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Instability of the Anomalies in the One-Dimensional Anderson Model at Weak Disorder

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We study the asymptotic behavior of the invariant measure, the Lyapunov exponent, and the density of states in the weak disorder limit in the case where the single-site potential distribution μ is not centered and for the special energies $E = \cos(\pi p/q)$. We also prove that in general the above quantities can be continuously extended to zero disorder as continuous functions in the disorder parameter for all energies $E \in (-1, 1)$.

KEY WORDS: Anderson model; invariant measure; instability of the anomalies; weak asymptotic expansion.

1. INTRODUCTION

The one-dimensional Anderson[1] model is given by the random Hamiltonian

$$H_\lambda = -\tfrac{1}{2}\Delta + \lambda V \quad \text{on} \quad l^2(\mathbb{Z}), \qquad \lambda \in \mathbb{R}$$

where for any $u \in l^2(\mathbb{Z})$

$$(\Delta u)(k) = u(k+1) + u(k-1), \qquad k \in \mathbb{Z}$$

and $\{V(k)\}_{k \in \mathbb{Z}}$ are independent identically distributed random variables with common probability distribution μ, whose characteristic function will be denoted by h, i.e., $h(t) = \int e^{-it(v)} \mu d(v)$.

For a given energy $E \in \mathbb{R}$ the eigenvalue equation associated with the operator H_λ is

$$u(k+1) + u(u-1) = 2[\lambda V(k) - E]\, u(k), \qquad k \in \mathbb{Z} \tag{1.1}$$

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If we define

$$z(k)=\frac{u(k)}{u(k-1)}\in\dot{\mathbb{R}}$$

where $\dot{\mathbb{R}}=R\cup\{\infty\}$ is the one-point compactification of $\dot{\mathbb{R}}$, we can rewrite (1.1) as

$$z(k+1)=2[\lambda V(k)-E]-\frac{1}{z(k)},\qquad k\in\mathbb{Z}\tag{1.2}$$

If μ is not concentrated on a single point, Furstenberg's Theorem (see B.II.4 of ref. 2) asserts that if $\lambda\neq 0$, there exists a unique invariant measure $\nu_{\lambda,E}$ on $\dot{\mathbb{R}}$ associated with the Markov process defined by (1.2), i.e.,

$$\int_{\dot{\mathbb{R}}} f(x)\,\nu_{\lambda,E}(dx)=\int \mu(dv)\int_{\dot{\mathbb{R}}} f\left(2(\lambda v-E)-\frac{1}{y}\right)\nu_{\lambda,E}(dy)\tag{1.3}$$

for all bounded measurable functions f. In addition, $\nu_{\lambda,E}$ is always a continuous measure and hence it can be viewed simply as a measure on $\mathbb{R}$.

A great deal of information about the properties of H_λ in the weak disorder region could be obtained through the study of the behavior of $\nu_{\lambda,E}$ as approaches zero (see ref. [2, 3, 4]).

Problems with a straightforward perturbation expansion in λ as proposed by Thouless[5] were first discovered in the case where μ has mean zero by Kappus and Wegner,[6] who noticed that the leading coefficient was inadequate in the center of band $E=0$ and that the differentiated density of states exhibited a discontinuity there. They called this phenomenon an anomaly. Derrida and Gardner,[4] looking at the invariant measure, extended this result. They found that at energies $E=\pm\frac{1}{2}$, the next-to-leading coefficient of the Thouless expansion was incorrect and they conjectured that such anomalies should indeed occur at all energies of the form $E=\cos(\pi p/q)$, with $p<q$ relatively prime integers. Bovier and Klein[3] gave a very detailed analysis of these anomalies and proved Derrida and Gardner's conjecture at the level of formal perturbation theory. They also derived for the case $E=\cos(\pi p/q)$ a modified perturbation expansion with finite coefficients at all orders; those differ from the naive ones only at order $\geqslant q-2$. Recently Campanino and Klein[7] proved that the modified expansion mentioned above is actually asymptotic to all orders.

In this paper we study the weak disorder limit of $\nu_{\lambda,E}$ for all energies $E\in(-1,1)$ and without the restriction that μ has mean zero (an essential assumption in all the works mentioned above.) It turns out, for example, that if the mean is not zero, the modified asymptotic expansion for the special energies is quite different than the one Bovier and Klein derived in ref. 3 and that exhibits anomalies only at order $\geqslant q-1$.

2. STATEMENT OF RESULTS

If μ is such that its characteristic function with its first derivative go to zero at infinity, Klein and Speis[8] have shown that the invariant measure $\nu_{\lambda,E}$ is absolutely continuous. Let $\varphi_{\lambda,E}$ be its density. Then (1.3) can be rewritten as

$$\varphi_{\lambda,E}(x)=\int \varphi_{\lambda,E}\left(\frac{1}{2(\lambda v-E)-x}\right)\frac{1}{[2(\lambda v-E)-x]^2}\,\mu(dv) \tag{2.1}$$

Let $K=L^2(\mathbb{R},(1+x^2)\,dx)$ and let T and $B_\lambda(E)$ be operators on K defined by

$$[Sf](x)=f\left(\frac{1}{-x}\right)\frac{1}{x^2}, \qquad f\in K, \quad x\in\mathbb{R}$$

and

$$[D_{\lambda,E}f](x)=\int f(x-2(\lambda v-E))\,\mu(dv), \qquad f\in K, \quad x\in\mathbb{R}$$

One can easily see that S is an isometry on K. Moreover, $D_{\lambda,E}$ is bounded on K and (2.1) can be written as

$$[D_{\lambda,E}S](\varphi_{\lambda,E})=\varphi_{\lambda,E} \tag{2.2}$$

Under very general assumptions for h, Klein and Speis[8] have shown that $D_{\lambda,E}S$ has one as a simple isolated eigenvalue for all $\lambda\neq 0$ and all $E\in\mathbb{R}$. However, if $\lambda=0$, this is no longer the case.

If the energy is of the form $E=\cos\pi\alpha$, where α is irrational, the equation

$$[D_{0,E}S](f)=f \tag{2.3}$$

still has a unique solution.[3] The obvious attempt in this case to find a weak-disorder expansion for $\varphi_{\lambda,E}$ is to formally write

$$\varphi_{\lambda,E}=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\varphi_E^n \tag{2.4}$$

and plug the equation above into (2.2) to obtain

$$(I-D_{0,E}S)(\varphi_E^n)=\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\partial^k}{\partial\lambda^k}D_{0,E}S\right)_{\lambda=0}(\varphi_E^{n-k}) \tag{2.5}$$

Since, as Bovier and Klein[3] showed, the set of equations in (2.5) have a unique (with the appropriate normalization) solution, it follows that if $\varphi_{\lambda,E}$ has an actual asymptotic expansion, its coefficients wil have to be equal to the ones specified by that solution.

If the energy is of the form $E=\cos(\pi p/q)$, where $p<q$ are relatively prime integers, $(D_{0,E}S)^q=I$,[3] so (2.5) is no longer adequate. However, one can rewrite (2.2) as

$$A_{\lambda,E}\varphi_{\lambda,E}=0 \tag{2.6}$$

where

$$A_{\lambda,E}=\frac{(D_{\lambda,E}S)^q-I}{\lambda}$$

and use (2.4) to obtain

$$\binom{n}{1}A_{0,E}\varphi_E^{n-1}(x)=-\sum_{k=2}^{n}\binom{n}{k}\left(\frac{\partial^k}{\partial\lambda^k}(D_{\lambda,E}S)^q\right)_{\lambda=0}\varphi_E^{n-k}(x) \tag{2.7}$$

If the mean of μ is not zero, we show that $iA_{0,E}$ (defined on an appropriate Hilbert space) is symmetric and that $-A_{0,E}^2$ extends to a positive self-adjoint operator with zero as a simple isolated eigenvalue. Combining this with the bounds we obtain in Section 3, we end up with the following theorem.

Theorem 2.1. Let μ be such that its characteristic function h is infinitely many times differentiable on $(0+\infty)$ with $h^{(i)}(t)=O[(1+t^2)^{-\alpha/2}]$ for all $i=0,1,2,...$ and some $\alpha>0$. If the first and second moments of μ exist and are not equal to zero, then for every energy of the form $E=\cos(\pi p/q)$ with $0<p<q$ relatively prime integers, the unique [up to normalization $\int\varphi_{\lambda,E}(x)\,dx=1$] solution of (2.7) forms a series which is asymptotic for $\mathbb{R}\ni\lambda\to\varphi_{\lambda,E}\in K$ to all orders at $\lambda=0$.

If $E=\cos\pi\alpha$, where α is irrational, the situation is quite different, since zero is now a simple eigenvalue of $D_{0,E}S$ imbedded in the continuum. This the question of whether the series given by the unique solution of (2.5) is indeed always asymptotic is still an open problem. We prove the following theorem.

Theorem 2.2. Let μ be such that its characteristic function h is three-times differentiable on $(0+\infty)$ with $h^{(i)}(t)=O[(1+t^2)^{-\alpha/2}]$ for all $i=0, 1, 2, 3$ and some $\alpha>0$. If the first and second moments of μ exist with the second one being different from zero, then the map $R\ni\lambda\mapsto\varphi_{\lambda,E}\in K$, where $\varphi_{\lambda,E}$ is normalized by $1=\int_{\mathbb{R}}\varphi_{\lambda,E}(x)\,dx$, is continuous for all $E\in(-1, 1)$. Moreover, if the first moment is different from zero as well, then

$$\lim_{\lambda\to 0}\varphi_{\lambda,E}(x)=\frac{1}{1-2Ex+x^2}=\varphi_{0,E}(x), \qquad x\in\mathbb{R}, \qquad E\in(-1, 1) \tag{2.8}$$

where one can recognize $\varphi_{0,E}$ as the unique solution of (2.3).

Remark 2.3. If the mean of μ is equal to zero, then (2.8) is still true provided that $E\neq 0$. However, if $E=0$, then $\lim_{\lambda\to 0}\varphi_{\lambda,0}(x)$ is equal to $1/(1+x^4)^{1/2}$, not to $1/(1+x^2)$.[3] This is a special case of the instability of anomalies mentioned in the introduction.

Note that since the Lyapunov exponent $\gamma_\lambda(E)$ and the integrated density at states $N_\lambda(E)$ can be obtained from the invariant measure by

$$\gamma_\lambda(E)=\int_{\mathbb{R}}\log|x|\;\nu_{\lambda,E}(dx)$$

$$N_\lambda(E)=\int_0^{+\infty}\nu_{\lambda,E}(dx)$$

one can derive statements similar to the ones in Theorems 2.1 and 2.2 for the quantities mentioned above as well.

Even though K seems to provide a simple and natural framework for the description of the properties of $\varphi_{\lambda,E}$ in the weak disorder limit, it turns out to be inadequate for the detailed technical estimates that the problem at hand calls for. We solve this problem by switching to a set of Hilbert spaces similar to the ones that were introduced by Campanino, Klein, and Speis[8–10] for the study of the supersymmetric transfer matrix and that are related in a precise way to K.[8] We then reexpress $D_{\lambda,E}$ and S as bounded operators $B_{\lambda,E}$ and T on these Hilbert spaces, where we show that $B_{\lambda,E}T$, $\lambda\neq 0$, has one as simple isolated eigenvalue whose eigenvector will be denoted by $\xi_{\lambda,E}$. Through a more detailed analysis of the dependence of the size of the gap around the eigenvalue one on the parameter λ, we are able to obtain a bound on the norm of $\xi_{\lambda,E}$ which is uniform in $\lambda\neq 0$. This, combined with a weak compactness argument, proves Theorem 2.2, which concludes Section 3.

Section 4 is devoted to the study of the operator $A_{0,E}$ for $E=\cos(\pi p/q)$, $0<p<q$ prime integers.

Finally, in Section 5 we combine the results from the two previous sections to prove that term by term the series associated with the solution of (2.7) is asymptotic and we discuss the anomalies of that expansion.

3. THE SUPERSYMMETRIC APPROACH

In this section we introduce an alternative form of S and $D_{\lambda,E}$ and we study them as operators defined on the Hilbert spaces mentioned in the previous section. The connection between this new approach and the one we used before will be made clear toward the end of this section.

We would like to point out that since for the benefit of the general audience we refrain from any use of superspaces and their geometry, several of our definitions might seems to be lacking any reasonable motivation. We refer the reader to the work of Klein and Speis[8,9] for more insight into the formalism and nomenclature used here.

Definition 3.1. Let $E \in \mathbb{R}$ and $\mathscr{S}(\mathbb{R}^2)$ be the usual Schwartz space over $\mathbb{R}^2$. We will denote the vector space $\mathscr{S}(\mathbb{R}^2) \times \mathscr{S}(\mathbb{R}^2)$ by $\mathscr{S}^2(\mathbb{R}^2)$. We introduce a sequence of multilinear functions q_E^n: $\mathscr{S}^2(\mathbb{R}^2) \times \mathscr{S}^2(\mathbb{R}^2) \to \mathscr{S}(\mathbb{R}^2)$ through the equations

$$\begin{aligned} q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right) = \sum_{i=1}^{2} \Bigg[& q_E^{n-1}\left(\begin{pmatrix} i\,\partial\varphi_i f_1 \\ M_{\varphi_i} f_1 \end{pmatrix}, \begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}\begin{pmatrix} i\,\partial\varphi_i g_1 \\ M_{\varphi_i} g_1 \end{pmatrix}\right) \\ & + q_E^{n-1}\left(\begin{pmatrix} i\,\partial\varphi_i f_2 \\ M_{\varphi_i} f_2 \end{pmatrix}, \begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}\begin{pmatrix} i\,\partial\varphi_i g_2 \\ M_{\varphi_i} g_2 \end{pmatrix}\right)\Bigg] \\ q_E^0\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right) = \bar{f}_1 g_2 + \bar{f}_2 g_2 & \end{aligned} \tag{3.1}$$

where $n = 0, 1, 2,\ldots$, f, $g \in \mathscr{S}(\mathbb{R}^2)$, $\varphi = (\varphi_1, \varphi_2) \in \mathbb{R}^2$, $\partial\varphi_1$ and $\partial\varphi_2$ denote the partial derivatives with respect to φ_1 and φ_2, and $M\varphi_1$ and $M\varphi_2$ stand for operator multiplications by φ_1 and φ_2, respectively.

Lemma 3.2. Let A be a compact subset of $(-1, 1)$.

1. We have

$$q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) \geqslant 0 \qquad \text{for all } \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathscr{S}^2(\mathbb{R}^2)$$

$n = 0, 1, 2,\ldots$, and all $E \in A$. Moreover,

$$q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) = 0 \Leftrightarrow \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$$

2. Let A, B be the two self-adjoint 2×2 matrices such that $A \leqslant B$. Then

$$q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, A\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) \leqslant q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, B\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) \quad \text{for all } \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathscr{L}^2(\mathbb{R}^2) \text{ and } E \in A$$

3. Let D be any 2×2 matrix. Then

$$q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, D\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) \leqslant \|D\| q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) \quad \text{for all } \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathscr{L}^2(\mathbb{R}^2) \text{ and } E \in A$$

and where $\|D\|$ denotes the norm of D.

4. Let n be any positive integer. Then there exists positive constants $XC_{A,n}$ and $C'_{A,n}$ which depend only on the set A and n such that

$$C_{A,n} q_0^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) \leqslant q_E^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right) \leqslant C'_{A,n} q_0^n\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right)$$

$$\text{for all } \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathscr{L}^2(\mathbb{R}^2) \quad \text{and} \quad E \in A$$

Proof. If $E \in A$, $\left(\begin{smallmatrix} 1 & E \\ E & 1 \end{smallmatrix}\right)$ is a positive-definite matrix which satisfies

$$C'_A I \leqslant \begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix} \leqslant C_A I, \qquad E \in A$$

where C_A and C'_A are positive constants that depend on the set A and I stands for the 2×2 identity matrix. The rest of the proof now follows from relations (3.1) and a simple induction argument. ∎

Definition 3.3. Let $R^+ = [0, +\infty)$ and let $\mathscr{S}(\mathbb{R}^+)$ be the usual Schwartz space over $\mathbb{R}^+$. Since we can always identify any element f of $\mathscr{S}(\mathbb{R}^+)$ with the function defined by

$$\mathbb{R}^2 \ni \varphi \mapsto f(\varphi^2) \in \mathbb{C} \tag{3.2}$$

where $\varphi^2 = \varphi \cdot \varphi$, we will be viewing $\mathscr{S}(\mathbb{R}^+)$ as a subspace of $\mathscr{S}(\mathbb{R}^2)$.

Let $E \in (-1, 1)$. We introduce a sequence of norms $\|\cdot\|_n^E$ on $\mathscr{S}(\mathbb{R}^+)$ through the equations

$$(\|f\|_n^E)^2 = \int_{R^2} q_E^{n-1}\left(\begin{pmatrix} 2if' \\ f \end{pmatrix}, \begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}\begin{pmatrix} 2if' \\ f \end{pmatrix}\right), \qquad n = 1, 2, 3, \ldots$$

$$\|f\|_0^E = |f(0)| \tag{3.3}$$

where f' denotes the derivative of f on $\mathbb{R}^+$.

We define the Hilbert spaces $H_{n,E}$ to be the completion of $\mathscr{S}(\mathbb{R}^+)$ under the norm $\|\cdot\|_{n,E}$, where

$$\|f\|_{n,E}^2 = \sum_{k=0}^{n} (\|f\|_k^E)^2, \qquad f \in \mathscr{S}(\mathbb{R}^+)$$

Remark 3.4. We would like to point out that the Hilbert spaces Campanino and Klein used in ref. 7 are somewhat smaller than ours. However, one can easily see through an explicit computation that they can be continuously embedded into the spaces defined here.

Let B_1 and B_2 be two Banach spaces. We will be using the notation $B_1 \hookrightarrow B_2$ to indicate that B_1 can be continuously embedded in B_2.

Proposition 3.5. Let A be a compact subset of $(-1, 1)$. Then:

1. $H_{n,E} \hookrightarrow H_{n',E}$ for all $E \in A$ and all $n, n' = 1, 2,\ldots,$ with $n \leqslant n'$.
2. Let n be any positive integer. There exist constants $C_{A,n}$ and $C'_{A,n}$ that depend only on A and n such that

$$C_{A,n} \|f\|_{n,0} \leqslant \|f\|_{n,E} \leqslant C'_{A,n} \|f\|_{n,0}$$

   for all $f \in \mathscr{S}(\mathbb{R}^+)$ and $E \in A$.
3. The unit sphere of $H_{n+2,E}$ is precompact in $H_{n,E}$ for all $E \in A$ and $n = 0, 1, 2,\ldots$ (also see ref. 7).

Proof. Parts 1 and 2 follow directly from Definition 3.3 and Lemma 3.2 through a simple induction argument.

Part 3. In view of parts 1 and 2 it is enough to prove the result in the case $E = 0$; for simplicity we will consider the case $n = 1$. The general case can be treated in the same way; one only needs to repeat the same argument several times.

Let $f \in \mathscr{S}(\mathbb{R}^2)$. One can easily show the following fundamental identity (see Th. 1.1.10/2 of ref. 11):

$$f(\varphi) = C \sum_{|\alpha| = 2} \int_{\mathbb{R}^2} \frac{(\varphi - \varphi')^\alpha}{(\varphi - \varphi')^2} \cdot D^\alpha f(\varphi')\, d^2\varphi', \qquad \varphi \in \mathbb{R}^2$$

where $\alpha = (\alpha_1, \alpha_2)$ is a two-dimensional multi-index with α_1, α_2 nonnegative integers, C is a positive constant, and we have used the conventions

$$\varphi^\alpha = \varphi_1^{\alpha_1} \varphi_2^{\alpha_2}, \qquad \varphi = (\varphi_1, \varphi_2) \in \mathbb{R}^2$$

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial \varphi_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \varphi_2^{\alpha_2}}$$

and

$$|\alpha| = \alpha_1 + \alpha_2$$

A straightforward estimation (also see Th. 1.4.5 of ref. 11) yields that if $|\alpha| = 2$,

$$\left|\frac{\varphi^\alpha}{\varphi^2}\right| \leqslant c, \qquad \varphi \in \mathbb{R}^2$$

and

$$\left|\frac{(\varphi+h)^\alpha}{(\varphi+h)^2} - \frac{\varphi^\alpha}{\varphi^2}\right| \leqslant c(h^2)^{1/2}(\varphi^2)^{-1/2}$$

for all $\varphi \in \mathbb{R}^2$ and $h \in \mathbb{R}^2$ such that $\varphi^2 \geqslant 9h^2$ and where c is a positive constant. Thus,

$$|f(\varphi+h) - f(\varphi)| \leqslant c \int_{(\varphi-\varphi')^2 \leqslant 9h^2} |D^\alpha f(\varphi')| \, d^2\varphi'$$
$$+ c(h^2)^{1/2} \int_{(\varphi-\varphi')^2 \geqslant 9h^2} |D^\alpha f(\varphi')| / [(\varphi-\varphi')^2]^{1/2} \, d^2\varphi'$$

Applying Hölder's inequality to both integrations of the right-hand side of the inequality above, we conclude that the intersection between $\mathscr{L}(\mathbb{R}^+)$ and the unit ball of $H_{3,0}$ form a uniformly equicontinuous family of functions over the compact subsets of $\mathbb{R}^2$.

Let $f \in \mathscr{L}(\mathbb{R}^+)$. Since $\nabla\varphi \cdot \varphi/2\varphi^2 = 0$ for $\varphi \neq 0$, where $\nabla\varphi$ is the usual gradient in $\mathbb{R}^2$ at the point φ, we get from Stokes' theorem that

$$f(r^2) = \int_{\mathbb{R}^2 - C_r} \frac{\varphi}{2\varphi^2} \cdot \nabla\varphi f(\varphi^2) \, d^2\varphi, \qquad r \in \mathbb{R}^+$$

where C_r is a disk of radius r centered at zero. Using the Hölder inequality once more, we obtain the inequality

$$|f(r)| \leqslant \frac{c}{r} \left[\int_{\mathbb{R}^2} |\varphi \cdot \nabla\varphi f(\varphi^2) \, d^2\varphi \right]^{1/2}$$

for some constant c and all $r \geqslant 1$.

Let us now consider a sequence $\{f_k\}_{k \in \mathbb{N}}$ at elements of $\mathscr{L}(\mathbb{R}^+)$ such that $\|f_k\|_{3,0} \leqslant M$ for all $k = 1, 2,...$ and some $M > 0$. From the dominated convergence theorem and a standard diagonalization argument we conclude that $\{f_k\}_{k\in\mathbb{N}}$ has a subsequence that converges in $L^2(\mathbb{R}^2\, d^2\varphi)$. Since we can repeat the same argument for $\{f'_k\}_{k\in\mathbb{N}}$, we can assume that the subsequence above converges in $H_{1,0}$ and the rest of the proof now follows from the fact that $\mathscr{S}(\mathbb{R}^+)$ is dense in $H_{3,0}$. ∎

We now give an alternative definition of S and $D_{\lambda,E}$ as operators defined on $H_{n,E}$, $n=0, 1, 2,\ldots$.

Definition 3.6. Let $E\in\mathbb{R}$ and let $\beta_{\lambda,E}$: $\mathbb{R}^2\to\mathbb{C}$ be the function defined by

$$\beta_{\lambda,E}(\varphi)=h(\lambda\varphi^2)e^{iE\varphi^2}, \qquad \varphi\in\mathbb{R}^2$$

where h is the characteristic function of the distribution of the potential defined in the introduction. We will denote by $B_{\lambda,E}$ the operator multiplication by $\beta_{\lambda,E}$.

A straightforward computation yields the following lemma.

Lemma 3.7.

1. Let $f\in\mathscr{S}(\mathbb{R}^2)$ and let $\alpha\in R$; then

$$\begin{pmatrix} i\,\partial\varphi_i(B_{0,\alpha}(f)) \\ M_{\varphi_i}(B_{0,\alpha}(f)) \end{pmatrix}=\begin{pmatrix} 1 & -2\alpha \\ \Omega & 1 \end{pmatrix}\begin{pmatrix} B_{0,\alpha}(i\,\partial\varphi_i f) \\ B_{0,\alpha}(M_{\varphi_i} f) \end{pmatrix}$$

for all $i=1, 2$.

2. The operator $B_{0,E}$ leaves $\mathscr{S}(\mathbb{R}^+)$ invariant and

$$\begin{pmatrix} 2i(B_{0,\alpha}(f))' \\ B_{0,\alpha}(f) \end{pmatrix}=\begin{pmatrix} 1 & -2\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2i(B_{0,\alpha}(f')) \\ B_{0,\alpha}(f) \end{pmatrix}$$

for all $f\in\mathscr{S}(\mathbb{R}^+)$.

Definition 3.8. We introduce T (the supersymmetric transfer matrix[7–10]) as the operator from $\mathscr{S}(\mathbb{R}^+)$ to $\mathscr{S}(\mathbb{R}^2)$ defined by

$$(T(f))(\varphi^2)=-\frac{1}{\pi}\int_{\mathbb{R}^2} e^{-i\varphi\cdot\varphi'} f'(\varphi'^2)\, d^2\varphi', \qquad \varphi\in\mathbb{R}^2 \tag{3.4}$$

We shall denote the ordinary Fourier transform on $\mathscr{S}(\mathbb{R}^2)$ by $\mathscr{F}$. Thus (3.4) can be rewritten as

$$(T(f))(\varphi^2)=-2(\mathscr{F}(f'))(\varphi), \qquad \varphi\in\mathbb{R}^2$$

Lemma 3.9.

1. Let n be any positive integer and let f be an element of $\mathscr{S}(\mathbb{R}^+)$. Then

$$2^k (T(f))^{(k)} \left(\sum_{i=1}^{n} \phi_i'^2 \right) = \frac{(-1)^b 2^{n-k}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \exp\left[-i \sum_{i=1}^{n} \phi_i \cdot \varphi_i' \right] f^{(n-k)} \left(\sum_{i=1}^{n} \phi_i^2 \right) \prod_{i=1}^{k} d^2\varphi_i \qquad (3.5)$$

for all $k = 0, 1, 2, \ldots$, with $k \leqslant n$ and all $(\varphi_1', \ldots, \varphi_n') \in \mathbb{R}^{2n}$.

2. The operator T leaves $\mathscr{S}(\mathbb{R}^+)$ invariant and

$$\begin{pmatrix} 2i(T(f))' \\ T(f) \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 2i\mathscr{F}(f') \\ \mathscr{F}(f) \end{pmatrix}$$

for all $f \in \mathscr{S}(\mathbb{R}^+)$.

3. Let $f \in \mathscr{S}(\mathbb{R}^2)$. Then

$$\begin{pmatrix} i\,\partial_{\varphi_i}(\mathscr{F}f) \\ M_{\varphi_i}(\mathscr{F}f) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathscr{F}(i\,\partial\varphi_i(f)) \\ \mathscr{F}(M\varphi_i(f)) \end{pmatrix}$$

for all $i = 1, 2$.

Proof. The derivation of Eq. (3.5) is an easy exercise in supersymmetric field theory (see Lemma II.5 of ref. 8 for a direct proof without the use of superspaces). The rest of the proof follows now from (3.5) and a straightforward computation. ▌

Proposition 3.10. Let A be a compact subset of $(-1, 1)$. Then:

1. $B_{0,E}T$ extends to an isometry on $H_{n,E}$ for all $n = 0, 1, 2, \ldots$, and $E \in A$.
2. Let h be n-times differentiable on $(0, +\infty)$ with bounded derivatives. Then $B_{\lambda,E'}$ extends to a bounded operator on $H_{n,E}$ for all $n = 0, 1, 2, \ldots$, $E \in (-1, 1)$, $E' \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Moreover, if the distribution μ has a second moment

$$\|B_{\lambda,\lambda m}(f)\|_{n,E} \leqslant E^{c_{A,n}\lambda^2} \|f\|_{n,E}, \qquad f \in H_{n,E}$$

for all $n = 0, 1, 2, \ldots$, $E \in A$, and $|\lambda| \leqslant 1$, where m is the mean of the distribution of the potential μ and $c_{A,n}$ is a positive constant that depends only on n and the set A.

Proof.

1. The proof of part 1 follows the fact that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2E & 1 \end{pmatrix}\begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}\begin{pmatrix} 1 & -2E \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}$$

for all $E \in \mathbb{R}$, Lemmas 3.7 and 3.8, and a simple induction argument.

2. Let $\alpha \in \mathbb{R}$ and let $E \in (-1, 1)$. Since

$$\begin{pmatrix} 1 & 0 \\ -2\alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}\begin{pmatrix} 1 & -2\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix} + \begin{pmatrix} 0 & -2\alpha \\ 2\alpha & 4\alpha^2 - 4\alpha E \end{pmatrix}$$

we can conclude from Lemmas 3.2 and 3.7 that if $f \in \mathscr{L}(\mathbb{R}^+)$

$$\|B_{0,\alpha}(f)\|_{n,E}^2 = \|f\|_{n,E}^2 + \alpha 0_{n,A}(\|f\|_{n,E}) + \alpha^2 O'_{n,A}(\|f\|_{n,E})$$

for all $n = 0, 1, 2, \ldots$, and $E \in A$, where $O_{n,A}(\cdot)$ and $O'_{n,A}(\cdot)$ are bounded functions which depend only on n, and A. On the other hand, using Jensen's inequality and Definition 3.6, we get that

$$\begin{aligned} \|B_{\lambda,\lambda m}(f)\|_{n,E}^2 &= \left\| \int_{\mathbb{R}} B_{0,\lambda(m-v)}(f)\, \mu(dv) \right\|_{n,E}^2 \\ &\leqslant \int_{\mathbb{R}} \|B_{0,\lambda(m-v)}(f)\, \mu(dv)\|_{n,E}^2\, \mu(dv) \end{aligned}$$

for all $n = 0, 1, 2, \ldots$ and the proof of part 2 follows. ▮

We now state a theorem that summarizes the spectral properties of $B_{\lambda,E}T$ for $\lambda \neq 0$.

Theorem 3.11. Let $E \in (-1, 1)$ and let h be n-times differentiable on $(0, +\infty)$ with bounded derivatives and such that $|h^{(k)}(r)| \to 0$ as $r \to +\infty$ for all $k \leqslant n$. Then:

1. $(B_{\lambda,E'}T)^2$ is compact on $H_{k,E}$ for all $k = 0, 1, \ldots, n$, $E' \in \mathbb{R}$, and $\lambda \neq 0$.
2. The spectral radius of $B_{\lambda,E'}T$ on $H_{k,E}$ is 1 for all $k = 0, 1, \ldots, n$, all $E' \in \mathbb{R}$, and $\lambda \neq 0$.
3. 1 is an algebraically simple eigenvalue of $B_{\lambda,E'}T$ on $H_{k,E}$ and it is the only eigenvalue of modulus 1 for all $k = 0, 1, \ldots, n$, all $E' \in \mathbb{R}$, and $\lambda \neq 0$.
4. Let $H_{k,E}^0 = \{f \in H_{k,E} \colon f(0) = 0\}$. Then $H_{k,E}^0$ is left invariant by $B_{\lambda,E'}T$ and the spectral radius of $B_{\lambda,E'}T$ on $H_{k,E}^0$ is strictly smaller that one for all $m = 0, 1, \ldots, n$, $E' \in \mathbb{R}$, and $\lambda \neq 0$.

Proof. The properties of $B_{\lambda,E}T$ described above are the key result used in the study of the density of states for one-dimensional Anderson models and a complete derivation in a more general case can be found in ref. 12. We only wish to point out to the interested reader that Klein *et al.*[12] used a sequence of Hilbert spaces that corresponds to the case $E=0$ and that their norms are equivalent to our $\|\cdot\|_{n,0}$'s. ∎

We will now study the behavior of $B_{\lambda,E}T$ as λ approaches zero.

Proposition 3.12. Let μ be such that h is n-times differentiable with $h(t)^{(i)}=O[(1+t^2)^{-\alpha/2}]$ for all $i=0, 1, 2,..., n$ and some $\alpha>0$. If the first and second moments of μ exist with the second one different from zero, then for every $E\in(-1, 1)$ and $i=0, 1,..., n$, there exist M, λ_0, and c strictly positive constants such that

$$\|(B_{\lambda,E}T)^k(f)\|_{i,E}\leqslant Me^{-ck\lambda^2}\|f\|_{i,E}$$

for all $|\lambda|<\lambda_0$, $k=0, 1, 2,...,$ and $f\in H^0_{i,E}$.

Proof. We first kind a bound for the operator norm of $(B_{\lambda,E}T)^k$ on $H_{i,E}$. Let $f\in\mathscr{L}(\mathbb{R}^+)$ and let m be the mean of the distribution μ. Since

$$[B_{\lambda,E}T](f)=[B_{\lambda,\lambda m}B_{0,E-\lambda m}T](f)$$

we can conclude from Proposition 3.10 that

$$\|(B_{\lambda,E}T)^k(f)\|_{n,E-\lambda m}\leqslant e^{k\lambda^2 c}\|f\|_{n,E-\lambda m}$$

for all $E\in(-1, 1)$, $k=0, 1,...,$ and $|\lambda|<\lambda_0$, when λ_0 and c are positive constants which depend only on n, E, and m. Thus, from Proposition 3.5, part 2, we get that

$$\|(B_{\lambda,E}T)^k(f)\|_{n,E}\leqslant M'e^{c'k\lambda^2}\|f\|_{n,E} \tag{3.6}$$

for all $k=0, 1, 2,...,$ $E\in(-1, 1)$, and $|\lambda|<\lambda_0$, where λ_0, M', and c' are positive constants that depend only on E, n, and m.

In view of Remark 3.4, the rest of the proof follows as in Lemma 3.2 of ref. 7. ∎

Corollary 3.13. Let μ satisfy the same conditions as in Proposition 3.12 and let $\xi_{\lambda,E}$ be the unique solution of

$$[B_{\lambda,E}T](\xi_{\lambda,E})=\xi_{\lambda,E}, \qquad \lambda\neq 0, \quad E\in(-1, 1) \tag{3.7}$$

normalized by $\xi_{\lambda,E}(0)=1$. Then

$$\|\xi_{\lambda,E}\|_{i,E}\leqslant M_{i,E}$$

for all $i=0, 1, 2,..., n$, $E\in(-1, 1)$, and $|\lambda|<\lambda_0$, where λ_0 and $M_{n,E}$ are positive constants that depend only on E and n.

Proof. Let ξ_0 be an element of $H_{n,E}$ such that $\xi_0(0)=1$. Then if $E\in(-1, 1)$

$$(B_{\lambda,E}T)^k\xi_0-\xi_0=\xi_{\lambda,E}-\xi_0+(B_{\lambda,E}T)^k(\xi_0-\xi_{\lambda,E})$$

for all $\lambda\neq 0$ and $k=0, 1,...$. Choosing $k=[k_0/\lambda^2]+1$, where k_0 is a positive constant to be determined later, we get from Proposition 3.12 and inequality (3.4)

$$\|\xi_{\lambda,E}-\xi_0\|_{n,E}\leqslant Me^{-k_0c}\|\xi_{\lambda,E}-\xi_0\|+M'(1+e^{+k_0c'})\|\xi_0\|_{n,E}$$

for all $n=0, 1, 2,...,$ and $|\lambda|<\lambda_0$, where λ_0 and M', M, C, and C' are positive constants that depend on n and E. Thus, by choosing k_0 sufficiently big, we have that the result follows. ∎

We now make the connection between the Hilbert space K defined in the introduction and the (supersymmetric) space $H_{1,0}$. We will simply state the results we use here and we refer the reader to Section III of ref. 8 for a complete discussion.

Definition 3.14. Let $\mathscr{L}^r(\mathbb{R}^+)$ be the real vector of all functions in $\mathscr{L}(\mathbb{R}^+)$ such that $f(0)\in\mathbb{R}$. We shall make use of the real Hilbert space $H^r_{1,0}$ which is obtained by taking the completion of $\mathscr{L}^r(\mathbb{R}^+)$ under the norm $\|\cdot\|_{1,0}$. We shall also make use of the operator F which acts on K and is defined by

$$(F(f))(r)=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-(i/2)rx}f(x)\,dx,\qquad f\in\mathscr{L}^r(\mathbb{R}^+)$$

Proposition 3.15.

1. $H^r_{1,0}=\{f\in H_{1,0}: f(0)\in\mathbb{R}\}$.
2. Let K^r be the real part of K. Then F extends to an orthogonal transformation from K^r to $H^r_{1,0}$ and its inverse is given by the formula

$$(F^{-1}g)(x)=\frac{1}{4\sqrt{\pi}}\,\mathrm{Re}\int_{-\infty}^{+\infty}e^{1/2)ixr}g(r)\,dr,\qquad g\in\mathscr{L}^r(\mathbb{R}^+)$$

3. T and S leave $H^r_{1,0}$ equivalently K^r invariant and

$$S=F^{-1}TF$$

4. Let μ be such that its characteristic function h is continuously differentiable on $(0, +\infty)$ and $h^{(i)}$ is bounded for all $i = 0, 1$; then $B_{\lambda,E}$ and $D_{\lambda,E}$ leave $H^r_{1,0}$ equivalently K^r invariant and
$$D_{\lambda,E} = F^{-1} B_{\lambda,E} F$$
on K^r.
5. Let μ be as in part 4. If in addition $|h^{(i)}(t)| \to 0$ as $t \to +\infty$ for all $i = 0, 1$, then the unique [up to the normalization $\int_{\mathbb{R}} \varphi_{\lambda,E}(x)\, dx = 1$] solution $\varphi_{\lambda,E}$ of (2.2) satisfies
$$\varphi_{\lambda,E} = \frac{2}{\sqrt{\pi}} F^{-1}(\xi_{\lambda,E})$$
where $\xi_{\lambda,E}$ is the unique [up to the normalization $\xi_{\lambda,E}(0) = 1$] solution of (3.7).

We finish this section by giving a proof for Theorem 2.2 in the case the energy E is not of the form $E = \cos(\pi p/q)$ with $p < q$ prime integers. Since continuity of $R \ni \lambda \mapsto \xi_{\lambda,E}$ for $\lambda \neq 0$ can be proven through standard arguments developed in refs. 8–10 and 12, we will concentrate on the case $\lambda = 0$.

An explicit computation shows that the function $\xi_{0,E}(\varphi^2) \in H_{n,0}$, $n = 0, 1, 2,\ldots$, defined by
$$\xi_{0,E}(\varphi^2) = \exp\{\tfrac{1}{2} i [E + i(1 - E^2)^{1/2}] \varphi^2\}, \qquad \varphi \in \mathbb{R}^2$$
is a solution of the equation
$$(B_{0,E} T)(\xi_{0,E}) = \xi_{0,E} \tag{3.8}$$
Moreover, in view of Proposition 3.15, we conclude from Lemma 3.1 of ref. 3 that if E is not of the special form mentioned above, $\xi_{0,E}$ is the unique [up to the normalization $\xi_{0,E}(0) = 1$] solution of (3.6). Let $f \in H_{3,0}$ and let $E \in (-1, 1)$. Using Taylor's theorem, we get that for all $g \in H_{3,0}$
$$\langle g, (B_{\lambda,E} T)(f) \rangle_{3,0} = \langle g, (B_{0,E} T)(f) \rangle_{3,0} + \lambda \left\langle g, \frac{d}{dh} (B_{h,E} T)\bigg|_{h=c} (f) \right\rangle_{3,0}$$
for some c with $|c| < \lambda$ and where $\langle\ ,\ \rangle_{3,0}$ denotes the inner product of $H_{3,0}$. Replacing f by $\xi_{\lambda,E}$, we get from Corollary 3.13 that
$$\|\xi_{\lambda,E} - (B_{0,E} T)(\xi_{\lambda,E})\|_{3,0} \leqslant M\lambda$$
for all $n = 0, 1, 2,\ldots$ and λ sufficiently small, where M is a positive constant independent of λ. Thus $\xi_{\lambda,E} - (B_{0,E} T)(\xi_{\lambda,E}) \to 0$ as $\lambda \to 0$ in $H_{3,0}$ for all $E \in (-1, 1)$. Now let $\{\lambda_k\}_{k \in \mathbb{N}}$ be any sequence of real numbers such that $\{\xi_{\lambda_k}\}_{k \in \mathbb{N}}$ has a weak limit in $H_{3,0}$ as $k \to +\infty$. The previous statement

implies that any such limit will have to satisfy (3.6). Therefore we conclude from Corollary 3.13 that $\xi_{\lambda,E}$ converges weakly in $H_{3,0}$ to $\xi_{0,E}$. Thus, $\xi_{\lambda,E}$ converges $\xi_{0,E}$ strongly in $H_{1,0}$ and the result follows from Proposition 3.15.

4. THE OPERATOR $A_{0,E}$

In this section we give a precise definition of the operator $A_{0,E}$, the key ingredient of Eq. (2.6), and we study its properties on the Hilbert spaces $H_{n,E}$. We will abuse the notation by writing $A_{0,E}$ instead of $\mathscr{F} A_{0,E} \mathscr{F}^{-1}$.

In view of relations (2.6) and Proposition 3.15 it seems natural to define $A_{0,E}$, at least on a dense subspace, through the equation

$$A_{0,E} = \lim_{\lambda \to 0} \frac{(B_{\lambda,E} T)^q - I}{\lambda} \tag{4.1}$$

Indeed, one can easily see that if the energy is of the special form $E = \cos(\pi p/q)$ with $p < q$ relatively prime integers, then $(B_{0,E} T)^q = I$.[3] Thus, we can conclude that $(1/\lambda)[(B_{\lambda,E} T)^q - I](f)$ converges in $H_{n,0}$ provided that $f \in H_{n+2,0}$ for all $n = 0, 1, \ldots$, and that

$$\lim_{\lambda \to 0} \left[\frac{(B_{\lambda,E} T)^q - I}{\lambda} \right] (f) = -im \left[\sum_{k=0}^{q-1} (B_{0,E} T)^k M_{\varphi_i^2} (B_{0,E} T)^{q-k} \right] (f)$$

where M_{φ^2} denotes the operator multiplication by φ^2.

An explicit computation shows that zero is an eigenvalue of $A_{0,E}$ for all E of the special form mentioned above with one of its eigenvectors being the function

$$\xi_{0,E}(\varphi^2) = \exp\{\tfrac{1}{2} i [E + i(1 - E^2)^{1/2}] \varphi^2\}, \qquad \varphi \in \mathbb{R}^2$$

Theorem 3.11 together with Proposition 3.12, however, suggests that the gap between the eigenvalue one and the rest of the spectrum of $B_{\lambda,E} T$ is of the order λ^2. Thus, (4.1) cannot be used to show that 0 is an isolated simple eigenvalue of $A_{0,E}$, a crucial fact in the proof of Theorem 2.1.

On the other hand, one can easily check that if the energy E is of the form mentioned above and $f \in H_{n+2,0}$, then $(1/\lambda)[(B_{0,E+i\lambda} T)^q - I](f)$ converges in $H_{n,0}$ and

$$\begin{aligned} A_{0,E}(f) &= im \lim_{\lambda \to 0} \left[\frac{(B_{0,E+i\lambda} T)^q - I}{\lambda} \right] (f) \\ &= -im \left[\sum_{k=0}^{q-1} (B_{0,E} T)^k M_{\varphi_i^2} (B_{0,E} T)^{q-k-1} \right] (f) \end{aligned} \tag{4.2}$$

for all $n = 0, 1, 2, 3, \ldots$.

It is easy to see now that considering $B_{0,E+i\lambda}$, $\lambda>0$ corresponds to the case where μ has the Cauchy distribution. Moreover, a careful review of the proof of Proposition 3.12 suggests that in this case the exponent of the right-hand side of the inequality proved there is of order λ. This indicates that (4.2) should be more suitable for the study of $A_{0,E}$. Indeed, one can modify the proofs of the previous section to accommodate the case of $B_{0,E+i\lambda}$, $\lambda>0$. However, we elect to present here a much simpler argument that is easily generalizable to higher-dimensional models and which is based on an explicit computation described by the following lemma.

Lemma 4.1. Let $f\in\mathscr{L}(\mathbb{R}^+)$, let $\lambda>0$, and let k be a positive integer. Then

$$\begin{aligned}&[(B_{0,E+i\lambda}T)^k(f)](\varphi^2)\\&\quad=\exp\{i(E+i\lambda)\varphi^2+\tfrac{1}{4}i\varphi^2G_{[2,k]}^{2,2,E+i\lambda}\}\\&\quad\quad\times[(TB_{0,(1/4)G_{[2,k]}^{k,k,E+i\lambda}})(f)](\varphi^2(\tfrac{1}{2}G_{[2,k]}^{2,k,E+i\lambda})^{-1}),\qquad\varphi\in\mathbb{R}^2\end{aligned}\tag{4.3}$$

where

$$G_{[2,k]}^{i,j,E+i\lambda}=\langle\delta_i|\,(-\tfrac{1}{2}\Delta_{[2,k]}-(E+i\lambda))^{-1}\,|\delta_j\rangle,\qquad i,j=2,3,\dots,k$$

$\Delta_{[2,k]}$ denotes the operator Δ restricted to $l^2([2,k])$ with Dirichlet boundary conditions outside $[2,k]$ and $\delta_i,\delta_j\in l^2([2,k])$ stand for the delta functions concentrated at the point i and j.

Proof. The derivation of relation (4.3) is a simple exercise in supersymmetric Gaussian integrals.[9] However, the proof can be done directly using arguments in principle similar to the ones used in Lemma 3.9, part 1, and it is left to the reader. ∎

We now state the proposition that contains the bounds for $B_{0,E+i\lambda}T$ that are necessary for the study of $A_{0,E}$.

Proposition 4.2. Let A be a compact set of $(-1,1)$ and let n be a positive integer. Then if $E\in A$:

1. $\|(B_{0,E+i\lambda}T)^k(f)\|_{n,E}\leqslant e^{c\lambda k}\|f\|_{n,E}$
   for all $f\in H_{n,E}$, $k=0,1,\dots$, and $\lambda>0$, where c is a positive constant that depends only on n and the set A.
2. $\|(B_{0,E+i\lambda}T)^k(f)\|_{n,E}\leqslant Me^{-c\lambda k}\|f\|_{n,E}$
   for all $f\in H^0_{n,E}$, $k=0,1,2,\dots$, and $\lambda>0$, where as before M and c are two positive constants that depend on n and the set A.

Proof.

Part 1. It is enough of course to show the inequality for $k=1$. One can easily check that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2(E+i\lambda) & 1 \end{pmatrix}\begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}\begin{pmatrix} 1 & -2(E+i\lambda) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$=\begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix}+\lambda G$$

for all $|\lambda|<1$, where G is a 2×2 square matrix whose norm is bounded in absolute value by a constant that depends on A. The result now follows from Lemmas 3.7 and 3.8 and a simple induction argument.

Part 2. In view of Proposition 3.5, it is enough to show the inequality for $E=0$. Let $\{p_k\}_{k\in\mathbb{N}}$ be the sequence of real numbers defined by

$$\begin{pmatrix} p_k \\ p_{k-1} \end{pmatrix}=\begin{pmatrix} -2(E+i\lambda) & -1 \\ 1 & 0 \end{pmatrix}\underbrace{\cdots\cdots}_{k-1 \text{ times}}\begin{pmatrix} -2(E+i\lambda) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

One can show[12] that

$$G^{i,k,E+i\lambda}_{[2,k]}=\frac{2p_{i-1}}{p_k}$$

for all $i=2, 3,..., k$ and $k=2, 3,...$.

Diagonalizing the matrix

$$\begin{pmatrix} -2(E+i\lambda) & -1 \\ 1 & 0 \end{pmatrix}$$

we get

$$\begin{pmatrix} p_{k+1} \\ p_k \end{pmatrix}=\frac{1}{(\alpha_{E,\lambda})^2-1}\begin{pmatrix} (\alpha_{E,\lambda})^{k+2} & -1/(\alpha_{E,\lambda})^k \\ (\alpha_{E,\lambda})^{k+1} & -1/(\alpha_{E,\lambda})^{k-1} \end{pmatrix}$$

where $\alpha_{E,\lambda}=-(E+i\lambda)-[(E+i\lambda)^2-1]^{1/2}$.

One can now easily verify that

$$e^{d_1\lambda}\leqslant|\alpha_{E,\lambda}|\leqslant e^{d_2\lambda}$$

for all $\lambda\geqslant 0$ and $E\in A$, where d_1, d_2 are two positive constants that depend only on A. Thus, we conclude that there exist constants c_0, M, $c>0$ such that if $k\lambda \geqslant c_0$, then

$$|G_{[2,k]}^{2,k,E+i\lambda}| \leqslant Me^{-ck\lambda}$$
$$|G_{[2,k]}^{2,2,E+i\lambda}| = |G_{[2,k]}^{k,k,E+i\lambda}| \leqslant M$$

and

$$\operatorname{Im} G_{[2,k]}^{2,2,E+i\lambda} = \operatorname{Im} G_{[2,k]}^{k,k,E+i\lambda} \geqslant c$$

for all $\lambda \geqslant 0$, $k=2, 3, 4,...,$ and $E \in A$.

Let f be an element of $H_{0,n}^0$. Using relation (3.5), we can rewrite (4.3) as

$$\begin{aligned}
&[(B_{0,E+i\lambda}T)^k(f)](\varphi^2)\\
&\quad = \exp\{i(E+i\lambda)\varphi^2 + \tfrac{1}{4}i\varphi^2 G_{[2,k]}^{2,2,E+i\lambda}\\
&\qquad \times \int_{\mathbb{R}^2} \left\{\exp\left[\frac{i}{2}\varphi\cdot\varphi_1 G_{[2,k]}^{2,k,E+i\lambda}\right] - 1\right\} [(B_{0,(1/4)G_{[2,k]}^{k,k,E+i\lambda}})(f)]'(\varphi_1^2)\, d_{\varphi_1}^2
\end{aligned} \tag{4.4}$$

Recalling the multi-index notation of the proof of Proposition 3.5, part 3, and using the bounds mentioned above, we conclude from the Hölder inequality and a straightforward computation that if $k\lambda > c_0$,

$$\left\| M_{\varphi_1^{k_1}} M_{\varphi_2^{k_2}} \frac{\partial^{k_3}}{\partial\varphi_1^{k_3}} \frac{\partial^{k_4}}{\partial\varphi_2^{k_4}} [(B_{0,E+i\lambda}T)^k(f)] \right\|_{L^2(\mathbb{R}^2, d^2\varphi)} \leqslant Me^{-c\lambda k} \tag{4.5}$$

for all $k_1, k_2, k_3, k_4 \in \mathbb{Z}^+$ with $k_1+k_2+k_3+k_4=n$ and $E \in A$, where M and C are two constants that depend on n and the set A. Using Lemmas 3.7 and 3.9 and relation (4.3), we can find an explicit formula for $[(B_{0,E+i\lambda}T)^k(f)]'$ similar to the ones described by (4.4). So we can use the same argument to conclude that $[(B_{0,E+i\lambda}T)^k(f)]'$ also satisfies an inequality similar to the one described by (4.5). Thus, we have shown the inequality for the case where $k\lambda$ is bigger than a fixed positive number and the rest of the proof follows from part 1. ∎

Let $E \in (-1, 1)$, let $\lambda > 0$, and $z \in \mathbb{C}$. We shall make use of the operators

$$R_{\lambda,E,z}^q \left[\frac{I-(B_{0,E+i\lambda}T)^q}{\lambda} - z\right]^{-1}$$

Lemma 4.3.

1. $\|R^q_{\lambda,E,z}(f)\|_{n,E} \leqslant M(\operatorname{Re} z + c)^{-1}$
for all $f \in H^0_{n,E}$, $n = 0, 1, 2,\ldots$, $0 < \lambda < 1$, where M and c are positive constants that depend on n and E.

2. $\|R^q_{\lambda,E,z}(f)\|_{n,E} \leqslant \dfrac{M'}{\operatorname{Re} z + \min(c, |\operatorname{Im} z|)}$
for all $f \in H_{n,E}$, $n = 0, 1, 2,\ldots$, where M' is a positive constant and c is the constant used in (1).

Proof.

Part 1. Using Proposition 4.2, part 2, we get that the power series

$$\frac{\lambda}{1-\lambda z} \sum_{k=0}^{\infty} \frac{(B_{0,E+i\lambda}T)^k}{(1-\lambda z)^k}$$

converges in $H_{n,E}$ as long as $\operatorname{Re} z > c$ for some positive constant c. Moreover, we can adjust c such that the convergence is uniform in λ, $0 < \lambda < 1$, and the proof of part 1 follows.

Part 2. Let $\lambda > 0$. Using Proposition 4.2, one can show that the vector defined by

$$\theta_{\lambda,E}(\varphi^2) = \exp\left(\frac{i}{2}\{(E+i\lambda) + [(E+i\lambda)^2 - 1]^{1/2}\}\,\varphi^2\right), \qquad \varphi \in \mathbb{R}$$

is the unique solution of the equation

$$(B_{0,E+i\lambda}T)\theta\lambda, E = \theta_{\lambda,E}$$

Lemma 3.9, part 1 implies[7,8] that the spectral projection

$$P_{\lambda,E} = \int_\gamma \frac{1}{B_{0,E+i\lambda}T - z}\,dz$$

where γ is an appropriate closed contour enclosing only the eigenvalue one, has the simple form

$$P_{\lambda,E}(f) = f(0)\theta_{\lambda,E}, \qquad f \in H_{n,E}$$

Thus, if $\operatorname{Re} z < 0$, we can write

$$R^q_{\lambda,E,z}(f) = -\frac{1}{z}\theta_{\lambda,E} + [(R^q_{\lambda,E,z})(I - P_{\lambda,E})](f), \qquad f \in H_{n,E}$$

and the proof of part 2 follows. ∎

Let $E = \cos(\pi p/q)$ with $p < q$ relatively prime integers and $\alpha \in \mathbb{R}$. From Proposition 3.10, part 1 we get that

$$\langle g, (B_{0,E+\alpha}T)^q(f)\rangle_{n,E+\alpha} = \langle (TB_{0,-(E+\alpha)})^q(g), f\rangle_{n,E+\alpha}$$

for all $f \in H_{n+2,E}$ and $n = 0, 1, 2,\dots$. Differentiating the relation above with respect to α and setting $\alpha = 0$, we get

$$\left\langle g, i\left[\sum_{k=0}^{q-1} (B_{0,E}T)^k M_{\varphi^2}(B_{0,E})^{q-k}\right](f)\right\rangle_{n,E}$$
$$= \left\langle -i\left[\sum_{k=0}^{q-1} (TB_{0,-E})^{q-k} M_{\varphi^2}(TB_{0,-E})^k\right](g), (f)\right\rangle_{n,E}$$

for all $f \in H_{n+2,E}$ and $n = 0, 1, 2,\dots$. However,

$$[TB_{0,-E}]^k = (B_{0,E}T)^{q-k}$$

for all $k = 0,\dots, q-1$. Thus, the unbounded operator $iA_{0,E}$ defined by

$$iA_{0,E}(f) = m\left[\sum_{k=0}^{q-1} (B_{0,E}T)^k M_{\varphi^2}(B_{0,E})^{q-k}\right](f), \qquad f \in H_{n+2}$$

is symmetric in $H_{n,E}$ for all $n = 0, 1, 2,\dots$.

Theorem 4.4. The Friedrichs extension of $-A_{0,E}^2$ is a positive self-adjoint operator on $H_{n,E}$ for all $n = 0, 1, 2,\dots$ with the following properties:

1. Its spectrum has the form $\{0\} \cup [c, +\infty]$ for some strictly positive constant c.
2. Zero is a simple eigenvalue with eigenvector

$$\xi_{0,E}(\varphi^2) = \exp\left\{\frac{i}{2}\,[E + i(1-E^2)^{1/2}]\,\varphi^2\right\}, \qquad \varphi \in \mathbb{R}^2$$

   while the rest of the spectrum is supported by the invariant hyperplane $H_{n,E}^0$.

Proof. Let n be a positive integer and let $f \in H_{n+2,E}^0$. Then

$$\langle f, -A_{0,E}^2(f)\rangle_{n,E} = \|A_{0,E}(f)\|_{n,E}^2 = \lim_{\lambda \to +0}\left\|\frac{I - (B_{0,E+i\lambda}T)^q}{\lambda}(f)\right\|_{n,E}^2$$

However, using Lemma 4.3, we can bound the right-hand side of the above equation by a positive constant times the $H_{n,E}$ norm of f and the rest of the proof now follows through an explicit computation. ∎

5. THE ASYMPTOTIC EXPANSION FOR THE SPECIAL ENERGIES

We start this section by giving a proof for Theorem 2.1. Let E be of the special form mentioned before, let $\xi_{\lambda,E}$ be the unique solution of (3.7) such that $\xi_{\lambda,E}(0)=1$, and let n be any positive integer. We will show that the function $\mathbb{R}\ni\lambda\to\xi_{\lambda,E}\in H_{l,0}$ has an asymptotic expansion of order n (see XII.3 of ref. 13) at $\lambda=0$ for all $l=0, 1, 2,\ldots$ and that its coefficients $\{\xi^{(i)}_{0,E}\}_{i=0,1,\ldots,n}$ around $\lambda=0$ satisfy the equations

$$\begin{aligned}
&\xi^{(0)}_{0,E}(0)=1, \qquad \xi^{(i)}_{0,E}(0)=0, \quad i=1, 2,\ldots, n\\
&A_{0,E}\xi^{(0)}_{0,E}=0 \qquad\qquad (5.1)\\
&(n+1)A_{0,E}\xi^{(n)}_{0,E}=-\sum_{k=2}^{n+1}\binom{n+1}{k}\left(\frac{d^k}{dh^k}(B_{h,E}T)^q\right)_{h=0}(\xi^{(n+1-k)}_{0,E}), \qquad n\geqslant 1
\end{aligned}$$

We will use induction in n.

$n=0$. Let $f\in H_{l+4,0}$. Using Taylor's theorem, we get that for all $g\in H_{l,0}$

$$\begin{aligned}
\langle g, (B_{\lambda,E}T)^q(f)\rangle_{l,0}&=\langle g, f\rangle_{l,0}+\lambda\langle g, A_{0,E}(f)\rangle_{l,0}\\
&\quad+\lambda^2\left\langle g, \left(\frac{d^2}{dh^2}(B_{h,E}T)^q\right)_{h=c}(f)\right\rangle_{l,0}
\end{aligned}$$

for some c with $|c|<|\lambda|$ and where $\langle\cdot,\cdot\rangle_{l,0}$ denotes the inner product of $H_{l,0}$. Replacing f by $\xi_{\lambda,E}$, we get from Corollary 3.13 that if $\xi_{0,E}$ is the eigenvector defined in (4.2), $\|A_{0,E}(\xi_{\lambda,E}-\xi_{0,E})\|_{l,0}$ converges to zero as λ approaches zero for all $l=0, 1, 2,\ldots$. Thus, from Theorem 4.4 we conclude that $\xi_{\lambda,E}\to\xi_{0,E}$ in $H_{l,0}$, $l=0, 1, 2,\ldots$, as $\lambda\to 0$, which finishes the proof for $n=0$.

$n\to n+1$. Let us assume that the result is true for n. Using Taylor's theorem as before, we get that for all $g\in H_{l,0}$,

$$\begin{aligned}
0&=\langle g, \lambda A_{0,E}\xi_{\lambda,E}\rangle_{l,0}+\left\langle g, \sum_{k=2}^{n+2}\frac{\lambda^k}{k!}\left(\frac{d^k}{dh^k}(B_{h,E}T)^q\right)_{h=0}(\xi_{\lambda,E})\right\rangle_{l,0}\\
&\quad+\left\langle g, \frac{\lambda^{n+3}}{(n+3)!}\left(\frac{d^{n+3}}{dh^{n+3}}(B_{h,E}T)^q\right)_{h=c}(\xi_{\lambda,E})\right\rangle_{l,0}
\end{aligned}$$

for some $|c|<|\lambda|$. Substituting

$$\xi_{\lambda,E}=\sum_{k=0}^{n}\frac{\lambda^k}{k!}\xi^{(k)}_{0,E}+o(\lambda^n)$$

into the equation above and regrouping terms, we get that for small λ

$$\left\langle g, \lambda A_{0,E}\xi_{\lambda,E} + \sum_{i=2}^{n+1} \frac{\lambda^i}{i!}\left[\sum_{j=2}^{i} \binom{i}{j}\left(\frac{d^j}{dh^j}(B_{h,E}T)^q\right)_{h=0} (\xi_{0,E}^{(i-j)})\right]\right\rangle_{l,0}$$
$$= -\left\langle g, \frac{\lambda^{n+2}}{(n+2)!}\sum_{j=2}^{n+2}\binom{n+2}{j}\left(\frac{d^j}{dh^j}(B_{h,E}T)^q\right)_{h=0}(\xi_{0,E}^{(n+2-j)}) + o(\lambda^{n+2})\right\rangle_{l,0} \tag{5.2}$$

where $o(\lambda^{n+2})$ is a vector-valued function which divided by λ^{n+2} converges to zero in norm in $H_{l,0}$. Using Eqs. (5.1), we can rewrite (5.2) as

$$\left\langle g, (n+2)!\, A_{0,E}\left[\frac{1}{\lambda^{n+1}}\left(\xi_{\lambda,E} - \sum_{i=0}^{n}\frac{\lambda^i}{i!}\xi_{0,E}^{(i)}\right)\right]\right\rangle_{l,0}$$
$$= \left\langle g, \sum_{j=2}^{n+2}\binom{n+2}{j}\left(\frac{d^j}{dh^j}(B_{h,E}T)^q\right)_{h=0}(\xi_{0,E}^{(n+2-j)}) + \frac{1}{\lambda}o(\lambda)\right\rangle_{l,0}$$

for some $|c'| < \lambda$. Thus, the norms

$$\left\|\frac{1}{\lambda^{n+1}}\left[\xi_{\lambda,E} - \sum_{i=0}^{n}\frac{\lambda^i}{i!}\xi_{0,E}^{(i)}\right]\right\|_{l,0}$$

are bounded by positive constants independent of λ for λ small enough for all $l = 0, 1, 2,...$. Since in view of Theorem 4.4 the solutions of (5.1) are unique, we conclude, as in the proof of Theorem 2.2, from a weak compactness argument and Proposition 3.5, part 3, that

$$\frac{1}{\lambda^{n+1}}\left[\xi_{\lambda,E} - \sum_{i=0}^{n}\frac{\lambda^i}{i!}\xi_{0,E}^{(i)}\right]$$

converges in $H_{l,0}$ as λ approaches zero for all $l = 0, 1, 2,...$, and its limit $\xi_{0,E}^{(n+1)}$ satisfies the equation

$$(n+2)A_{0,E}\xi_{0,E}^{(n+1)} = -\sum_{k=2}^{n+2}\binom{n+2}{k}\left(\frac{d^k}{dh^k}(B_{h,E}T)^q\right)_{h=0}\xi_{0,E}^{(n+2-j)}$$

This completes our induction. The proof of Theorem 2.1 now follows from Proposition 3.15. ∎

We finish this section by discussing the instability of the anomalies method in the introduction.

Let $E = \pi\alpha$; since the action of $D_{0,E}S$ is equivalent to an ergodic map on the circle, one can show[3] that the set of equations

$$(I - D_{0,E}S)(\varphi_E^n) = \sum_{k=1}^{n}\binom{n}{k}\left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}S)\right)_{\lambda=0}(\varphi_E^{n-k}) \tag{5.2}$$

has a unique solution up to normalization $\int_{\mathbb{R}} \varphi_E^0(x)\,dx = 1$. Moreover, Bovier and Klein[3] have shown that as E approaches a special energy of the form $E_0 = \cos(\pi p/q)$ with $p < q$ relatively prime integers, φ_E^n have limits for all $n = 0, 1, 2, \ldots, q-1$. We will show that these limits have to be equal to the components of the solution to the modified equations

$$\binom{n}{1} A_{0,E}(\varphi_E^{n-1})) = -\sum_{k=2}^{n} \binom{n}{k} \left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}S)^q\right)_{\lambda=0} (\varphi_E^{n-k}) \tag{5.3}$$

for all $n = 0, \ldots, q-2$.

Let $\{\varphi_E^i\}_{i=0,1,\ldots}$ be a solution of (5.2). Applying $D_{0,E}S$ to both sides of the equation, we get

$$\begin{aligned}(D_{0,E}S)^2(\varphi_E^n) = \varphi_E^n &- \sum_{k=1}^{n} \binom{n}{k}\left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}S)\right)_{\lambda=0} (\varphi_E^{n-k}) \\ &- \left[(D_{0,E}S)\left[\sum_{k=1}^{n}\binom{n}{k}\left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}S)\right)_{\lambda=0}\right]\right](\varphi_E^{n-k})\end{aligned}$$

Using again (5.2), we can rewrite the above equation as

$$\begin{aligned}(D_{0,E}S)^2(\varphi_E^n) = \varphi_E^n &- \sum_{k=1}^{n}\binom{n}{k}\left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}T)\right)_{\lambda=0} \\ &\times\left[\sum_{j=0}^{n-k}\binom{n-k}{j}\left(\frac{d^j}{d\lambda^j}(D_{\lambda,E}S)\right)_{\lambda=0}(\varphi_E^{n-k-j})\right] \\ &- (D_{0,E}S)\left[\sum_{k=1}^{n}\binom{n}{k}\left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}S)\right)_{\lambda=0}(\varphi_E^{n-k})\right]\end{aligned}$$

or

$$\begin{aligned}(D_{0,E}S)^2(\varphi_E^n) = \varphi_E^n &- \sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{i}\binom{n-i}{k-i}\left(\frac{d^i}{d\lambda^i}(D_{\lambda,E}S)\right)_{\lambda=0} \\ &\times\left[\left(\frac{d^{k-i}}{d\lambda^{k-i}}(D_{\lambda,E}S)\right)_{\lambda=0}(\varphi_E^{n-k})\right] \\ &- (D_{0,E}S)\left[\sum_{k=1}^{n}\binom{n}{k}\left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}S)\right)_{\lambda=0}(\varphi_E^{n-k})\right]\end{aligned}$$

or

$$(D_{0,E}S)^2(\varphi_E^n) = \varphi_E^n - \sum_{k=1}^{n}\binom{n}{k}\left(\frac{d^k}{d\lambda^k}(D_{\lambda,E}S)^2\right)_{\lambda=0}(\varphi_E^{n-k})$$

Performing the same calculation q times, we get

$$(D_{0,E}S)^q(\varphi_E^n) = \varphi_E^n - \sum_{k=1}^{n} \binom{n}{k} \left(\frac{d^k}{d\lambda^k} (D_{\lambda,E}S)^q \right)_{\lambda=0} (\varphi_E^{n-k}) \tag{5.4}$$

Let $\tilde{\varphi}_{E_0}^n = \lim_{E \to E_0} \varphi_E^n$, $n = 0, 1, 2,..., q-1$ (see Lemma 3.2 of ref. 3). Relation (5.4) implies that

$$\binom{q-1}{1} A_{0,E_0} \tilde{\varphi}_{E_0}^{q-2} = -\sum_{k=2}^{q-1} \binom{q-1}{k} \left(\frac{d^k}{d\lambda^k} (D_{\lambda,E_0}S)^q \right)_{\lambda=0} \varphi_{E_0}^{q-1-k}$$

Since the solution of the equation above are unique (up to the usual normalization) the result follows.

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