

Existence Theorems for Multidimensional Lagrange Problems¹

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Abstract. Existence theorems are proved for multidimensional Lagrange problems of the calculus of variations and optimal control. The unknowns are functions of several independent variables in a fixed bounded domain, the cost functional is a multiple integral, and the side conditions are partial differential equations, not necessarily linear, with assigned boundary conditions. Also, unilateral constraints may be prescribed both on the space and the control variables. These constraints are expressed by requiring that space and control variables take their values in certain fixed or variable sets which are assumed to be closed but not necessarily compact.

1. Introduction

In the present paper, we consider multidimensional Lagrange problems of the calculus of variations of various forms, in particular, Lagrange problems where we seek the minimum of a functional of the form

$$I[x, u] = \int_G f_0(t, x, u) dt$$

in certain classes of pairs

$$x(t) = (x^1, \dots, x^n), \quad u(t) = (u^1, \dots, u^m), \quad t = (t^1, \dots, t^v) \in G \subset E_v,$$

satisfying (a) a system of partial differential equations each of the form

$$\partial^h x^i / \partial^{\alpha_1} t^1 \cdots \partial^{\alpha_v} t^v = f_{i\alpha}(t, x, u), \quad \alpha = (\alpha_1, \dots, \alpha_v), \quad h = \alpha_1 + \cdots + \alpha_v,$$

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(b) constraints of the form

$$(t, x(t)) \in A \subset E_\nu \times E_n, \quad x(t) \in U(t, x(t)) \subset E_m$$

(where A is a given fixed set, and $U(t, x)$, $(t, x) \in A$, is a given variable set depending on t and x), and (c) a suitable system of boundary conditions concerning the values of the functions x^i and a number of their partial derivatives on the boundary ∂G of the fixed open set G . Here, x is said to be the space variable and u the control variable.

These problems are called Pontryagin problems when the sets $U(t, x)$ are all compact and contained in a fixed bounded part of the u -space. In the generality above, in particular when the sets $U(t, x)$ are closed and not necessarily compact subsets of E_m , these problems are called Lagrange problems with unilateral constraints on the control variable u . If $U(t, x) = E_m$ for all t and x , then the same problems are simply called Lagrange problems (or problems without unilateral constraints in u).

In Ref. 1, we have already given existence theorems for optimal solutions for Pontryagin problems. In the present paper, we give existence theorems for optimal solutions for Lagrange problems with or without unilateral constraints. Other existence theorems will be given elsewhere. As in Ref. 1, we seek the optimal solutions in classes of pairs of functions x, u , with x belonging to suitable Sobolev spaces and u measurable. For the purpose of proving existence theorems for Lagrange problems with or without unilateral constraints, we use a technique which is analogous to the one we used in previous papers for unidimensional ($\nu = 1$) Pontryagin and Lagrange problems (Refs. 2 and 3), and for multidimensional ($\nu > 1$) Pontryagin problems (U compact) (Ref. 1). The present, more difficult situation ($\nu > 1$, U closed but not necessarily compact) has necessitated a more subtle analysis.

2. First Closure Theorem

Let G be a bounded open subset of the t -space E_ν , $t = (t^1, \dots, t^\nu)$; let $x = (x^1, \dots, x^n)$ denote a vector variable in E_n and $u = (u^1, \dots, u^m)$ a vector variable in E_m . As usual, we shall denote by $\text{cl } G$ and by $\text{bd } G = \partial G$ the closure and the boundary of G . We shall also denote by $\text{co } H$ the convex hull of a set H , and thus $\text{cl co } H$ is the closed convex hull of H . For every $t \in \text{cl } G$, let $A(t)$ be a given nonempty subset of E_n , and let A be the set of all (t, x) with $t \in \text{cl } G$, $x \in A(t)$. For every $(t, x) \in A$, let $U(t, x)$ be a subset of E_m and let M be the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$.

The set A defined above is a subset of $E_v \times E_n$, and its projection on E_v is $\text{cl } G$. The set M defined above is a subset of $E_v \times E_n \times E_m$ and its projection on $E_v \times E_m$ is A .

We shall assume below that G is bounded by a surface S which is a regular boundary in the sense of Sobolev (see Ref. 4, Ch. 1, p. 72) and, for the sake of simplicity, we shall say that G is of class K_l . Thus, S can be decomposed into a finite number of manifolds S_1, \dots, S_J of dimension $n - 1$ (and corresponding boundaries), each S_j having the property that it can be mapped into a hyperplane π_j by means of a transformation of coordinates T_j defined on a part G_j of G and continuous with continuous derivatives up to l th order, $j = 1, \dots, J$.

We shall denote by $x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^m)$, $t \in G$, vector functions of t in G . For every $i = 1, \dots, n$, we shall denote by $\{\alpha\}_i$ a given finite system of nonnegative integral indices $\alpha = (\alpha_1, \dots, \alpha_v)$, $0 \leq |\alpha| \leq l_i \leq l$, with $|\alpha| = \alpha_1 + \dots + \alpha_v$. We shall assume that each component $x^i(t)$ of $x(t)$ is L_{p_i} -integrable in G and possesses the generalized partial derivatives $D^\alpha x^i(t)$ of the orders $\alpha \in \{\alpha\}_i$, all L_{p_i} -integrable in G for certain $p_i \geq 1$, $i = 1, \dots, n$. We shall assume that each component $u^j(t)$ of $u(t)$ is measurable in G .

Let N denote the total number of indices α contained in the n systems $\{\alpha\}_i$, $i = 1, \dots, n$, and let $f(t, x, u) = (f_{i\alpha})$ denote an N -vector function whose components are real-valued functions $f_{i\alpha}(t, x, u)$ defined on M . We shall consider the system of N partial differential equations in G

$$D^\alpha x^i = f_{i\alpha}(t, x, u), \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n,$$

or, briefly,

$$Dx = f(t, x, u).$$

We are interested in pairs x, u of vector functions $x(t), u(t)$, $t \in G$, as above, satisfying the constraints

$$(t, x(t)) \in A, \quad u(t) \in U(t, x(t)) \quad \text{a.e. in } G,$$

and the system of partial differential equations

$$D^\alpha x^i(t) = f_{i\alpha}(t, x(t), u(t)) \quad \text{a.e. in } G, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n,$$

or, briefly,

$$Dx(t) = f(t, x(t), u(t)) \quad \text{a.e. in } G.$$

Given $\delta > 0$ and a point $(t_0, x_0) \in A$, we shall denote by closed neighborhood

$N_\delta(t_0, x_0)$ of radius δ of (t_0, x_0) in A the set of all $(t, x) \in A$ at a distance $\leq \delta$ from (t_0, x_0) . Then, by $U(t_0, x_0; \delta)$ we shall denote the set

$$U(t_0, x_0; \delta) = \bigcup U(t, x),$$

where \bigcup ranges over all $(t, x) \in N_\delta(t_0, x_0)$. We shall say that $U(t, x)$ satisfies property (U) at a point $(t_0, x_0) \in A$, provided

$$U(t_0, x_0) = \bigcap_{\delta > 0} \text{cl } U(t_0, x_0; \delta),$$

that is,

$$U(t_0, x_0) = \bigcap_{\delta} \text{cl } \bigcup_{(t, x) \in N_\delta(t_0, x_0)} U(t, x).$$

We shall say that $U(t, x)$ satisfies property (U) in A if $U(t, x)$ satisfies property (U) at every point $(t_0, x_0) \in A$. A set $U(t, x)$ satisfying property (U) is necessarily closed as the intersection of closed sets. Property (U) is the so-called property of upper semicontinuity used for different purposes by G. Choquet (Ref. 5), C. Kuratowski (Ref. 6), and E. Michael (Ref. 7).

We shall consider the sets

$$Q(t, x) = f(t, x, U(t, x)) = [z \mid z = f(t, x, u), u \in U(t, x)] \subset E_N$$

or other analogous sets, and assume that they are convex. We shall say that a set $Q(t, x)$ satisfies property (Q) at a point $(t_0, x_0) \in A$, provided

$$Q(t_0, x_0) = \bigcap_{\delta > 0} \text{cl co } Q(t_0, x_0; \delta),$$

that is,

$$Q(t_0, x_0) = \bigcap_{\delta} \text{cl co } \bigcup_{(t, x) \in N_\delta(t_0, x_0)} Q(t, x).$$

We shall say that $Q(t, x)$ satisfies property (Q) in A if $Q(t, x)$ satisfies property (Q) at every point $(t_0, x_0) \in A$. A set $Q(t, x)$ satisfying property (Q) is necessarily closed and convex as the intersection of closed and convex sets.

2.1. Closure Theorem 1. Let G be a bounded open set of the t -space E_ν of some class K_l , $l \geq 1$, for every $t \in \text{cl } G$ let $A(t)$ be a nonempty subset of the x -space E_n , and assume that the set A of all $(t, x) \in E_\nu \times E_n$ with $t \in \text{cl } G$, $x \in A(t)$ is closed. For every $(t, x) \in A$, let $U(t, x)$ be a nonempty closed subset of the u -space E_m and assume that $U(t, x)$ satisfies property (U)

in A . Let M be the set of all $(t, x, u) \in E_p \times E_n \times E_m$ with $(t, x) \in A$, $u \in U(t, x)$. For every $i = 1, \dots, n$, let $\{\alpha\}_i$ be a finite system of nonnegative integral indices $\alpha = (\alpha_1, \dots, \alpha_p)$, $1 \leq |\alpha| \leq l_i \leq l$, and let N be the total number of elements $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$. Let $f(t, x, u) = (f_{i\alpha}, \alpha \in \{\alpha\}_i, i = 1, \dots, n)$ be a continuous N -vector function on the set M , and assume that the set $Q(t, x) = f(t, x, U(t, x))$ is a convex closed subset of the z -space E_N for every $(t, x) \in A$, and that $Q(t, x)$ satisfies property (Q) in A . Let x, u , and $x_k, u_k, k = 1, 2, \dots$, be pairs of vector functions $x(t) = (x^1, \dots, x^n), u(t) = (u^1, \dots, u^m), x_k(t) = (x_k^1, \dots, x_k^n), u_k(t) = (u_k^1, \dots, u_k^m), t \in G, x^i, x_k^i \in L_1(G), i = 1, \dots, n, u^j, u_k^j$ measurable in $G, j = 1, \dots, m$. Assume that each component x^i, x_k^i of x, x_k possesses generalized partial derivatives $D^\alpha x^i, D^\alpha x_k^i \in L_1(G)$ of the orders $\alpha \in \{\alpha\}_i, i = 1, \dots, n, k = 1, 2, \dots$. Assume that

$$(t, x_k(t)) \in A, \quad u_k(t) \in U(t, x_k(t)) \quad \text{a.e. in } G, \quad (1)$$

$$D^\alpha x_k^i(t) = f_{i\alpha}(t, x_k(t), u_k(t)) \quad \text{a.e. in } G, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n, \quad (2)$$

and that

$$x_k^i(t) \rightarrow x^i(t) \text{ strongly in } L_1(G) \text{ as } k \rightarrow \infty, \quad (3)$$

$$D^\alpha x_k^i(t) \rightarrow D^\alpha x^i(t) \text{ weakly in } L_1(G) \text{ as } k \rightarrow \infty \quad (4)$$

for all $\alpha \in \{\alpha\}_i, i = 1, \dots, n$. Then, there is a measurable vector function $u(t) = (u^1, \dots, u^m), t \in G$, such that

$$(t, x(t)) \in A, \quad u(t) \in U(t, x(t)) \quad \text{a.e. in } G, \quad (5)$$

$$D^\alpha x^i(t) = f_{i\alpha}(t, x(t), u(t)) \quad \text{a.e. in } G, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n.$$

The strong convergence (3) can be replaced by pointwise convergence almost everywhere in G . We omit the proof of this statement since Closure Theorem 1 is a particular case of Closure Theorem 2. A proof of Closure Theorem 1 can be obtained by obvious simplifications in the one of Closure Theorem 2.

3. Second Closure Theorem

Let us denote by $y = (x^1, \dots, x^s)$ the s -vector made up of certain components, say $x^1, \dots, x^s, 0 \leq s \leq n$, of the n -vector $x = (x^1, \dots, x^n)$, and by z the complementary $(n-s)$ -vector $z = (x^{s+1}, \dots, x^n)$ of x , so that we shall write $x = (y, z)$. We shall assume that $f(t, y, u)$ depends only on the coordinates x^1, \dots, x^s of x . If $x(t), t \in G$, is any vector function, we shall denote by $x(t) =$

$(y(t), z(t))$ the corresponding decomposition of $x(t)$ in its coordinates $y(t) = (x^1, \dots, x^s)$ and $z(t) = (x^{s+1}, \dots, x^n)$. We shall denote by A a closed subset of the ty -space $E_v \times E_s$, and then $\bar{A} = A \times E_{n-s}$ will be a closed subset of the tx -space $E_v \times E_n$. We shall consider n -vector functions $x(t) = (y(t), z(t))$ for which the first s components, or vector $y(t) = (x^1, \dots, x^s)$, possess certain generalized partial derivatives $D^\alpha x^i(t)$ as in Closure Theorem 1. On the remaining $n - s$ components, or vector $z(t) = (x^{s+1}, \dots, x^n)$, we shall have a different set of assumptions.

We shall consider countable systems $\{I\}$ of intervals $I = [a, b] \subset G$, $a = (a^1, \dots, a^v)$, $b = (b^1, \dots, b^v)$, $a^j < b^j$, $j = 1, \dots, v$, with the property that, for any point $t \in G$, there are hypercubes $I \in \{I\}$ with $t \in I$ and $\text{diam } I$ as small as we want. For any given function $z(t)$, $t \in G$, we shall consider the usual differences of order ν relative to the 2^ν vertices of I , say

$$\Delta z = \Delta_1 z = z(b) - z(a) \quad \text{if } \nu = 1,$$

$$\Delta z = \Delta_2 z = z(b^1, b^2) - z(b^1, a^2) - z(a^1, b^2) + z(a^1, a^2) \quad \text{if } \nu = 2,$$

and so on. A nonconstant function $z(t)$, $t \in G$, is said to be singular in G with respect to the system $\{I\}$ provided $\Delta_1 z$ is singular, that is, provided for almost all $t_0 \in G$ we have $(\text{meas } I)^{-1} \Delta z \rightarrow 0$ as $\text{diam } I \rightarrow 0$, with I a hypercube, $I \in \{I\}$, $t_0 \in I$. As we know (Ref. 8), any interval function $\varphi(I)$ of bounded variation possesses a decomposition $\varphi(I) = \Phi(I) + S(I)$, where Φ is absolutely continuous and S is singular. If $\varphi(I)$ is nonnegative, then both Φ and S are nonnegative.

3.1. Closure Theorem 2. Let G be a bounded open set of the t -space E_v of some class K_l , $l \geq 1$, let $A(t)$ be a nonempty subset of the y -space E_s defined for every $t \in \text{cl } G$, and assume that the set A of all $(t, y) \in E_v \times E_s$ with $t \in \text{cl } G$, $y \in A(t)$, is closed. Let \bar{A} be the closed set $\bar{A} = A \times E_{n-s} \subset E_v \times E_n$. Let $U(t, y)$ denote a closed subset of E_n for every $(t, y) \in A$, and let M be the set of all $(t, y, u) \in E_v \times E_s \times E_n$ with $(t, y) \in A$, $u \in U(t, y)$. For every $i = 1, \dots, n$, let $\{\alpha\}_i$ be a finite system of indices $\alpha = (\alpha_1, \dots, \alpha_v)$, $1 \leq |\alpha| \leq l_i \leq l$, and let \bar{N} be the total number of elements $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$. We assume that, for every $i = s + 1, \dots, n$, $\{\alpha\}_i$ is made up of only one element $\alpha = \alpha_0 = (1, 1, \dots, 1)$. If N denotes the number of elements $\alpha \in \{\alpha\}_i$, $i = 1, \dots, s$, then $\bar{N} = N + (n - s)$. Let $f(t, y, u) = (f_{i\alpha})$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$, be a continuous \bar{N} -vector function on M , which we may write in the form $\bar{f} = (f, f_0)$, where f is the N -vector of its first N components, and f_0 the $(n - s)$ -vector of its remaining components. Let $\bar{z} = (z^1, \dots, z^{\bar{N}}) = (z, z_0)$ with $z = (z^1, \dots, z^N)$, $z_0 = (z^{N+1}, \dots, z^{\bar{N}})$ and, for every $(t, y) \in A$, let $\bar{Q}(t, y)$

denote the subset of the τ -space $E_{\bar{N}}$ defined by $z = f(t, y, u)$, $z^i \geq f_{i\alpha_0}(t, y, u)$, $i = N + 1, \dots, \bar{N}$, $u \in U(t, y)$. Let us assume that $\bar{Q}(t, y)$ is a closed convex subset of $E_{\bar{N}}$ satisfying property (Q) in A . Also, assume that $f_{i\alpha}(t, y, u) \geq -c_{i\alpha}$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, s$, for all $(t, y, u) \in A$ and some constants $c_{i\alpha} \geq 0$. Let $x = (y, z)$, u , $x_k = (y_k, z_k)$, u_k , $k = 1, 2, \dots$, be functions defined on G , all components of x and x_k of class $L_1(G)$ and all components of u and u_k measurable. Assume that each component x^{k_i} and x_k^i of x and x_k has generalized partial derivatives $D^\alpha x^i$, $D^\alpha x_k^i$ of the orders $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$, all of class $L_1(G)$, and that

$$(t, y_k(t)) \in A, \quad u_k(t) \in U(t, y_k(t)) \quad \text{a.e. in } G, \quad (6)$$

$$D^\alpha x_k^i(t) = f_{i\alpha}(t, y_k(t), u_k(t)) \quad \text{a.e. in } G, \quad (7)$$

with

$$\alpha \in \{\alpha\}_i, \quad i = 1, \dots, n, \quad k = 1, 2, \dots$$

Assume that, as $k \rightarrow \infty$, we have

$$x_k^i(t) \rightarrow x^i(t) \text{ strongly in } L_1(G), \quad i = 1, \dots, s, \quad (8)$$

$$D^\alpha x_k^i(t) \rightarrow D^\alpha x^i(t) \text{ weakly in } L_1(G), \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, s. \quad (9)$$

Assume that there is a countable system of real numbers $[t]$ everywhere dense in E_1 such that, for all points $\{t_\rho\}$, $t_\rho \in G$, of the form $t_\rho = (t^1, \dots, t^v)$, $t^j \in [t]$, $j = 1, \dots, v$, we have

$$x_k^i(t) \rightarrow x^i(t) \text{ at every } t \in \{t_\rho\} \text{ as } k \rightarrow \infty, \quad i = s + 1, \dots, n, \quad (10)$$

and for all intervals $\{I\}$, $I \subset G$, with vertices in $\{t_\rho\}$, we have

$$\int_I D^\alpha x_k^i(t) dt \rightarrow \int_I D^\alpha x^i(t) dt \text{ as } k \rightarrow \infty \quad (11)$$

with

$$\alpha = \alpha_0 \in \{\alpha\}_i, \quad i = s + 1, \dots, n.$$

Assume that there is a decomposition $x^i(t) = X^i(t) + S^i(t)$, $i = s + 1, \dots, n$, of each x^i , $i = s + 1, \dots, n$, into two parts X^i , S^i , both of class $L_1(G)$, $X^i(t)$ with generalized partial derivative $D^{\alpha_0} X^i$ of class $L_1(G)$ and $S^i(t)$ singular. Let $Z(t)$ denote the $(n - s)$ -vector $Z(t) = (X^{s+1}, \dots, X^n)$. Then, there is a measurable vector function $u(t) = (u^1, \dots, u^m)$, $t \in G$, such that, a.e. in G , we have

$$(t, y(t)) \in A, \quad u(t) \in U(t, y(t)), \quad (12)$$

$$D^\alpha x^i(t) = f_{i\alpha}(t, y(t), u(t)), \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, s, \quad (13)$$

$$D^\alpha X^i(t) = f_{i\alpha}(t, y(t), u(t)), \quad \alpha = \alpha_0 \in \{\alpha\}_i, \quad i = s + 1, \dots, n. \quad (14)$$

Strong convergence (8) can be replaced by pointwise convergence almost everywhere in G .

3.2. Remark. Closure Theorem 2 reduces to Closure Theorem 1 when $s = n$, and then $A = \tilde{A}$, $f = \tilde{f}$, $x = y$.

3.3. Proof. For every $\alpha \in \{\alpha_j^i\}$, $i = 1, \dots, n$, let $\varphi_{i\alpha}$, $\varphi_{i\alpha k}$ denote the real-valued functions defined a.e. in G

$$\begin{aligned}\varphi_{i\alpha k}(t) &= D^{\alpha x_k^i}(t) = f_{i\alpha}(t, y_k(t), u_k(t)), \\ \varphi_{i\alpha}(t) &= D^{\alpha x^i}(t), \quad i = 1, \dots, s, \\ \varphi_{i\alpha}(t) &= D^{\alpha X^i}(t), \quad i = s + 1, \dots, n,\end{aligned}$$

and let φ , φ_k denote the \tilde{N} -vectors

$$\begin{aligned}\varphi_k(t) &= (\varphi_{i\alpha k}(t), \alpha \in \{\alpha_j^i\}, i = 1, \dots, n), \quad k = 1, 2, \dots, \\ \varphi(t) &= (\varphi_{i\alpha}(t), \alpha \in \{\alpha_j^i\}, i = 1, \dots, n).\end{aligned}$$

By hypothesis, A is a closed subset of $E_v \times E_s$, $U(t, y)$ is a closed subset of E_m for every $(t, y) \in A$, and $U(t, y)$ satisfies property (U) in A . By Ref. 1, Section 4, M is a closed subset of $E_v \times E_s \times E_m$.

For every $t_0 \in G$, $t_0 = (t_0^1, \dots, t_0^v)$, let $\delta_0 = \delta_0(t_0)$ denote the distance of t_0 from ∂G and let $q = q_h = [t_0, t_0 + h]$ denote any closed hypercube $q = [\tilde{t}^j \leq t^j \leq \tilde{t}^j + h, j = 1, \dots, v]$, where $\tilde{t}^j, \tilde{t}^j + h \in [t]$, h is a positive number with $0 < h < \delta_0/v$, $\tilde{t}^j \leq t_0^j \leq \tilde{t}^j + h$, so that $t_0 \in q \subset G$; and we have denoted here, for the sake of simplicity, by h also the v -vector (h, \dots, h) . For almost every $t_0 \in G$, we have, as $h \rightarrow 0^+$,

$$h^{-v} \int_q x^i(t) dt = x^i(t_0), \quad i = 1, \dots, s, \quad (15)$$

$$h^{-v} \int_q D^{\alpha x^i}(t) dt \rightarrow D^{\alpha x^i}(t_0), \quad \alpha \in \{\alpha_j^i\}, \quad i = 1, \dots, s, \quad (16)$$

$$h^{-v} \int_q X^i(t) dt \rightarrow X^i(t_0), \quad i = s + 1, \dots, n, \quad (17)$$

$$h^{-v} \Delta_q S^i \rightarrow 0, \quad i = s + 1, \dots, n. \quad (18)$$

Also, for almost all $t \in G$, we have $(t, y_k(t)) \in A$ for all $k = 1, 2, \dots$. The convergence $x_k^i \rightarrow x^i$ in $L_1(G)$ as $k \rightarrow \infty$, $i = 1, \dots, s$, as stated in (8) (that is, $y_k \rightarrow y$ in $L_1(G)$), implies convergence in measure in G , and hence

there is a subsequence $[y_{k_r}(t)]$ which converges pointwise a.e. in G . Let G_0 be the set of all $t \in G$ where the relations (15) through (18) hold, where $(t, y_k(t)) \in A$ for all k , and $y_{k_r}(t) \rightarrow y(t)$ as $r \rightarrow \infty$. Then, G_0 is measurable with $\text{meas } G_0 = \text{meas } G$. For every $t \in G_0$, we have now $(t, y_{k_r}(t)) \in A$, where A is a closed set, and as $r \rightarrow \infty$, then $(t, y(t)) \in A$ for $t \in G_0$, that is, $(t, y(t)) \in A$ a.e. in G .

Because of the convergence $y_k(t) \rightarrow y(t)$ in $L_1(G)$, and hence in measure, and consequent pointwise convergence $y_{k_r}(t) \rightarrow y(t)$ everywhere in G_0 with $\text{meas } G_0 = \text{meas } G$, we know that there are closed sets C_λ , $\lambda = 1, 2, \dots$, with $C_\lambda \subset G_0$, $C_\lambda \subset C_{\lambda+1}$, $\text{meas } C_\lambda > \text{meas } G_0 - \lambda^{-1}$, such that $y(t)$ is continuous on C_λ and $y_{k_r}(t) \rightarrow y(t)$ uniformly on C_λ as $r \rightarrow \infty$ for every $\lambda = 1, 2, \dots$. Since G is bounded, and $C_\lambda \subset G_0 \subset G$, each set C_λ is compact, and hence $y(t)$, $y_{k_r}(t)$, $r = 1, 2, \dots$, are continuous, uniformly continuous, and equicontinuous on each C_λ .

Let λ be any fixed integer with $\lambda > (\text{meas } G)^{-1}$; hence, $\text{meas } C_\lambda > 0$. Let $\epsilon > 0$ be an arbitrary number. There is some $\delta'_0 = \delta'_0(\epsilon, \lambda) > 0$ such that $|t - t'| \leq \delta'_0$, with $t, t' \in C_\lambda$, implies $|y(t) - y(t')| \leq \epsilon$, $|y_{k_r}(t) - y_{k_r}(t')| \leq \epsilon$ for every $r = 1, 2, \dots$. Also, there exists some $k = k(\epsilon, \lambda)$ such that $k_r \geq k(\epsilon, \lambda)$, $t \in C_\lambda$, implies $|y_{k_r}(t) - y(t)| \leq \epsilon$.

Let $c > 0$ be any number such that $0 \leq c_{i\alpha} < c$, or $0 \leq c_{i\alpha}/c < 1$, $\alpha = \alpha_0 \in \{\alpha\}_i$, $i = s + 1, \dots, n$.

Let $\chi_\lambda(t)$, $\chi_\lambda^*(t)$, $t \in G$, be the characteristic functions of the sets C_λ and $G - C_\lambda$, so that $\chi_\lambda + \chi_\lambda^* = 1$ everywhere in G . All functions $\chi_\lambda(t)$ and $\chi_\lambda(t) D^\alpha x^i(t)$, $\alpha \in \{\alpha\}_i$, $i = 1, \dots, s$, are of class $L_1(G)$, and for every $t_0 \in C_\lambda$ we have

$$\chi_\lambda(t_0) = 1, \quad \chi_\lambda^*(t) D^\alpha x^i(t_0) = 0.$$

Then, for almost all $t_0 \in C_\lambda$, we have also, as $h \rightarrow 0^+$,

$$h^{-\nu} \int_q \chi_\lambda(t) dt \rightarrow \chi_\lambda(t_0) = 1, \tag{19}$$

and

$$h^{-\nu} \int_q \chi_\lambda^*(t) D^\alpha x^i(t) dt \rightarrow \chi_\lambda^*(t_0) D^\alpha x^i(t_0) = 0, \tag{20}$$

with

$$\alpha \in \{\alpha\}_i, \quad i = 1, \dots, s.$$

Let C'_λ be the subset of C_λ where this occurs. Then, C_λ is measurable, $C'_\lambda \subset C_\lambda \subset G_0 \subset G$, $\text{meas } C'_\lambda = \text{meas } C_\lambda > \text{meas } G_0 - \lambda^{-1} > 0$, and, finally,

$$\text{meas} \left(\bigcup_\lambda C'_\lambda \right) = \text{meas } G_0 = \text{meas } G. \tag{21}$$

Let $\eta > 0$ be any positive number independent of ϵ . Let t_0 be any point of C'_λ , let $y_0 = y(t_0)$, and let u_0 be any point of $U(t_0, y_0)$. Let $M_1 > 0$ be any constant such that

$$|f_{i\alpha}(t_0, y_0, u_0)| \leq M_1, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n.$$

Let us fix $h > 0$ so small that

$$h < \epsilon/\nu, \quad h < \delta_0/\nu, \quad h < \delta'_0/\nu,$$

where

$$\delta_0 = \delta_0(t_0), \quad \delta'_0 = \delta'_0(\epsilon, \lambda),$$

and also so small that

$$\left| D^{\alpha} X^i(t_0) - h^{-\nu} \int_q D^{\alpha} X^i(t) dt \right| < \eta \tilde{N}^{-1}, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, s, \quad (22)$$

$$h^{-\nu} \text{meas}(q \cap C_\lambda) = h^{-\nu} \int_q \chi_\lambda(t) dt \geq \max[2^{-1}, 1 - \eta \tilde{N}^{-1} M_1, 1 - \eta \tilde{N}^{-1} c^{-1}], \quad (23)$$

$$\left| h^{-\nu} \int_q \chi_\lambda^*(t) D^{\alpha} X^i(t) dt \right| < \eta \tilde{N}^{-1}, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, s, \quad (24)$$

$$|h^{-\nu} \Delta_q S^i| < \eta \tilde{N}^{-1}, \quad i = s + 1, \dots, n, \quad (25)$$

$$\left| D^{\alpha} X^i(t_0) - h^{-\nu} \int_q D^{\alpha} X^i(t) dt \right| \leq \eta \tilde{N}^{-1}, \quad \alpha = \alpha_0 \in \{\alpha\}_i, \quad i = s + 1, \dots, n. \quad (26)$$

This is possible because of expressions (15)–(20). Let H and H^* be the sets

$$H = q \cap C_\lambda, \quad H^* = q - H = q - q \cap C_\lambda = q - C_\lambda.$$

Then, we have

$$h^{-\nu} \text{meas } H \geq \max[2^{-1}, 1 - \eta \tilde{N}^{-1} M_1^{-1}, 1 - \eta \tilde{N}^{-1} c^{-1}], \quad (27)$$

$$h^{-\nu} \text{meas } H^* \leq \min[2^{-1}, \eta \tilde{N}^{-1} M_1^{-1}, \eta \tilde{N}^{-1} c^{-1}]. \quad (28)$$

For any k , let $\bar{\varphi}_{i\alpha k}(t)$, $t \in q$, be the function defined by

$$\begin{aligned} \bar{\varphi}_{i\alpha k}(t) &= \varphi_{i\alpha k}(t) = f_{i\alpha}(t, y_k(t), u_k(t)) && \text{for } t \in H, \\ \bar{\varphi}_{i\alpha k}(t) &= f_{i\alpha}(t_0, y_0, u_0) && \text{for } t \in H^* = q - H, \end{aligned}$$

and let $\bar{\varphi}_k(t)$ be the \tilde{N} -vector

$$\varphi_k(t) = (\varphi_{i\alpha k}(t), \alpha \in \{\alpha\}_i, i = 1, \dots, n).$$

For $t \in H$ and $k = k_r \geq k(\epsilon, \lambda)$, we have now

$$\begin{aligned} |t - t_0| &\leq \nu h \leq \min[\epsilon, \delta_0, \delta'_0], \\ |y_k(t) - y_0| &= |y_k(t) - y_k(t_0)| + |y_k(t_0) - y(t_0)| \leq \epsilon + \epsilon = 2\epsilon, \end{aligned}$$

and, hence,

$$(t, y_k(t)) \in N_{3\epsilon}(t_0, y_0) \quad \text{for } t \in H, \quad k = k_r \geq k(\epsilon, \lambda).$$

Thus,

$$\begin{aligned} \bar{\varphi}_k(t) &= f(t, y_k(t), u_k(t)) = (f_{i\alpha}(t, y_k(t), u_k(t)), \alpha \in \{\alpha\}_i, i = 1, \dots, n), \quad t \in H, \\ \bar{\varphi}_k(t) &= f(t_0, y_0, u_0) = (f_{i\alpha}(t_0, y_0, u_0), \alpha \in \{\alpha\}_i, i = 1, \dots, n), \quad t \in q - H, \end{aligned}$$

and, hence, for all $t \in q$, we have

$$\bar{\varphi}_k(t) \in \tilde{Q}(t_0, y_0; 3\epsilon), \quad t \in q, \quad k = k_r \geq k(\epsilon, \lambda). \quad (29)$$

Finally,

$$h^{-\nu} \int_q \bar{\varphi}_k(t) dt \in \text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon), \quad k = k_r \geq k(\epsilon, \lambda), \quad (30)$$

since the last set is convex and closed.

Note that the relations

$$|f_{i\alpha}(t_0, y_0, u_0)| \leq M_1, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n,$$

certainly imply that

$$\begin{aligned} \left| h^{-\nu} \int_{H^*} \bar{\varphi}_{i\alpha k}(t) dt \right| &= \left| h^{-\nu} \int_{H^*} f_{i\alpha}(t_0, y_0, u_0) dt \right| \\ &\leq M_1 h^{-\nu} \text{meas } H^* \leq M_1 h^{-\nu} (h^\nu \eta \tilde{N}^{-1} M_1^{-1}) = \eta \tilde{N}^{-1} \end{aligned} \quad (31)$$

with

$$\alpha \in \{\alpha\}_i, \quad i = 1, \dots, n.$$

We shall now obtain certain estimates first for $i = 1, \dots, s$ and then for $i = s + 1, \dots, n$.

Let $\chi_{\lambda q}(t)$, $\chi_{\lambda q}^*(t)$, $t \in G$, be the characteristic functions of the sets H and H^* in G . Since $H \cap H^* = \emptyset$, $H \cup H^* = q$, we have

$$\begin{aligned} \chi_{\lambda q} + \chi_{\lambda q}^* &= 1 \quad \text{everywhere in } q, \\ \chi_{\lambda q} = \chi_{\lambda q}^* &= 0 \quad \text{everywhere in } G - q. \end{aligned}$$

Then, as $k \rightarrow \infty$, hypothesis (9) implies that

$$\begin{aligned} \int_G \chi_{\lambda q}(t) \varphi_{i\alpha k}(t) dt &\rightarrow \int_G \chi_{\lambda q}(t) \varphi_{i\alpha}(t) dt, \\ \int_G \chi_{\lambda q}^*(t) \varphi_{i\alpha k}(t) dt &\rightarrow \int_G \chi_{\lambda q}^*(t) \varphi_{i\alpha}(t) dt, \end{aligned}$$

with

$$\alpha \in \{\alpha\}_i, \quad i = 1, \dots, s,$$

that is,

$$\int_H \varphi_{i\alpha k} dt \rightarrow \int_H \varphi_{i\alpha} dt, \quad \int_{H^*} \varphi_{i\alpha k} dt \rightarrow \int_{H^*} \varphi_{i\alpha} dt.$$

Note that the sum of these relations yields

$$\int_q \varphi_{i\alpha k} dt \rightarrow \int_q \varphi_{i\alpha} dt$$

as $k \rightarrow \infty$. We can now determine an integer $k' = k'(t_0, \epsilon, \lambda, \eta) \geq k(\epsilon, \lambda)$ such that, for $k = k_r$, $k \geq k'(t_0, \epsilon, \lambda, \eta)$, we have

$$\begin{aligned} \left| \int_H \varphi_{i\alpha k} dt - \int_H \varphi_{i\alpha} dt \right| &\leq \tilde{N}^{-1} h^\nu \eta, \\ \left| \int_{H^*} \varphi_{i\alpha k} dt - \int_{H^*} \varphi_{i\alpha} dt \right| &\leq \tilde{N}^{-1} h^\nu \eta, \end{aligned} \tag{32}$$

and, of course, also

$$\left| \int_q \varphi_{i\alpha k} dt - \int_q \varphi_{i\alpha} dt \right| \leq 2\tilde{N}^{-1} h^\nu \eta. \tag{33}$$

Now, for

$$k = k_r, \quad k = k'(t_0, \epsilon, \lambda, \eta), \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, s,$$

we have

$$\begin{aligned} &\left| D^\alpha x^i(t_0) - h^{-\nu} \int_q \bar{\varphi}_{i\alpha k}(t) dt \right| \\ &= \left| \varphi_{i\alpha}(t_0) - h^{-\nu} \int_q \varphi_{i\alpha k} dt + h^{-\nu} \int_{H^*} \varphi_{i\alpha k} dt - h^{-\nu} \int_{H^*} \bar{\varphi}_{i\alpha k} dt \right| \\ &\leq \left| \varphi_{i\alpha}(t_0) - h^{-\nu} \int_q \varphi_{i\alpha} dt \right| + \left| h^{-\nu} \int_q \varphi_{i\alpha} dt - h^{-\nu} \int_q \varphi_{i\alpha k} dt \right| \\ &\quad + \left| h^{-\nu} \int_{H^*} \varphi_{i\alpha k} dt - h^{-\nu} \int_{H^*} \varphi_{i\alpha} dt \right| + \left| h^{-\nu} \int_{H^*} \varphi_{i\alpha} dt \right| \\ &\quad + \left| h^{-\nu} \int_{H^*} \bar{\varphi}_{i\alpha k} dt \right| = d_1 + d_2 + d_3 + d_4 + d_5. \end{aligned} \tag{34}$$

By (22) we have $d_1 \leq \tilde{N}^{-1}\eta$, by (33) we have $d_2 \leq 2\tilde{N}^{-1}\eta$, by (32) we have $d_3 \leq \tilde{N}^{-1}\eta$, by (24) we have $d_4 \leq \tilde{N}^{-1}\eta$, and by (31) we have $d_5 \leq \tilde{N}^{-1}\eta$. Thus, (34) yields

$$\left| D^\alpha x^i(t_0) - h^{-\nu} \int_q \bar{\varphi}_{i\alpha k}(t) dt \right| \leq 6\tilde{N}^{-1}\eta, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, s, \quad (35)$$

for $k = k_r$, $k \geq k'(t_0, \epsilon, \lambda, \eta)$.

We shall now obtain analogous estimates for $i = s + 1, \dots, n$. For these values of i , we have $f_{i\alpha}(t, y, u) \geq -c_{i\alpha}$ and, hence,

$$\begin{aligned} h^{-\nu} \int_{H^*} \varphi_{i\alpha k} dt &= h^{-\nu} \int_{H^*} f_{i\alpha}(t, y_k(t), u_k(t)) dt \\ &\geq -h^{-\nu} c_{i\alpha} \text{meas } H^* \geq -h^{-\nu} c_{i\alpha} (h^\nu \tilde{N}^{-1} c^{-1} \eta) > -\tilde{N}^{-1}\eta \end{aligned} \quad (36)$$

for all k and where $\alpha = \alpha_0 \in \{\alpha\}_i$, $i = s + 1, \dots, n$. Since $\alpha = \alpha_0 = (1, 1, \dots, 1)$, we have

$$\begin{aligned} h^{-\nu} \int_q \varphi_{i\alpha k}(t) dt &= h^{-\nu} \Delta_q x_k^i, \quad i = s + 1, \dots, n, \quad k = 1, 2, \dots, \quad (37) \\ h^{-\nu} \int_q \varphi_{i\alpha}(t) dt &= h^{-\nu} \Delta_q X^i, \quad i = s + 1, \dots, n. \end{aligned}$$

On the other hand, $x^i = X^i + S^i$ and, by (25),

$$h^{-\nu} \Delta_q x^i = h^{-\nu} \Delta_q X^i + h^{-\nu} \Delta_q S^i, \quad |h^{-\nu} \Delta_q S^i| \leq \tilde{N}^{-1}\eta \quad (38)$$

Also, by force of (10), we can determine $k'(t_0, \epsilon, \lambda, \eta)$ above so that, for $k = k_r \geq k'(t_0, \epsilon, \lambda, \eta)$, we have

$$|h^{-\nu} \Delta_q x_k^i - h^{-\nu} \Delta_q x^i| \leq \tilde{N}^{-1}\eta, \quad i = s + 1, \dots, n. \quad (39)$$

Finally, (37), (38), and (39) yield

$$\begin{aligned} \left| h^{-\nu} \int_q \varphi_{i\alpha k}(t) dt - h^{-\nu} \int_q \varphi_{i\alpha}(t) dt \right| &= |h^{-\nu} \Delta_q x_k^i - h^{-\nu} \Delta_q X^i| \\ &\leq |h^{-\nu} \Delta_q x_k^i - h^{-\nu} \Delta_q x^i| + |h^{-\nu} \Delta_q S^i| \leq \tilde{N}^{-1}\eta + \tilde{N}^{-1}\eta = 2\tilde{N}^{-1}\eta. \end{aligned} \quad (40)$$

For $\alpha = \alpha_0 \in \{\alpha\}_i$, $i = s + 1, \dots, n$, we have then, as in (34),

$$\begin{aligned} D^\alpha X^i(t_0) - h^{-\nu} \int_q \bar{\varphi}_{i\alpha k}(t) dt &= \varphi_{i\alpha}(t_0) - h^{-\nu} \int_q \varphi_{i\alpha k} dt + h^{-\nu} \int_{H^*} \varphi_{i\alpha k} dt - h^{-\nu} \int_{H^*} \bar{\varphi}_{i\alpha k} dt \\ &\geq - \left| \varphi_{i\alpha}(t_0) - h^{-\nu} \int_q \varphi_{i\alpha} dt \right| - \left| h^{-\nu} \int_q \varphi_{i\alpha} dt - h^{-\nu} \int_q \varphi_{i\alpha k} dt \right| \\ &\quad + h^{-\nu} \int_{H^*} \varphi_{i\alpha k} dt - \left| h^{-\nu} \int_{H^*} \bar{\varphi}_{i\alpha k} dt \right| \\ &= d_{01} + d_{02} + d_{03} + d_{04}. \end{aligned} \quad (41)$$

By (26) we have $d_{01} \geq -\tilde{N}^{-1}\eta$, by (40) we have $d_{02} \geq -2\tilde{N}^{-1}\eta$, by (36) we have $d_{03} > -\tilde{N}^{-1}\eta$, by (31) we have $d_{04} > -\tilde{N}^{-1}\eta$, and hence (41) yields

$$D^s X^i(t_0) - h^{-\nu} \int_q \bar{\varphi}_{i\alpha k}(t) dt \geq -5\tilde{N}^{-1}\eta, \quad \alpha = \alpha_0 \in \{\alpha\}_i, \quad i = s+1, \dots, n, \quad (42)$$

for every $k = k_r \geq k'(t_0, \epsilon, \lambda, \eta)$.

Let us denote now, for the sake of simplicity, by y and Z the s -vector $y = (x^1, \dots, x^s)$ and $(n-s)$ -vector $Z = (X^{s+1}, \dots, X^n)$ respectively and, hence, by Dy and DZ the N -vector and $(n-s)$ -vector

$$\begin{aligned} Dy &= (D^s x^i, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, s), \\ DZ &= (D^s X^i, \quad \alpha = \alpha_0 \in \{\alpha\}_i, \quad i = s+1, \dots, n). \end{aligned}$$

Also, let z_1 and z_2 denote the N -vector and the $(n-s)$ -vector

$$\begin{aligned} z_1 &= \left(h^{-\nu} \int_q \bar{\varphi}_{i\alpha k}(t) dt, \alpha \in \{\alpha\}_i, i = 1, \dots, s \right), \\ z_2 &= \left(h^{-\nu} \int_q \bar{\varphi}_{i\alpha k}(t) dt, \alpha = \alpha_0 \in \{\alpha\}_i, i = s+1, \dots, n \right). \end{aligned}$$

Then, relations (35) and (42) can be written in the simple form

$$Dy(t_0) = z_1 + \xi_1, \quad DZ(t_0) = z_2 + \xi_2 + \xi_2^+,$$

where $|\xi_1| \leq 6\eta$, $|\xi_2| \leq 5\eta$, and all components of ξ_2^+ are nonnegative; also, by (30), we obtain

$$\begin{aligned} (Dy(t_0), DZ(t_0)) &= (z_1 + \xi_1, z_2 + \xi_2 + \xi_2^+), \\ (z_1, z_2) &= h^{-\nu} \int_q \bar{\varphi}_k(t) dt \in \text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon). \end{aligned}$$

Since all components of ξ_2^+ are nonnegative and because of the particular definition of \tilde{Q} , we have also

$$(z_1, z_2 + \xi_2^+) \in \text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon)$$

and, hence,

$$(z_1 + \xi_1, z_2 + \xi_2 + \xi_2^+) \in (\text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon))_{12\eta},$$

or

$$(Dy(t_0), DZ(t_0)) \in (\text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon))_{12\eta}.$$

Here, t_0 is a fixed point and, hence,

$$\begin{aligned} (Dy(t_0), DZ(t_0)) &\in \bigcap_{\eta > 0} (\text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon))_{12\eta}, \\ (Dy(t_0), DZ(t_0)) &\in \text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon), \end{aligned}$$

since the last set is closed. Also,

$$(Dy(t_0), DZ(t_0)) \in \bigcap_{\epsilon > 0} \text{cl co } \tilde{Q}(t_0, y_0; 3\epsilon) = Q(t_0, y_0)$$

with $y_0 = y(t_0)$, because of property (Q). Here, t_0 is an arbitrary point of C'_λ . Hence,

$$(Dy(t), DZ(t)) \in \tilde{Q}(t, y(t))$$

for all points $t \in G'' = \cup_\lambda C'_\lambda$ and, by (21), $\text{meas } G'' = \text{meas } G$. Thus, for almost all $t \in G$, there is some $\bar{u}(t) \in U(t, y(t))$ such that

$$\begin{aligned} D^{\alpha}x^i(t) &= f_{i\alpha}(t, y(t), \bar{u}(t)), & \alpha \in \{\alpha\}_i, & \quad i = 1, \dots, s, \\ D^{\alpha}X^i(t) &= f_{i\alpha}(t, y(t), \bar{u}(t)), & \alpha = \alpha_0 \in \{\alpha\}_i, & \quad i = s + 1, \dots, n. \end{aligned}$$

The existence of at least one measurable function $u(t)$ as above follows by the same argument used for $\nu = 1$ in Ref. 2, Section 5 (for a general statement to the same effect, see also Ref. 9). Closure Theorem 2 is hereby proved.

3.4. Remark. We mention here variants of Closure Theorems 1 and 2 which are of some interest. One of these actually will be applied in Section 5. We may assume, for instance, that G is made up of components G_1, \dots, G_μ and that, in each of these, there is a different system of $\{\alpha\}_i, i = 1, \dots, s$, and of functions $f_{i\alpha}$. Another situation is of interest. Assume that $G_\sigma, \sigma = 1, \dots, \mu$, are finitely many open bounded subsets of E_ν and that, for each G_σ , there is a given set $\{\alpha\}_{i\sigma}, i = 1, \dots, n, \sigma = 1, \dots, \mu$, and a system $f_{i\alpha}$ of functions f . Now, let us consider all possible nonempty intersections $F_r = G_{\sigma_1} \cap G_{\sigma_2} \cap \dots \cap G_{\sigma_p}$ of p of the sets $G_\sigma, 1 \leq p \leq \mu$. These sets $F_r, r = 1, \dots, N$, are finitely many and each is a nonempty bounded open subset of E_ν . For each of these N sets $F_r = G_{\sigma_1} \cap \dots \cap G_{\sigma_p}$, we shall consider the vector function $f^{(r)}(t, x, u)$ whose components are all those of the functions $f = (f_{i\alpha}), \alpha \in \{\alpha\}_{i\sigma}, i = 1, \dots, n, \sigma = \sigma_1, \dots, \sigma_p$. We shall then require that, for each r , the set $\tilde{Q}^{(r)}(t, x)$ corresponding to $f^{(r)}$ is a convex subset of the relative Euclidean space $E_{M_r+(n-s)}$. Here, M_r is the total number of all distinct indices $\alpha \in \{\alpha\}_{i\sigma}, i = 1, \dots, n, \sigma = \sigma_1, \dots, \sigma_p$. In other words, in each

G_σ , we have a different system of partial differential equations $D^{\alpha}x^i = f_{i\alpha}$ and, in each nonempty intersection $F_r = G_{\sigma_1} \cap \dots \cap G_{\sigma_p}$, we consider the logical union of the various differential systems. We assume that these systems are compatible, and then the hypothesis that $\tilde{Q}^{(r)}(t, x)$ is convex corresponds to the usual hypothesis for these composite differential systems. A further extension can be obtained by assuming that the functions $f_{i\alpha}$ are sectionally continuous in each set G_σ but, for instance, coincide on each set F_r with functions which are continuous on the closure of F_r . These variants of Closure Theorems 1 and 2 are proved exactly by the same argument.

4. More Notations for the Existence Theorems

We shall use the same general notations as in Section 2. Besides the N -vector $f(t, x, u) = (f_{i\alpha})$, we now consider a scalar function $f_0(t, x, u)$ continuous on M , and we shall denote by $\hat{f}(t, x, u)$ the $(N + 1)$ -vector function $f(t, x, u) = (f_0, f_{i\alpha})$ continuous on M . Concerning the n -vector functions $x(t) = (x^1, \dots, x^n)$, $t \in G$, we shall require that each component $x^i(t)$, $t \in G$, belongs to a Sobolev class $W_{p_i}^{l_i}(G)$ for given l_i and p_i , $1 \leq l_i \leq l$, $p_i > 1$, $i = 1, 2, \dots, n$. By force of Sobolev's imbedding theorems (Ref. 4), each function x^i and each derivative $D^{\alpha}x^i$ of order $\alpha = (\alpha_1, \dots, \alpha_p)$, $0 \leq |\alpha| \leq l_i - 1$, has boundary values φ_{α}^i defined almost everywhere on the boundary $S = \partial G$ of G , and each φ_{α}^i is of class L_{p_i} on S .

We shall now require a set (B) of boundary conditions involving the boundary values of the functions x^i and their derivatives $D^{\alpha}x^i$, $0 \leq |\alpha| \leq l_i - 1$. On the boundary conditions (B) , we assume only the following closure property (P_1) : If $x(t) = (x^1, \dots, x^n)$, $x_k(t) = (x_k^1, \dots, x_k^n)$, $t \in G$, $k = 1, 2, \dots$, are vector functions whose components x^i , x_k^i belong to the Sobolev class $W_{p_i}^{l_i}(G)$, if $D^{\beta}x_k^i(t) \rightarrow D^{\beta}x^i(t)$ as $k \rightarrow \infty$ strongly in $L_{p_i}(G)$ for every β with $0 \leq |\beta| \leq l_i - 1$, if $D^{\beta}x_k^i(t) \rightarrow D^{\beta}x^i(t)$ as $k \rightarrow \infty$ weakly in $L_{p_i}(G)$ for every β with $|\beta| = l_i$, and if the boundary values $\varphi_{k\alpha}^i$ of $x_k^i(t)$, $i = 1, \dots, n$, $0 \leq |\alpha| \leq l_i - 1$, on ∂G satisfy the boundary conditions (B) , then the boundary values φ_{α}^i of $x^i(t)$, $i = 1, \dots, n$, $0 \leq |\alpha| \leq l_i - 1$, on ∂G satisfy the boundary conditions (B) .

For instance, if the boundary conditions (B) are defined by stating that some of the boundary values $\varphi_{k\alpha}^i(t)$ coincide with preassigned continuous functions φ_{α}^i on certain parts of $S = \partial G$, then, by force of Sobolev's imbedding theorems (Ref. 4), we know that property (P_1) is valid.

We shall need a further property of boundary conditions (B) , say (P_2) : If $x(t) = (x^1, \dots, x^n)$, $t \in G$, denotes any vector function satisfying boundary

conditions (B), whose components $x^i(t) \in W_{p_i}^{l_i}(G)$, $p_i > 1$, $1 \leq l_i \leq l$, satisfy

$$\int_G |D^\beta x^i(t)|^{p_i} dt \leq M_{i\beta}$$

for all $\beta = (\beta_1, \dots, \beta_\nu)$ with $|\beta| = l_i, i = 1, \dots, n$, and constants $M_{i\beta}$, then there are constants $M_{i\alpha}$ such that

$$\int_G |D^\alpha x^i(t)|^{p_i} dt \leq M_{i\alpha}$$

for all $\alpha = (\alpha_1, \dots, \alpha_\nu)$ with $0 \leq |\alpha| \leq l_i - 1, i = 1, \dots, n$, where the constants $M_{i\alpha}$ depend only on p_i, ν , all $M_{i\beta}$, G , and boundary conditions (B), but not on the vector function $x(t)$ above.

For instance, the boundary conditions (B) defined by preassigning the continuous boundary value functions φ_α^i on ∂B of all derivatives $D^\alpha x^i(t)$, $\alpha = (\alpha_1, \dots, \alpha_\nu)$, $0 \leq |\alpha| \leq l_i - 1, i = 1, \dots, \nu$, satisfy condition (P₂).

A pair $x(t) = (x^1, \dots, x^n), u(t) = (u^1, \dots, u^m), t \in G$, with $x^i \in W_{p_i}^{l_i}(G), u^j$ measurable in G , satisfying $(t, x(t)) \in A, u(t) \in U(t, x(t)), D^\alpha x^i(t) = f_{i\alpha}(t, x(t), u(t)), \alpha \in \{\alpha\}_i, i = 1, \dots, n$, a.e. in G , and $f_0(t, x(t), u(t)) \in L_1(G)$, is said to be *admissible*. A class Ω of admissible pairs is said to be *complete* if, for any sequence $x_k, u_k, k = 1, 2, \dots$, of pairs all in Ω and any admissible pair x, u such that $x_k \rightarrow x$ in the sense described under P₁, the pair x, u belongs to Ω . The class of all admissible pairs is obviously complete.

5. Existence Theorems

5.1. Existence Theorem 1. Let G be a bounded open set of some class $K_l, l \geq 1$, of the t -space E_ν , for every $t \in \text{cl } G$ let $A(t)$ be a nonempty subset of the x -space E_n , and assume that the set A of all $(t, x) \in E_\nu \times E_n$ with $t \in \text{cl } G, x \in A(t)$, is closed. For every $(t, x) \in A$, let $U(t, x)$ be a nonempty closed subset of the u -space E_m , and assume that $U(t, x)$ satisfies property (U) in A . Let M be the set of all $(t, x, u) \in E_\nu \times E_n \times E_m$ with $(t, x) \in A$ and $u \in U(t, x)$. For every $i = 1, \dots, n$, let $\{\alpha\}_i$ be a given finite system of indices $\alpha = (\alpha_1, \dots, \alpha_\nu), 0 \leq |\alpha| \leq l_i \leq l$, and let N be the total number of elements $\alpha \in \{\alpha\}_i, i = 1, \dots, n$. Let $\tilde{f}(t, x, u) = (f_0, f_{i\alpha}, \alpha \in \{\alpha\}_i, i = 1, \dots, n) = (f_0, f)$ be a continuous $(N + 1)$ -vector function on M and assume that the set $\tilde{Q}(t, x) = \tilde{f}(t, x, U(t, x))$ is a convex closed subset of E_{N+1} for every $(t, x) \in A$ and satisfies property (Q) in A . Let (B) be a system of boundary conditions satisfying properties (P₁) and (P₂).

Let Ω be a nonempty complete class of pairs $x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^m)$, $t \in G$, $x^i(t) \in W_{p_i}^{l_i}(G)$, $1 \leq i \leq n$, $p_i > 1$, $i = 1, \dots, n$, $u^j(t)$ measurable in G , $j = 1, \dots, m$, satisfying (a) the constraints

$$(t, x(t)) \in A, \quad u(t) \in U(t, x(t)) \quad \text{a.e. in } G,$$

(b) the system of partial differential equations

$$D^\alpha x^i(t) = f_{i\alpha}(t, x(t), u(t)) \quad \text{a.e. in } G, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n,$$

(c) the boundary conditions (B) on the boundary ∂G of G concerning the boundary values of the functions $x^i(t)$ and their generalized partial derivatives $D^\beta x^i(t)$ of orders β , $0 \leq |\beta| \leq l_i - 1$, $i = 1, \dots, n$, and (d) the system of inequalities

$$\int_G |D^\beta x^i(t)|^{p_i} dt \leq N_{i\beta} \text{ for all } \beta \text{ with } |\beta| = l_i, \quad i = 1, \dots, n,$$

$$\int_G |f_0(t, x(t), u(t))|^{p_0} dt \leq N_0,$$

where $N_{i\beta}$, N_0 are given constants, and p_0 , $p_i > 1$ are given. Then, the cost functional

$$I[x, u] = \int_G f_0(t, x(t), u(t)) dt$$

possesses an absolute minimum and an absolute maximum in Ω .

5.2. Proof. Let R be an interval $R = [a, b]$ containing $\text{cl } G$ in its interior, say $\text{cl } G \subset \text{int } R \subset E_\nu$. It is not restrictive to assume $R = [0, b]$ where 0 and b represent the ν -vectors $(0, \dots, 0)$ and (b, \dots, b) . Let x^0 denote a new variable and $x = (x^0, x) = (x^0, x^1, \dots, x^n)$. Let $l_0 = \nu$ and let $\tilde{f}_0(t, x, u)$ be a new scalar function defined by

$$\tilde{f}_0(t, x, u) = 0 \quad \text{for } t \in R - \text{cl } G, \quad x \in E^n, \quad u \in E^m,$$

$$\tilde{f}_0(t, x, u) = f_0(t, x, u) \quad \text{for } t \in \text{cl } G, \quad (t, x, u) \in M.$$

We shall denote by $x^0(t)$, $t \in R$, a new scalar function which is L -integrable in R , possesses the generalized partial derivative $\partial^\nu x^0 / \partial t^1 \cdots \partial t^\nu$, or $D^\alpha x^0$ with $\alpha = \alpha_0 = (1, \dots, 1)$, which is also L_1 -integrable in R , satisfies the partial differential equation

$$D^\alpha x^0 = f_0(t, x(t), u(t)) \quad \text{a.e. in } R, \quad \alpha = (1, \dots, 1),$$

and the boundary conditions

$$x^0(t'_i, 0) = 0, \quad (t'_i, 0) \in \partial R, \quad i = 1, \dots, \nu$$

where t'_i denotes the $(\nu - 1)$ -vector $t'_i = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_\nu)$. Then, we have

$$x^0(b) = \int_R f_0(t, x(t), u(t)) dt = \int_G f_0(t, x(t), u(t)) dt = I[x, u],$$

$$x^0(t) = \int_0^t f_0(\tau, x(\tau), u(\tau)) d\tau, \quad t \in R,$$

where the last integral is taken over the interval $[0 \leq \tau^i \leq t^i, i = 1, \dots, \nu]$, and then

$$\int_R |D^\alpha x^0(t)|^{p_0} dt \leq N_0, \quad \alpha = \alpha_0 = (1, \dots, 1).$$

Also, if $q_0^{-1} + p_0^{-1} = 1$, then

$$|x^0(b)| = |I[x, u]| \leq \left(\int_R dt \right)^{1/q_0} \left(\int_R |f_0|^{p_0} dt \right)^{1/p_0} \leq b^{\nu/q_0} N_0^{1/p_0}$$

and thus $I[x, u]$ is bounded above and below in Ω . As usual, we shall use the vector variable $\tilde{x} = (x^0, x)$. If i denotes the infimum of $I[x, u]$ in Ω , let $x_k, u_k, k = 1, 2, \dots$, be a sequence of pairs all in Ω with $I[x_k, u_k] = x_k^0(b) \rightarrow i$ as $k \rightarrow \infty$. By the weak compactness of the unit ball in $W_{p_i}^{1,i}(G), i = 1, \dots, n$, and in $L_{p_0}(G)$, there exists a subsequence, say still $[x_k, u_k]$, and limit functions $x = (x^1, \dots, x^n)$ and ψ , such that, as $k \rightarrow \infty$, we have

$$D^\alpha x_k^i \rightarrow D^\alpha x^i \text{ weakly in } L_{p_i}(G), \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n,$$

$$x_k^i \rightarrow x^i \text{ strongly in } L_{p_i}(G), \quad i = 1, \dots, n,$$

$$D^\alpha x_k^0 \rightarrow \psi \text{ weakly in } L_{p_0}(G), \quad \alpha = \alpha_0 = (1, \dots, 1).$$

If $x^0(t), t \in G$, is defined by

$$x^0(t) = \int_0^t \psi(\tau) d\tau,$$

where the integration is taken in the interval $[0, t]$, or $0 \leq \tau^i \leq t^i, i = 1, \dots, \nu$, then

$$\psi(t) = D^\alpha x^0(t) \quad \text{a.e. in } G, \quad \alpha = \alpha_0 = (1, \dots, 1),$$

and since $D^{\alpha x_k^0} = \bar{f}_0$ is zero in $R - \text{cl } G$, we know that also $\psi = 0$ in $R - \text{cl } G$. Finally, the weak convergence $D^{\alpha x_k^0} \rightarrow D^{\alpha x^0}$ in $L_{p_0}(R)$, $\alpha = \alpha_0 = (1, \dots, 1)$, implies that, as $k \rightarrow \infty$,

$$\int_R \chi_t(\tau) D^{\alpha x_k^0}(\tau) d\tau \rightarrow \int_R \chi_t(\tau) D^{\alpha x^0}(\tau) d\tau,$$

where $\chi_t(\tau)$ is the characteristic function of the interval $[0, t]$ in R . In other words, as $k \rightarrow \infty$,

$$\int_0^t D^{\alpha x_k^0}(\tau) d\tau \rightarrow \int_0^t D^{\alpha x^0}(\tau) d\tau, \quad \alpha = \alpha_0 = (1, \dots, 1),$$

or $x_k^0(t) \rightarrow x^0(t)$ pointwise as $k \rightarrow \infty$ for every $t \in R$. We can now apply Closure Theorem 1, with pointwise convergence $x_k^p \rightarrow x^0$ replacing strong convergence, with R replacing G for the component x^0 as in the remark at the end of Section 3, and with \bar{f}_0 continuous in G and in $R - \text{cl } G$, precisely, sectionally continuous in R . Thus, by Closure Theorem 1, there exists a measurable function $u(t) = (u^1, \dots, u^m)$, $t \in G$, such that

$$\begin{aligned} (t, x(t)) \in A, \quad u(t) \in U(t, x(t)) \quad \text{a.e. in } G, \\ D^{\alpha x^i} = f_{i\alpha}(t, x(t), u(t)) \quad \text{a.e. in } G, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n, \\ D^{\alpha x^0} = \bar{f}_0(t, x(t), u(t)) \quad \text{a.e. in } R, \quad \alpha = \alpha_0. \end{aligned}$$

Hence,

$$x^0(b) = I[x, u] = \int_R \bar{f}_0 dt = \int_G f_0 dt = \lim_{k \rightarrow \infty} x_k^0(b) = i,$$

and the pair x, u belongs to Ω , because Ω is complete. This proves that $I[x, u]$ has an absolute minimum in Ω . The same proof holds for the maximum.

5.3. Existence Theorem 2. Let G be a bounded open set of class K_l , $l > 1$, of the t -space E_v , for every $t \in \text{cl } G$ let $A(t)$ be a nonempty subset of the x -space E_n , and assume that the set A of all $(t, x) \in E_v \times E_n$ with $t \in \text{cl } G$, $x \in A(t)$, is closed. For every $(t, x) \in A$, let $U(t, x)$ be a nonempty closed subset of the u -space E_m , and assume that $U(t, x)$ satisfies property (U) in A . Let M be the set of all $(t, x, u) \in E_v \times E_n \times E_m$ with $(t, x) \in A$, $u \in U(t, x)$. For every $i = 1, \dots, n$, let $\{\alpha\}_i$ be a given finite system of indices $\alpha = (\alpha_1, \dots, \alpha_v)$, $0 \leq |\alpha| \leq l_i \leq l$, and let N be the total number of elements $\alpha \in \{\alpha\}_i$, $i = 1, \dots, n$. Let $\bar{f}(t, x, u) = (f_0, f_{i\alpha}, \alpha \in \{\alpha\}_i, i = 1, \dots, n) = (f_0, f)$

be a continuous $(N + 1)$ -vector function on M , and assume that the set $\tilde{Q}(t, x)$ of all $\tilde{z} = (z^0, z^1, \dots, z^n) = (z^0, z) \in E_{N+1}$ with $z^0 \geq f_0(t, x, u)$, $z = f(t, x, u)$, $u \in U(t, x)$ is a convex closed subset of E_{N+1} for every $(t, x) \in A$ and satisfies property (Q) in A . Also, assume that $f_0(t, x, u) \geq -M_0$ for all $(t, x, u) \in M$ and some constant $M_0 \geq 0$. Let (B) be a system of boundary conditions satisfying properties (P_1) and (P_2) . Let Ω be a nonempty complete class of pairs $x(t) = (x^1, \dots, x^n)$, $u(t) = (u^1, \dots, u^m)$, $t \in G$, $x^i(t) \in W_{p_i}^{l_i}(G)$, $1 \leq l_i \leq l$, $p_i > 1$, $i = 1, \dots, n$, $u^j(t)$ measurable in G , $j = 1, \dots, m$, satisfying (a) the constraints

$$(t, x(t)) \in A, \quad u(t) \in U(t, x(t)) \quad \text{a.e. in } G,$$

(b) the system of partial differential equations

$$D^\alpha x^i(t) = f_{i\alpha}(t, x(t), u(t)) \quad \text{a.e. in } G, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n,$$

(c) the boundary conditions (B) on the boundary ∂G of G concerning the boundary values of the functions $x^i(t)$ and their generalized partial derivatives $D^\beta x^i(t)$ of orders β , $0 \leq |\beta| \leq l_i - 1$, $i = 1, \dots, n$, and (d) the system of inequalities

$$\int_G |D^\beta x^i(t)|^{p_i} dt \leq N_{i\beta}$$

for all β with $|\beta| = l_i$, $\beta \notin \{\alpha\}_i$, $i = 1, 2, \dots, n$, where $N_{i\beta}$ are given constants. Assume that $f_0(t, x(t), u(t))$ is L_1 -integrable in G . Finally, assume that whenever

$$\int_G f_0(t, x(t), u(t)) dt \leq L_0 \tag{43}$$

for some constants L_0 and pairs $x(t)$, $u(t)$ in Ω , then for the same pairs we have also

$$\int_G |D^\alpha x^i(t)|^{p_i} dt \leq L_{i\alpha}, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n, \tag{44}$$

for constants $L_{i\alpha}$ depending only on L_0 , Ω , p_i , l_i on the boundary conditions (B) , but not on the particular pair x , u . Then, the cost functional

$$I[x, u] = \int_G f_0(t, x(t), u(t)) dt$$

possesses an absolute minimum in Ω . If, for given i and β as in (d), it happens that, for any $L_0 \geq 0$ sufficiently large, there is some $N_{i\beta}$ such that $I[x, u] \leq L_0$ implies $\int_G |D^\beta x^i(t)|^{p_i} dt \leq N_{i\beta}$, then the absolute minimum still exists, even if in defining Ω we disregard the corresponding requirement in (d); the absolute minimum, however, may change.

5.4. Proof. By hypothesis, we have $f_0(t, x, u) + M_0 \geq 0$ for all $(t, x, u) \in M$. For every pair x, u of the class Ω , we have now

$$+\infty > I[x, u] = \int_G f_0 dt \geq -M_0 \text{ meas } G > -\infty.$$

Then, the infimum i of $I[x, u]$ in the class Ω is finite. Let $x_k(t), u_k(t), t \in G, k = 1, 2, \dots$, be a sequence of pairs all in Ω such that $I[x_k, u_k] \rightarrow i$ as $k \rightarrow \infty$. We may assume

$$i \leq I[x_k, u_k] = \int_G f_0(t, x_k(t), u_k(t)) dt \leq i + k^{-1} \leq i + 1, \quad k = 1, 2, \dots$$

Then, by hypothesis, there are constants $L_{i\alpha}$ such that

$$\int_G |D^\alpha x_k^i(t)|^{p_i} dt \leq L_{i\alpha}, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n, \quad k = 1, 2, \dots, \quad (45)$$

and, by (d),

$$\int_G |D^\beta x_k^i(t)|^{p_i} dt \leq N_{i\beta} \text{ for all } |\beta| = l_i, \quad \beta \notin \{\alpha\}_i, \quad i = 1, \dots, n, \quad k = 1, 2, \dots, \quad (46)$$

Relations (45) and (46) show that the sequence $[x_k^i]$ belongs to a well determined balls of the Sobolev space $W_{p_i}^{l_i}(G), i = 1, \dots, n$. By the weak compactness of such balls, there is a subsequence, say still $[x_k]$ for the sake of simplicity, converging weakly toward a vector function $x(t) = (x^1, \dots, x^n), t \in G$, with $x^i \in W_{p_i}^{l_i}(G)$, and, precisely, by Sobolev's imbedding theorems (Ref. 4),

$$\begin{aligned} D^\alpha x_k^i &\rightarrow D^\alpha x^i \text{ strongly in } L_{p_i}(G), & |\alpha| \leq l_i - 1, & \quad i = 1, \dots, n, \\ D^\alpha x_k^i &\rightarrow D^\alpha x^i \text{ weakly in } L_{p_i}(G), & |\alpha| = l_i, & \quad i = 1, \dots, n. \end{aligned}$$

By properties (P_1) and (P_2) , we conclude that x satisfies the boundary conditions (B) on ∂G .

Let R be an interval containing $\text{cl } G$ in its interior, $G \subset \text{cl } G \subset \text{int } R \subset R$. We can assume $R = [0, b]$, where 0 and b denote $(0, \dots, 0)$ and (b, \dots, b) . Let $\varphi_k(t) = f_0(t, x_k(t), u_k(t))$ for $t \in G, \varphi_k(t) = 0$ for $t \in R - G$, and let $\varphi^-(t) = -M_0, t \in R$, and $\varphi_k^+(t) = \varphi_k(t) + M_0, t \in R$, so that $\varphi_k^+ + \varphi_k^- = \varphi_k, \varphi_k^+(t) \geq 0, \varphi^-(t) \leq 0$ everywhere in R . Let x_k^0, x_k^+, x^- be the functions

$$x_k^0(t) = \int_0^t \varphi_k(\tau) d\tau, \quad x_k^+ = \int_0^t \varphi_k^+(\tau) d\tau, \quad x^-(t) = \int_0^t \varphi^-(\tau) d\tau, \quad t \in R,$$

where the integrals range on the interval $[0, t]$, or $0 \leq \tau^i \leq t^i$, $i = 1, \dots, \nu$. Hence,

$$x_k^0(b) = \int_R \varphi_k(t) dt = \int_G \varphi_k(t) dt = I[x_k, u_k] \leq i + 1,$$

$$x_k^0(t) = x^-(t) + x_k^+(t), \quad t \in R, \quad k = 1, 2, \dots.$$

For every integral $I \subset R$, we denote as usual by $\Delta_I x_k^0$, $\Delta_I x_k^+$, $\Delta_I x^-$ the differences of order ν of the indicated functions relative to the 2^ν vertices of I , and we have $\Delta_I x_k^0 = \Delta_I x_k^+ + \Delta_I x^-$, with

$$\Delta_I x_k^0 = \int_I \varphi_k(t) dt, \quad \Delta_I x_k^+ = \int_I \varphi_k^+(t) dt, \quad \Delta_I x^- = \int_I \varphi^-(t) dt = -M_0 \text{ meas } I.$$

If $t = (t^1, \dots, t^\nu)$, $t' = (t'^1, \dots, t'^\nu)$ are any two points of R , then obviously $|t^i - t'^i| \leq |t - t'|$, $i = 1, \dots, \nu$. Also, we denote by $t_0 = t_1, t_2, \dots, t_\nu = t'$, the points of R defined by $t_j = (t^1, \dots, t^{\nu-j}, t'^{\nu-j+1}, \dots, t'^\nu)$, $j = 0, 1, \dots, \nu$, and by r_j the intervals

$$r_j = [0, t^1] \times \dots \times [0, t^{\nu-j+1}] \times [t^{\nu-j}, t'^{\nu-j}] \times [0, t'^{\nu-j+1}] \times \dots \times [0, t'^\nu],$$

$j = 0, 1, \dots, \nu - 1.$

Then,

$$x_k^0(t) - x_k^0(t') = \Sigma_j [x_k^0(t_j) - x_k^0(t_{j+1})],$$

where Σ ranges over all $j = 0, 1, \dots, \nu - 1$ and

$$x_k^0(t_j) - x_k^0(t_{j+1}) = \int_{r_j} \varphi_k(t) dt,$$

and analogous formulas hold for x_k^+ and x^- . Since

$$\text{meas } r_j \leq b^{\nu-1} |t^j - t'^j| \leq b^{\nu-1} |t - t'|$$

and $\varphi^- = -M_0$, we have

$$|x^-(t) - x^-(t')| \leq \nu M_0 b^{\nu-1} |t - t'|, \quad t, t' \in R.$$

On the other hand, $\varphi_k^+ \geq 0$ and, hence, $\Delta_I x_k^+ \geq 0$ for every $I \subset R$, $k = 1, 2, \dots$.

If $[I_1, \dots, I_L]$ denotes any finite subdivision of R into nonoverlapping intervals I_s , $s = 1, \dots, L$, then

$$\Sigma_s | \Delta_{I_s} x^- | = -\Sigma_s \Delta_{I_s} x^- = M_0 \Sigma_s \text{meas } I_s = M_0 b^\nu,$$

$$\Sigma_s | \Delta_{I_s} x_k^+ | = \Sigma_s \Delta_{I_s} x_k^+ = \Sigma_s \Delta_{I_s} x_k^0 - \Sigma_s \Delta_{I_s} x^- = x_k^0(b) + M_0 b^\nu \leq i + 1 + M_0 b^\nu,$$

and, thus,

$$\Sigma_s | \Delta_{I_s} x_k^0 | \leq \Sigma_s | \Delta_{I_s} x_k^+ | + \Sigma_s | \Delta_{I_s} x^- | \leq i + 1 + 2M_0 b^\nu.$$

We have proved that the interval functions $\Delta_I x^-$, $\Delta_I x_k^+$, $\Delta_I x_k^0$ are of bounded variation, namely, of equibounded variation in R . In addition, $\Delta_I x^-$ is obviously absolutely continuous in R . Note that, if $I_i = [0, t]$ denotes the interval $0 \leq \tau^i \leq t^i$, $i = 1, \dots, \nu$, in R , then,

$$| x_k^0(t) | = | \Delta_{I_i} x_k^0 | \leq i + 1 + 2M_0 b^\nu, \quad t \in R,$$

that is, the functions $x_k^0(t)$ are equibounded in R , and so are the functions $x_k^+(t)$.

Let us consider the countably many lattice points $\{t_\rho\}$, or $t = b\rho \in R$, $\rho = (\rho_1, \dots, \rho_\nu)$, where ρ_1, \dots, ρ_ν denote all rational numbers, $0 \leq \rho_j \leq 1$, $j = 1, \dots, \nu$. Let $\{I\}$ be the countable system of intervals $I \subset R$ whose vertices are points $t \in \{t_\rho\}$. We may order the points t_ρ into a sequence. Since the functions $x_k^0(t)$, $x_k^+(t)$, $t \in R$, are uniformly bounded in R (and, hence, at each $t = t_\rho$), we can successively select subsequences which are convergent at $t = t_\rho$ and then, by the diagonal process, we can select a unique subsequence $\{k_s\}$ of integers k such that the limits $x_{k_s}^0(t) \rightarrow x^0(t)$, $x_{k_s}^+(t) \rightarrow x^+(t)$ as $s \rightarrow \infty$ exist, for every $t \in \{t_\rho\}$. Here, $x^-(t)$ does not depend on k , and $\Delta_I x_{k_s}^0 = \Delta_I x_{k_s}^+ + \Delta_I x^-$. Thus, as $k = k_s \rightarrow \infty$, $\Delta_I x_{k_s}^0$ and $\Delta_I x_{k_s}^+$ have limits, say $\Delta_I x^0$, $\Delta_I x^+$, and $\Delta_I x^0 = \Delta_I x^+ + \Delta_I x^-$ for every $I \in \{I\}$, and $\Delta_I x^0$, $\Delta_I x^+$, $\Delta_I x^-$ are additive interval functions in $\{I\}$. Moreover, $\Delta_I x^-$ is absolutely continuous, $\Delta_I x^+ \geq 0$, $\Delta_I x^+$ is of bounded variation and, hence, $\Delta_I x^0$ is also of bounded variation in $\{I\}$.

By Ref. 8, we know that $\Delta_I x^0$ is then the difference Δ of a function $x^0(t)$ defined everywhere in R , with a Lebesgue decomposition $x^0(t) = X(t) + S(t)$. Hence, $\Delta_I x^0 = \Delta_I X + \Delta_I S$, where

$$x^0(t'_i, 0) = X(t'_i, 0) = S(t'_i, 0) = 0, \quad (t'_i, 0) \in \partial R, \quad i = 1, \dots, \nu,$$

and where $\Delta_I X$ is absolutely continuous and $\Delta_I S$ is singular. Consequently, for every $t_0 \in R$ and interval $I \in \{I\}$ with $t_0 \in I$, we can form the quotients

$\Delta_I S/\text{meas } I$. We know that for I a hypercube, $I \in \{I\}$, $t_0 \in I$, and almost all $t_0 \in R$, we have $\Delta_I S/\text{meas } I \rightarrow 0$ as $\text{diam } I \rightarrow 0$. On the other hand, the same process applied to $\Delta_I X$ yields $\Delta_I X/\text{meas } I \rightarrow z(t_0)$. Now, $z(t)$ is defined a.e. in R , is L_1 -integrable in R , and

$$\Delta_I X = \int_I z(t) dt, \quad I \in \{I\}.$$

In addition, the decomposition $x_k^0 = x_k^+ + x^-$, with $\Delta_I x_k^+ \geq 0$, $\Delta_I x^- \leq 0$, x^- independent of k and $\Delta_I x^-$ absolutely continuous, implies $x^0 = X + S = X^+ + x^-$, with $\Delta_I X^+ \geq 0$, $\Delta_I x^- \leq 0$. By the uniqueness of the Lebesgue decomposition, we conclude that $\Delta_I X^+$ has the Lebesgue decomposition $\Delta_I X^+ = \Delta_I(X - x^-) + \Delta_I S$, where $\Delta_I(X - x^-)$ is absolutely continuous and $\Delta_I S$ is singular. Then $\Delta_I X^+ \geq 0$ implies $\Delta_I(X - x^-) \geq 0$, $\Delta_I S \geq 0$, in particular $S(b) = \Delta_R S \geq 0$.

We can now apply Closure Theorem 2 with s replaced by n , n replaced by $n + 1$, and $n - s$ replaced by 1; also, x^1, \dots, x^s is replaced by x^1, \dots, x^n and x^{s+1}, \dots, x^n is replaced by x^0 . By Closure Theorem 2 with $x^0 = X + S$, we conclude that there exists some measurable function $u(t) = (u^1, \dots, u^m) \in U(t, x(t))$, $x(t) = (x^1, \dots, x^n)$, such that

$$\begin{aligned} (t, x(t)) \in A, \quad u(t) \in U(t, x(t)) \quad & \text{a.e. in } G, \\ D^\alpha x^i(t) = f_{i\alpha}(t, x(t), u(t)) \quad & \text{a.e. in } G, \quad \alpha \in \{\alpha\}_i, \quad i = 1, \dots, n, \\ D^\alpha X(t) = f_0(t, x(t), u(t)) \quad & \text{a.e. in } G, \quad \alpha = \alpha_0. \end{aligned}$$

On the other hand, $x(t), u(t), t \in G$, is in Ω since Ω is complete and

$$i \leq I[x, u] = X(b) = x^0(b) - S(b) = \lim_{s \rightarrow \infty} x_k^0(b) - S(b) = i - S(b) \leq i.$$

Thus, the equality sign holds everywhere in this relation, $S(b) = 0$, $I[x, u] = i$, and, hence, $I[x, u]$ possesses an absolute minimum in Ω .

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