(r)$, upon substituting the explicit form of $w(r|r')$ from Eq. (18) in Eq. (17), then simplifies

$$\begin{aligned} F_n(r) &= \frac{1}{2}\int_{\max(R,r-1)}^{r+1} F_{n-1}(r')dr' \\ &= \int_R^{\infty} F_{n-1}(r')f(r - r')dr' \end{aligned} \tag{19}$$

where $f(y)$ corresponds to the uniform probability density over the interval $y \in [-1, 1]$, i.e. $f(y) = 1/2$ if $-1 \leq y \leq 1$ and $f(y) = 0$ otherwise. The recursion in Eq. (19) now starts with the initial condition, $F_0(r) = r\theta(r - R)$. One can simplify Eq. (19) further by introducing a shift, i.e. defining $z = r - R$, and writing $F_n(r) = F_n(z + R) = Q_n(z)$, Eq. (19) becomes, for all $z > 0$,

$$Q_n(z) = \int_0^{\infty} Q_{n-1}(z')f(z - z')\,dz', \qquad \text{starting with} \quad Q_0(z) = R + z \tag{20}$$

Defining the generating function, $\tilde{Q}(z, s) = \sum_{n=0}^{\infty} Q_n(z)s^n$, one obtains an identical Wiener-Hopf integral equation as in Eq. (15),

$$\tilde{Q}(z, s) = s\int_0^{\infty} \tilde{Q}(z', s)f(z - z')dz' + Q_0(z), \tag{21}$$

the only difference is in the inhomogeneous term $Q_0(z) = R + z$ that is set by the initial condition. One then needs to solve this integral equation to obtain $Q_n(z)$, from which one can read off the density profile at step n

$$\rho_n(r) = \frac{P_n(r)}{4\pi r^2} = \frac{\rho_0}{r} F_n(r) = \frac{\rho_0}{r} Q_n(r - R). \tag{22}$$

The flux to the sphere at time step n can then be computed from the following relation[4]

$$\Phi(n) = \int_R^\infty dr' \int_0^R dr\, w(r\,|\,r') P_{n-1}(r') \tag{23}$$

$$= \pi\rho_0 \int_R^{R+1} [R^2 - (r' - 1)^2] F_{n-1}(r') dr' \tag{24}$$

$$= \pi\rho_0 \int_0^1 [2R(1 - z) - (1 - z)^2]\, Q_{n-1}(z)\, dz \tag{25}$$

In going from Eq. (23) to (24) we have used the explicit form of $w(r|r')$ in Eq. (18).

We end this section with one remark. For a continuous-time Brownian motion it is quite standard[6,12] that, using the transformation $\rho(r, t) = \rho_0 F(r, t)/r$, the 3-d diffusion equation for the density field $\rho(r, t)$ can be reduced to a 1-d diffusion equation; the same trick naturally works for the 3-d Schrödinger equation as well. Based on this fact, it is natural that a similar transformation $\rho_n(r) = \rho_0 F_n(r)/r$ would also work for the discrete-time problem. However, the fact, that the reduced 1-d problem satisfies exactly the same integral equation (albeit with a different initial condition) with the same *uniform* kernel as the 1-d maximum displacement problem, is hard to guess apriori without the explicit calculation as presented here.

4. WIENER-HOPF PROBLEM

We have seen from the previous sections that the two *a priori* different problems (I) maximum of a random walker hoping on a line and (II) flux to a spherical trap in three dimensions can be both recast as the same Wiener-Hopf integral equation problem, albeit with different inhomogeneous terms arising due to the difference in the initial conditions of the two problems. The general mathematical problem then is to solve the following half-space inhomogeneous integral equation for $z > 0$

$$\psi(z, s) = s \int_0^\infty \psi(z', s)\, f(z - z') dz' + J(z). \tag{26}$$

where the inhomogeneous source term $J(z)$ is different for the two problems

$$J(z) = 1 \qquad \text{for Problem I} \tag{27}$$

$$= R + z \qquad \text{for Problem II.} \tag{28}$$

Even though in both of these problems the kernel $f(z - z')$ corresponds to the uniform jump density, i.e.

$$f(x) = \frac{1}{2} \qquad \text{for} \quad -1 \leq x \leq 1$$

$$= 0 \qquad \text{otherwise,} \tag{29}$$

it is useful to study the integral Eq. (26) with a general continuous and symmetric kernel $f(z) = f(-z)$ that is normalized $\int_{-\infty}^{\infty} f(z)dz = 1$ and has a finite second moment $\sigma^2 = \int_{-\infty}^{\infty} z^2 f(z)dz$.

The explicit solution $\psi(z, s)$ of Eq. (26) with different source terms as in Eqs. (27) and (28) will then provide the solutions to the two problems. In Problem I, $\psi(z, s) = \tilde{q}(z, s) = \sum_{n=0}^{\infty} q_n(z)s^n$ provides the generating function for the cumulative distribution of the maximum of the random walk up to n steps. In Problem II, $\psi(z, s) = \tilde{Q}(z, s) = \sum_{n=0}^{\infty} Q_n(z)s^n$ gives the generating function for the density profile $\rho_n(r) = \rho_0 Q_n(z = r - R)/r$ of the particles outside the sphere of radius R in three dimensions. It turns out, as will be shown later explicitly, that the difference in the source term in Eqs. (27) and (28) actually leads to completely different types of solutions to the integral equation (26). In Problem I, the solution $q_n(z)$ depends explicitly on n even at late times, i.e. for large n, and does not have an n-independent stationary solution. Rather it has a scaling solution involving both z and n. In contrast, the solution $Q_n(z)$ in Problem II approaches an n-independent stationary solution.

4.1. Explicit Solution for Exponential Kernel

Before providing the general solution for arbitrary continuous and symmetric kernel $f(z)$, it is instructive to derive the explicit solutions with the two different source terms for a special kernel $f(z) = \exp[-|z|]/2$. This will clearly bring out how the different source terms lead to different behavior of the same integral equation. The exponential kernel $f(z) = \exp[-|z|]/2$ is special since one can recast the integral equation (26) into a differential equation by using the identity $f''(z) = f(z) - \delta(z)$, where $f''(z) = d^2 f/dz^2$. Differentiating Eq. (26) twice with respect to z and using the above identity, one gets for all $z > 0$

$$\frac{d^2\psi}{dz^2} = (1 - s)\psi(z, s) + J''(z) - J(z). \tag{30}$$

Consider first Problem I where $J(z) = 1$. Then the most general solution of Eq. (30) is given by,

$$\psi_{\rm I}(z,s) = \frac{1}{(1-s)} + A_1(s)\, e^{-\sqrt{1-s}\,z} + B_1(s)\, e^{\sqrt{1-s}\,z}, \tag{31}$$

where $A_1(s)$ and $B_1(s)$ are two arbitrary z independent constants. Since the solution cannot diverge exponentially as $z \to \infty$, one gets $B_1(s) = 0$. We need to still determine the constant $A_1(s)$. Here we use a method that is slightly different from that used in ref. 3. We substitute the solution in Eq. (31) into the integral equation (26). Performing the integration explicitly, one finds that the solution in Eq. (31) satisfies the integral equation if and only if $A_1(s) = -[1-\sqrt{1-s}]/(1-s)$. Thus, the full solution is given by[3]

$$\psi_{\rm I}(z,s) = \sum_{n=0}^{\infty} q_n(z) s^n = \frac{1}{(1-s)} - \frac{1-\sqrt{1-s}}{1-s}\, e^{-\sqrt{1-s}\,z}. \tag{32}$$

One can then get the expected maximum from this explicit solution by an integration and an expansion in powers of s[3]

$$E[M_n] = \frac{2}{\sqrt{\pi}}\frac{\Gamma(n+3/2)}{\Gamma(n+1)} - 1 \simeq 2\sqrt{\frac{n}{\pi}} - 1 \qquad \text{as} \qquad n \to \infty, \tag{33}$$

which is of the same general form as in Eq. (4) with $\sigma = \sqrt{2}$ and $c = 1$. In addition, it is also instructive to derive the solution $q_n(z)$ for large n by analysing its generating function in Eq. (32) in the vicinity of $s = 1$. Taking the limits $s \to 1$ and $z \to \infty$ but keeping $z\sqrt{1-s}$ fixed, one can replace the generating function by a Laplace transform, and inverting the Laplace transform one gets the scaling solution valid for large n

$$q_n(z) \simeq \text{erf}\left(\frac{z}{\sqrt{4n}}\right) + \frac{1}{\sqrt{\pi n}}\, e^{-z^2/4n}, \tag{34}$$

where $\text{erf}(z) = 2\pi^{-1/2}\int_0^z e^{-u^2}\,du$ is the error function. Note that the first term on the rhs of Eq. (34) corresponds to the continuum solution of the diffusion equation for a particle starting at $z > 0$ and staying above an absorbing boundary at 0, which is also the same as the cumulative probability that the maximum of a continuous-time Brownian motion stays below z up to time t, provided one makes the standard correspondence $\sigma^2 n = 2Dt$ between the discrete step number n and the continuous time t, D being the diffusion constant for the Brownian motion. The second term on the rhs of Eq. (34) corresponds to the leading correction due to discrete jumps and indeed is responsible for the constant c in Eq. (4). This can be seen by substituting the scaling solution in Eq. (34) in the exact relation, $E[M_n] = \int_0^\infty (1 - q_n(z))dz$. Upon integrating, one recovers the large-n asymptotic solution in Eq. (33) and one sees explicitly that indeed the constant $c = 1$ in Eq. (33) arises from the integration of the second term in Eq. (34).

We now turn to Problem II where $J(z) = R + z$. Proceeding exactly as in the first case one finds that the explicit solution of the differential equation (30) is given by

$$\psi_{\mathrm{II}}(z, s) = \frac{R + z}{1 - s} + A_2(s)\, e^{-\sqrt{1-s}\, z}, \tag{35}$$

where we have used the boundary condition that the solution cannot diverge exponentially as $z \to \infty$. Substituting this solution in the integral equation (26) fixes the constant $A_2(s)$ and we get the full solution,

$$\psi_{\mathrm{II}}(z, s) = \sum_{n=0}^{\infty} Q_n(z) s^n = \frac{R + z}{1 - s} + (1 - R)\, \frac{1 - \sqrt{1 - s}}{(1 - s)}\, e^{-\sqrt{1-s}\, z}. \tag{36}$$

The behavior of $Q_n(z)$ for large n can be derived by analysing the generating function near $s = 1$. In this case, one finds that for large n,

$$Q_n(z) \simeq (z + 1) + (R - 1)\, \mathrm{erf}\left(\frac{z}{\sqrt{4n}}\right) + \frac{R - 1}{\sqrt{\pi n}}\, e^{-z^2/4n}. \tag{37}$$

Comparison with the asymptotic solution of Problem I in Eq. (34) shows that in Problem II, the solution approaches an n-independent stationary solution as $n \to \infty$

$$Q_\infty(z) = z + 1 \qquad \text{for all} \quad z \geq 0. \tag{38}$$

This solution is also independent of R; all terms containing R in Eq. (37) disappear in the long-time limit. The corrections to this stationary solution for large n have the scaling forms similar to Problem I.

In fact, we will show in the next section that quite generically, i.e. for arbitrary continuous and symmetric kernel $f(z)$, the solution $Q_n(z)$ for Problem II always approaches a stationary solution $Q_\infty(z)$ which is, generically, a nontrivial function of z. However, for large z, we will show that this stationary solution has a rather simple asymptotic,

$$Q_\infty(z) \simeq z + c' \qquad \text{as} \quad z \to \infty, \tag{39}$$

where the constant c' will be shown to be exactly equal to c in Eq. (5). In particular, for the uniform distribution $f(z)$ given in Eq. (29) where one can relate back to the original 3-d flux problem, we will show that indeed this same constant $c' = c$ appears as the extrapolation length in the expression for flux in Eq. (9). In the particular example of the exponential kernel $f(z) = \exp[-|z|]/2$, we see explicitly that indeed $c' = c = 1$ by inspecting Eqs. (38) and (33). This thus proves a special case of the general result $c' = c$ valid for arbitrary continuous and symmetric kernel $f(z)$. Note also that for this special case of the exponential kernel, the stationary solution $Q_\infty(z)$ in Eq. (38) actually retains its asymptotic form in Eq. (39) all the way down to $z = 0$. This property, however, is rather special to the exponential kernel. For a generic continuous and symmetric kernel, $Q_\infty(z)$ has a nontrivial form for small z as will be shown in the next section.

Another quantity of interest, as we will see later in a more general context, is the transient behavior of $Q_n(0)$ for Problem II. It follows by subsituting $z = 0$ in Eq. (36)

$$\sum_{n=0}^{\infty} Q_n(0)s^n = \frac{1}{1-s} + \frac{R-1}{\sqrt{1-s}}. \tag{40}$$

Expanding the rhs of Eq. (40) in powers of s one gets

$$Q_n(0) = 1 + (R-1)\binom{2n}{n}\frac{1}{2^{2n}} \simeq 1 + \frac{(R-1)}{\sqrt{\pi n}} \qquad \text{as} \qquad n \to \infty. \tag{41}$$

We will see later that this transient behavior for the exponential kernel confirms, as a special case, the validity of a general result in Eq. (96) proved for arbitrary continuous and symmetric kernels.

5. GENERAL SOLUTION TO THE WIENER-HOPF PROBLEM

In this section, we present an explicit solution to the integral equation (26) for the two different inhomogeneous terms in Eqs. (27) and (28). Our result is valid for any arbitrary continuous and symmetric kernel $f(z)$ that is normalized, $\int_{-\infty}^{\infty} f(z)dz = 1$ and with a finite second moment $\sigma^2 = \int_{-\infty}^{\infty} z^2 f(z)dz$. Our method relies on a beautiful general formalism developed by Ivanov[13] to deal with half-space problems in the context of photon scattering. Let us first summarize this formalism. Consider the integral equation (26) with an arbitrary source term $J(z)$. There are three steps to obtain the solution.

1. The first step is to define a Green's function $G(z, z_1, s)$ that satisfies the same integral equation but with a delta function source term, i.e.

$$G(z, z_1, s) = s\int_0^{\infty} G(z', z_1, s)f(z-z')dz' + \delta(z - z_1). \tag{42}$$

It is then easy to see that the solution of the inhomogeneous equation (26) is given by

$$\psi(z, s) = \int_0^{\infty} G(z, z_1, s)J(z_1)dz_1. \tag{43}$$

2. The next step is to determine the Green's function $G(z, z_1, s)$ that satisfies Eq. (42). One first defines the double Laplace transform,

$$\tilde{G}(\lambda, \lambda_1, s) = \int_0^\infty dz\, e^{-\lambda z} \int_0^\infty dz_1\, e^{-\lambda_1 z_1}\, G(z, z_1, s). \tag{44}$$

Ivanov showed that this double Laplace transform can be determined in closed form by solving Eq. (42) and is given by a simple form[13]

$$\tilde{G}(\lambda, \lambda_1, s) = \frac{\phi(s, \lambda)\, \phi(s, \lambda_1)}{\lambda + \lambda_1}, \tag{45}$$

where the function $\phi(s, \lambda)$ is the following Laplace transform

$$\phi(s, \lambda) = \int_0^\infty dz\, e^{-\lambda z}\, G(z, 0, s) \tag{46}$$

3. The third step is to obtain an explicit expression[13] for the function $\phi(s, \lambda)$

$$\phi(s, \lambda) = \exp\left[-\frac{\lambda}{\pi} \int_0^\infty \frac{\ln(1 - s\hat{f}(k))}{\lambda^2 + k^2}\, dk \right], \tag{47}$$

where $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$ is the Fourier transform of the kernel $f(x)$.

Substituting the explicit expression for $\phi(s, \lambda)$ from Eq. (47) into Eq. (45), one has an explicit expression for the double Laplace transform $G(\lambda, \lambda_1, s)$. By inverting this double transform, one can obtain the Green's function $G(z, z_1, s)$, at least in principle. Subsequently, by performing the integral in Eq. (43) one obtains the required solution $\psi(z, s)$. In practice, however, these last two steps are difficult to carry out explicitly in general. However, for the two special source terms in Eqs. (27) and (28), we show below that one can make progress.

5.1. General Solution for Problem I

Consider first Problem I where $J(z) = 1$. Then, Eq. (43) gives

$$\psi_{\mathrm{I}}(z, s) = \int_0^\infty G(z, z_1, s) dz_1. \tag{48}$$

Let us define the Laplace transform

$$\tilde{\psi}_{\mathrm{I}}(\lambda, s) = \int_0^\infty \psi_{\mathrm{I}}(z, s)\, e^{-\lambda z}\, dz. \tag{49}$$

Taking the Laplace transform with respect to z in Eq. (48) we get

$$\tilde{\psi}_{\rm I}(\lambda, s) = \int_0^\infty dz\, e^{-\lambda z} \int_0^\infty G(z, z_1, s)\, dz_1 = \tilde{G}(\lambda, 0, s). \tag{50}$$

Eqs. (45) and (47) then give

$$\tilde{\psi}_{\rm I}(\lambda, s) = \frac{1}{\lambda}\, \phi(s,\lambda)\, \phi(s, 0) \tag{51}$$

where $\phi(s,\lambda)$ is given in Eq. (47). Let us first evaluate $\phi(s, 0)$. Note that one cannot naïvely put $\lambda = 0$ in the expression in Eq. (47) since the integral multiplying λ inside the exponential in Eq. (47) diverges as $\lambda \to 0$. Hence one needs to extract the value of $\phi(s, 0)$ carefully. To achieve this, an alternate expression for $\phi(s,\lambda)$ that was obtained in ref 3 turns out to be useful. It was shown in ref. 3 that $\phi(s,\lambda)$ in Eq. (47) can also be written as

$$\phi(s,\lambda) = \frac{1}{\left[\sqrt{1-s} + \sigma\lambda\sqrt{s/2}\right]} \exp\left[-\frac{\lambda}{\pi}\int_0^\infty \frac{dk}{\lambda^2 + k^2} \ln\left(\frac{1 - s\hat{f}(k)}{1 - s + s\sigma^2 k^2/2}\right)\right]. \tag{52}$$

This representation is useful to derive the properties of $\phi(s,\lambda)$ near $\lambda = 0$. On the other hand, the representation in Eq. (47) is useful to extract the asymptotic behavior of $\phi(s,\lambda)$ for large λ. Taking $\lambda \to 0$ limit in Eq. (52) one gets

$$\phi(s, 0) = \frac{1}{\sqrt{1-s}}, \tag{53}$$

which, when substituted in Eq. (51) gives

$$\tilde{\psi}_{\rm I}(\lambda, s) = \sum_{n=0}^{\infty} s^n \int_0^\infty q_n(z)\, e^{-\lambda z}\, dz = \frac{1}{\lambda\sqrt{1-s}}\, \phi(s,\lambda) \tag{54}$$

where $\phi(s,\lambda)$ is defined in Eq. (47) and has also an alternative expression as in Eq. (52). This result in Eq. (54) goes by the name of the Pollaczek-Spitzer formula which was originally derived using completely different methods[14,15]. This result was subsequently utilized in ref. 3 to extract the constant c in Eq. (4) appearing as a subleading term for large n in the expected maximum $E[M_n]$ of a random walker.

5.2. General Solution for Problem II

We now turn to Problem II where $J(z) = R + z$. We get from Eq. (43)

$$\psi_{\rm II}(z, s) = \int_0^\infty (R + z_1) G(z, z_1, s) dz_1. \tag{55}$$

Taking the Laplace transform, $\tilde{\psi}_{\mathrm{II}}(\lambda, s) = \int_0^\infty \psi_{\mathrm{II}}(z, s)\, e^{-\lambda z}\, dz$ gives

$$\tilde{\psi}_{\mathrm{II}}(\lambda, s) = \int_0^\infty dz\, e^{-\lambda z} \int_0^\infty (R + z_1)\, G(z, z_1, s)\, dz_1$$
$$= R\, \tilde{G}(\lambda, 0, s) - \frac{\partial \tilde{G}(\lambda, \lambda_1, s)}{\partial \lambda_1}|_{\lambda_1=0}. \tag{56}$$

Eqs. (45) and (47) then give

$$\tilde{\psi}_{\mathrm{II}}(\lambda, s) = \frac{1}{\lambda}\left[\left(R + \frac{1}{\lambda}\right)\phi(s, 0) - \frac{\partial \tilde{\phi}(s, \lambda_1)}{\partial \lambda_1}|_{\lambda_1=0}\right]\phi(s, \lambda) \tag{57}$$

where $\phi(s, \lambda)$ is given in Eq. (47) or alternately in Eq. (52). Using the representation in Eq. (52) one gets

$$\frac{\partial \tilde{\phi}(s, \lambda_1)}{\partial \lambda_1}|_{\lambda_1=0} = -\frac{\sigma}{(1-s)}\sqrt{\frac{s}{2}} - \frac{1}{\pi\sqrt{1-s}}\int_0^\infty \frac{dk}{k^2}\ln\left(\frac{1 - s\hat{f}(k)}{1 - s + s\sigma^2 k^2/2}\right). \tag{58}$$

Substituting this result in Eq. (57) and using $\psi(s, 0) = 1/\sqrt{1-s}$ from Eq. (53) gives

$$\tilde{\psi}_{\mathrm{II}}(\lambda, s) = \frac{1}{\lambda}\left[\frac{1}{\sqrt{1-s}}\left(R + \frac{1}{\lambda}\right) + \frac{\sigma}{(1-s)}\sqrt{\frac{s}{2}} + \frac{1}{\pi\sqrt{1-s}}\right.$$
$$\left.\times \int_0^\infty \frac{dk}{k^2}\ln\left(\frac{1 - s\hat{f}(k)}{1 - s + s\sigma^2 k^2/2}\right)\right]\phi(s, \lambda). \tag{59}$$

5.3. Analysis of the Results

Thus the Laplace transform of the solution to the integral equation (26) for both problems respectively with $J(z) = 1$ (Problem I) and $J(z) = R + z$ (Problem II) can be written as a product of two functions

$$\tilde{\psi}(\lambda, s) = W(s, \lambda)\, \phi(s, \lambda) \tag{60}$$

where $\phi(s, \lambda)$ is given in Eq. (47) or in Eq. (52) and is independent of the source term. The function $W(s, \lambda)$, however, depends explicitly on the source term, i.e. on the initial conditions of the original recursion relations in Eqs. (14) and (20) and has different expressions for the two problems. While for Problem I it is rather simple

$$W_I(s, \lambda) = \frac{1}{\lambda\sqrt{1-s}}, \tag{61}$$

for Problem II it has a more complicated expression

$$W_{II}(s,\lambda) = \frac{1}{\lambda}\left[\frac{1}{\sqrt{1-s}}\left(R+\frac{1}{\lambda}\right) + \frac{\sigma}{(1-s)}\sqrt{\frac{s}{2}} + \frac{1}{\pi\sqrt{1-s}} \right.$$
$$\left. \times \int_0^\infty \frac{dk}{k^2} \ln\left(\frac{1-s\hat{f}(k)}{1-s+s\sigma^2k^2/2}\right)\right] \tag{62}$$

Since the function $\phi(s,\lambda)$ appears in both problems, it is useful to first list its asymptotic properties for small and large λ. For small λ, it is convenient to use the representation in Eq. (52). On the other hand, for large λ, the representation in Eq. (47) turns out to be more convenient. We find

$$\phi(s,\lambda) \simeq \frac{1}{\sqrt{1-s}} - \alpha(s)\lambda + O(\lambda^2) \qquad \text{as} \quad \lambda \to 0 \tag{63}$$

$$\simeq 1 - \frac{\beta(s)}{\lambda} + O(\lambda^{-2}) \qquad \text{as} \quad \lambda \to \infty \tag{64}$$

where the two functions $\alpha(s)$ and $\beta(s)$ are given by

$$\alpha(s) = \frac{\sigma}{(1-s)}\sqrt{\frac{s}{2}} + \frac{1}{\pi\sqrt{1-s}}\int_0^\infty \frac{dk}{k^2}\ln\left(\frac{1-s\hat{f}(k)}{1-s+s\sigma^2k^2/2}\right) \tag{65}$$

$$\beta(s) = \frac{1}{\pi}\int_0^\infty dk\, \ln\left[1-s\hat{f}(k)\right]. \tag{66}$$

Now we are ready to obtain the constants c in Problem I and c' in Problem II and show that indeed they are same, as they both emerge from the properties of the function $\phi(s,\lambda)$ that is common to both the problems. We consider the two problems separately.

Expected maximum in Problem I: Consider first Problem I. Consider the Laplace transform of the distribution of the maximum, $E[e^{-\lambda M_n}] \equiv \int_0^\infty e^{-\lambda z} q_n'(z)\, dz = \lambda \int_0^\infty e^{-\lambda z} q_n(z)\, dz$. It follows from Eq. (54)

$$\sum_{n=0}^{\infty} s^n E[e^{-\lambda M_n}] = \frac{1}{\sqrt{1-s}}\,\phi(s,\lambda). \tag{67}$$

We expand both sides in λ for small λ and use Eq. (63) for the expansion of the rhs of Eq. (67). Comparing the term linear in λ one gets

$$\sum_{n=0}^{\infty} s^n E[M_n] = \frac{\alpha(s)}{\sqrt{1-s}}, \tag{68}$$

where $\alpha(s)$ is given in Eq. (65). To extract the large n behavior of $E[M_n]$, we need to analyse the generating function in Eq. (68) near $s \to 1$. Expanding the rhs of

Eq. (68) near $s = 1$ one gets

$$\sum_{n=0}^{\infty} s^n E[M_n] \simeq \frac{\sigma}{\sqrt{2}\,(1-s)^{3/2}} + \frac{1}{\pi(1-s)} \int_0^\infty \frac{dk}{k^2} \ln\left[\frac{(1-\hat{f}(k))}{\sigma^2 k^2/2}\right] + O((1-s)^{-1/2}). \tag{69}$$

Inverting one gets the large n result[3]

$$E[M_n] = \sigma\sqrt{\frac{2n}{\pi}} - c + O(n^{-1/2}), \tag{70}$$

where the constant c is given by the integral in Eq. (5).

Steady-state density profile in Problem II: Turning now to Problem II, we first show that for large n, $Q_n(z)$ approaches a stationary solution $Q_\infty(z)$. From the definition, we have

$$\sum_{n=0}^{\infty} s^n \int_0^\infty Q_n(z)\, e^{-\lambda z}\, dz = \tilde{\psi}_{\mathrm{II}}(\lambda, s), \tag{71}$$

where $\tilde{\psi}_{\mathrm{II}}(\lambda, s)$ is given in Eq. (59). Thus, if $Q_n(z) \to Q_\infty(z)$ as $n \to \infty$, then the lhs of Eq. (71) will behave as

$$\sum_{n=0}^{\infty} s^n \int_0^\infty Q_n(z)\, e^{-\lambda z}\, dz \simeq \frac{1}{1-s} \int_0^\infty Q_\infty(z) e^{-\lambda z}\, dz \qquad \text{as} \quad s \to 1. \tag{72}$$

We now expand the rhs of Eq. (71) near $s = 1$ using the explicit expression of $\tilde{\psi}_{\mathrm{II}}(\lambda, s)$ in Eq. (59). We find from Eq. (59) that as $s \to 1$, the leading order term behaves as

$$\tilde{\psi}_{\mathrm{II}}(\lambda, s) \simeq \frac{\sigma\, \phi(1, \lambda)}{\sqrt{2}\lambda\,(1-s)}. \tag{73}$$

Comparing Eqs. (72) and (73) gives the exact Laplace transform of the stationary solution

$$\int_0^\infty Q_\infty(z) e^{-\lambda z}\, dz = \frac{\sigma\, \phi(1, \lambda)}{\lambda\sqrt{2}}, \tag{74}$$

where $\phi(1, \lambda)$ can be obtained by putting $s = 1$ either in Eq. (47) or alternately in Eq. (52). Both expressions are equivalent and one gets

$$\int_0^\infty Q_\infty(z) e^{-\lambda z}\, dz = \frac{\sigma}{\lambda\sqrt{2}} \exp\left[-\frac{\lambda}{\pi} \int_0^\infty \frac{dk}{\lambda^2 + k^2} \ln(1 - \hat{f}(k))\right] \tag{75}$$

$$= \frac{1}{\lambda^2} \exp\left[-\frac{\lambda}{\pi} \int_0^\infty \frac{dk}{\lambda^2 + k^2} \ln\left(\frac{1-\hat{f}(k)}{\sigma^2 k^2/2}\right)\right]. \tag{76}$$

As a check, one can verify that for the exponential kernel $f(x) = \exp[-|x|]/2$ so that $\hat{f}(k) = 1/(1+k^2)$ and $\sigma = \sqrt{2}$, Eq. (76) gives

$$\int_0^\infty Q_\infty(z) e^{-\lambda z}\, dz = \frac{1}{\lambda^2} + \frac{1}{\lambda} \tag{77}$$

which, when inverted, gives $Q_\infty(z) = z + 1$, in agreement with Eq. (38).

For a general kernel $f(z)$, it is difficult to invert the Laplace transform in Eqs. (75) or (76). However, one can easily extract the asymptotic behavior for large and small z. Consider first the large z behavior. In this case, the expression in Eq. (76) is more convenient. Expanding the rhs of Eq. (76) for small λ we get

$$\int_0^\infty Q_\infty(z) e^{-\lambda z}\, dz \simeq \frac{1}{\lambda^2} - \frac{1}{\lambda \pi} \int_0^\infty \frac{dk}{k^2} \ln\left[\frac{1-\hat{f}(k)}{\sigma^2 k^2/2}\right]. \tag{78}$$

Inverting the Laplace transform, one gets

$$Q_\infty(z) \simeq z + c' \qquad \text{as} \quad z \to \infty \tag{79}$$

where

$$c' = -\frac{1}{\pi} \int_0^\infty \frac{dk}{k^2} \ln\left[\frac{1-\hat{f}(k)}{\sigma^2 k^2/2}\right] = c \tag{80}$$

thus proving one of the main results of this paper. In particular, for the uniform kernel in Eq. (29) such that $\hat{f}(k) = \sin(k)/k$, we get

$$c' = c = 0.29795219028\ldots \tag{81}$$

The asymptotic exact result $Q_\infty(z) \to z + c'$ as $z \to \infty$ with $c' = 0.29795219028\ldots$, when substituted in Eq. (22) in the steady-state $n \to \infty$ limit, provides the exact steady-state density profile in the original 3-d flux problem at distance $r \gg R$,

$$\rho_\infty(r \gg R) = \frac{\rho_0}{r} Q_\infty(r - R) = \rho_0 \left(1 - \frac{R - c'}{r}\right), \tag{82}$$

in perfect agreement with the numerically observed[4] behavior in Eq. (11) (note that we have set $l = 1$ in Eq. (11)).

Similarly, one can work out the small z behavior of $Q_\infty(z)$ by analysing the large λ behavior of the Laplace transform. In this case, the expression in Eq. (75) is more convenient. Expanding Eq. (75) for large λ one finds

$$\int_0^\infty Q_\infty(z) e^{-\lambda z}\, dz = \frac{\sigma}{\sqrt{2}} \left[\frac{1}{\lambda} - \frac{1}{\pi \lambda^2} \int_0^\infty dk\, \ln(1 - \hat{f}(k)) + O(\lambda^{-3})\right] \tag{83}$$

Inverting the Laplace transform, we get

$$Q_\infty(z) \simeq \frac{\sigma}{\sqrt{2}} + bz \qquad \text{as} \quad z \to 0, \tag{84}$$

where b is a new constant given by

$$b = -\frac{\sigma}{\pi\sqrt{2}} \int_0^\infty dk \, \ln(1 - \hat{f}(k)). \tag{85}$$

For the exponential case, $\hat{f}(k) = 1/(k^2+1)$ and $\sigma = \sqrt{2}$ we get from Eq. (85)

$$b = \frac{1}{\pi} \int_0^\infty dk \, \ln\left[\frac{(k^2+1)}{k^2}\right] = 1. \tag{86}$$

Thus from Eq. (84), for the exponential kernel, we get $Q_\infty(z) = z + 1$ as $z \to 0$, in agreement with Eq. (38). For the uniform case, $\hat{f}(k) = \sin(k)/k$ and $\sigma = 1/\sqrt{3}$, we get

$$b = -\frac{1}{\pi\sqrt{6}} \int_0^\infty dk \, \ln\left[1 - \frac{\sin(k)}{k}\right] = 0.653857\ldots \tag{87}$$

Thus, for the uniform kernel, the small z behavior of $Q_\infty(z)$ from Eq. (84) reads as

$$Q_\infty(z) \simeq \frac{1}{\sqrt{6}} + (0.653857\ldots)z \qquad \text{as} \quad z \to 0 \tag{88}$$

This is in perfect agreement with the numerical results obtained from the work of ref. 4, $Q_\infty(z) \approx 0.408245 + 0.6538z$ as $z \to 0$.[4] Substituting the result from Eq. (88) in Eq. (22) in the limit $n \to \infty$, we find the steady-state density profile in the original 3-d flux problem near the surface $r \to R$

$$\rho_\infty(r \to R) = \frac{\rho_0}{r} Q_\infty(r - R) \approx \frac{\rho_0}{R}\left(\frac{1}{\sqrt{6}} + (0.653857\cdots)(r - R)\right). \tag{89}$$

In particular, on the surface, the steady-state density approaches to a constant value

$$\rho_\infty(R) = \frac{1}{\sqrt{6}} \frac{\rho_0}{R}, \tag{90}$$

thus proving the numerically observed[(4)] relation in Eq. (12) and identifying the constant 0.408245 as $1/\sqrt{6}$ (note that we have set $l = 1$ in Eq. (12)).

Transient behavior in Problem II: Another quantity that was investigated numerically in ref. 4 is the transient behavior of $Q_n(0)$ for large n, where it was

[4] The numerical value of the slope at $z = 0$ in this formula was determined from the data collected in the work in ref. 4, but not published in that paper. We deduced the value given here before we determined the theoretical result given in Eq. (88).

found that for the uniform kernel in Eq. (29)

$$Q_n(0) \approx 0.408245 + \left(\frac{R}{\sqrt{\pi}} - 0.17\right) n^{-1/2} \tag{91}$$

Our exact solution reproduces this result also. To see this, we investigate our explicit solution in Eq. (59) for a general continuous and symmetric kernel. By taking $\lambda \to \infty$ limit in Eq. (71) we see that the lhs behaves as

$$\sum_{n=0}^{\infty} s^n \int_0^{\infty} Q_n(z)\, e^{-\lambda z}\, dz \simeq \frac{1}{\lambda} \sum_{n=0}^{\infty} Q_n(0) s^n \qquad \text{as} \quad \lambda \to \infty. \tag{92}$$

On the other hand, expanding the exact expression of $\tilde{\psi}_{\rm II}(\lambda, s)$ in Eq. (59) for large λ, we find that the rhs of Eq. (71) behaves as

$$\tilde{\psi}_{\rm II}(\lambda, s) \simeq \frac{1}{\lambda}\left[\alpha(s) + \frac{R}{\sqrt{1-s}}\right] \qquad \text{as} \quad \lambda \to \infty \tag{93}$$

where $\alpha(s)$ is given in Eq. (65). Equating Eqs. (92) and (93) and using the expression for $\alpha(s)$ from Eq. (65) we get an exact expression for the generating function

$$\sum_{n=0}^{\infty} Q_n(0) s^n = \frac{\sigma}{(1-s)}\sqrt{\frac{s}{2}} + \frac{1}{\pi\sqrt{1-s}} \int_0^{\infty} \frac{dk}{k^2} \ln\left(\frac{1 - s\hat{f}(k)}{1 - s + s\sigma^2 k^2/2}\right) + \frac{R}{\sqrt{1-s}}. \tag{94}$$

Now, analysing the behavior near $s = 1$ of the rhs of Eq. (94), one can get the leading asymptotic behavior of $Q_n(0)$ for large n. One gets the leading behavior near $s = 1$,

$$\sum_{n=0}^{\infty} Q_n(0) s^n \simeq \frac{\sigma}{\sqrt{2}\,(1-s)} + \frac{(R-c)}{\sqrt{1-s}} \tag{95}$$

where c is the same constant as in Eq. (5). Inverting, one obtains an exact asymptotic result for large n

$$Q_n(0) \simeq \frac{\sigma}{\sqrt{2}} + \frac{(R-c)}{\sqrt{\pi}}\, n^{-1/2} \tag{96}$$

In particular, for the uniform kernel $\hat{f}(k) = \sin(k)/k$ with $\sigma = 1/\sqrt{3}$ and $c = 0.29795219028\ldots$ from Eq. (81) we get

$$Q_n(0) \simeq \frac{1}{\sqrt{6}} + \frac{(R - 0.29795219028\ldots)}{\sqrt{\pi}}\, n^{-1/2} = 0.40824829\ldots + \left(\frac{R}{\sqrt{\pi}} - 0.168101522\ldots\right) n^{-1/2}, \tag{97}$$

in excellent agreement with the numerical results[4] stated in Eq. (91).

6. SPARRE ANDERSEN THEOREM FOR PROBLEM I AND ITS ANALOGUE FOR PROBLEM II

The recursion relation in Eq. (14) satisfied by the $q_n(z)$'s in Problem I has an explicit solution given in Eq. (54). From this explicit solution, one can easily extract $q_n(0)$, the probability that a random walker starting at the origin stays positive (or negative) up to n steps. Indeed, the integral $\int_0^\infty q_n(z)e^{-\lambda z}dz \to q_n(0)/\lambda$ in the $\lambda \to \infty$ limit. On the other hand, the rhs of Eq. (54) tends to $1/\lambda\sqrt{1-s}$ as $\lambda \to \infty$ since $\phi(s,\infty) = 1$. Matching the lhs and the rhs gives,

$$\sum_{n=0}^{\infty} q_n(0)s^n = \frac{1}{\sqrt{1-s}}. \tag{98}$$

Expanding the rhs of Eq. (98) in powers of s and identifying the coefficient of s^n on both sides, one gets for all n

$$q_n(0) = \binom{2n}{n}\frac{1}{2^{2n}}. \tag{99}$$

The amazing fact is that the result in Eq. (99) is *universal* for all n, i.e. it does not depend on the density function $f(z)$ as long as $f(z)$ is continuous and symmetric. This, in fact, is the celebrated Sparre Andersen theorem which was originally proved using combinatorial methods[16] and has since been reproduced by various other methods[15,17]. In particular, one notes from Eq. (99) that in the limit of large n,

$$\lim_{n\to\infty}[\sqrt{n}q_n(0)] = \frac{1}{\sqrt{\pi}} = \text{a universal constant} \tag{100}$$

A natural question is if there is an analogue of the universality à la Sparre Andersen theorem for Problem II. The recursion relation for Problem II in Eq. (20) is identical to that of Problem I in Eq. (14), except that the initial condition $Q_0(z) = R + z$ is different from that in Eq. (14). The question is whether $Q_n(0)$ still remains universal with this different initial condition. Indeed, the answer to this question is evident from our exact result in Eq. (94). It is clear from Eq. (94) that unlike in Problem I, $Q_n(0)$ in Problem II is not universal for all n as it depends explicitly on the density function $\hat{f}(k)$. However, one recovers universality (up to a constant scale factor σ) asymptotically, i.e. in the limit of large n. Indeed, it follows from Eq. (96)

$$\lim_{n\to\infty}\frac{Q_n(0)}{\sigma} = \frac{1}{\sqrt{2}} = \text{a universal constant} \tag{101}$$

Thus $Q_n(0)$, scaled by σ, approaches a universal constant $1/\sqrt{2}$ as $n \to \infty$, independently of the density function $f(z)$ as long as $f(z)$ is continuous, symmetric and has a finite second moment $\sigma^2 = \int_{-\infty}^{\infty} z^2 f(z)\, dz$. The result in Eq. (101) for

Problem II can be thought of as an asymptotic analogue of the Sparre Andersen result in Eq. (100) for Problem I.

7. CONCLUSIONS

We have shown that the two apparently different discrete-time random walk problems, the expected maximum in one dimension and the three-dimensional flux to a trap, are intimately related in that they end up satisfying the same recursion relation, but with different initial conditions. We confirmed that the constant c' in the flux problem is identical to the constant c in the expected-maximum problem, and thus provide an exact derivation for this constant which was previously found only numerically, fourteen years ago. We also find the surprisingly simple result that the steady-state density in the flux problem reaches a constant at the boundary of the sphere, $\rho_\infty(R) = \rho_0 l/(\sqrt{6}R)$ or equivalently $Q_\infty(0) = 1/\sqrt{6}$, and find explicitly both the asymptotic time-dependent approach to that value, Eq. (97), and the slope of the density at the wall at $z = 0$ in the steady state, Eq. (88).

For the flux problem, c' represents the extrapolation length inside the boundary where the steady-state solution far from the sphere $\rho_\infty(r) = \rho_0(r - R + c'\ell)/r$ goes to zero. That is, the solution far from the sphere assumes the form of the solution to the diffusion equation, but with the effective boundary somewhat inside the actual boundary. Putting this expression into the formula for the flux, $\Phi = 4\pi r^2 D(d/dr)\rho_\infty(r)$, yields the leading term in Eq. (9) (with $l = 1$). We evaluate Φ for large r where the solution is valid, and the result is naturally independent of r because it is in steady state.

In this paper, we have presented explicitly the steady-state density profile for the 3-d discrete flux problem. As mentioned above, this solution suffices to predict immediately the leading behavior of the flux $\Phi(t)$ as $t \to \infty$ in Eq. (9). To obtain the subleading time-dependent term given in Eq. (9), it is necessary to study the large time asymptotic behavior of the integral in Eq. (25). Indeed, using the asymptotic solution for $Q_n(z)$ derived here, it is possible to calculate this subleading term explicitly; we have not presented this calculation here and it will be published elsewhere(18). It is worth remarking that in the result found for the flux in Eq. (9), the combination $R - c'l$ enters in the time-dependent correction in the same form as in the steady-state term—that is, the extrapolation-length idea also figures in the asymptotically large-time behavior of the system.

We have considered the three-dimensional flux problem here, in which particles undergo a Rayleigh flight of unit step length. The resulting equation, Eq. (20), can also be interpreted as representing a one-dimensional flux problem where the particles undergo a uniformly distributed jump, with an adsorbing boundary at $z = 0$. For the one-dimensional interpretation, however, the initial condition is uniform rather than linear as in (28). (Call this Problem II'). Then Problems I and II' become identical, with $E[M_n]$ of the former, Eq. (13), corresponding to the total integrated flux up to time n of the latter. For a uniform jump distribution, Eq. (3) thus gives the accumulated flux up to time n. In ref. 4, the quantity $a^{(2)}(n)$ of Eq. (10a) is equal to six times the flux at time n for precisely this one-dimensional problem. Summing the data for $a^{(2)}(n)/6$ up to time n (some of which data is presented in Table I of that paper), we indeed find that Eq. (3) (including c) is satisfied, with the next correction approximately equal to $0.0921n^{-1/2}$. In fact, the latter agrees with the prediction $(1/5)\sqrt{2/3\pi n} = 0.09213177\ldots n^{-1/2}$ given in ref. 2 and this prediction also agrees with the conjectured behavior of $a^{(2)}(n)$ given in Eq. (21c) of ref. 4. Thus, we have verified an additional conjecture of ref. 4.

On the mathematical side, we have discussed explicit solutions to the Wiener-Hopf type integral equations for continuous and symmetric kernel with a finite second moment σ^2. This is because both the physics problems that we were interested in this paper have kernels that satisfy these properties. Mathematically it is interesting to ask how the solutions would change if the kernel is asymmetric or for example, does not have a finite second moment. This later problem with diverging second moment corresponds to discrete-time Lévy flights and at least for the expected maximum problem (Problem I), exact results using similar techniques have recently been obtained[3]. On the other hand, while general solutions to the Wiener-Hopf type integral equations with asymmetric kernels can, in principle, be obtained[13], they are mostly not explicit. Finding explicit solutions with asymmetric kernels remains a hard and challenging problem.

Finally, our solution of the recursion relation in Eq. (14) with a constant initial condition includes, as a special case, a simple derivation of the Sparre Andersen theorem that states that $q_n(0)$, i.e. the probability that starting at 0 a random walker's path stays above (or below) 0 up to n steps, is independent of the jump density function as long as it is continuous and symmetric. However, we have proved that for the same recursion relation, but with a linear initial condition as in Eq. (20), the universality in $Q_n(0)$ holds only asymptotically for large n up to a constant factor, i.e. the ratio $Q_n(0)/\sigma$ (where σ is the standard deviation associated with the jump density function) approaches to a universal constant $1/\sqrt{2}$, irrespective of the details of the jump density function as long as it is continuous and symmetric. This paper provides a rigorous proof of this new theorem.

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