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Growth of Resistance with Density of Scatterers in One Dimensional Wave Propagation

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We study wave propagation in a one-dimensional disordered array of scattering potentials. We consider three different ensembles of scatterer configurations: an N-ensemble with a fixed number N of scatterers, an L-ensemble with a varying number of scatterers distributed over a fixed length L, and an NL-ensemble where both N and L are fixed. The latter ensemble allows a detailed study of the mean resistance and its variance for a fixed length L as the number of scatterers N increases. We find that the Landauer result, which predicts an exponential increase of the mean resistance with N, is valid only in the low-density regime. At high density the mean resistance grows exponentially with $\sqrt{N}$ and the concept of optical potential applies. In the crossover regime we find an interesting resonance.

KEY WORDS: One-dimensional conduction; mean; variance of resistance.

1. INTRODUCTION

We consider wave propagation in a one-dimensional disordered array of scatterers. It has been argued by Landauer[1] that the resistance of such an array is given by the ratio of the reflected to the transmitted intensity of an incident plane wave. Buttiker *et al.*[2] have recently discussed the physical assumptions underlying this expression for the resistance. As is well known, the resistance of a one-dimensional array is not proportional to the length of the sample. Rather, Landauer[1] has shown that on average the resistance grows exponentially with the number of scatterers. It was later realized that the average resistance is not a very meaningful quantity, since the width of the distribution of resistance grows even more rapidly.

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Anderson *et al.*[3] argued that the logarithm of the resistance would have a better behaved probability distribution and they proposed a scaling theory.

In this article we study the average resistance and its variance as a function of the density of scatterers and show that these quantities behave qualitatively differently in the low- and high-density regime. Explicit calculations are carried out for a model with identical δ-function scatterers distributed with Poisson statistics, the so-called Pδ-model. We consider ensembles of scatterer configurations where either the number N of scatterers, the length L of the array, or both N and L are fixed. We call these ensembles the N-ensemble, the L-ensemble, and the NL-ensemble, respectively. In the NL-ensemble we find that for $N \ll kL$, where k is the wavenumber, the mean resistance grows exponentially with N as predicted by Landauer, whereas for $N \gg kL$ the mean grows exponentially with $\sqrt{N}$ at fixed L. The latter behavior can be understood on the basis of the concept of optical potential. We also show that for $N \ll kL$ the variance grows faster than the mean squared, whereas for $N \gg kL$ the variance grows at most at the same rate. This suggests that the probability distribution of the resistance has qualitatively different behavior in the two regimes. It seems possible that in the high-density regime the relative variance actually decreases with N at fixed L, indicating a sharpening distribution on the scale of the mean.

As shown by Erdös and Herndon,[4] the resistance may be expressed as an element of a three-dimensional transfer matrix. Similarly, its square may be expressed as an element of a five-dimensional transfer matrix. In an earlier article[5] (referred to as I), one of us gave a streamlined derivation of these expressions. For the Pδ-model in the L-ensemble Frisch and Lloyd[6] have developed an entirely different theory. We show how their analysis is related to the transfer matrix method.

2. TRANSFER MATRICES

We consider the time-independent Schrödinger equation describing wave propagation through a one-dimensional disordered array of identical nonoverlapping scatterers,

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + \sum_{j=1}^{N} V(x - x_j)\varphi = E\varphi \tag{2.1}$$

With appropriate transcription the theory applies to classical problems, such as acoustic or electromagnetic wave propagation. We assume that the scattering centers are ordered $x_1 < x_2 < \cdots < x_N$ with x_1 located at the origin. The transmission of a wave incident from the left through the array is described by the transfer relation

$$\begin{pmatrix} T \\ 0 \end{pmatrix} = \mathsf{W}(N) \begin{pmatrix} 1 \\ R \end{pmatrix} \tag{2.2}$$

where T is the transmission coefficient and R is the reflection coefficient. The transfer matrix $\mathsf{W}(N)$ is given by the product

$$\mathsf{W}(N) = \mathsf{G}^*(x_N)\, \mathsf{M}\mathsf{G}(x_N - x_{N-1})\mathsf{M} \cdots \mathsf{M}\mathsf{G}(x_2)\mathsf{M} \tag{2.3}$$

where M describes the effect of a scatterer and the matrix $\mathsf{G}(x_{j+1} - x_j)$ describes the free propagation between scatterers. The transfer matrix M may be written in the form

$$\mathsf{M} = \begin{pmatrix} \alpha & -i\beta \\ i\beta & \alpha^* \end{pmatrix} \tag{2.4}$$

with complex α and real β, which depend on the energy E and are related by

$$|\alpha|^2 - \beta^2 = 1 \tag{2.5}$$

The propagation matrix has the form

$$\mathsf{G}(\xi_j) = \begin{pmatrix} \exp(ik\xi_j) & 0 \\ 0 & \exp(-ik\xi_j) \end{pmatrix}, \qquad \xi_j = x_j - x_{j-1} \tag{2.6}$$

for wavenumber $k = (2mE/\hbar^2)^{1/2}$.

The resistance ρ of the array is defined by $\rho = |R|^2/|T|^2$. We have shown previously[5] that the wave propagation may be mapped onto the motion of a two-dimensional harmonic oscillator which is perturbed parametrically by hits occurring at instants $x_1, \ldots, x_N$. In this mapping the resistance is related to the energy of the oscillator after the last hit by

$$\rho = \tfrac{1}{2}\mathscr{E}(X) - \tfrac{1}{2} \tag{2.7}$$

The energy $\mathscr{E}(X)$ is given by a matrix element of a three-dimensional transfer matrix

$$\mathscr{E}(X) = [\mathsf{K}_3 \mathsf{G}_3(\xi_N) \cdots \mathsf{G}_3(\xi_2)\, \mathsf{K}_3]_{22} \tag{2.8}$$

where K_3 and $\mathsf{G}_3(\xi)$ are three-dimensional matrices and $[\cdots]_{ij}$ denotes the

ij element of the matrix in square brackets. The collision matrix K_3 is given by

$$\mathsf{K}_3 = \begin{pmatrix} \alpha^2 & -i\alpha\beta & -\beta^2 \\ 2i\alpha\beta & |\alpha|^2+\beta^2 & -2i\alpha^*\beta \\ -\beta^2 & i\alpha^*\beta & \alpha^{*2} \end{pmatrix} \tag{2.9}$$

and the propagation matrix G_3 by

$$\mathsf{G}_3(\xi) = \begin{pmatrix} e^{2ik\xi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2ik\xi} \end{pmatrix} \tag{2.10}$$

We shall be interested in evaluating the average resistance and its variance for a probability distribution of scatterer configurations. For the calculation of the variance it is useful to write the square of the energy $\mathscr{E}^2(X)$ as a matrix element of a five-dimensional transfer matrix. For a fixed configuration of scatterers we have[5]

$$\mathscr{E}^2(X) = \tfrac{2}{3}[\mathsf{K}_5 \mathsf{G}_5(\xi_N) \cdots \mathsf{G}_5(\xi_2)\, \mathsf{K}_5]_{33} + \tfrac{1}{3} \tag{2.11}$$

The five-dimensional collision matrix K_5 is given by

$$\mathsf{K}_5 = \begin{pmatrix} \alpha^4 & -2i\alpha^3\beta & -\alpha^2\beta^2 & 2i\alpha\beta^3 & \beta^4 \\ 2i\alpha^3\beta & \alpha^2(|\alpha|^2+\beta^2) & -i\alpha\beta(|\alpha|^2+\beta^2) & -\beta^2(3\,|\alpha|^2+\beta^2) & 2i\alpha^*\beta^3 \\ -6\alpha^2\beta^2 & 6i\alpha\beta(|\alpha|^2+\beta^2) & |\alpha|^4+4\,|\alpha|^2\,\beta^2+\beta^4 & -6i\alpha^*\beta(|\alpha|^2+\beta^2) & -6\alpha^{*2}\beta^2 \\ -2i\alpha\beta^3 & -\beta^2(3\,|\alpha|^2+\beta^2) & i\alpha^*\beta(|\alpha|^2+\beta^2) & \alpha^{*2}(|\alpha|^2+3\beta^2) & -2i\alpha^{*3}\beta \\ \beta^4 & -2i\alpha^*\beta^3 & -\alpha^{*2}\beta^2 & 2i\alpha^{*3}\beta & \alpha^{*4} \end{pmatrix} \tag{2.12}$$

and the propagation matrix $\mathsf{G}_5(\xi)$ by

$$\mathsf{G}_5(\xi) = \begin{pmatrix} e^{4ik\xi} & & & & \\ & e^{2ik\xi} & & & 0 \\ & & 1 & & \\ 0 & & & e^{-2ik\xi} & \\ & & & & e^{-4ik\xi} \end{pmatrix} \tag{2.13}$$

The expression (2.11) has the advantage over the square of the expression (2.8) that its average is more easily evaluated.

3. PROBABILITY DISTRIBUTIONS

In choosing the probability distribution of scattering configurations we demand ease of calculation as well as relevance to physically realizable situations. Thus we assume that the probability distribution may be described as a random walk in which the probability that scatterer $j+1$ is located between $x_j+\xi$ and $x_j+\xi+d\xi$, on condition that scatterer j is located at x_j, equals $f(\xi)\,d\xi$, where $f(\xi)$ is a normalized distribution. The probability for a particular configuration of N scatterers is then given by a Markov chain with probability

$$P_N(x_1,..., x_N)=\delta(x_1)\prod_{j=2}^{N} f(x_j-x_{j-1}) \tag{3.1}$$

Averages over this distribution will be denoted by $\langle\cdots\rangle_N$. Probability distributions of the type (3.1) occur in particular in the pressure ensemble of fluids with nearest neighbor pair interactions.[7]

We shall also wish to average over a subensemble of N scatterers with fixed total length L. The corresponding probability distribution is given by

$$P_{N,L}(x_1,..., x_N)=\delta(x_1)\,\delta(x_N-L)\prod_{j=2}^{N} f(x_j-x_{j-1})/F_N(L) \tag{3.2}$$

with the length distribution

$$F_N(L)=\langle\delta(x_N-L)\rangle_N=\left\langle\delta\left(L-\sum_{j=2}^{N}\xi_j\right)\right\rangle_N \tag{3.3}$$

The Laplace transform of the length distribution is

$$\hat{F}_N(s)=\int_0^\infty e^{-sL}F_N(L)\,dL=[\hat{f}(s)]^{N-1} \tag{3.4}$$

where $\hat{f}(s)$ is the Laplace transform of the neighbor distribution $f(\xi)$. In the pressure ensemble, real, positive values of s are identified with p/θ, where p is the pressure and θ is Boltzmann's constant times absolute temperature. Averages over the probability distribution (3.2) will be denoted by $\langle\cdots\rangle_{N,L}$. Evaluation of such averages will involve an inverse Laplace transform of a quantity $\langle\cdots\rangle_{N,s}$ calculated at fixed N and s. We shall refer to the probability distribution (3.1) as the N-ensemble and to the distribution (3.2) as the NL-ensemble.

In the case of fluids in thermal equilibrium the length distribution (3.3) is related to the canonical configurational partition function $Q_N(L)$ by

$$F_N(L)=e^{-pL/\theta}Q_N(L)/\Phi_N(p) \tag{3.5}$$

where $\Phi_N(p)$ is the normalization factor. We also wish to consider the grand canonical ensemble with a fluctuating number of particles on a fixed length L. The corresponding probability distribution is

$$P_L(x_1,..., x_N) = z^{N-2} Q_N(L)\, P_{N,L}(x_1,..., x_N)/\Xi(z, L) \qquad (3.6)$$

where z is the activity and $\Xi(z, L)$ is the grand canonical partition function

$$\Xi(z, L) = \sum_{N=2}^{\infty} z^{N-2} Q_N(L) \qquad (3.7)$$

We have taken account of the fact that particles 1 and N are fixed at the ends. We shall refer to the probability distribution (3.6) as the L-ensemble and denote averages by $\langle \cdots \rangle_L$. We may relate the activity z to the pressure p occurring in the N-ensemble by defining the chemical potential as $\mu = \theta \log z$ and using the thermodynamic relation $L/\langle N \rangle_L = (\partial \mu / \partial p)_\theta$.

In our explicit calculation we shall consider the simplest possible probability distribution. This corresponds to shot noise, or equivalently to a Poisson process. In the language of fluids one deals with ideal gas statistics, completely characterized by the average number density n. The nearest neighbor distribution is given by

$$f(\xi) = n e^{-n\xi} \qquad (3.8)$$

This has the Laplace transform

$$\hat{f}(s) = \frac{n}{n+s} \qquad (3.9)$$

and from (3.4) one easily finds that the length distribution is given by

$$F_N(L) = n e^{-nL} \frac{(nL)^{N-2}}{(N-2)!} \qquad (3.10)$$

For the ideal gas the canonical and grand canonical partition functions are

$$Q_N(L) = \frac{L^{N-2}}{(N-2)!}, \qquad \Xi(z, L) = e^{zL} \qquad (3.11)$$

The equation of state $p = n\theta$ yields $z = n$, so that the L-ensemble (3.6) becomes

$$P_L(x_1,..., x_N) = n^{-1} F_N(L)\, P_{N,L}(x_1,..., x_N) \qquad (3.12)$$

Integrating over positions and using (3.10), we see that the probability $P_L(N)$ of finding $N-2$ particles on the line between 0 and L is given by the Poisson distribution

$$P_L(N) = n^{-1} F_N(L) = \frac{(nL)^{N-2}}{(N-2)!} e^{-nL} \tag{3.13}$$

The above equations completely specify the probability distributions for the three ensembles in the case of shot noise.

4. MEAN RESISTANCE

Using the probability distributions defined in the preceding section, we can now evaluate the mean resistance for the various ensembles. It follows from (2.7) that alternatively we may consider the mean energy of a parametrically perturbed oscillator. From (2.8) we find by averaging over the probability distribution (3.1) for the mean energy

$$\langle \mathscr{E} \rangle_N = [\mathsf{K}_3(\langle \mathsf{G}_3 \rangle \mathsf{K}_3)^{N-1}]_{22} \tag{4.1}$$

where $\langle \mathsf{G}_3 \rangle$ is the average of the propagation matrix $\mathsf{G}_3(\xi)$ in (2.10) over the neighbor distribution $f(\xi)$. The asymptotic behavior of $\langle \mathscr{E} \rangle_N$ for large N is dominated by the largest positive real root of the characteristic equation $|\Lambda \mathsf{I} - \langle \mathsf{G}_3 \rangle \mathsf{K}_3| = 0$, which reads explicitly

$$\Lambda^3 - A(f_2^*)\Lambda^2 + A(f_2^{-1})\,|f_2|^2 \Lambda - |f_2|^2 = 0 \tag{4.2}$$

with the abbreviations

$$f_2 = \int_0^\infty e^{-2ik\xi} f(\xi)\, d\xi, \qquad A(z) = |\alpha|^2 + \beta^2 + 2\,\mathrm{Re}(\alpha^2 z) \tag{4.3}$$

In the case of Poisson statistics we find from (3.9)

$$f_2 = \frac{n}{n+2ik} \tag{4.4}$$

For δ-function scatterers with potential $V(x-x_j) = V_0 \delta(x-x_j)$

$$\alpha = 1 - i\beta, \qquad \beta = mV_0/\hbar^2 k \tag{4.5}$$

In this case the cubic equation (4.2) may be written

$$(v^2+4)\Lambda^3 - (3v^2 + 8\beta v + 8\beta^2 + 4)\Lambda^2 + (3v^2 + 8\beta v)\Lambda - v^2 = 0 \tag{4.6}$$

where we have introduced the dimensionless variable $v = n/k$. We denote the roots of this equation as $\Lambda_i(v, \beta)$ for $i = 1, 2, 3$. There is always one real root larger than unity, which we shall denote as $\Lambda_1(v, \beta)$. The asymptotic behavior of the mean resistance $\langle \rho \rangle_N$ for large N is given by

$$\langle \rho \rangle_N \sim \exp(N \ln \Lambda_1) \tag{4.7}$$

The average resistance for the probability distribution (3.2) at fixed N and L is found via the Laplace transform

$$\langle \mathscr{E} \rangle_{N,s} = [\mathsf{K}_3(\langle \hat{\mathsf{G}}_3(s) \rangle \, \mathsf{K}_3)^{N-1}]_{22} \tag{4.8}$$

where the matrix $\langle \hat{\mathsf{G}}_3(s) \rangle$ is defined by

$$\langle \hat{\mathsf{G}}_3(s) \rangle = \int_0^\infty e^{-s\xi} f(\xi) \, \mathsf{G}_3(\xi) \, d\xi \tag{4.9}$$

The average resistance at fixed N and L follows from (2.7) with the mean energy

$$\langle \mathscr{E} \rangle_{N,L} = \frac{1}{2\pi i F_N(L)} \int e^{sL} \langle \mathscr{E} \rangle_{N,s} \, ds \tag{4.10}$$

where the integration path goes from $-i\infty$ to $+i\infty$ in the complex s-plane to the right of all singularities of the integrand. The expression (4.8) is of the form

$$\langle \mathscr{E} \rangle_{N,s} = \sum_{i=1}^{3} C_i(s) \exp[N \ln \lambda_i(s)] \tag{4.11}$$

with certain amplitudes $C_i(s)$ and roots $\lambda_i(s)$ of the cubic equation $|\lambda \mathsf{I} - \langle \hat{\mathsf{G}}_3(s) \rangle \, \mathsf{K}_3| = 0$. In the case of δ-function scatterers this equation reads explicitly

$$\lambda^3 + \hat{c}_2 \lambda^2 + \hat{c}_1 \lambda + \hat{c}_0 = 0 \tag{4.12}$$

with coefficients

$$\begin{aligned}
\hat{c}_0 &= -\hat{f}_2 \hat{f}_0 \hat{f}_{-2} \\
\hat{c}_1 &= (1 + 2\beta^2) \hat{f}_2 \hat{f}_{-2} + (1 - \beta^2) \hat{f}_0 (\hat{f}_2 + \hat{f}_{-2}) + 2i\beta \hat{f}_0 (\hat{f}_2 - \hat{f}_{-2}) \\
\hat{c}_2 &= -(1 + 2\beta^2) \hat{f}_0 - (1 - \beta^2)(\hat{f}_2 + \hat{f}_{-2}) - 2i\beta(\hat{f}_2 - \hat{f}_{-2})
\end{aligned} \tag{4.13}$$

with the definition

$$\hat{f}_j = \int_0^\infty e^{-s\xi - ijk\xi} f(\xi) \, d\xi = \hat{f}(s + ijk) \tag{4.14}$$

In the case of Poisson statistics

$$\hat{f}_j = \frac{n}{n+s+ijk} = \frac{n}{n'} \frac{n'}{n'+ijk} \tag{4.15}$$

where $n' = n + s$. Hence it follows that in that case the roots of (4.12) are related to those of (4.6) by the simple transformation

$$\lambda_i(n, k, \beta, s) = \frac{n}{ky} \Lambda_i(y, \beta), \qquad i = 1, 2, 3 \tag{4.16}$$

where $y = n'/k$. This relation will allow us to calculate $\langle \rho \rangle_{N,L}$ for large N and L in a relatively simple manner.

5. RELATION TO FRISCH–LLOYD EQUATION

We shall denote the model with Poisson statistics and δ-function scatterers as the Pδ-model. This model has been studied in great detail by Frisch and Lloyd,[6] who evaluated the density of states. Their method was based on a mapping of the eigenstates onto the motion of a one-dimensional oscillator suffering random hits. They derived a kinetic equation for the probability distribution in the phase space of the oscillator. The method is easily extended to the study of scattering solutions with nonvanishing current density. As shown in I, in that case a mapping onto the motion of a two-dimensional oscillator is more appropriate. Assuming that the first hit occurs at $x_1 = 0$, we choose two real, standard solutions of the Schrödinger equation with the properties

$$q_1(x) = \cos kx, \qquad q_2(x) = \sin kx, \qquad x < 0 \tag{5.1}$$

We identify $q_1(x)$ and $q_2(x)$ as the position coordinates of a two-dimensional oscillator at time $t = kx$. Hence the momentum $\mathbf{p} = (p_1, p_2)$ is given by the equations

$$p_1(x) = k^{-1} \frac{dq_1}{dx}, \qquad p_2(x) = k^{-1} \frac{dq_2}{dx} \tag{5.2}$$

The corresponding kinetic equation in the Pδ-model is given by

$$\frac{\partial f(\mathbf{q}, \mathbf{p}, t)}{\partial t} + \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{q}} - \mathbf{q} \cdot \frac{\partial f}{\partial \mathbf{p}} = nk^{-1}[f(\mathbf{q}, \mathbf{p} - 2\beta\mathbf{q}, t) - f(\mathbf{q}, \mathbf{p}, t)] \tag{5.3}$$

the left-hand side describing the harmonic oscillator motion and the right-hand side the effect of the random hits. The last hit occurs at $x = L$ and the kinetic equation (5.3) is valid on the interval $0 < x < L$. The motion specified by (5.1) and (5.2) corresponds to initial conditions $\mathbf{q}(0-) = (1, 0)$ and $\mathbf{p}(0-) = (0, 1)$, so that the distribution function starts as

$$f(\mathbf{q}, \mathbf{p}, 0-) = \delta(q_1 - 1)\,\delta(q_2)\,\delta(p_1)\,\delta(p_2 - 1) \tag{5.4}$$

It is easily seen that $J = q_1 p_2 - q_2 p_1$ is a constant of the motion for the kinetic equation (5.3) with value $J = 1$. The energy $\mathscr{E} = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)$ is not conserved. Its initial value at $0-$ equals unity and the equation describes how the energy varies stochastically. The time evolution of the average energy is found from the moment equations

$$\begin{aligned}
\frac{d}{dt}\langle q_i q_j\rangle &= \langle q_i p_j + p_i q_j\rangle \\
\frac{d}{dt}\langle q_i p_j + p_i q_j\rangle &= -2(1-\zeta)\langle q_i q_j\rangle + 2\langle p_i p_j\rangle \\
\frac{d}{dt}\langle p_i p_j\rangle &= 2\beta\zeta\langle q_i q_j\rangle - (1-\zeta)\langle q_i p_j + p_i q_j\rangle
\end{aligned} \tag{5.5}$$

where $(i, j) = 1, 2$ and we have abbreviated $\zeta = 2n\beta/k$. The average is over the time-dependent distribution $f(\mathbf{q}, \mathbf{p}, t)$. For $i = j$ these equations are identical to the moment equations (41) derived by Frisch and Lloyd,[6] who also gave the explicit solution of the equations. There are solutions with exponential time dependence $\sim e^{zt}$ if z satisfies the cubic equation

$$z^3 + 4(1-\zeta)z - 4\beta\zeta = 0 \tag{5.6}$$

There is always one positive root, say z_1, and roots z_2 and z_3 either both negative or complex conjugates with negative real part. Hence, the exponential growth of the energy is dominated by the root z_1. The asymptotic behavior of the mean resistance $\langle\rho\rangle_L$ at large L is given by

$$\langle\rho\rangle_L \sim e^{kLz_1} \tag{5.7}$$

Before discussing the relationship to the method of the preceding sections, we note that the original Frisch–Lloyd equation for a one-dimensional oscillator may be recovered from (5.3) by integration over either (q_1, p_1) or (q_2, p_2). Furthermore, we remark that Eq. (5.3) may be derived by Ubbink's method,[8] as discussed by van Kampen.[9]

The average energy after the last hit $\langle \mathscr{E} \rangle_L$ may alternatively be calculated from the average of $\langle \mathscr{E} \rangle_{N,L}$ over the number of hits occurring in the interval $(0, L)$,

$$\langle \mathscr{E} \rangle_L = \sum_{N=2}^{\infty} P_L(N) \langle \mathscr{E} \rangle_{N,L} \tag{5.8}$$

For Poisson statistics we find from (3.13) and (4.10)

$$\langle \mathscr{E} \rangle_L = \frac{1}{2\pi i n} \int e^{sL} \langle \mathscr{E} \rangle_s \, ds \tag{5.9}$$

where $\langle \mathscr{E} \rangle_s$ is given by

$$\langle \mathscr{E} \rangle_s = \sum_{N=2}^{\infty} \langle \mathscr{E} \rangle_{N,s} \tag{5.10}$$

Substituting the result (4.8) obtained by the transfer matrix method, we find

$$\langle \mathscr{E} \rangle_s = [\mathsf{K}_3 [\mathsf{I} - \langle \hat{\mathsf{G}}_3(s) \rangle \, \mathsf{K}_3]^{-1} \langle \hat{\mathsf{G}}_3(s) \rangle \, \mathsf{K}_3]_{22} \tag{5.11}$$

It follows from (5.9) and (5.11) that the asymptotic behavior of $\langle \mathscr{E} \rangle_L$ for large L is dominated by the root s_1 of the characteristic equation $|\mathsf{I} - \langle \hat{G}_3(s) \rangle \, \mathsf{K}_3| = 0$ with largest real part. It follows from (4.12) with $\lambda = 1$ and from (4.13) and (4.15) that this is a cubic equation in s, which is identical to (5.6) with $z = s/k$.

Thus we have shown how the Frisch–Lloyd equation is related to the transfer matrix method. The FL equation allows one to evaluate averages at fixed L, but with fluctuating number of scatterers, and for such an ensemble the method is quite powerful. The rationale for using the two-dimensional FL equation (5.3) is that in this way one can evaluate averages of observables involving cross products of quantities expressed in (q_1, p_1) and (q_2, p_2), respectively, for instance, powers of the resistance ρ. We now return to the transfer matrix method, which allows one to evaluate more detailed ensemble averages.

6. MEAN RESISTANCE AT FIXED *N* AND *L*

We consider the mean resistance $\langle \rho \rangle_{N,L}$ for a fixed number N of scatterers with $x_1 = 0$ and $x_N = L$. This is expressed by (2.7) in terms of the mean energy $\langle \mathscr{E} \rangle_{N,L}$, which in turn is given by (4.10). We evaluate the integral in (4.10) asymptotically for large N and L by the saddle point method.

For positive real s the average $\langle \mathscr{E} \rangle_{N,s}$ in (4.11) will be dominated for large N by the root λ_1, which corresponds via (4.16) to the root $\Lambda_1(v, \beta)$ of the cubic equation (4.6). Introducing $x = y^{-1}\Lambda(y, \beta)$, we transform to the equation

$$y(y^2+4)\,x^3 - (3y^2+8\beta y+8\beta^2+4)\,x^2 + (3y+8\beta)x - 1 = 0 \qquad (6.1)$$

For positive y this equation has a root $x_1(y)$ with the behavior

$$x_1(y) \approx \frac{1+2\beta^2}{y} \quad \text{for } y \to 0, \qquad x_1(y) \approx \frac{1}{y} \quad \text{for } y \to \infty \qquad (6.2)$$

For small β the root is $x_1(y) \approx y^{-1}$ for all positive y. The corresponding exponent in (4.11) is given by

$$sL + N \ln \lambda_1(s) = (y - v)\,kL + N \ln[vx_1(y)] \qquad (6.3)$$

Hence we find the saddle point $s_0 = ky_0 - n$, with y_0 determined from the equation

$$\eta R(y_0) = 1 \qquad (6.4)$$

where $\eta = N/kL$ and where the function

$$R(y) = -\frac{d}{dy} \ln x_1(y) \qquad (6.5)$$

may be obtained in explicit form from (6.1). Using (3.10) in Stirling's approximation, we find from (4.10) that the asymptotic growth of the average resistance is given by

$$\langle \rho \rangle_{N,L} \sim \exp[y_0 kL - N + N \ln(\eta x_1(y_0))] \qquad (6.6)$$

Explicit formulas may be obtained with ease only in the low- and high-density limits corresponding to $\eta \ll 1$ and $\eta \gg 1$, respectively. In both limits one finds by use of (6.2) that $y_0 \approx \eta$. In the low-density limit this leads to

$$\langle \rho \rangle_{N,L} \sim \exp[N \ln(1+2\beta^2)], \qquad N \ll kL \qquad (6.7)$$

This agrees with Landauer's formula[1] for $\langle \rho \rangle_N$ when specialized to δ-function scatterers. In the high-density limit the exponent in (6.6) vanishes to lowest order and we must consider higher order corrections.

Rather than using the explicit expression for the roots of the cubic

(6.1), it is simpler to work directly from the equation. Introducing the variable ξ by putting $x=(1+\xi)/y$, we transform to the simpler equation

$$(y^2+4)\,\xi^3+8(1-\beta^2-\beta y)\,\xi^2+4(1-4\beta^2-2\beta y)\xi-8\beta^2=0 \quad (6.8)$$

We find the asymptotic expansion for the root $\xi_1(y)$ for large y by substituting a sum of powers of $y^{-1/2}$ and comparing coefficients. The first few terms of the corresponding expansion of $x_1(y)$ read

$$x_1(y)=\frac{1}{y}+\frac{(8\beta)^{1/2}}{y^{3/2}}+\frac{9}{2}\frac{\beta}{y^2}+O(y^{-5/2}) \quad (6.9)$$

In the same manner we find an expansion valid for small y by substituting a sum of powers of y in (6.8) or directly in (6.1). This yields

$$x_1(y)=\frac{1+2\beta^2}{y}+A_0+A_1\,y+O(y^2) \quad (6.10)$$

with coefficients

$$A_0=\frac{4\beta^3}{1+2\beta^2}, \qquad A_1=\frac{2\beta^4(4-\beta^2-2\beta^4)}{(1+2\beta^2)^3} \quad (6.11)$$

From the first two terms in (6.9) we find for the average resistance in the high-density limit

$$\langle\rho\rangle_{N,L}\sim\exp(8\beta kNL)^{1/2}, \qquad N\gg kL \quad (6.12)$$

For completeness we also give the first correction terms to the results (6.7) and (6.12). In the low-density limit we find from (6.10)

$$\langle\rho\rangle_{N,L}\sim\exp\left[N\ln(1+2\beta^2)+N\frac{A_0}{1+2\beta^2}\eta\right], \qquad N\ll kL \quad (6.13)$$

In the high-density limit we find from (6.9)

$$\langle\rho\rangle_{N,L}\sim\exp[(8\beta kNL)^{1/2}-\tfrac{1}{2}\beta kL], \qquad N\gg kL \quad (6.14)$$

The most remarkable feature of the above results is embodied in (6.7) and (6.12). These expressions show that for fixed wavenumber k and fixed length L the mean resistance at first grows exponentially with the number of scatterers N, as predicted by Landauer,[1] but finally for large N grows only with the exponential of $\sqrt{N}$. Growth of the average resistance $\langle\rho\rangle_{N,L}$ as $\exp(a\sqrt{N})$ for fixed length L has been found in numerical studies by Eberle and Erdös (see Ref. 4).

We can write the asymptotic expression (6.6) in the form

$$\langle \rho \rangle_{N,L} \sim \exp[kL\Gamma(\eta)] \tag{6.15}$$

with the growth rate

$$\Gamma(\eta) = y_0 - \eta + \eta \log[\eta x_1(y_0)] \tag{6.16}$$

Correspondingly, we define

$$S(\eta) = d \log \Gamma(\eta)/d \log \eta \tag{6.17}$$

which is a measure for the rate of increase of the growth with N. From the asymptotic behavior derived above it follows that $S(\eta)$ equals unity for small η and tends to $\frac{1}{2}$ for large η. We have studied the crossover behavior numerically. It is convenient to choose a value of y and calculate $x_1(y)$ from (6.1) and η from (6.4) and (6.5). In Fig. 1 we plot $S(\eta)$ versus $\log \eta$ for $\beta = 0.1$. Evidently there is a strong increase of resistance in the crossover region. In Fig. 2 we show the same plot for $\beta = 0.01$. There is a pronounced resonance, which sharpens as β decreases. At the same time the position of the maximum shifts to larger values of η.

We attempt to locate the position of the maximum. From the Landauer formula for the resistance in the low-density regime one deduces a

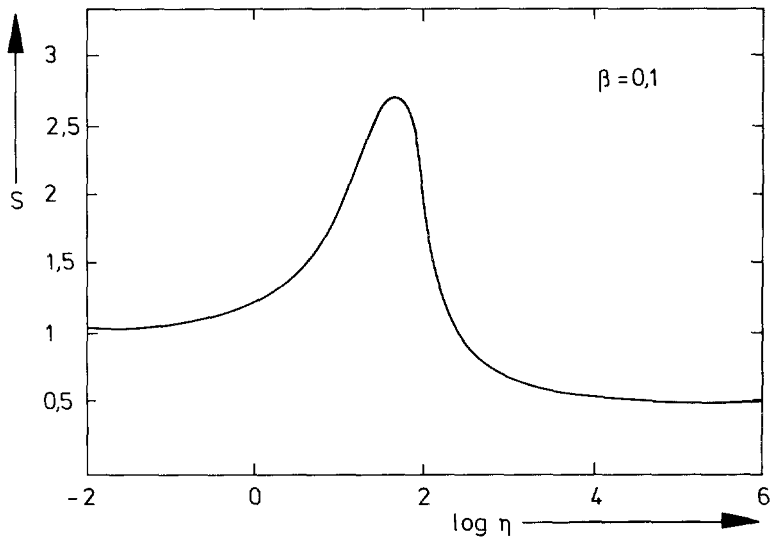

Fig. 1. Plot of the rate of increase of the growth $S(\eta)$ defined in (6.17) versus $\log \eta$, where $\eta = N/kL$, for Poisson-distributed δ-function scatterers of strength $\beta = 0.1$. For the critical density $\eta_0 = 1/(2\beta)$ found from the optical potential one has $\log \eta_0 = 1.61$.

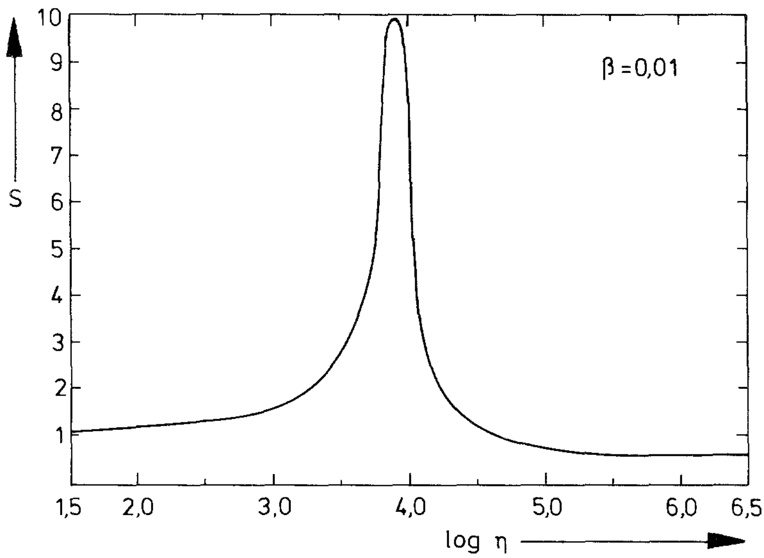

Fig. 2. Same plot as in Fig. 1, but for $\beta = 0.01$. The critical density corresponds to $\log \eta_0 = 3.91$.

mean free path $l = 2/[n \log(1 + 2\beta^2)]$ or $l = 1/n\beta^2$ for small β. The criterion for localization is $kl = 1$, and the corresponding critical density is $n_l = k/\beta^2$, or equivalently $\eta_l = 1/\beta^2$. For repulsive scatterers ($\beta > 0$) one obtains another critical density by equating the energy $E = \hbar^2 k^2/2m$ to the optical potential. In the Pδ-model with scatterers $V_0 \delta(x - x_j)$ the optical potential is $U = \eta k V_0$. Using (4.5), we find the critical value $\eta_0 = 1/(2\beta)$. Numerically we find that the maximum in $S(\eta)$ is located near η_0. In other words, the value of the optical potential determines the location of the transition.

7. VARIANCE OF THE RESISTANCE

Next we consider the variance of the resistance for the different ensembles. It follows from (2.7) that the variance is related to that of the energy of the oscillator by

$$\langle \rho^2 \rangle - \langle \rho \rangle^2 = \tfrac{1}{4} \langle \mathscr{E}^2 \rangle - \tfrac{1}{4} \langle \mathscr{E} \rangle^2 \tag{7.1}$$

From (2.11) we find by averaging over the probability distribution (3.1) for the average of the square energy

$$\langle \mathscr{E}^2 \rangle_N = \tfrac{2}{3} [\mathsf{K}_5 (\langle \mathsf{G}_5 \rangle \, \mathsf{K}_5)^{N-1}]_{33} + \tfrac{1}{3} \tag{7.2}$$

where K_5 is given by (2.12) and $\langle \mathsf{G}_5 \rangle$ is the average of the propagation matrix $\mathsf{G}_5(\xi)$ in (2.12) over the neighbor distribution $f(\xi)$. The asymptotic behavior of $\langle \mathscr{E}^2 \rangle_N$ for large N is dominated by the largest positive real root of the characteristic equation $|\Lambda \mathsf{I} - \langle \mathsf{G}_5 \rangle \mathsf{K}_5| = 0$, which reads explicitly

$$\Lambda^5 - B(f_2^*, f_4^*)\,\Lambda^4 + C(f_2^*, f_4^*)\,\Lambda^3 - C(f_2^{-1}, f_4^{-1})\,|f_2|^2|f_4|^2\,\Lambda^2 + B(f_2^{-1}, f_4^{-1})\,|f_2|^2|f_4|^2\,\Lambda - |f_2|^2|f_4|^2 = 0 \tag{7.3}$$

with $f_2 = \hat{f}(2ik)$ and $f_4 = \hat{f}(4ik)$ given by (4.14) and the functions $B(z_1, z_2)$ and $C(z_1, z_2)$ defined by

$$\begin{aligned} B(z_1, z_2) &= 1 + 6\beta^2 + 6\beta^4 + 2(1 + 4\beta^2)\,\mathrm{Re}(\alpha^2 z_1) + 2\,\mathrm{Re}(\alpha^4 z_2) \\ C(z_1, z_2) &= (1 + 10\beta^2 + 24\beta^4 + 16\beta^6)\,|z_1|^2 \\ &\quad + (1 + 4\beta^2 + 6\beta^4 + 4\beta^6)\,|z_2|^2 \\ &\quad + 2(1 + 4\beta^2 + 6\beta^4)\,\mathrm{Re}(\alpha^2 z_1) + 2(1 + 6\beta^2)\,\mathrm{Re}(\alpha^4 z_2) \\ &\quad + 2\,\mathrm{Re}(\alpha^6 z_1 z_2) + 2(1 + 3\beta^2)^2\,\mathrm{Re}(\alpha^2 z_1^* z_2) \end{aligned} \tag{7.4}$$

Equation (7.3) is a slight generalization of an equation derived by Erdös and Herndon.[4] For the Pδ-model the quintic equation (7.3) may be written more explicitly, but we shall not give the lengthy expression. The coefficients in the equation depend on $v = n/k$ and on β. We denote the roots as $\Lambda_i(v, \beta)$ for $i = 1,..., 5$, with $\Lambda_1(v, \beta)$ the real root larger than unity.

The variance of the resistance for a fixed number N of scatterers with $x_1 = 0$ and $x_N = L$ is related as in (7.1) to the variance of the energy. From (2.11) we find for the average of the squared energy

$$\langle \mathscr{E}^2 \rangle_{N,L} = \frac{1}{3} + \frac{1}{2\pi i F_N(L)} \int e^{sL} \left\langle \mathscr{E}^2 - \frac{1}{3} \right\rangle_{N,s} ds \tag{7.5}$$

where in analogy to (4.8) the integrand is given by

$$\langle \mathscr{E}^2 - \tfrac{1}{3} \rangle_{N,s} = \tfrac{2}{3}[\mathsf{K}_5(\langle \hat{\mathsf{G}}_5(s) \rangle \mathsf{K}_5)^{N-1}]_{33} \tag{7.6}$$

The latter has a form analogous to (4.11) with roots $\lambda_i(s)$ of the quintic equation $|\lambda \mathsf{I} - \langle \hat{\mathsf{G}}_5(s) \rangle \mathsf{K}_5| = 0$. For the P$\delta$-model the roots are related to those of (7.3) by the relation (4.16) for $i = 1,..., 5$. Finally, the integral in (7.5) may be evaluated asymptotically for large N and L by the saddle point method.

We shall discuss only the limiting cases of low and high density of scatterers in the Pδ-model. As in Section 6, we introduce the variable $x = y^{-1}A(y, \beta)$. For small y the quintic equation may be approximated by

$$64yx^5 - 64(1+6\beta^2+6\beta^4)\,x^4 + 128\beta(2+9\beta^2)\,x^3 - 4(5+106\beta^2)\,x^2 + 40\beta x - 1 = 0 \tag{7.7}$$

so that the root $x_1(y)$ is given approximately by

$$x_1(y) \approx \frac{1+6\beta^2+6\beta^4}{y} \qquad \text{for} \quad y \to 0 \tag{7.8}$$

Hence we find, in analogy to (6.7),

$$\langle \rho^2 \rangle_{N,L} \sim \exp[N \ln(1+6\beta^2+6\beta^4)], \qquad N \ll kL \tag{7.9}$$

in agreement with (8.3) of I by virtue of $\beta^2 = |r|^2/(1-|r|^2)$. Comparison with (6.7) shows that in the low-density limit the relative variance $(\langle \rho^2 \rangle_{N,L} - \langle \rho \rangle_{N,L}^2)/\langle \rho \rangle_{N,L}^2$ grows exponentially with N at fixed L.

The high-density limit corresponds to large values of y. As in Section 6, we introduce the variable ξ by putting $x = (1+\xi)/y$. The quintic equation becomes approximately

$$y^2\xi^5 - 40\beta y\xi^4 - 40\beta y\xi^3 + 344\beta^2\xi^2 + 256\beta^2\xi = 0 \tag{7.10}$$

The corresponding root $x_1(y)$ is given by

$$x_1(y) = \frac{1}{y} + \frac{(32\beta)^{1/2}}{y^{3/2}} + O(y^{-2}) \tag{7.11}$$

Hence we find in analogy to (6.12)

$$\langle \rho^2 \rangle_{N,L} \sim \exp(32\beta kNL)^{1/2}, \qquad N \gg kL \tag{7.12}$$

Comparison with (6.12) shows that in the high-density limit $\langle \rho^2 \rangle_{N,L}$ grows exponentially with $\sqrt{N}$ at precisely the same rate as $\langle \rho \rangle_{N,L}^2$. It seems likely that the relative variance actually decreases with N. This would be in agreement with computer simulations of Eberle and Erdös (see Ref. 4).

8. DISCUSSION

By investigating the behavior of the mean resistance $\langle \rho \rangle_{N,L}$ in the NL-ensemble we have found remarkable deviations from the behavior predicted by Landauer.[1] At fixed L the average resistance indeed grows exponentially with N as long as the number of scatterers per wavelength is much less than unity, but then the behavior changes and finally when $\eta = N/kL$ is much larger than unity the resistance grows only with the exponential of $\sqrt{N}$. As shown in Figs. 1 and 2, in the crossover region there is an enhanced resistance and the rate of increase of the growth with N shows a resonance which becomes more pronounced for weaker scatterers. We have found the above features by explicit calculation in the Pδ-model (δ-function scatterers distributed with Poisson statistics), but qualitatively the same results should hold for a wider class of models.

For $\eta \gg 1$ one expects that the concept of an optical potential gains validity. In the Pδ-model with scatterers $V_0 \delta(x - x_j)$ the optical potential is $U = \eta k V_0$. Calculating the resistance for a slab of thickness L with potential U, one finds

$$\rho_U = \frac{1 + |C|^2}{4 \operatorname{Re} C} - \frac{1}{2} \tag{8.1}$$

where C is given by

$$C = \frac{k}{k'} \frac{k' + k + (k' - k)\, e^{2ik'L}}{k' + k - (k' - k)\, e^{2ik'L}} \tag{8.2}$$

with $k' = (k^2 - 2mU/\hbar^2)^{1/2}$. For large η we may put $k' = i\kappa$ with $\kappa = (2mU/\hbar^2)^{1/2} \gg k$, which leads to

$$\rho_U \approx \frac{\kappa^2}{16k^2} e^{2\kappa L} \tag{8.3}$$

Using (4.5), we may rewrite the exponential as

$$\exp(2\kappa L) = \exp[(8\beta k N L)^{1/2}] \tag{8.4}$$

which is precisely the expression found in (6.12) for the asymptotic behavior of $\langle \rho \rangle_{N,L}$. From (6.9) and the relation $x_1 = y^{-1} A_1(y, \beta)$ we find $A_1(y, \beta) \approx 1 + (8\beta/y)^{1/2}$ for large y, so that (4.7) yields

$$\langle \rho \rangle_N \sim \exp[N(8\beta k/n)^{1/2}] \tag{8.5}$$

which is the same exponential as in (8.4) if we replace $n = N/\langle L \rangle_N$ by N/L. Finally we note from (5.6) that for large $\zeta = 2n\beta/k$ the root z_1 is given by $z_1 \approx 2\sqrt{\zeta}$. From (5.7) we therefore find for high density

$$\langle \rho \rangle_L \sim \exp[L(8n\beta k)^{1/2}] \tag{8.6}$$

which is again the same exponential if we replace $n = \langle N \rangle_L / L$ by N/L. Hence, in the high-density limit the various ensembles lead effectively to the same result, which is identical to that found from the optical potential. We have shown in (7.12) that in this limit $\langle \rho^2 \rangle_{N,L}$ grows with N at the same rate as $\langle \rho \rangle_{N,L}^2$. All this suggests that in the high-density limit the probability distribution of the resistance becomes relatively sharp and independent of the ensemble with a mean that may be found from the optical potential.

In the low-density limit the probability distribution of ρ broadens rapidly with increasing N, as may be seen from comparison of (6.7) and (7.9). It is easily shown from (4.7) and (6.10) that for $\eta \ll 1$ the averages $\langle \rho \rangle_N$ and $\langle \rho \rangle_{N,L}$ are both given by the Landauer expression (6.7). In this limit one finds from (5.6) the root $z_1 \approx \beta\zeta$, so that (5.7) yields

$$\langle \rho \rangle_L \sim \exp(2n\beta^2 L) \tag{8.7}$$

which differs from (6.7) when n is replaced by N/L. The reason is that configurations with large N from the tail of the probability distribution $P_L(N)$ strongly influence the mean.

Consideration of the NL-ensemble provides detailed information on the density dependence of the resistance. The transition from the dilute scatterer regime, corresponding to the Boltzmann limit in three dimensions, to the high-density regime where the concept of optical potential applies may be followed in detail. We have shown that in the one-dimensional Pδ-model the transition occurs when the energy equals the optical potential. Our study is far from being exhaustive and both the resonance in the growth rate of the mean resistance and the corresponding behavior of the variance merit further investigation.

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