THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

THE ONSET OF LAMINAR NATURAL CONVECTION IN A FLUID WITH HOMOGENEOUSLY DISTRIBUTED HEAT SOURCES

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1959

June, 1959

IP-374

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UMR 6604

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ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Professors John A. Clark and Chia-Shun Yih who, along with the other members of his doctoral committee, generously gave suggestions and encouragement during the course of the work presented herein. A debt of gratitude is owed to Dean John S. McNown, University of Kansas, who as a professor at the University of Michigan shared with the students his wide knowledge of fluid mechanics and stimulated the author to study further in the subject. The help and resourcefulness of the technicians from the Department of Mechanical Engineering and the Department of Engineering Mechanics during the construction of the experimental apparatus is much appreciated.

The author was assisted financially in his work by a Rackham Fellowship and a research assistantship for a project sponsored by the Office of Ordnance Research under Contract No. DA-20-018-ORD-15556 with the University of Michigan.

The two copper plates used in the experimental work were kindly donated by the American Brass Company of Detroit, Michigan.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONT'D)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES

APPENDICES

CHAPTER I

INTRODUCTION

1. Statement of the Problem.

The study of the thermal instability (onset of natural convection) of a fluid having homogeneously distributed heat sources is of interest as it has some bearing to the design of nuclear reactors. Of equal importance is the fact that when results of such a study are compared with other investigations, a conclusion as to the qualitative influence of the variation of the temperature gradient on the general problem can be obtained.

In the present work the stability of a viscous fluid confined between two horizontal boundaries and with homogeneously distributed heat sources is examined. The fluid under consideration has the state of rest as an equilibrium state, but can become unstable if a critical value of the Rayleigh number is exceeded. The ensuing motion is one of maintained natural convection. The Rayleigh number is defined to be $g\alpha b^3(\Delta \Theta)/\nu\kappa$, in which g is the gravitational constant, α the coefficient of thermal expansion, b the depth of the fluid layer, $\Delta \Theta$ the temperature difference between the center of the fluid and one of its boundaries, ν the kinematic viscosity, and κ the thermal diffusivity. The value of the critical Rayleigh number is shown to depend upon the nature of the fluid's boundaries (e.g., free or rigid; conducting or adiabatic).

A mathematical solution of the problem has been achieved by employing an approximation technique, and an experiment has been performed for one boundary configuration to verify the analysis.

2. <u>Historical Background.</u>

Stability problems in the field of fluid mechanics have occupied the attention of a host of authors since the latter part of the nineteenth century. The scope of the investigations covers the many aspects of secondary laminar flow and initial transition to turbulence. The influence of velocity profile, boundary shape, acceleration, pressure gradient, temperature variation and magnetic fields, are but some of the facets of the general problem which have received consideration. Specific applications of the obtained results are employed by aeronautical engineers and naval architects. The improvement of lift and the lessening of drag through boundary layer control has been the result of extensive experimental and mathematical work dealing with the stability of the flow in the boundary layer. Our knowledge of convective heat transfer has been broadened by including the influence of temperature variation on the fluid motion and studying the resulting stability problem. In this case the temperature variations cause a density variation which in turn can bring about natural convection. Meterologists are combining the thermal and rotational effects on secondary flow to gain an insight into the mechanisms of storm systems and the jet stream. Scientists interested in the earth's magnetic field are studying the secondary flows that could be induced in the earth's molten core. These geophysicists, along with astrophysicists studying the atmosphere around the planets and physicists engaged in developing nuclear reactors utilizing an ionized plasma have done much to develop the science of magnetohydrodynamics.

The present thesis deals with a problem which is akin to that studied by Bénard, Jeffreys, Low, Rayleigh and others. The subject of that problem is the maintained convective motion in a fluid that is heated from below (hereafter referred to as the Bénard problem). Pellew and Southwell (1) were able to obtain an exact solution for the mathematical formulation of the problem, while Chandra (2) and Schmidt and Milverton (3) previously obtained experimental data which are in agreement with the analytic solution. The equations for the Bénard problem are directly related to the classical problem for the stability of viscous flow between rotating cylinders, first studied by G. I. Taylor (4), for the case in which the difference in radii of the two cylinders is small compared totheir mean, and the speeds of rotation of the cylinders are nearly equal. Chandrasekhar (5) solved the resultant equations for these problems by an approximation technique - even though the exact solution of Pellew and Southwell was already available - to show the power of the method and the rapid convergence of the attendant solution. In a recent paper by Reid and Harris (6) the various exact and approximate methods of solution for the Bénard problem are discussed with a view to giving some insight into the relative merits of the approximative methods that are available for solving those problems which do not have an exact solution.

For the problem that is presented in this thesis the method of Chandrasekhar (5) is utilized for solving the governing differential equations. These equations do not lend themselves to an exact solution of the type given by Pellew and Southwell.

A study of the Bénard problem was of value in the work of this thesis not only because the method of solution is relevant but also because the quantitative results of that solution can be used to reveal the role played by the temperature distribution upon the stability of the fluid. For the case of the fluid being heated from below the mean temperature variation is linear; whereas for distributed heat sources the mean temperature variation is parabolic if the thermal diffusivity is considered to be constant.

3. Related Literature.

It was stated at the beginning of the previous section that problems of hydrodynamic stability have attracted the interest of investigators in many fields of the physical sciences. To present a list of all the work that has been done in this area is not only impractical but also unnecessary. The book by Lin (7) is an excellent introduction to the subject and contains an extensive bibliography of 258 entries. A paper by Ostrach (8) deals with convection phenomena and makes reference to 38 other papers. A short summary of the work in heat transfer, including natural convection, that has been done in the last two years is presented by Eckert, Hartnett, and Irvine (9). The subject of magnetohydrodynamics has recently been outlined in book form by Cowling (10). In this field there have been new papers by Nisbet (11) and Yih (12). The work of Malkus (13) and Stuart (14) is significant for they obtain solutions for the equations governing hydrodynamic stability in which the non-linear terms have been retained. With this work available, it is now possible (as suggested by Pellew and Southwell) to predict the shape of the convection cell, a fact that cannot be ascertained if the equations are linearized, and which, therefore, must be arbitrarily assumed.

CHAPTER II

MATHEMATICAL SOLUTION OF THE PROBLEM

1. The Governing Equations

Consider a horizontal layer of fluid with a depth b, confined between two parallel planes $x_3 = 0$ and $x_3 = b$. These planes are solid, heat conducting and of equal temperature. The fluid has uniformly distributed heat sources. The equations of motion and heat conduction applicable to this problem are

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\delta_{i3}(g\rho) - \frac{\partial}{\partial x_i}\left(p - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \rho\nu\Delta u_i \tag{1}$$

and

$$\frac{1}{K}\frac{D\Theta}{Dt} - \frac{q}{k} = \Delta\Theta \tag{2}$$

in which the subscripts in the Navier-Stokes equations have values of 1, 2, and 3 for the coordinate directions, and the repetition of a particular suffix implies the summation convention of tensor analysis; u_i is the component of the fluid velocity in the i^{th} coordinate direction; ρ is the mass density of the fluid; λ, a Lamé constant; g the gravitational constant; p the pressure in the fluid; μ and ν, the dynamic and kinematic viscosities; δ_{i3} the Kronecker delta, being zero for $i \neq 3$ and unity for $i = 3$; and Δ the symbol for the Laplacian operator. Also, in the foregoing equations Θ is the absolute temperature of the fluid, κ is the thermal diffusivity; q is the time rate of heat generation per unit volume of fluid, k is the thermal conductivity, and $\frac{D}{Dt}$ is the substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}$$

The coordinate system employed for this problem is presented in Fig. 1 along with the mean temperature distribution in the fluid layer.

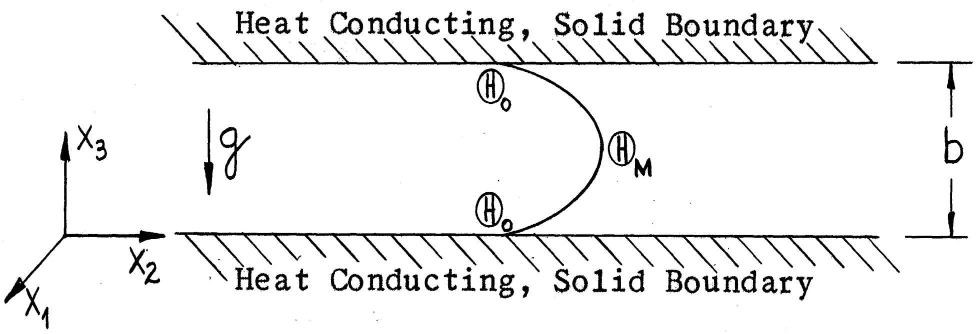


Fig. 1. Schematic Drawing of Fluid Layer with the Mean Temperature Distribution and the Coordinate System

In writing equation (2) the dissipation of mechanical energy into heat is considered small and therefore neglected. This assumption will be reviewed later.

The density of the fluid can be expressed as

$$\rho = \rho_0 \left\{ 1 - \alpha \left(\Theta - \Theta_0 \right) \right\} \tag{3}$$

if the temperature differences in the fluid are small.
In equation (3) ρ_0 is the density at the reference temperature, Θ_0 ; and α is the coefficient of thermal expansion.

Under equilibrium conditions the fluid layer is quiescent and the heat liberated by the fluid is carried to the boundaries by conduction alone. Accordingly, equations (1) and (2) can be solved for this condition and by assuming constant thermal properties one has

$$\bar{u}_i = 0 \tag{4}$$

$$\bar{\Theta} = \frac{q}{2k}(b x_3 - x_3^2) + \Theta_0 \tag{5}$$

$$\bar{\Theta}_m = \frac{q b^2}{8k} + \Theta_0 \tag{6}$$

$$\frac{\partial \bar{p}}{\partial x_i} = -\delta_{i3}(g\bar{\rho}) \tag{7}$$

in which the bar over a symbol denotes a mean quantity, and hence the value associated with the quiescent state. Θ_m is the temperature at the middle of the fluid layer while pure conduction is taking place, and Θ_0 is the temperature of the upper and lower boundaries of the fluid. The boundary temperatures will be forced to remain constant regardless of the motion of the fluid. The symmetry of the temperature about the mid-plane of the fluid is a consequence of the assumed constant thermal properties. This constancy could result from the nature of the fluid itself or from the fact that only small temperature variations are being considered. With equations (3) and (5) it is possible to write the mean density as

$$\bar{\rho} = \rho_o \left\{ 1 - \frac{\alpha q}{2k} \left(b x_3 - x_3^2 \right) \right\}. \tag{8}$$

Now consider small departures in the values of the temperature, pressure, density, and velocity from those existing during the conduction state. The values of these quantities will be

$$\Theta = \bar{\Theta} + \Theta', \quad p = \bar{p} + p'$$

$$\rho = \bar{\rho} + \rho', \quad u_i = u_i' \tag{9}$$

if a linear perturbation is employed, and the primed quantities are these perturbations.

The perturbed variables (9) are then substituted into equations (1) and (2) and equations result from which a solution for the perturbations can be obtained. The solution will be examined to determine the conditions under which a perturbated quantity can have a steady-state value other than zero. The equation resulting from the substitution of (9) into equation (1) is

$$\frac{\partial u_i}{\partial t} = -\delta_{i3}\left(g\frac{\rho'}{\rho}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu \Delta u_i \, . \tag{10}$$

In writing this equation the primed quantities are considered to be small and all products of perturbation quantities can be ignored because their magnitude will be very much smaller in comparison with the other terms in the equation. The result of this simplification is three scalar equations which are linear.

By introducing $\textcircled{H}_0$ from (9) into equation (3) and by comparison with the expression for ρ given in (9), one can conclude that

$$\rho' = \rho_0\left(-\alpha \textcircled{H}'\right) . \tag{11}$$

Recalling the series expansion for the reciprocal of $(1-X)$ it is possible to write $\frac{1}{\rho}$ as

$$\frac{1}{\rho} \approx \frac{1}{\rho_0}\left\{1 + \frac{\alpha g}{2k}\left(bX_3 - X_3^2\right)\right\} .$$

For the case that the term $\frac{\alpha g}{2k}$ is small and the fluid depth b is also small, the second term in the brackets can be neglected in comparison with unity and equation (10) reduces to

$$\frac{\partial u_i}{\partial t} = \delta_{i3}(g\alpha\Theta') - \frac{1}{\rho_0}\frac{\partial}{\partial x_i}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu\Delta u_i \cdot$$

(12)

The value of ν in the above equation is considered to be constant for the small variation in temperature which will be encountered.

The equation of continuity, or mass conservation, is

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = 0$$

(13)

and can be written as

$$\frac{\partial \rho'}{\partial t} + \overbrace{u_i\frac{\partial \bar{\rho}}{\partial x_i} + \bar{\rho}\frac{\partial u_i}{\partial x_i}}^{\frac{\partial}{\partial x_i}(\bar{\rho} u_i)} = 0$$

(14)

if the perturbated variables of (9) are employed and the terms consisting of products of two perturbation quantities are neglected as before.

Substitution of equations (8) and (11) for $\bar{\rho}$ and ρ', respectively, into equation (14) gives the following result for the divergence of the perturbated velocity field.

$$\frac{\partial u_i}{\partial x_i} = \frac{\alpha\left\{\frac{\partial\Theta}{\partial t} + \frac{u_3 q}{2k}(b - 2x_3)\right\}}{1 - \frac{\alpha q}{2k}(bx_3 - x_3^2)}$$

(15)

For the case that α is sufficiently small, $\frac{\partial u_i}{\partial x_i}$ is small in comparison with $\frac{\partial\Theta}{\partial t}$ and u_3, if Θ and

U_3 are of the same order of magnitude. One can draw the conclusion from (15) that for time-independent solutions, $\frac{\partial u_i}{\partial x_i}$ is small compared to U_3. The equation

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{16}$$

can be used in the sense that

$$\left|\frac{\partial u_i}{\partial x_i}\right| \ll |U_3|.$$

By introducing the perturbated quantities into equation (2) to account for heat conduction, it is possible to write after neglecting the perturbation terms of the second power

$$\frac{1}{k}\left(\frac{\partial \Theta'}{\partial t} + u_i \frac{\partial \bar{\Theta}}{\partial x_i}\right) = \Delta \Theta' \tag{17}$$

Correspond to Yih's eqn 28 (44)

since

$$\frac{\partial \bar{\Theta}}{\partial t} = 0$$

and

$$\frac{u_i}{k}\frac{\partial \bar{\Theta}}{\partial x_i} = \Delta \bar{\Theta} + \frac{q}{k}$$

from equation (5).

Equations (12) and (17) can be used to solve the stability problem if the terms associated with the pressure contained in (12) can be eliminated. This is achieved by cross-differentiation of the three scalar equations of motion and making appropriate combinations of the terms in conjunction with the continuity equation given in equation (16). Thus, writing out the scalar equations, one obtains

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial x_1}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu \Delta u_1 \tag{12a}$$

$$\frac{\partial u_2}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial x_2}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu \Delta u_2 \tag{12b}$$

$$\frac{\partial u_3}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial x_3}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu \Delta u_3 + g\alpha \Theta' \tag{12c}$$

By operating on equations (12c), (12a), (12c) and (12b) with

$$\frac{\partial^2}{\partial x_1^2}, \quad \frac{\partial^2}{\partial x_3 \partial x_1}, \quad \frac{\partial^2}{\partial x_2^2}, \quad \frac{\partial^2}{\partial x_3 \partial x_2},$$

respectively, one obtains

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u_3}{\partial x_1^2}\right) = g\alpha \frac{\partial^2 \Theta'}{\partial x_1^2} - \frac{1}{\rho_0}\frac{\partial^3}{\partial x_3 \partial x_1^2}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu \frac{\partial^2}{\partial x_1^2}\left(\Delta u_3\right) \tag{18}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u_1}{\partial x_3 \partial x_1}\right) = -\frac{1}{\rho_0}\frac{\partial^3}{\partial x_3 \partial x_1^2}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu \frac{\partial^2}{\partial x_1 \partial x_3}\left(\Delta u_1\right) \tag{19}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u_3}{\partial x_2^2}\right) = g\alpha\frac{\partial^2 \Theta'}{\partial x_2^2} - \frac{1}{\rho_0}\frac{\partial^3}{\partial x_3 \partial x_2^2}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu\frac{\partial^2}{\partial x^2}\left(\Delta u_3\right)$$

(20)

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u_2}{\partial x_3 \partial x_2}\right) = -\frac{1}{\rho_0}\frac{\partial^3}{\partial x_3 \partial x_2^2}\left(p' - (\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right) + \nu\frac{\partial^2}{\partial x_3 \partial x_2}\left(\Delta u_2\right)$$

(21)

A subtraction of equation (19) from (18) and equation (21) from (20), and then adding the two remainders gives

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 u_3}{\partial x_1^2} - \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial^2 u_3}{\partial x_2^2} - \frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_3}\right) - \underline{\frac{\partial^2 u_3}{\partial x_3^2}} + \underline{\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_3}\right)}\right] = g\alpha\Delta_1\Theta'$$

$$+\nu\left[\left(\frac{\partial^2}{\partial x_1^2}\left(\Delta u_3\right) + \frac{\partial^2}{\partial x_2^2}\left(\Delta u_3\right) + \underline{\frac{\partial^2}{\partial x_3^2}\left(\Delta u_3\right)}\right)\right.$$

$$\left. - \frac{\partial}{\partial x_3}\left(\frac{\partial}{\partial x_1}\left(\Delta u_1\right) + \frac{\partial}{\partial x_2}\left(\Delta u_2\right) + \underline{\frac{\partial}{\partial x_3}\left(\Delta u_3\right)}\right)\right].$$

(22)

One notes that the terms which are underlined in the above equation have been introduced for convenience and do not change the value of the expressions. Equation (22) reduces to

$$\frac{\partial}{\partial t}\left(\Delta u_3 - \frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)\right) = g\alpha\Delta_1\Theta' + \nu\Delta\left(\Delta u_3 - \frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)\right)$$

(23)

in which $\Delta_1 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$.

From equation (15) it was possible to conclude that $\partial u_i/\partial x_i$ is an infinitesimal of an order higher than u_3. Thus, while $\frac{\partial u_i}{\partial x_i}$ may not be exactly zero, it is assumed that $\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)$ will be small in comparison with Δu_3, and may be neglected in terms that also contain Δu_3. The validity of this assumtion will be examined when the solution for the velocity components is obtained. After neglecting the derivative of the divergence of the perturbed velocity, equation (23) becomes

$$\frac{\partial}{\partial t}\left(\Delta u_3\right) = g\alpha\Delta_1\Theta' + \nu\Delta\Delta u_3. \tag{23a}$$

A rearrangement of the above result gives the second of the two desired governing equations as

$$\left[\frac{\partial}{\partial t} - \nu\Delta\right]\Delta u_3 = g\alpha\Delta_1\Theta'. \tag{24}$$

The solution of equations (17) and (24) - along with the appropriate boundary conditions - is necessary for the determination of a stability criterion for the flow under study. The attendant boundary conditions are

$$\Theta' = 0 \quad \text{at } x_3 = 0, b; \tag{25i}$$

$$u_3 = 0 \quad \text{at } x_3 = 0, b; \tag{25ii}$$

$$\frac{\partial u_3}{\partial x_3} = 0 \quad \text{at } x_3 = 0, b. \tag{25iii}$$

The first four of these conditions result from the fact that the upper and lower boundaries are isothermal and rigid, hence allow no variation of the surface temperatures, and no velocity components perpendicular to the boundary. From the fact that u_1 and u_2 are zero everywhere on the boundary due to the presence of viscosity, the last two boundary conditions can be obtained with the help of the continuity equation.

It is possible to proceed directly with the solution of the problem given by equations (17) and (24) in conjunction with the restrictions of (25), but it is more convenient, and the results more general, if the equations are non-dimensionalized by setting

$$\tau = \frac{tk}{b^2}, \quad (x, y, z) = \left(\frac{x_1}{b}, \frac{x_2}{b}, \frac{x_3}{b}\right)$$
$$w = \frac{u_3 b}{k}, \quad T = \frac{\Theta'}{\Theta_m - \Theta_0} = \frac{\beta k \Theta'}{g b^2}. \tag{26}$$

By employing these new variables in equations (17) and (24) the following is obtained

$$\left(\frac{\partial}{\partial\tau} - \nabla^2\right)T = -4w(1-2z) \quad \text{- heat} \tag{27}$$

and

$$\left(\frac{\kappa}{\nu}\frac{\partial}{\partial\tau} - \nabla^2\right)\nabla^2 w = \frac{g\alpha b^3}{\nu\kappa}\left(\Theta_m - \Theta_o\right)\nabla_1^2 T \,, \quad \text{motion} \tag{28}$$

in which ∇^2 and ∇_1^2 are now the Laplacian operators with respect to the new coordinates (x,y,z) and (x,y), respectively.

The boundary conditions (25) become

$$T = 0 \quad \text{at } z = 0, 1\,; \tag{29i}$$

$$w = 0 \quad \text{at } z = 0, 1\,; \tag{29ii}$$

$$\frac{\partial w}{\partial z} = 0 \quad \text{at } z = 0, 1. \tag{29iii}$$

For the solution of partial differential equations of the type given by equations (27) and (28), the method of separation of variables can be used. If w and T are assumed to have the separable form

$$w = f\hat{w}e^{\sigma\tau} \tag{30}$$

$$T = f\hat{T}e^{\sigma\tau} \tag{31}$$

in which f is a function of x and y only, the functions $\hat{w}$ and $\hat{T}$ are dependent only on z, and σ has real and imaginary parts σ_r and σ_i, respectively. For σ_r greater than zero the solution (representing the disturbance) will grow exponentially with time, and for σ_r less than zero, the perturbation will decay. If σ_i is non-zero the perturbation quantities will be oscillatory.

Upon substitution of the assumed form of the solutions into equation (27) one has

$$e^{\sigma\tau}\left[\sigma f\hat{T} - \left(f_{xx}\hat{T} + f_{yy}\hat{T} + f\hat{T}_{zz}\right)\right] = e^{\sigma\tau}\left(-4f\hat{w}\right)\left(1-2z\right)$$

in which the subscripts denote partial differentiation in accordance with standard practice. The above result can be written as

$$-\frac{f_{xx}}{f} - \frac{f_{yy}}{f} = -\frac{4\hat{w}}{\hat{T}}\left(1-2z\right) + \frac{\hat{T}_{zz}}{\hat{T}} - \sigma = a^2 \tag{32}$$

in which a is a dimensionless constant characterizing the mode of the disturbance. This constant is called the "cell number". The last equation yields

$$f_{xx} + f_{yy} + a^2 f = 0 \tag{33}$$

and

$$\left[\sigma - \left(D^2 - a^2\right)\right]\hat{T} = -4\hat{w}\left(1-2z\right), \tag{34}$$

in which $D = \frac{\partial}{\partial z}$.

The substitution of the assumed form of the solution, equations (30) and (31), into equation (28) yields, with the aid of equation (33),

$$\left[\frac{\kappa}{\nu}\sigma - \left(D^2 - a^2\right)\right]\left[D^2 - a^2\right]\hat{W} = -\hat{T}a^2 R, \tag{35}$$

in which

$$R = \frac{g\alpha b^3(\Theta_m - \Theta_o)}{\nu\kappa} = \frac{g\alpha b^5}{8\nu\kappa}\frac{q}{k} \tag{36}$$

At the threshold of instability it is clear that the solution neither grows nor decays. Hence σ_r must be zero. The value of σ_i cannot be assigned <u>a priori</u> but in certain problems (5) and (15) it has been shown that at neutral stability σ_i must be zero. In these cases the differential system is self-adjoint. For a large class of problems the governing differential equations are not self-adjoint and it is impossible to show mathematically that σ_i is zero without extensive and detailed calculation. It is assumed in those cases for which the value of σ_i cannot be determined simply, that σ_i is zero, when σ_r is zero. This is the assumption of the "principle of the exchange of stabilities". For the present problem this principle will be invoked and σ is taken as being zero at neutral stability.

Hence the system of ordinary differential equations which must be solved is simplified to

$$(D^2-a^2)\hat{T} = \overset{4}{\cancel{2}}\,\hat{w}\,(1-2z) \tag{37}$$

$$(D^2-a^2)^2\hat{w} = \hat{T}a^2\mathbb{R}. \tag{38}$$

The boundary conditions for these equations are

$$\hat{T}=0 \quad \text{at} \quad z=0,1; \tag{39i}$$

$$\hat{w}=0 \quad \text{at} \quad z=0,1; \tag{39ii}$$

$$\frac{d\hat{w}}{dz}=0 \quad \text{at} \quad z=0,1. \tag{39iii}$$

2. <u>Method of Solution.</u>

The solution of the governing differential equations can be obtained readily by the method of Chandrasekhar. This numerical technique provides a means of solving problems for which there is no simple exact solution. The procedure is straightforward and is not too involved, so that one need not resort to electronic computing devices to obtain the solution. A brief outline of the method is presented in this section.

The thermal boundary conditions (39i) will be satisfied by a function $\hat{T}$ having a Fourier series expansion of the form

$$\hat{T} = \sum_{n=1}^{\infty} A_n \sin n\pi z \qquad n = 1, 2, 3, \cdots . \tag{40}$$

Using the above expression in conjunction with equation (38) permits one to solve for $\hat{W}$.

$$\hat{W} = a^2 R \sum_{n=1}^{\infty} A_n \left\{ B_n \cosh az + C_n \sinh az + D_n z \sinh az + E_n z \cosh az + \frac{1}{N_n^2} \sin n\pi z \right\} \tag{41}$$

in which

$$N_n = (n\pi)^2 + a^2. \tag{42}$$

The velocity boundary conditions (39ii) and (39iii) when applied to equation (41) and its derivative require that

$$B_n = 0 \tag{43}$$

$$C_n = \frac{n\pi}{N_n^2} \left[\frac{a + (-1)^n \sinh a}{\sinh^2 a - a^2} \right] \tag{44}$$

$$D_n = -E_n \coth a - C_n \tag{45}$$

$$E_n = -\frac{n\pi a}{N_n^2} \left[\frac{a + (-1)^n \sinh a}{\sinh^2 a - a^2} + \frac{1}{a} \right]. \tag{46}$$

$\left[\frac{a + \sinh a \quad + \sinh^2 a - a^2}{a(\quad)}\right]$

Substituting the series equivalents for $\hat{T}$ and $\hat{W}$, equations (40) and (41), into equation (37), one gets

$$\sum_{n=1}^{\infty}(-N_n)A_n \sin n\pi z = 4a^2 R \sum_{n=1}^{\infty} A_n\Big\{C_n(1-2z)\sinh az$$

$$+ D_n(z-2z^2)\sinh az + E_n(z-2z^2)\cosh az + \frac{(1-2z)}{N_n^2}\sin n\pi z\Big\}. \tag{47}$$

If this equation is to be true, the coefficients of $\sin n\pi z$ on both sides of the equation must be equal for all n. (Note: the terms on the right hand side of the equation could be expressed as an infinite series of sine terms). The method for equating these coefficients consists of multiplying both sides of equation (47) by $\sin m\pi z$ and integrating from zero to one, the range of z. This method is the standard one for determining the coefficients of a Fourier series. The equation resulting from the prescribed integration is then

$$-\sum_{n=1}^{\infty} N_n A_n \int_0^1 \sin n\pi z \sin m\pi z\, dz = 4a^2 R \sum_{n=1}^{\infty} A_n\Big\{C_n\int_0^1 (1-2z)\sin m\pi z \sinh az\, dz$$

$$+ D_n \int_0^1 (z-2z^2)\sin m\pi z \sinh az\, dz$$

$$+ E_n \int_0^1 (z-2z^2)\sin m\pi z \cosh az\, dz$$

$$+ \frac{1}{N_n^2}\int_0^1 (1-2z)\sin n\pi z \sin m\pi z\, dz\Big\}. \tag{48}$$

Because of the orthogonality of the sine function, the above equation can be written upon integration as

$$0=\sum_{n=1}^{\infty} A_n\Bigg[C_n\left\{\frac{m\pi}{M_m}(-1)^m\sinh a-\frac{4am\pi}{M_m^2}\left[(-1)^m\cosh a-1\right]\right\}$$

$$+D_n\left\{-\frac{2am\pi}{M_m^2}\left[3(-1)^m\cosh a+1\right]+\frac{m\pi}{M_m}(-1)^m\sinh a\left[1+\frac{16a^2}{M_m^2}-\frac{4}{M_m}\right]\right\}$$

$$+E_n\left\{\frac{m\pi}{M_m}(-1)^m\cosh a-6\frac{am\pi}{M_m^2}(-1)^m\sinh a\right.$$

$$\left.+\frac{4m\pi}{M_m^2}\left[(-1)^m\cosh a-1\right]\left[\frac{4a^2}{M_m}-1\right]\right\}+\frac{X_{mn}}{N_m^2}+\frac{\delta_{mn}N_n}{8a^2R}\Bigg],\tag{49}$$

in which

$$M_m=(m\pi)^2+a^2;$$

$$\begin{aligned}X_{mn}&=0 && \text{if } m=n,\\ &=0 && \text{if } m+n \text{ is even } \quad m\neq n,\\ &=\frac{8mn}{\pi^2(m^2-n^2)^2} && \text{if } m+n \text{ is odd};\end{aligned}$$

$$\begin{aligned}\delta_{mn}&=1 \text{ for } m=n\\ &=0 \text{ for } m\neq n.\end{aligned}$$

(The integration of the terms on the right hand side of equation (48) is lengthy, but in no way difficult. The details of these integrations are given in Appendix I.)

The non-trivial solution of this system of m homogeneous equations with n terms requires that the determinant of the coefficients of A_n vanish:

$$\left| \begin{array}{l} C_n\left\{\frac{m\pi}{M_m}(-1)^m \sinh \bar{a} - \frac{4\bar{a}m\pi}{M_m^2}\left[(-1)^m \cosh \bar{a} - 1\right]\right\} \\[2ex] + D_n\left\{-\frac{2\bar{a}m\pi}{M_m^2}\left[3(-1)^m \cosh \bar{a} + 1\right] + \frac{m\pi}{M_m}(-1)^m \sinh \bar{a}\left[1 + \frac{16\bar{a}^2}{M_m^2} - \frac{4}{M_m}\right]\right\} \\[2ex] + E_n\left\{\frac{m\pi}{M_m}(-1)^m \cosh \bar{a} - \frac{6\bar{a}m\pi}{M_m^2}(-1)^m \sinh \bar{a} + \frac{4m\pi}{M_m^2}\left[(-1)^m \cosh \bar{a} - 1\right]\left[\frac{4\bar{a}^2}{M_m} - 1\right]\right\} \\[2ex] + X_{mn}\left(\frac{1}{N_n^2}\right) + \frac{\delta_{mn} N_n}{8\bar{a}^2 R} \end{array} \right| = 0$$

C^M

(50)

The works of Chandrasekhar (5) and Yih (15) have shown that the solution of the infinite order secular equation can be closely approximated by evaluating the determinant for a finite n . In essence, the series solutions, equations (40) and (41), are truncated after n terms and the solution is determined from the finite number of terms that are retained. In order to simplify the subsequent writing of the secular determinant, let

$$C^m = \frac{m\pi}{M_m}(-1)^m \sinh \bar{a} - \frac{4\bar{a}m\pi}{M_m^2}\left[(-1)^m \cosh \bar{a} - 1\right] , \tag{51}$$

$$D^m = -\frac{2am\pi}{M_m^2}\left[3(-1)^m \cosh a + 1\right] + \frac{m\pi}{M_m}(-1)^m\left[1 + \frac{16a^2}{M_m^2} - \frac{4}{M_m}\right]\sinh a\,, \tag{52}$$

$$E^m = \frac{m\pi}{M_m}(-1)^m \cosh a - \frac{6am\pi}{M_m^2}(-1)^m \sinh a + \frac{4m\pi}{M_m^2}\left[(-1)^m \cosh a - 1\right]\left[\frac{4a^2}{M_m} - 1\right] \tag{53}$$

and

$$C_n C^m + D_n D^m + E_n E^m + \frac{X_{mn}}{N_n^2} + \frac{\delta_{mn} N_n}{8a^2 R} = A_{mn}. \tag{54}$$

With this notation, equation (50) becomes

$$\left|A_{mn}\right| = 0. \tag{55}$$

In his original paper which described the method of solution, Chandrasekhar found that the solution could be approximated closely by solving the n by n determinant (55) for n equal to one! The accuracy of the solution improves as n increases but in the case of the Bénard problem and one considered by Yih (15) the improvement is less than 10 per cent. However, in the present problem the solution for $A_{11} = 0$ indicates that the system is always stable, whereas higher approximations give a finite Rayleigh number as the stability criterion. This is due to the lack of symmetry of the convection so that the first term (associated with A_{11}) is inadequate to describe the convection. The difficulty with using $A_{11} = 0$ as a possible solution becomes apparent as the required numerical calculations are carried out, but it follows directly from the definitions of C_n, C^m, D_n, D^m, E_n, E^m, and X_{mn}. Then A_{11} becomes $\dfrac{A_1}{8a^2\mathbb{R}}$ and setting this first element of the determinant equal to zero results in an infinite value for $\mathbb{R}$, for a non-zero wave number a. It can also be shown that $A_{mn}=0$ for $m+n$ an even number ($m \neq n$), and $A_{mn}=\dfrac{N_n}{8a^2\mathbb{R}}$ for $m=n$.

These results simplify the evaluation of the secular equation and are given in Appendix II.

Therefore, one is forced to consider the solution of a 2 by 2 determinant in order to get a first approximation to the particular problem at hand. To obtain the required solution - the minimum value of $\mathbb{R}$ as a function of a in equation (50) - one assumes values of a and solves for $\mathbb{R}$. A graphical presentation of these solutions determines the value of the minimum, or critical, Rayleigh number. Fig. 2 shows the result of such a series of calculations. The numerical procedure was carried out for a 2 by 2, 3 by 3 and 4 by 4 determinant in order to improve the accuracy of the solution. A summary of the pertinent conclusions are given in Table I.

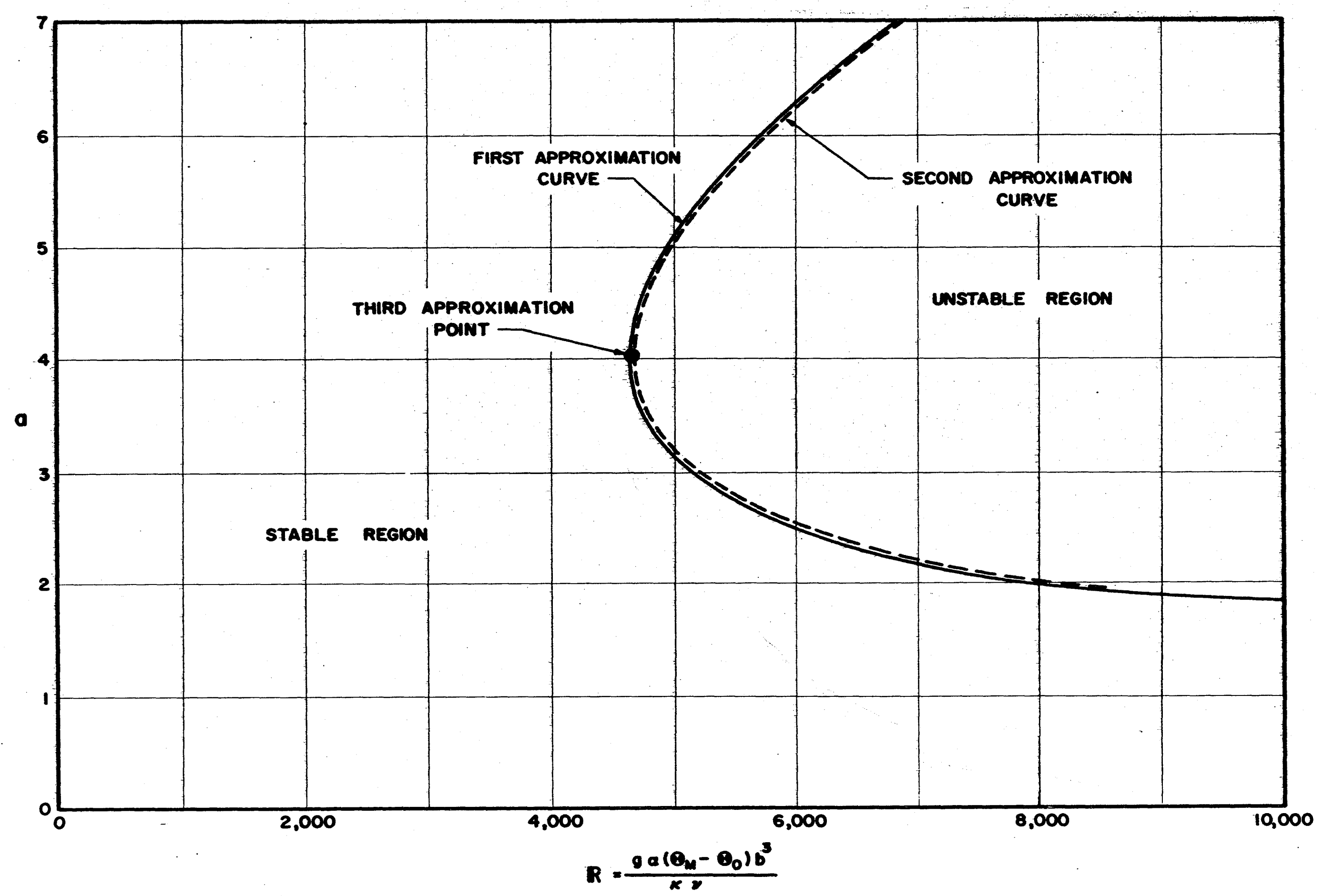


Figure 2. Neutral-stability curve for laminar natural convection in a fluid with homogeneously distributed heat sources contained between horizontal boundaries that are rigid and isothermal.

TABLE I

Summary of Computations of Neutral-Stability Curve for a Fluid Having Homogeneously Distributed Heat Sources Contained Between Horizontal Boundaries That are Rigid and Isothermal

	2.0	2.5	3.0	3.5	4.0	4.5
a	2.0	2.5	3.0	3.5	4.0	4.5
R (First Mode)						
First Approximation	7818	6000	5143	4885	4654	4731
Second Approximation			5201		4684	
Third Approximation					4669	
$\frac{R\ \text{(First Approx.)}}{R\ \text{(Second Approx.)}}$					0.9936	
$\frac{R\ \text{(Second Approx.)}}{R\ \text{(Third Approx.)}}$					1.0031	
$\frac{R\ \text{(First Approx.)}}{R\ \text{(Third Approx.)}}$					0.9967	

3. Temperature and Velocity Distribution in the Fluid Layer.

The solution just carried out for the critical Rayleigh number involved the non-dimensional temperature and velocity functions $\hat{T}$ and $\hat{W}$, defined by equations (40) and (41). These functions depend only on the vertical coordinate, z. The nature of $\hat{T}$ and $\hat{W}$ can be seen by examining Fig. 3. The curves presented in this Figure were obtained in the following manner:

If the series in equation (40) is truncated at $n=4$, one can write four equations involving A_1, A_2, A_3, and A_4. These equations are, in the notation of equation (54),

$$A_1A_{11} + A_2A_{12} + A_3A_{13} + A_4A_{14} = 0 \tag{56}$$

$$A_1A_{21} + A_2A_{22} + A_3A_{23} + A_4A_{24} = 0 \tag{57}$$

$$A_1A_{31} + A_2A_{32} + A_3A_{33} + A_4A_{34} = 0 \tag{58}$$

$$A_1A_{41} + A_2A_{42} + A_3A_{43} + A_4A_{44} = 0 \tag{59}$$

This is the set of homogeneous equations that were used to solve for the critical Rayleigh number in the last approximation achieved. The last three equations can be used to solve A_2, A_3, and A_4 in terms of A_1. After this is done the expressions for $\hat{W}$ and $\hat{T}$ will contain only one arbitrary constant, A_1, which is the amplitude of the perturbation. The magnitude of A_1 cannot be determined

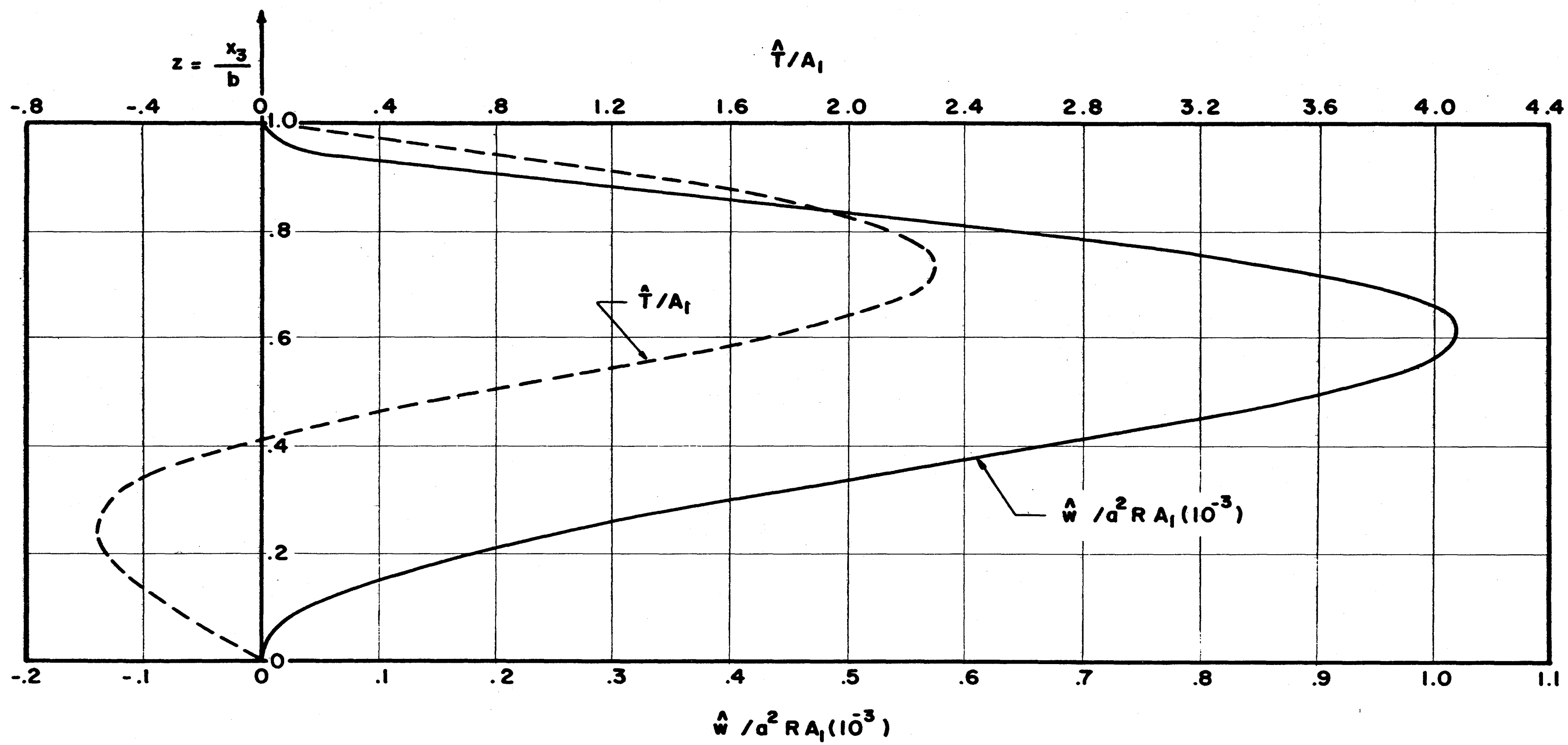


Figure 3. The variation of the nondimensional temperature and velocity functions, $\hat{T}$ and $\hat{w}$, with the nondimensional vertical coordinate, z.

if a linear analysis is used. The recent non-linear work of Malkus (13) and Stuart (14) is capable of giving a solution for A_1. The coefficients A_2, A_3, and A_4 are

$$A_2 = \frac{-A_1 \begin{vmatrix} A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \\ A_{41} & A_{43} & A_{44} \end{vmatrix}}{\Delta}, \tag{60}$$

$$A_3 = \frac{-A_1 \begin{vmatrix} A_{22} & A_{21} & A_{24} \\ A_{32} & A_{31} & A_{34} \\ A_{42} & A_{41} & A_{44} \end{vmatrix}}{\Delta}, \tag{61}$$

$$A_4 = \frac{-A_1 \begin{vmatrix} A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \\ A_{42} & A_{43} & A_{41} \end{vmatrix}}{\Delta}, \tag{62}$$

and

$$\Delta = \begin{vmatrix} A_{22} & A_{23} & A_{24} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{vmatrix}.$$

These determinants can be slightly simplified by recalling that A_{31}, A_{24}, and A_{42} are equal to zero. The values for the various A_{mn} are obtained by substituting the critical Rayleigh number and the associated critical cell number into equations (44)-(46) and (51)-(54).

For this particular problem the critical Rayleigh and cell numbers are obtained from Fig. 2 and are 4672 and 4.0, respectively. The solutions to the three equations (60), (61), and (62) are then

$$A_2 = - A_1 \, (1.429777501) \tag{63}$$

$$A_3 = - A_1 \, (0.2251148661) \tag{64}$$

and

$$A_4 = - A_1 \, (0.05253886715) \tag{65}$$

In view of the above, $\hat{T}$, the part of the non-dimensional perturbation temperature that is z-dependent, is

$$\hat{T} = A_1 \left[\sin \pi z - 1.429777501 \sin 2\pi z + 0.2251148661 \sin 3\pi z + 0.05253886715 \sin 4\pi z \right]. \tag{66}$$

This is the expression that is plotted in Fig. 3 for $\hat{T}$.

With the evaluation of the first four A_n's in terms of A_1 the first four terms in the series for $\hat{W}$ become -cf. equation (41)

$$\frac{\hat{w}}{a^2 \mathbb{R}} = A_1\left[C_1 \sinh az + D_1 z \sinh az + E_1 z \cosh az + \frac{1}{N_1^2}\sin \pi z\right]$$

$$-A_1\left[(1.42\cdots)C_2 \sinh az + (1.42\cdots)D_2 z \sinh az + (1.42\cdots)E_2 z \cosh az + \frac{(1.42\cdots)}{N_2^2}\sin 2\pi z\right]$$

$$+A_1\left[(0.22\cdots)C_3 \sinh az + (0.22\cdots)D_3 z \sinh az + (0.22\cdots)E_3 z \cosh az + \frac{(0.22\cdots)}{N_3^2}\sin 3\pi z\right]$$

$$+A_1\left[(0.05\cdots)C_4 \sinh az + (0.05\cdots)D_4 z \sinh az + (0.05\cdots)E_4 z \cosh az + \frac{(0.05\cdots)}{N_4^2}\sin 4\pi z\right]$$

in which only the first two decimal places of the constants given in equations (63), (64), and (65) have been written for the sake of economy.

This expression for $\frac{\hat{w}}{a^2 \mathbb{R}}$ can be rearranged for more efficient evaluation as

$$\frac{\hat{w}}{a^2 R} = A_1\Big[\Big[\{\sinh az\}\{C_1-(1.42\cdots)C_2+(0.22\cdots)C_3+(0.05\cdots)C_4\}$$
$$+\{z\sinh az\}\{D_1-(1.42\cdots)D_2+(0.22\cdots)D_3+(0.05\cdots)D_4\}$$
$$+\{z\cosh az\}\{E_1-(1.42\cdots)E_2+(0.22\cdots)E_3+(0.05\cdots)E_4\}$$
$$+\frac{\sin\pi z}{N_1^2}-\frac{(1.42\cdots)\sin 2\pi z}{N_2^2}+\frac{(0.22\cdots)\sin 3\pi z}{N_3^2}+\frac{(0.05\cdots)\sin 4\pi z}{N_4^2}\Big]\Big] \tag{67}$$

in which $R = 4672$ and $a = 4$ as before, and the coefficients C_1, C_2, C_3, C_4, D_1, D_2, etc., are determined from equations (44), (45), and (46). The evaluation of the above equation for $\hat{w}$ as a function of z is shown in Fig. 3.

The solution that has been obtained for $\hat{w}$ is deficient in one regard. The direction of $\hat{w}$ at the center of the cell is indeterminate. Whether the flow is upward or downward at the cell's center could be learned from an experiment. For the Bénard problem the flow is downward at the center and upward along the sides of the cell. The recently developed non-linear analysis is capable of predicting the flow direction for the Bénard problem -

and does so in agreement with the experimental observation!

After $\hat{W}$ and $\hat{T}$ have been determined, there remains only the specification of f in equations (30) and (31) to completely describe W and T (up to the undetermined amplitude, A_1). The quantity f must satisfy equation (33) which is rewritten here for convenience:

$$f_{xx} + f_{yy} + a^2 f = 0 \tag{33}$$

This equation is amenable to solution if f is considered to be a separable function of x and y, i.e.,

$$f = gh, \quad g = g(x), \quad h = h(y). \tag{68}$$

(Only the spirit of the solution for f will be given here so that the form of the f that was assumed will be clear).

When f, from equation (68) is incorporated in equation (33) one obtains

$$-\frac{g''}{g} = \frac{h''}{h} + a^2 = c^2$$

or

$$g'' + c^2 g = 0, \quad g = P_1 \sin cx + P_2 \cos cx, \tag{69}$$

and

$$h'' + (a^2 - c^2)h = 0, \quad h = S_1 \sin\sqrt{a^2-c^2}\,y + S_2 \cos\sqrt{a^2-c^2}\,y. \tag{70}$$

Thus

$$f = \left[P_1 \sin cx + P_2 \cos cx\right]\left[S_1 \sin\sqrt{a^2-c^2}\,y + S_2 \cos\sqrt{a^2-c^2}\,y\right].$$

The specification of the constants in this equation is achieved by imposing the necessary boundary conditions on f. These conditions can be obtained from a knowledge of the desired velocity field.

The conditions on f are determined by considering equation (31) in conjunction with equation (16). This latter equation can be rewritten in non-dimensional form by recalling

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \tag{16}$$

and

$$x = \frac{x_1}{b}, \text{ etc} \qquad u = \frac{u_1 b}{k}, \text{ etc}$$

whence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 .$$

At the conditions of neutral stability (i.e., $\sigma = 0$)

$$w = f\hat{w}$$

and equation (33) is valid so that

$$\frac{\partial}{\partial x}\left[\frac{1}{a^2}\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{1}{a^2}\frac{\partial f}{\partial y}\right] + f = 0 .$$

Multiplying this last equation by $\frac{d\hat{w}}{dz}$, one has

$$\frac{\partial}{\partial x}\left[\frac{1}{a^2}\frac{\partial f}{\partial x}\frac{d\hat{w}}{dz}\right]+\frac{\partial}{\partial y}\left[\frac{1}{a^2}\frac{\partial f}{\partial y}\frac{d\hat{w}}{dz}\right]+f\frac{d\hat{w}}{dz}=0 \tag{71}$$

which, upon comparison with equation (16), becomes

$$\frac{\partial(u)}{\partial x}+\frac{\partial(v)}{\partial y}+\frac{\partial(w)}{\partial z}=0$$

if

$$u=\frac{1}{a^2}\frac{\partial f}{\partial x}\frac{d\hat{w}}{dz} \tag{72i}$$

and

$$v=\frac{1}{a^2}\frac{\partial f}{\partial y}\frac{d\hat{w}}{dz} \tag{72ii}$$

Because there is no coupling of u, v, and w in the equations of motion (10), u and v can be chosen in the manner above.

One must prescribe the location of the cell boundaries in the x-y plane and the required vanishing of the components of u and v perpendicular to those boundaries will create the related restrictions on $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ so as to specify the function f. In solving for f the separation constants a and c enter into the problem via equations (69) and (70). As a consequence of this fact, a ratio exists between the dimensions of the cell in the $x-y$ plane and the height of the cell in the z direction. It seems logical, therefore, to call a the cell number in view of the role it plays in the cell's geometry.

The solution of f for the case of a hexagonal tesselation was obtained by Christopherson (16). For this configuration f has the form

$$f = \frac{1}{3} f_0 \left[\cos \frac{2p\pi}{3L} (\sqrt{3}\, x_1 + x_2) + \cos \frac{2p\pi}{3L} (\sqrt{3}\, x_1 - x_2) + \cos \frac{4p\pi}{3L} x_2 \right]$$

in which f_0 is a constant, L is the length of one side of the hexagon, and p is an integer. The equation relating L/b and a is

$$a = \frac{4p\pi b}{3L} .$$

Thus f can be written as

$$f = \frac{1}{3} f_0 \left[\cos \frac{a}{2} (\sqrt{3}\, x + y) + \cos \frac{a}{2} (\sqrt{3}\, x - y) + \cos a y \right] . \tag{73}$$

It is this function of f that was utilized to complete the evaluation of W and T in equations (30) and (31). A hexagonal cell shape was specified because that shape was experimentally observed by Bénard for the problem bearing his name. However, when experimentally studying the Bénard problem for cases in which the fluid layer is confined between solid vertical boundaries that are closely spaced, it is found that the shape of the boundary influences the shape of the cell. Thus, for a horizontal region that has a diameter of 8 inches, the cell shape consists of concentric rings; for a square boundary with a side of 6 inches the cell shape consists of many cells which are approximately square in shape. In view of this, the form of any streamlines that are constructed by assuming *a priori* an f, may not have any physical counterpart in a particular experiment. It should be noted, however, that the indeterminatecy of f for the present problem, which was treated by a linearizing process, in no way affects the stability criterion that was obtained.

The streamlines in the vertical planes of symmetry of the cell can be found with the help of equations (71), (72), and (73). For the motion in the plane for which $y=0$, $v=0$ because of the symmetry. Consequently, assuming a stream function, ψ, such that

$$u=-\frac{\partial\psi}{\partial z}, \quad w=\frac{\partial\psi}{\partial x},$$

one obtains, upon comparison with equation (69),

$$\psi=-\frac{1}{a^2}\hat{w}\frac{\partial f}{\partial x}\bigg|_{y=0}=-\frac{1}{a^2}\frac{\partial w}{\partial x}\bigg|_{y=0} \tag{74}$$

In the same manner the stream function, Ψ, which exists in the plane $x=0$, is

$$\Psi = \frac{1}{a^2}\frac{\partial w}{\partial y}\bigg|_{x=0}. \tag{75}$$

Equations (67), (73), and (74) are now employed to determine the streamlines in the $y=0$ plane:

$$\psi = \frac{f_c \hat{W} a\sqrt{3}}{6}\left\{\sin\frac{a}{2}(\sqrt{3}x+y) + \sin\frac{a}{2}(\sqrt{3}x+y)\right\}_{y=0}$$

$$= \frac{f_c \hat{W} a\sqrt{3}}{3}\left\{\sin\frac{a}{2}\sqrt{3}x\right\},$$

whence

$$\psi \sim \hat{W}\sin\frac{a}{2}\sqrt{3}x$$

and for $a=4$

$$\psi \sim \hat{W}\sin\sqrt{3}x\cos\sqrt{3}x. \tag{76}$$

The streamlines in the plane of symmetry $y=0$ are plotted in Fig. 4 by using equation (76).

It can be seen from this figure that the mean parabolic temperature distribution has resulted in a flow that is asymmetrical with respect to the mid-plane of the fluid layer, whereas in the Bénard problem the flow is symmetrical about this plane.

4. Heat Transfer Aspects of the Solution.

Once the temperature distribution T is known for

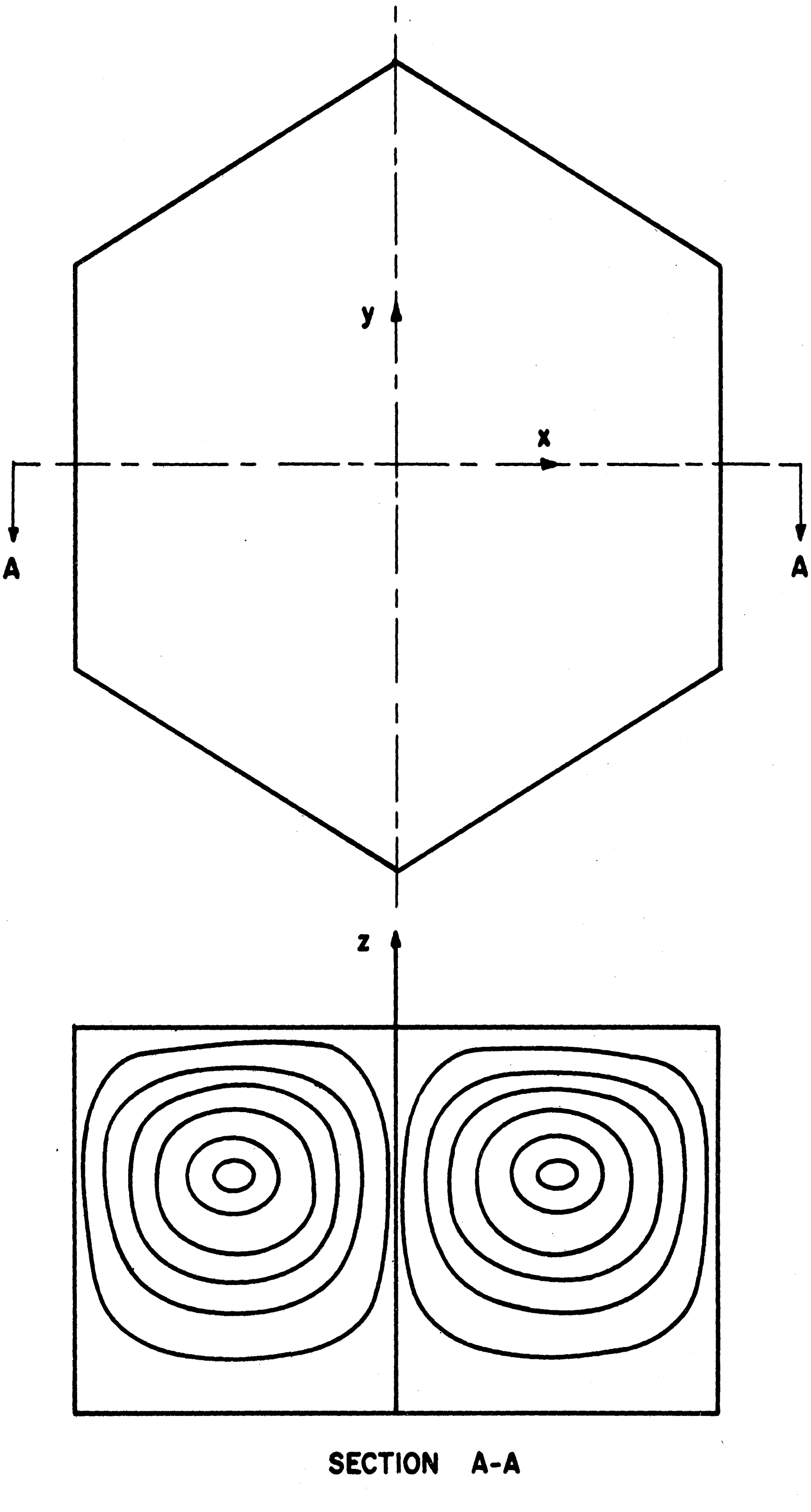


Figure 4. The cell pattern in one plane of symmetry at the onset of convection for the first mode and for an assumed hexagonal tesselation.

the convection (or unstable) state the temperature gradient can be found along the upper and lower boundaries of the fluid layer. By integrating this temperature gradient over the entire horizontal fluid boundary, it is possible to determine the fraction of the total quantity of heat transferred across each of the two horizontal boundaries. Hence, to evaluate the quantity of heat flowing across the lower boundary one writes

$$Q_L = k \int_A \left(\frac{\partial \textcircled{H}}{\partial X_3} \right)_{X_3=0} dA \tag{77}$$

in which the integration is carried out over the hexagonal lower boundary of one convection cell, and the subscript on Q denotes this lower boundary. From the definition of $\textcircled{H}$, the vertical temperature gradient is

$$\frac{\partial \textcircled{H}}{\partial X_3} = \frac{\partial \bar{\textcircled{H}}}{\partial X_3} + \frac{\partial \textcircled{H}'}{\partial X_3} = \frac{\partial \bar{\textcircled{H}}}{\partial X_3} + \frac{(\textcircled{H}_m - \textcircled{H}_o)}{b} \frac{\partial T}{\partial z} ;$$

$$\frac{\partial \textcircled{H}}{\partial X_3} = \frac{\partial \bar{\textcircled{H}}}{\partial X_3} + \frac{(\textcircled{H}_m - \textcircled{H}_o)}{b} (f) \frac{\partial \hat{T}}{\partial z} . \tag{78}$$

The heat transferred, equation (77), can be rewritten as

$$Q_L = k \iint_A \frac{\partial \bar{\textcircled{H}}}{\partial X_3} dX_1 dX_2 + k \frac{(\textcircled{H}_m - \textcircled{H}_o)}{b} \left(\frac{\partial \hat{T}}{\partial z} \right)_{z=0} b^2 \iint_A f \, dx dy$$

in which the first integral represents the amount of heat transferred by the mean temperature distribution - a value which because of the symmetry of the temperature distribution is the same at the upper boundary - and the second integral represents the increase (or decrease) in the amount of heat transferred at the boundary due to the perturbation.

If the second integral is examined in more detail its magnitude can be obtained without the necessity of a formal evaluation. With the aid of equation (33), this integral becomes

$$\iint_A f \, dx\, dy = -\frac{1}{a^2}\iint_A \left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right\} dx\, dy$$

and by means of Green's Theorem the area integral can be written as a line integral such that

$$\iint_A f \, dx dy = -\frac{1}{a^2}\oint_C \frac{\partial f}{\partial x}\, dy - \frac{\partial f}{\partial y}\, dx$$

in which the integral on the right hand side is evaluated on the hexagonal closed curve comprising the boundary of the cell in the x-y plane. The line integral can be given a vector interpretation which is convenient for its evaluation -- cf., Kaplan (17). Consider a vector

$$\vec{F} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

in which $\vec{i}$ and $\vec{j}$ are base vectors in the x and y directions, respectively. A unit vector normal to the hexagonal boundary can be written as

$$\vec{n} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$$

in which s is the distance along the boundary. Consequently,

$$\vec{F}\cdot\vec{n} = \frac{\partial f}{\partial x}\frac{dy}{ds} - \frac{\partial f}{\partial y}\frac{dx}{ds} = F_n$$

in which F_n is the component of $\vec{F}$ in the $\vec{n}$ direction. Then

$$\oint\left(\frac{\partial f}{\partial x}\frac{dy}{ds} - \frac{\partial f}{\partial y}\frac{dx}{ds}\right)ds = \oint F_n\,ds\,.$$

It will be noted that the vector defined as $\vec{F}$ is the temperature gradient in the x-y plane (the lower fluid boundary for the situation under discussion). The integral just obtained requires that the component of the temperature gradient which is perpendicular to the hexagonal boundary be integrated along the boundary. Examination of the dimensionless perturbation temperature, T, shows that, for a prescribed value of z, the isotherms are everywhere perpendicular to the hexagonal boundary. Hence, the temperature gradient has no component perpendicular to the boundary and

$$\oint F_n\,ds = 0\,.$$

One concludes from this result that there is no net change in the amount of heat transferred at the lower fluid boundary as a result of the onset of convection.

By direct analogy it is apparent that this same conclusion holds for the upper fluid boundary. The fact that the sum of the heat transferred to <u>both</u> boundaries must remain a constant, regardless of whether conduction or convection takes place is to be expected, and required, since the amount of heat liberated by a unit volume of the fluid in an interval of time is a property of the fluid that is considered to be independent of the type of motion. However, it seems reasonable that under convection conditions there would be more heat transferred to the upper fluid boundary than the lower boundary if both were kept at the same temperature. The question can evidentally not be resolved by a linear theory. It can be speculated that a non-linear analysis might bear out this conjecture.

5. Review of Assumptions

In the course of developing the governing equations a number of assumptions were made which resulted in a set of linear differential equations that were subsequently solved. These assumptions were the following:

a. The thermal properties of the fluid would remain constant, despite the presence of small temperature differences in the fluid.

b. The viscosity of the fluid would remain a constant over the range of temperatures existing in the fluid layer.

c. The Laplacian of u_3 would be large compared to $\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)$ with the result that the latter could be neglected by comparison with the former.

d. The conversion of mechanical energy to heat could be neglected in writing the energy equation.

The validity of the first of these assumptions can be verified with a list of the physical properties of the fluid in question. Water, which was used in the experimental work, has thermal properties which satisfy this assumption. These are -- cf., Handbook of Chemistry and Physics (18).

Temperature (^{o}C)	15^{o}	20^{o}
Thermal conductivity, k (Cal/cm/sec/^{o}C)	0.00144	0.00143
Specific heat, C_p (Cal/g/^{o}C)	0.99976	0.99883
Density, ρ (g/cm^3)	0.999099	0.998203

The coefficient of thermal expansion for water is also constant for small temperature variations. Its value

At 20° C is $2(10^{-4})/^{\circ}$ C.

The validity of the assumption regarding the constancy of the viscosity is borne out less well. For the temperature range 15-20° C the dynamic and kinematic viscosities vary approximately 12 per cent. However, if the temperature variation in the fluid were about 2° C, the resultant variation in viscosity would have a minor effect on the problem.

When considering the third assumption it is convenient to employ $\nabla^2 w$ in the calculation of Δu_3 .

Thus

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = f_{xx}\hat{w} + f_{yy}\hat{w} + f\hat{w}_{zz}$$

and from equation (33)

$$f_{xx}\hat{w} + f_{yy}\hat{w} = -a^2 f\hat{w}.$$

Therefore

$$\nabla^2 w = -a^2 f\hat{w} + f\hat{w}_{zz} = w\left[\frac{\hat{w}_{zz} - a^2\hat{w}}{\hat{w}}\right].$$

It follows that from (67)

$$\hat{w}_{zz} = a^2\hat{w} + a^2 R A_1 \left[2aS\cosh az + 2aT\sinh az \right.$$

$$\left. - \frac{\sin \pi z}{N_1} - \frac{A_2}{A_1}\frac{\sin 2\pi z}{N_2} - \frac{A_3}{A_1}\frac{\sin 3\pi z}{N_3} - \frac{A_4}{A_1}\frac{\sin 4\pi z}{N_4}\right],$$

in which the constants S and T are the coefficients of $z \sinh az$ and $z \cosh az$ given in equation (67). The above equation can be written more simply as

$$\hat{w}_{zz} = a^2 \hat{w} + a^2 \mathbb{R} A_1 [P] ,$$

in which

$$P = 2aS \cosh az + 2aT \sinh az - \frac{\sin \pi z}{N_1} - \frac{A_2}{A_1} \frac{\sin 2\pi z}{N_2} - \frac{A_3}{A_1} \frac{\sin 3\pi z}{N_3} - \frac{A_4}{A_1} \frac{\sin 4\pi z}{N_4} .$$

Thus

$$\nabla^2 w = \left(a^2 \mathbb{R} A_1 \right) P f$$

and

$$\Delta u_3 = \frac{K}{b^3} \left(a^2 \mathbb{R} A_1 \right) P f .$$

From equation (15) one can write for steady-state conditions

$$\frac{\partial u_i}{\partial x_i} \approx \frac{\alpha q}{2k} u_3 \left(b - 2x_3 \right) ,$$

and

$$\frac{\partial}{\partial x_3} \left(\frac{\partial u_i}{\partial x_i} \right) \approx \frac{\alpha q}{2k} \left[-2u_3 + \left(b - 2x_3 \right) \frac{\partial u_3}{\partial x_3} \right] ,$$

or

$$\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right) \approx \frac{\alpha q_T K}{2kb}\left[-2w + (1-2z)\frac{\partial w}{\partial z}\right] .$$

To compare the magnitude of Δu_3 and $\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)$

one can calculate their ratio as

$$\frac{\Delta u_3}{\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)} = \frac{2k}{b^2 \alpha q}\left\{\frac{P}{-2\hat{w}/\bar{a}^2 \mathbb{R} A_1 + (\partial\hat{w}/\partial z)(1-2z)/\bar{a}^2 \mathbb{R} A_1}\right\} .$$

The denominator contained in the bracket will be referred to henceforth as L . Thus

$$\frac{\Delta u_3}{\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)} = \frac{2k}{b^2 \alpha q}\,\frac{P}{L} .$$

The function L can be readily computed once $\frac{\partial\hat{w}}{\partial z}$

is determined. Again using equation (67) one has

$$\frac{1}{\bar{a}^2 \mathbb{R} A_1}\left(\frac{\partial\hat{w}}{\partial z}\right) = (\bar{a}Q + T)\cosh\bar{a}z + S\sinh\bar{a}z + \bar{a}Sz\cosh\bar{a}z$$

$$+\bar{a}Tz\sinh\bar{a}z + \pi\left[\frac{\cos\pi z}{N_1^2} + \frac{2A_2}{A_1}\cos 2\pi z + \frac{3A_3}{A_1}\cos 3\pi z + \frac{4A_4}{A_1}\cos 4\pi z\right]$$

in which Q is the coefficient of $\sinh \partial \zeta$ in equation (67).

The ratio P/L was computed for $0 \leq \zeta \leq 1$ and the results are shown in Fig. 5. By using the physical constants for water, it can be seen that only in the minute regions $0.86 < \zeta < 0.87$ and $0.288 < \zeta < 0.291$ does the ratio $|\Delta u_3| / \left|\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)\right|$ fall below the value of 200. In fact, in almost the entire region the ratio exceeds 1000. Hence, it can be concluded that, except for a small region at $\zeta = 0.865$ and a smaller region at $\zeta = 0.290$, the assumption $|\Delta u_3| \gg \left|\frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)\right|$ is justified.

In writing the equation for heat conduction the term representing the viscous dissipation of mechanical energy to heat (Rayleigh Dissipation Function) was omitted. This omission is reasonable in a problem for which the velocities and the velocity gradients are small. Not only is it to be expected that the dissipation function will be small but its effect will be further reduced by multiplying it by the dynamic viscosity and dividing by the Joule constant.

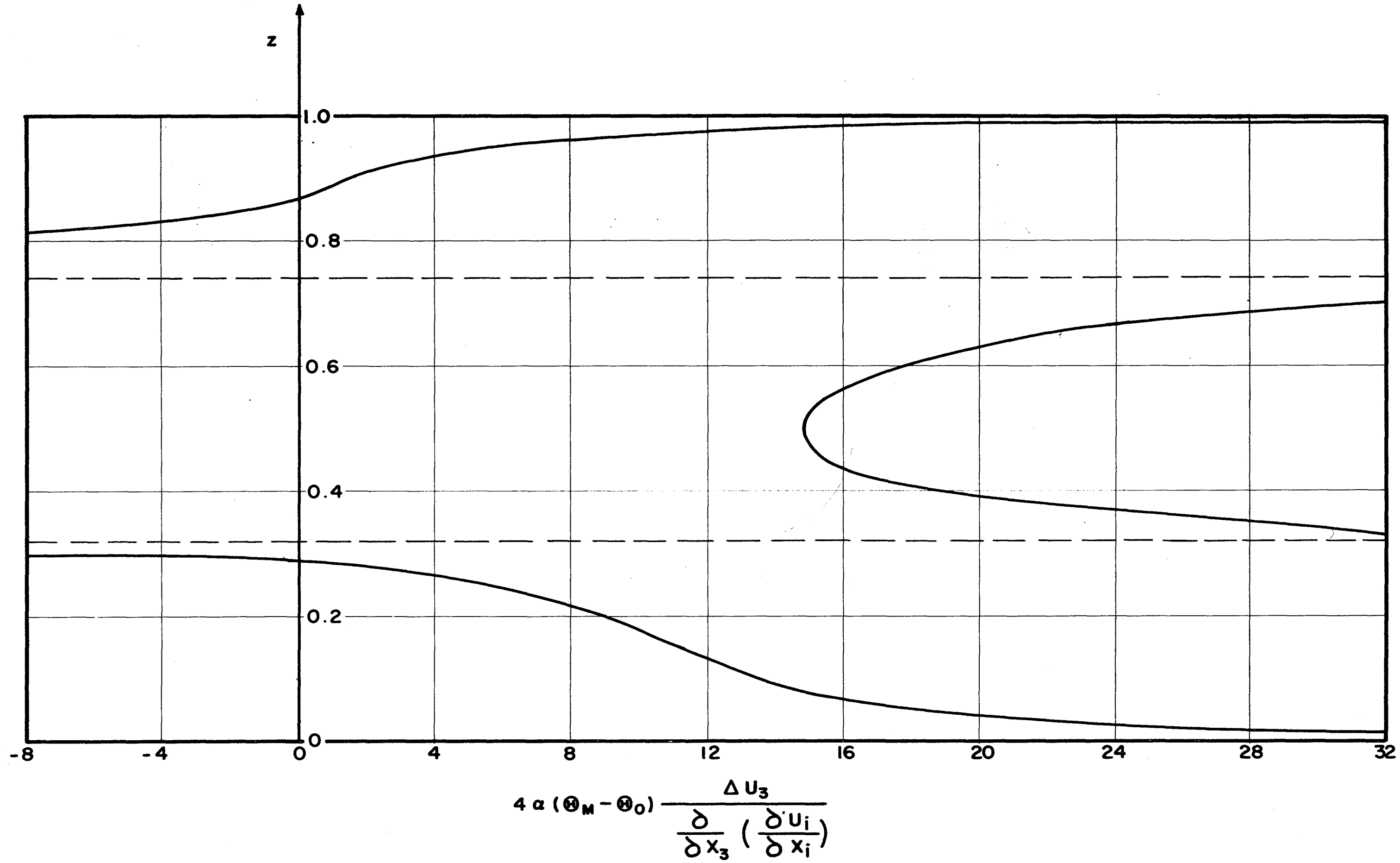


Figure 5. The ratio of the Laplacian of the vertical velocity component to the derivative, in the vertical direction, of the divergence of the velocity.

CHAPTER III

MATHEMATICAL FORMULATION AND SOLUTION OF RELATED STABILITY PROBLEMS

The stability of a fluid with homogeneously distributed heat sources that is confined between two horizontal boundaries can be studied for boundary conditions other than those used in Chapter II. The mathematical boundary conditions on the velocity and temperature that can be imposed at the upper and lower surfaces of the fluid corresponds to the various physical conditions which can exist at these surfaces. These new boundary conditions will result in different solutions, even though the governing equations are the same or similar, because the solution must satisfy the differential equation <u>and</u> the applicable boundary conditions. In this chapter, two additional solutions are obtained for the thermal stability of a heat generating fluid. In treating these two problems the notation and assumptions adopted in Chapter II are retained.

1. <u>Rigid Thermally Conducting Upper Boundary and Rigid Thermally Insulating Lower Boundary.</u>

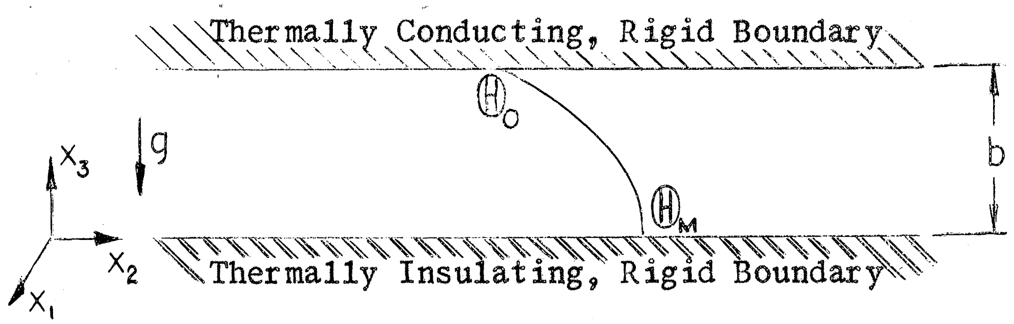


Fig.6 Schematic Drawing Of Fluid Layer With The Mean Temperature Distribution And The Coordinate System.

For the first problem to be considered, Fig.6 , the mean temperature is given by

$$\bar{\Theta} = \frac{q}{2k}\left(b^2 - x_3^2\right) + \Theta_0 .$$

(79)

The method of deriving the governing equations is identical to that previously employed:

a. Perturbation variables for u_i , Θ , p and ρ are introduced into the equations of motion (1) and heat conduction (2).

b. All products of perturbation variables and their derivatives are neglected as being small.

c. The equations of motion are cross-differentiated and selectively added and subtracted to eliminate p.

c. The term $\frac{\partial}{\partial x_3}\left[\frac{\partial u_j}{\partial x_j}\right]$ is neglected by comparison with Δu_3 in conjunction with magnitude considerations arising out of the continuity equation.

The governing differential system that results from these operations is

$$\left[\frac{\partial}{\partial t} - \nu\Delta\right]\Delta u_3 = g\alpha\Delta_1\Theta'$$

(80)

and

$$\left[\frac{\partial}{\partial t} - \kappa\Delta\right]\Theta' = \left(\frac{q x_3}{k}\right)u_3 .$$

(81)

The boundary conditions are

$$u_3 = 0 \quad \text{at} \quad x_3 = 0,\ b\ ; \tag{82i}$$

$$\frac{\partial u_3}{\partial x_3} = 0 \quad \text{at} \quad x_3 = 0,\ b\ ; \tag{82ii}$$

$$\frac{\partial \Theta'}{\partial x_3} = 0 \quad \text{at} \quad x_3 = 0\ ; \tag{82iii}$$

$$\Theta' = 0 \quad \text{at} \quad x_3 = b. \tag{82iv}$$

The dimensionless variables, (9), can be introduced so that the governing equations are

$$\left[\frac{\kappa}{\nu}\frac{\partial}{\partial \tau} - \nabla^2\right]\nabla^2 w = \frac{g\alpha b^3(\Theta_m - \Theta_0)}{\kappa\nu}\nabla_1^2 T \tag{83}$$

and

$$\left[\frac{\partial}{\partial \tau} - \nabla^2\right] T = 2wz, \tag{84}$$

with

$$w = 0 \quad \text{at} \quad z = 0,\ 1\ ; \tag{85i}$$

$$\frac{\partial w}{\partial z} = 0 \quad \text{at} \quad z = 0,\ 1\ ; \tag{85ii}$$

$$\frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = 0\ ; \tag{85iii}$$

$$T=0 \quad \text{at} \quad z=1. \tag{85iv}$$

The solution of the problem posed by equations (83) and (84) is achieved by the method of separation of variables so that

$$w=f\hat{w}e^{\sigma\tau},$$

$$T=f\hat{T}e^{\sigma\tau},$$

in which f is a function of x and y and the functions $\hat{w}$ and $\hat{T}$ are z-dependent. With the velocity and temperature so defined the governing equations and associated boundary conditions are

$$\left[\frac{K}{\nu}\sigma-(D^2-a^2)\right](D^2-a^2)\hat{w}=\frac{g\alpha b^3}{K\nu}(\Theta_m-\Theta_0)(-a^2\hat{T})=-a^2 R\hat{T} \tag{86}$$

and

$$\left[\sigma-(D^2-a^2)\right]\hat{T}=2z\hat{w}, \tag{87}$$

with

$$\hat{w}=0 \quad \text{at} \quad z=0,1; \tag{88i}$$

$$\frac{\partial\hat{w}}{\partial z}=0 \quad \text{at} \quad z=0,1; \tag{88ii}$$

$$\frac{\partial \hat{T}}{\partial z} = 0 \quad \text{at} \quad z = 0\,; \tag{88iii}$$

$$\hat{T} = 0 \quad \text{at} \quad z = 1\,. \tag{88iv}$$

The boundary condition (88iii) results from the prescription of an insulated lower surface.

The "principle of exchange of stabilities" is employed as before so that $\sigma = 0$, and the equations which must be solved reduce to

$$\left(D^2 - a^2\right)^2 \hat{w} = a^2 R \hat{T} \tag{89}$$

and

$$\left(D^2 - a^2\right)\hat{T} = -2z\hat{w}. \tag{90}$$

The solution of the problem stated by (88), (89), and (90) is achieved by the approximative method used in Chapter II. The assumed form for $\hat{T}$ is

$$\hat{T} = \sum_{n}^{\infty} A_n \cos \frac{n\pi z}{2} \qquad n = 1, 3, 5, \cdots \tag{91}$$

allows the thermal boundary conditions to be satisfied. Equations (89) and (91) are combined to yield

$$\hat{w} = a^2 R \sum_{n}^{\infty} A_n \left[B_n \sinh az + C_n \cosh az + D_n z \sinh az + E_n z \cosh az + \frac{\cos \frac{n\pi z}{2}}{N_n^2} \right], \qquad n = 1, 3, 5, \cdots \tag{92}$$

in which

$$N_n = \left(\frac{n\pi}{2}\right)^2 + a^2.$$

The velocity boundary conditions require

$$B_n = \frac{\sinh a \left\{ -\left(\frac{n\pi}{2}\right) \sin \frac{n\pi}{2} + \cosh a \right\} + a}{N_n^2 (\sinh^2 a - a^2)}, \tag{93}$$

$$C_n = -\frac{1}{N_n^2}, \tag{94}$$

$$D_n = -B_n - (C_n + E_n)(\coth a), \tag{95}$$

$$E_n = -a B_n. \tag{96}$$

In order that the two series given by (91) and (92) can show the equality stated in (90) the following secular determinant must be satisfied:

$$\left| B_n B^m + C_n C^m + D_n D^m + E_n E^m + \frac{X_{mn}}{N_n^2} - \frac{\delta_{mn} N_n}{4\bar{a}^2 R} \right| = 0 \; , \tag{97}$$

in which

$$B^m = \frac{(m\pi/2)(-1)^{\frac{m+3}{2}}}{M_m}\left[\sinh \bar{a} - \frac{2\bar{a}\cosh \bar{a}}{M_m}\right] , \tag{98}$$

$$C^m = (m\pi/2)(-1)^{\frac{m+3}{2}}\left[\cosh \bar{a} - \frac{2\bar{a}\sinh \bar{a}}{M_m}\right] + \frac{1}{M_m} - \frac{2(m\pi/2)^2}{M_m^2} , \tag{99}$$

$$D^m = \frac{(m\pi/2)(-1)^{\frac{m+3}{2}}}{M_m}\sinh \bar{a}\left[1 - 2\frac{(m^2\pi^2/4 - \bar{a}^2)}{M_m^2} + \frac{4\bar{a}^2}{M_m^2}\right] - 4\bar{a}\frac{(m\pi/2)(-1)^{\frac{m+3}{2}}}{M_m^2}\cosh \bar{a} + \frac{4\bar{a}}{M_m^3}(m\pi/2)^2 + \frac{2\bar{a}}{M_m^3}(m^2\pi^2/4 - \bar{a}^2) , \tag{100}$$

$$E^m = \frac{(m\pi/2)(-1)^{\frac{m+3}{2}}}{M_m}\cosh \bar{a}\left[1 + \frac{6\bar{a}^2 - 2m^2\pi^2/4}{M_m^2}\right] - \frac{4\bar{a}}{M_m^2}(m\pi/2)(-1)^{\frac{m+3}{2}}\sinh \bar{a} \; , \tag{101}$$

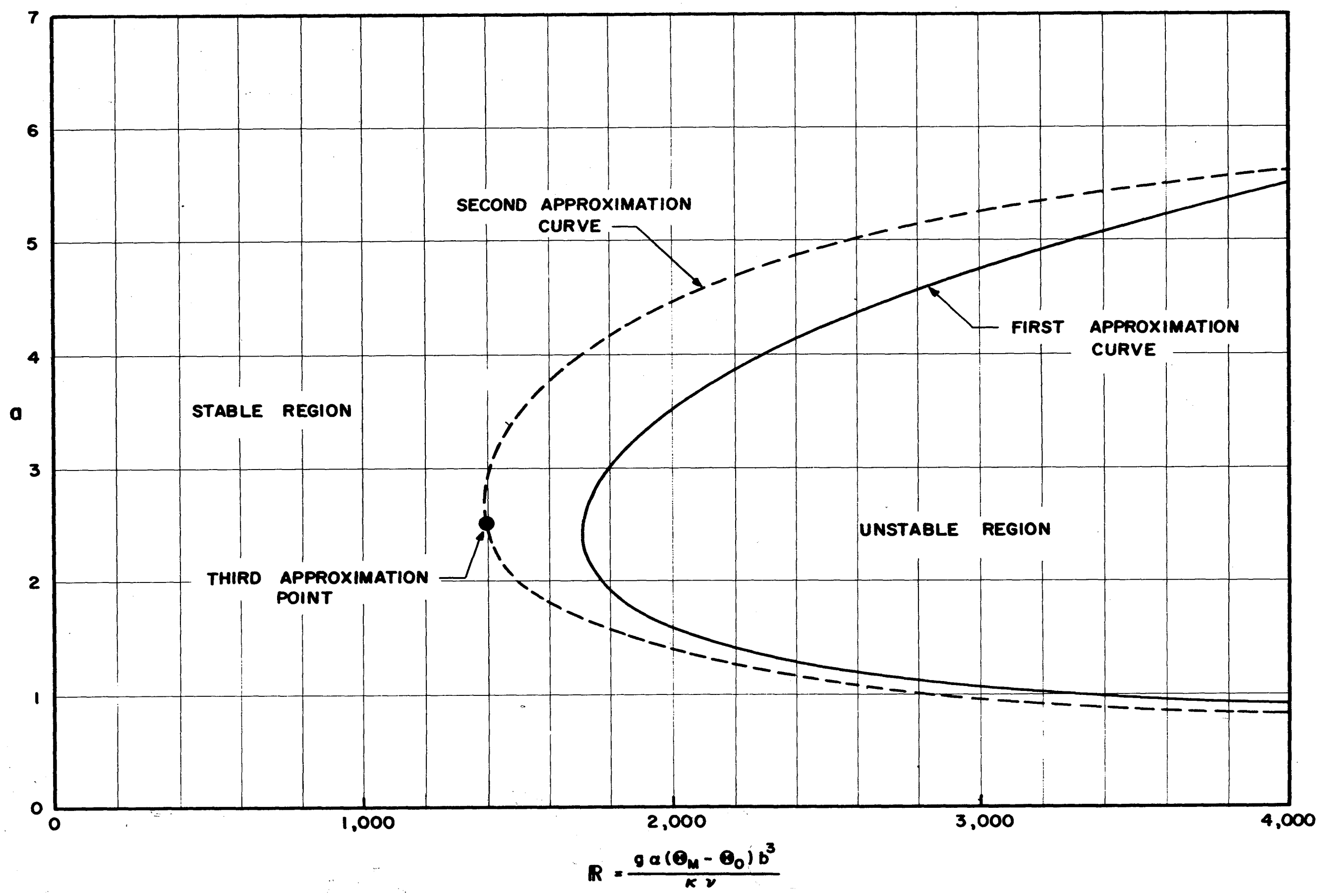


Figure 7. Neutral-stability curve for laminar natural convection in a fluid with homogeneously distributed heat sources contained between a lower adiabatic rigid boundary and an upper rigid isothermal boundary.

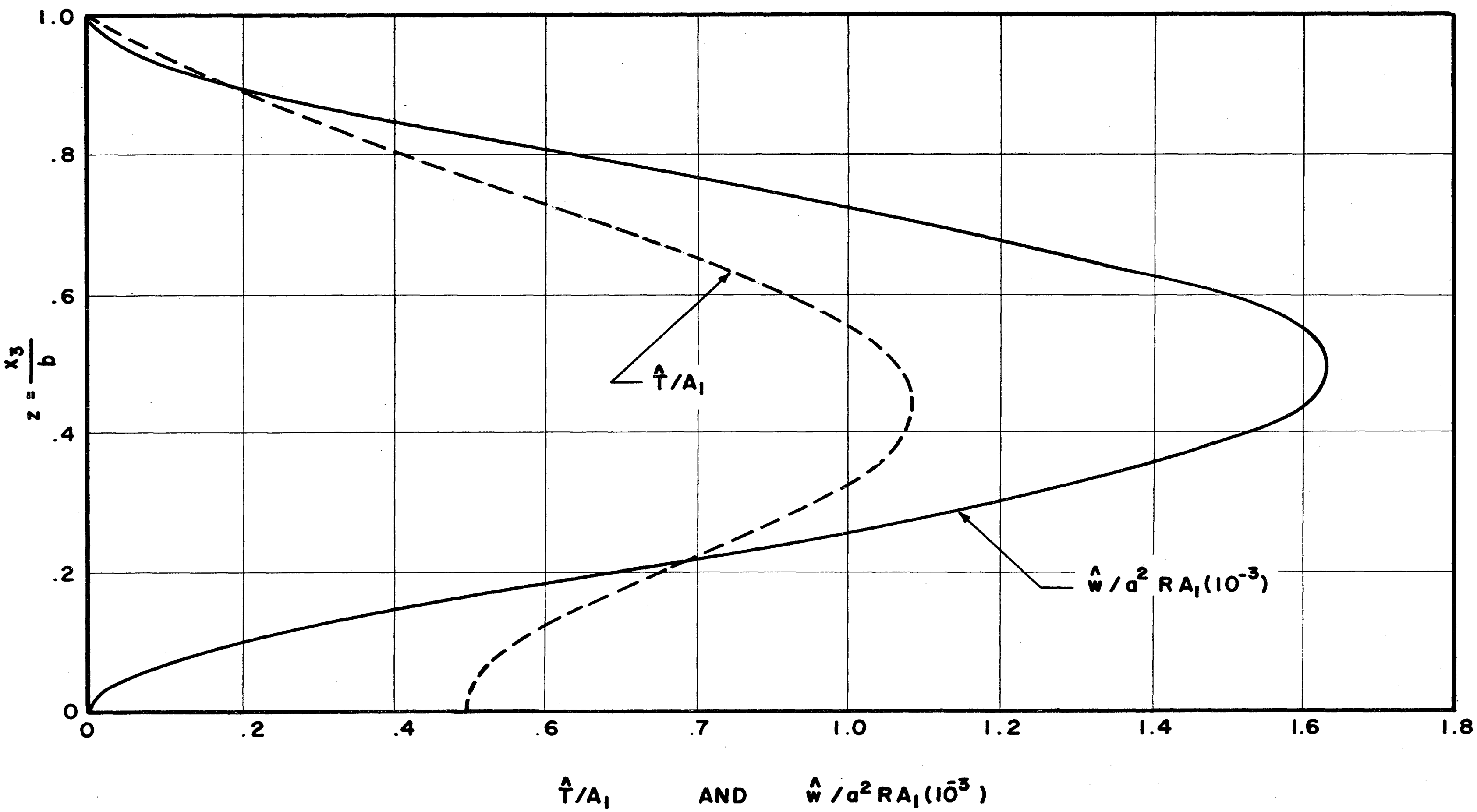


Figure 8. The variation of the nondimensional temperature and velocity functions, $\hat{T}$ and $\hat{w}$, with the nondimensional vertical coordinate, z.

$$\chi_{mn} = \frac{1}{4} - \frac{1}{(m\pi)^2} \quad \text{for } m=n\,,$$

$$= \frac{-4}{\pi^2(n-m)^2} \quad \text{for } \frac{n+m}{2} \text{ even and } m \neq n\,,$$

$$= \frac{-4}{\pi^2(n+m)^2} \quad \text{for } \frac{n+m}{2} \text{ odd and } m \neq n\,, \tag{102}$$

$$\delta_{mn} = 1 \quad \text{for } n=m\,,$$
$$= 0 \quad \text{for } n \neq m\,, \tag{103}$$

and

$$M_m = (m\pi/2)^2 + \bar{a}^2.$$

The relationship between $\bar{a}$ and $\mathbb{R}$ which must be maintained for a solution of (97) is obtained by solving successively this determinant for m = n = 1, m = n = 3, m = n = 5, etc. The results of such computations are given in Fig. 7 . The functions $\hat{T}$ and $\hat{W}$ are plotted in Fig. 8.

2. <u>Free Thermally Conducting Upper Surface and Rigid Thermally Insulating Lower Boundary.</u>

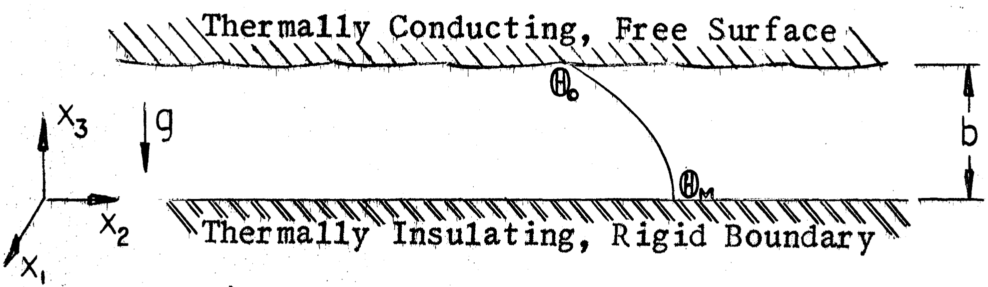


Fig. 9 Schematic Drawing Of Fluid Layer With The Mean Temperature Distribution And The Coordinate System.

The problem described by Fig. 9 can be solved in exactly the same manner as the two previously discussed. Indeed, the governing equations are those given by (89) and (90), i.e.,

$$(D^2 - a^2)^2 \hat{W} = a^2 \mathbb{R} \hat{T}$$

and

$$(D^2 - a^2)\hat{T} = -2z\hat{W}.$$

The boundary conditions for this particular case are

$$\hat{W} = 0 \quad \text{at } z = 0, 1; \tag{104i}$$

$$\frac{\partial \hat{W}}{\partial z} = 0 \quad \text{at } z = 0; \tag{104ii}$$

$$\frac{\partial^2 \hat{W}}{\partial z^2} = -a^2 \hat{W} = 0 \quad \text{at } z = 1; \tag{104iii}$$

$$\frac{\partial \hat{T}}{\partial z} = 0 \quad \text{at } z = 0; \tag{104iv}$$

$$\hat{T} = 0 \quad \text{at } z = 1. \tag{104v}$$

Boundary condition (104iii) results from the free surface which requires that there be no shear present.

The variables $\hat{T}$ and $\hat{W}$ have the forms

$$\hat{T} = \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\frac{n\pi z}{2}\right) \tag{105}$$

$$\hat{w} = \bar{a}^2 \mathbb{R} \sum_n^{\infty} A_n \Big[B_n \sinh \bar{a}z + C_n \cosh \bar{a}z + D_n z \sinh \bar{a}z$$

$$+ E_n z \cosh \bar{a}z + \frac{\cos\left(\frac{n\pi}{2}z\right)}{N_n^2} \Big], \qquad n = 1, 3, 5, \cdots \tag{106}$$

in which

$$N_n = (n\pi/2)^2 + \bar{a}^2,$$

$$B_n = D_n \frac{\coth \bar{a}}{\bar{a}}, \tag{107}$$

$$C_n = -\frac{1}{N_n^2}, \tag{108}$$

$$D_n = \frac{\bar{a} \sinh 2\bar{a}}{N_n^2 (\sinh 2\bar{a} - 2\bar{a})}, \tag{109}$$

$$E_n = -\bar{a} B_n . \tag{110}$$

The secular determinant which must be solved is

$$\left| B_n B^m + C_n C^m + D_n D^m + E_n E^m + \frac{X_{mn}}{N_n^2} - \frac{\delta_{mn} N_n}{4\bar{a}^2 \mathbb{R}} \right| = 0 \tag{111}$$

in which the B_n, etc., are obtained from (107)-(110) and the B^m, etc., are those given by (98)-(103).

The result of solving this determinant for $n = 1, 3,$ and 5 is given in Fig. 10 .

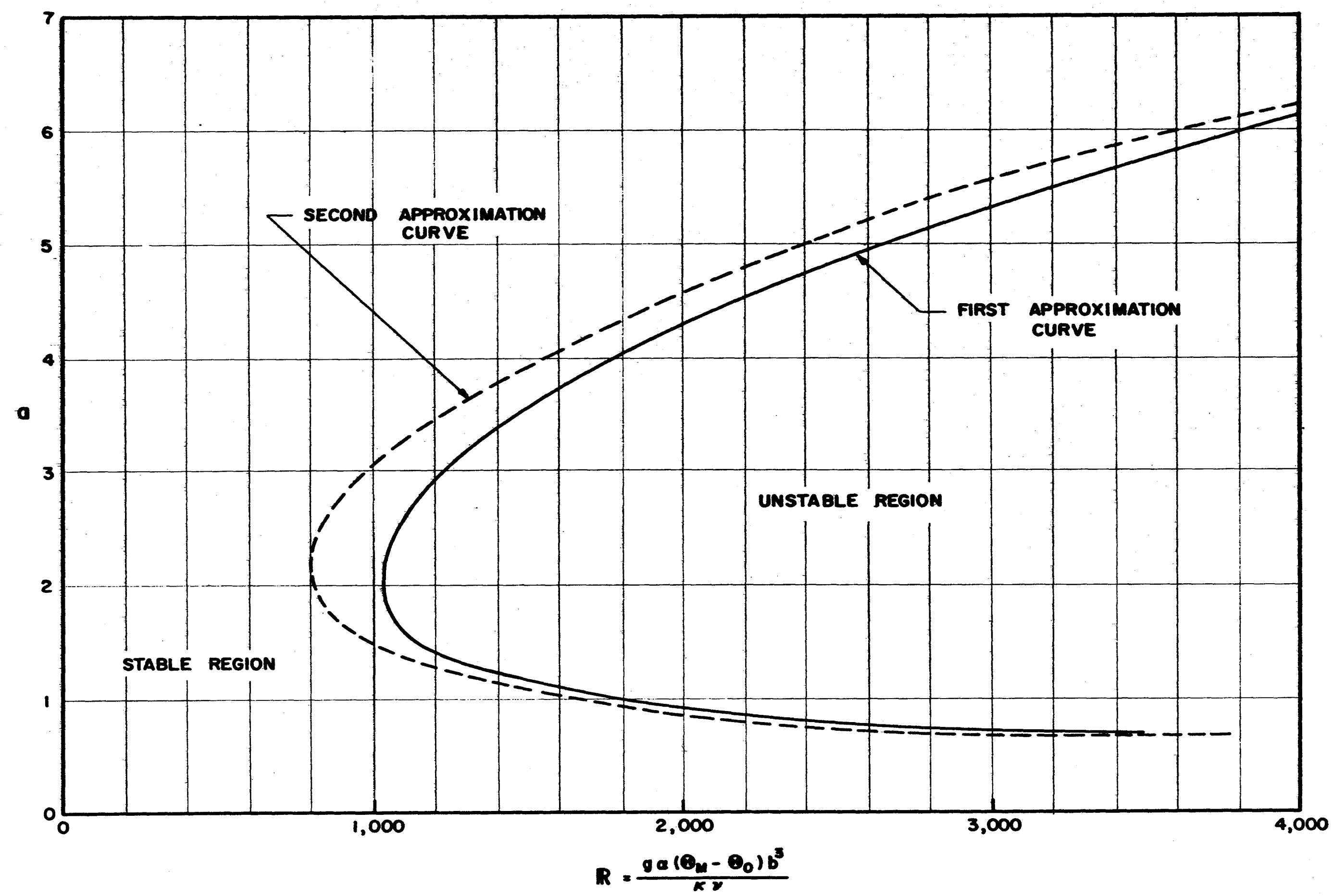


Figure 10. Neutral-stability curve for laminar natural convection in a fluid with homogeneously distributed heat sources contained between a lower adiabatic rigid boundary and an upper free isothermal boundary.

3. Other Related Problems.

In Chapter II, one stability problem for a fluid with homogeneously distributed heat sources that is confined between horizontal boundaries is discussed in detail. Two additional problems have been briefly dealt with in the present chapter. These last two problems were the result of imposing different mathematical boundary conditions at the upper and lower surfaces of the fluid layer. These boundary conditions correspond to different physical situations, and additional cases can be studied. A different solution would result from having:

a. Conducting free surfaces at the upper and lower boundaries,

b. Conducting free surface at the upper boundary and insulated free surface at the lower boundary,

c. Conducting free surface at the upper boundary and conducting rigid lower boundary,

d. Conducting rigid surface at the upper boundary and conducting free surface at the lower boundary,

e. Conducting rigid surface at the upper boundary and insulated free surface at the lower boundary.

All of these cases would permit a time-independent, stable state to exist. The solution to these problems could be obtained in the same manner as those treated in this and the preceeding chapter. For those cases in which both surfaces are heat conducting, equations (37) and (38) would be the governing relationships. The $B^{m'}s$, etc., to be used in (50) would be the same as those listed by equations (51), (52), and (53). These coefficients are listed in Appendix III. The various $B_n's$, etc., would come from the particular boundary conditions.

The cases for which the lower boundary is insulated could be studied by using equations (89) and (90). Once again the boundary conditions would determine the $B_n's$, etc., but the $B^{m'}s$, etc., are independent of these conditions and are found also in Appendix III.

4. <u>Problems Similar To Those Discussed.</u>

A set of problems which are related to those described by Figs. 6 and 9 consists of studying the thermal instability (in the sense of secondary flow) of water at 4^{o} Celsius. At this point the density-temperature relationship has a first derivative of zero which results in the well-known fact that water at 0^{o} C. is less dense than at 4^{o} C. The density-temperature relationship is approximately parabolic in the neighborhood of its maximum point as can be seen from Fig. 11. It is this fact which makes the problems similar to those just studied since the density variation induced in the heat generating fluid is also parabolic. However, one item keeps the two, seemingly similar, situations from being equivalent. For the heat generating fluid there is a term in the heat conduction equation that is not present in the problem of water at

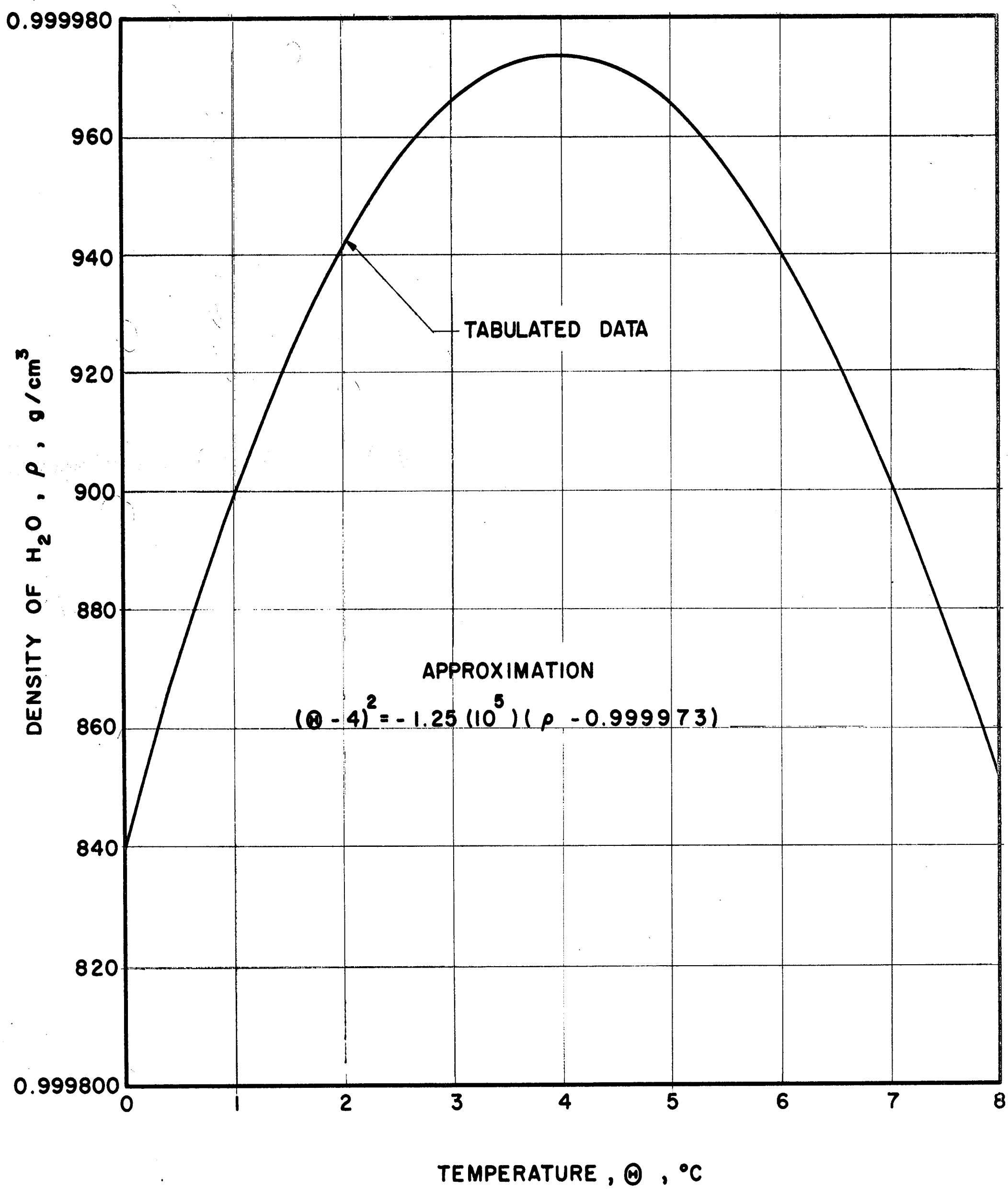


Figure 11. Temperature-density relationship for water near 4°C.

4^{o} C. Consequently, it can be expected that the equations governing the stability of water near its freezing point will be different from those given by (37) and (38) or (89) and (90). Because a discussion of this new stability problem is germain, but not completely relevant, to the stability problems of a heat generating fluid, the solution is given in Appendix IV .

CHAPTER IV

EXPERIMENTAL VERIFICATION OF ANALYSIS

An experiment was performed to verify the critical Rayleigh number which is a consequence of the analysis in Chapter II. The result of this experiment is in substantial agreement with that given by the analysis.

1. Description of Apparatus

The apparatus, shown schematically in Figure 12 , consists of the following:

a. Two copper plates, 11.5 inches square and 0.5 inches thick.

b. Three plastic plugs of predetermined height which separate the copper plates and thereby form the test chamber.

c. A lower cooling chamber in which cooling water is circulated so as to keep the plate at a constant and uniform temperature.

d. An upper cooling chamber which functions as c, above.

e. An acrylic plastic housing which contains the copper plates and cooling chambers and permits viewing of the test region.

f. Water, having suitable electrical conductivity, placed in the test region between the copper plates.

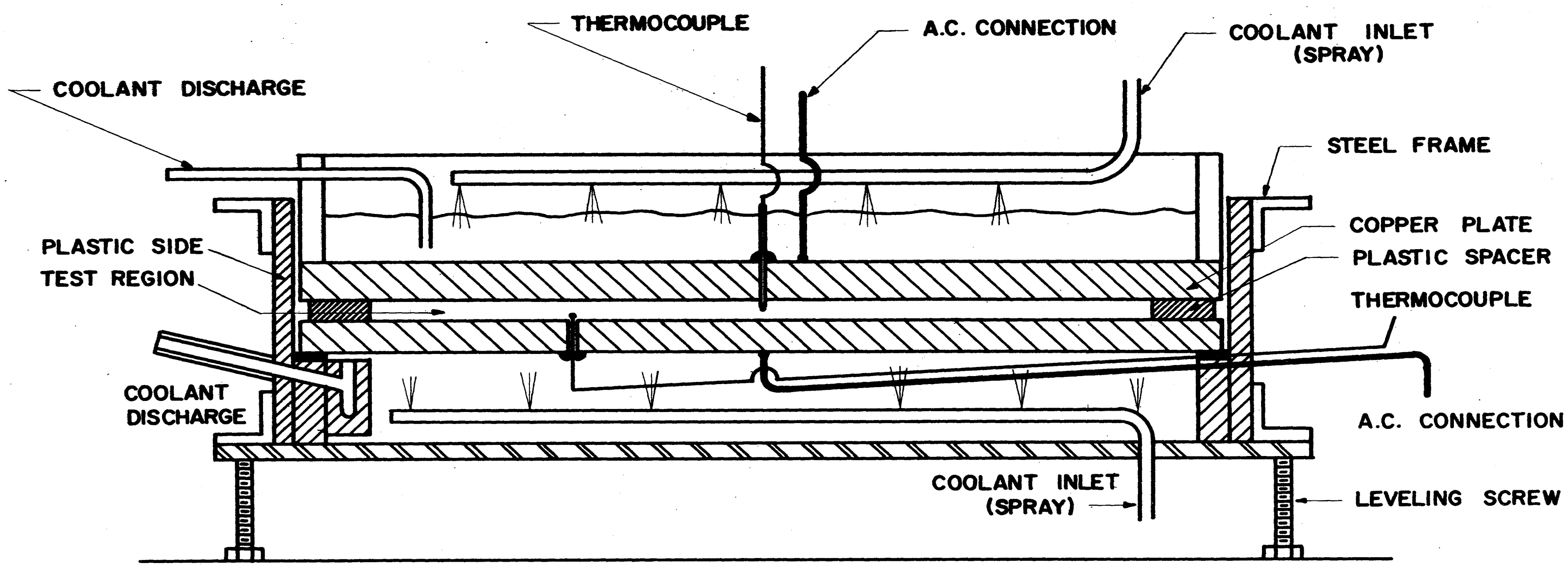


Figure 12. Schematic diagram of experimental apparatus.

Also employed in the work were a variable output voltage transformer (Variac), copper-constantan thermocouples (#36 wire), a potentiometer (Leeds & Northrup, K-2), a voltmeter and an ammeter.

The copper plates were hand-lapped so that they were smooth and flat to within 0.5 inches of each edge. These edges were slightly rounded as a result of the shearing operation which cut the plates to size. It was the opinion of Assistant Professor K. Moltrecht, Department of Mechanical Engineering, that any machining which would be extensive enough to result in having the plates flat over their entire surface could result in serious warpage of the cold-rolled plates. Consequently, the plates were hand-lapped only where necessary and the surface was flat over an area 10.5 by 10.5 inches.

The fluid layer that existed between the copper plates was heated by passing an alternating current through it by means of the transformer. This transformer allowed the power dissipated in the film, and consequently the temperature distribution, to be varied.

The surface temperature of each plate was measured and in the case of the lower plate, at least three thermocouples were installed so as to be sure that the plate was at a uniform temperature. These thermocouples recorded the temperature at the fluid-plate surface. In addition, the temperature of the fluid at the mid-plane of the layer was measured and also at a point intermediate to the copper surface and the mid-plane. The location of these thermocouples was determined with a depth gauge prior to the experiment.

The thermocouples were made of #36 wire, each strand of which was nylon coated and the two insulated wires jacketed together in nylon. These thermocouples were calibrated against a steam and an ice point. In addition, a test was made to determine the characteristics of the thermocouples between 0° and 30° C. These tests were conducted in the Sohma Laboratory, College of Engineering, University of Michigan. The results of these calibrations were used in interpreting the potentiometer readings during the stability test.

Commercially distilled water was used in the test section of the apparatus. This fluid has sufficient electrical conductivity to permit its use and its well-tabulated values for viscosity and thermal conductivity made it a desirable medium.

The cooling chambers that kept the plates at a constant and uniform temperature contained a spray system so that fine jets of water were directed toward the plates. In this manner the coolant was kept in an agitated state and any dissolved air that came out of solution would be swept away before it had an opportunity to become a large insulating bubble. The fluid was removed from each cooling chamber at 4 locations which were near the copper plates.

This was done to facilitate air removal and to increase the "scrubbing" action of the coolant.

In order to observe the fluid motion, prior and subsequent to convection, manometer fluid with a specific gravity of 1.000 at 20° C was injected into the test region. A #25 hypodermic needle was used for this injection, and the small bore of the needle resulted in extremely fine oil particles. Prior to convection, these particles would eventually move to one of the copper plates to which they would attach themselves. This migration resulted from the minute difference in density between the oil and the distilled water in the test section. After the onset of convection the oil particles moved in a regular and continuous flow pattern.

In order to observe the particle motion the test chamber was lighted from the back side so as to silhouette the particles.

2. Experimental Procedure

Cooling water was allowed to circulate so as to bring the entire apparatus to a uniform temperature. Readings of temperature were taken to confirm this uniformity. A small potential difference was applied across the plates and after a 10 minute interval the thermocouple outputs were recorded. These readings were repeated 5 minutes later to determine whether equilibrium had been achieved. Oil was injected and the particles observed to determine whether they had any persistent and regular motion. If the particles eventually came to rest it was concluded that convection had not commenced. Under these conditions the voltage across the plates (and consequently the power dissipated in heating) was increased a small amount and readings were recorded. This procedure continued until convection was observed.

After convection state was established, the power was slowly reduced until the conduction state reappeared. In the course of lowering the power dissipation, thermocouple readings were taken so as to establish that equilibirum conditions (in the mean) were present whenever the nature of the fluid motion was being observed. Several experiments were conducted to verify the analytical work and the fluid depth b was varied as a parameter. It will be recalled that this dimension enters the Rayleigh number to the third power.

3. Results of Experiment

The results of the experiment shown in Table II, are in substantial agreement with the analytical work. It can be noted that the experimentally determined critical Rayleigh numbers are higher than predicted. The inability to observe the exact instability point by means of dye or the oil particles and also the

TABLE II

SUMMARY OF EXPERIMENTAL DATA

(Data Associated with the Onset of Natural Convection in a Fluid with Homogeneously Distributed Heat Sources which is Confined Between Two Rigid, Isothermal, Horizontal Boundaries)

Plate Spacing, b (Inches)	0.250	0.500	0.745
Fluid Temperature at Mid-plane, Θ_M (°F)	69.0	63.8	61.4
Fluid Temperature at Solid boundary, Θ_b (°F)	65.5	63.2	61.1
Average Coolant Temperature (°F)	64.8	--	61.1
Applied Voltage Volts	40.0	24	11.0
Current (Amperes)	3.5	1.2	0.4
Kinematic Viscosity (Ft^2/Sec)	$1.135(10^{-5})$	$1.17(10^{-5})$	$1.20(10^{-5})$
Coefficient of Thermal Expansion (1/°F)	$1(10^{-4})$	$1(10^{-4})$	$1(10^{-4})$
Thermal Diffusivity (Ft^2/Sec)	$(1.55)(10^{-6})$	$(1.55)(10^{-6})$	$(1.55)(10^{-6})$
Rayleigh Number			
Experimental	5800	7700	12,600
Theoretical	4672	4672	4672

influence of the vertical boundaries on the instability would tend to give a critical value that was higher than predicted. A more sensitive test for the onset of conduction could be employed by using an apparatus similar to that of Schmidt and Milverton (3). This equipment utilized the defraction of a light beam that passed through the fluid. Under conduction conditions the light beam was defracted uniformly across the plate but during convection the light beam was defracted non-uniformly across the plate as a result of the temperature variations that are associated with the convection cell. It is anticipated that the apparatus that was constructed can be used to test and explore other convection phenomena, including the behavior of non-Newtonian fluids. For such experimental work a light-technique should be developed and employed.

CHAPTER V

RESULTS OF THEORETICAL ANALYSIS AND EXPERIMENTAL INVESTIGATION

The theoretical analysis for the various cases treated shows that there is a critical Rayleigh number that marks the transition to laminar natural convection from a quiescent state. This Rayleigh number is different for each set of velocity and temperature conditions that can exist at the horizontal fluid boundaries.

a. For the case of two rigid and isothermal boundaries the critical Rayleigh number is 4672 and the associated cell number is 4.0.

b. If the lower rigid boundary is adiabatic while the upper rigid boundary is isothermal, the critical Rayleigh number is 1393 and the concomitant cell number is 2.5.

c. The critical Rayleigh number is 810 and the attendant cell number is 2.25 if the lower boundary is rigid and adiabatic, and the upper boundary is free and isothermal.

From the above results one can see that a free surface at the upper boundary of the fluid, case (c), is less stabilizing than a rigid boundary, case (b). It is possible to interpret the results from the case of the heat generating fluid contained between isothermal horizontal boundaries, case (a), in terms of the role played by a free surface on stability. The mid-plane of this layer can be thought of as being a free surface, below which there is a fluid with a stable temperature distribution (i.e., positive gradients). Above the mid-plane there is a fluid with an unstable temperature distribution. This layer has a depth of d which is one half of the full distance between the plates, b. At the interface of these two fluids, the mid-plane, neither the velocity nor the temperature is prescribed. By using the half depth, d, as the characteristic length for the upper layer, the criterion for stability of this layer is one-eigth of that found for case (a). The result is a Rayleigh number of 584.

Hence, a free upper surface condition is destabilizing and a free lower surface upon which neither the temperature nor its gradient is specified is also destabilizing.

The analysis carried out in Chapter III for two problems with non-positive temperature gradient shows that the critical Rayleigh number for those problems is less than 1709, the value associated with the Bénard problem. (Indeed, this result holds for the problem of Chapter II if our attention is directed to the layer in the top half of the region for which, as was discussed previously, the critical Rayleigh number is 584). Thus, it can be concluded that the mean temperature distribution, and not just the average temperature difference between the upper and lower surfaces, affects the thermal stability of the fluid. From this result one can conjecture that the thermal instability of a fluid which has a changing temperature distribution (e.g., transient heating) may not be solely governed by the overall temperature differences incurred.

It should be pointed out that the boundary condition on the perturbation temperature in Chapter III is not the same as that for the Bénard problem. In the cases which were examined the vertical temperature gradient was required to remain zero (i.e., an adiabatic surface) and in the Bénard problem, as well as for all isothermal boundary conditions, the perturbation temperature must remain zero at the boundary. However, a calculation (not presented herein) was performed for the problem described in Fig. 6 but with the artificial thermal boundary condition at the lower surface which required that the perturbation temperature be zero there. This calculation also yielded a critical Rayleigh number which was less than 1709 and thereby substantiated the belief that the higher local temperature gradients associated with a non-linear temperature distribution tend to destabilize the system.

For the problem in Chapter II which has a positive mean temperature gradient over the lower half of the region, the critical Rayleigh number is increased over that of the Bénard problem, if the characteristic length is the full fluid depth, b . Thus, a local positive temperature gradient, favorably placed can have a stabilizing effect on the system.

The experimental work described in Chapter IV and summarized in Table II is in agreement with the analytical predictions. Therefore, the assumption of the "principle of exchange of stabilities" was justified in the solutions. The cell shape has not been measured.

CHAPTER VI

CONCLUSIONS

1. For the case of a fluid with homogeneously distributed heat sources contained within horizontal boundaries, the onset of laminar natural convection is governed by a critical Rayleigh number, the magnitude of which depends upon the velocity and temperature conditions existing at the horizontal boundaries. A free upper surface is destabilizing and a lower surface upon which neither the temperature nor its gradient is specified is also destabilizing. A local positive temperature gradient, favorably placed, can be stabilizing.

2. The Rayleigh number associated with the onset of instability for the fluid under consideration is determined not only by the overall mean temperature difference but also by the mean temperature distribution in the fluid.

3. Detailed relationships between the Rayleigh number and the cell number have been obtained by an approximative method for three problems dealing with a fluid having homogeneously distributed heat sources. The results of these solutions are presented herein via graphs.

4. An experimental study of one of the problems which was treated analytically yielded a Rayleigh number which was in general agreement with the analytical prediction. The result of this experiment justified the use of the "principle

of exchange of stabilities" in the mathematical solutions.

APPENDIX I

EVALUATION OF INTEGRALS WITH TRIGONOMETRIC AND HYPERBOLIC INTEGRANDS

Note: $\quad sh\,az = \sinh az \;, \quad ch\,az = \cosh az$

1. $$\int sh\,az \sin bz\, dz = \left(\frac{a}{a^2+b^2}\right) ch\,az \sin bz - \left(\frac{b}{a^2+b^2}\right) sh\,az \cos bz$$

2. $$\int sh\,az \cos bz\, dz = \left(\frac{a}{a^2+b^2}\right) ch\,az \cos bz + \left(\frac{b}{a^2+b^2}\right) sh\,az \sin bz$$

3. $$\int ch\,az \cos bz\, dz = \left(\frac{a}{a^2+b^2}\right) sh\,az \cos bz + \left(\frac{b}{a^2+b^2}\right) ch\,az \sin bz$$

4. $$\int ch\,az \sin bz\, dz = \left(\frac{a}{a^2+b^2}\right) sh\,az \sin bz - \left(\frac{b}{a^2+b^2}\right) ch\,az \cos bz$$

5. $$\begin{aligned}\int z\, sh\,az \sin bz\, dz = {} & z\left\{\left(\frac{a}{a^2+b^2}\right) ch\,az \sin bz - \left(\frac{b}{a^2+b^2}\right) sh\,az \cos bz\right\} \\ & - \left(\frac{a}{a^2+b^2}\right)\left\{\left(\frac{a}{a^2+b^2}\right) sh\,az \sin bz - \left(\frac{b}{a^2+b^2}\right) ch\,az \cos bz\right\} \\ & + \left(\frac{b}{a^2+b^2}\right)\left\{\left(\frac{b}{a^2+b^2}\right) sh\,az \sin bz + \left(\frac{a}{a^2+b^2}\right) ch\,az \cos bz\right\}\end{aligned}$$

6. $$\begin{aligned}\int z\, sh\,az \cos bz\, dz = {} & z\left\{\left(\frac{a}{a^2+b^2}\right) ch\,az \cos bz + \left(\frac{b}{a^2+b^2}\right) sh\,az \sin bz\right\} \\ & - \left(\frac{a}{a^2+b^2}\right)\left\{\left(\frac{a}{a^2+b^2}\right) sh\,az \cos bz + \left(\frac{b}{a^2+b^2}\right) ch\,az \sin bz\right\} \\ & - \left(\frac{b}{a^2+b^2}\right)\left\{\left(\frac{a}{a^2+b^2}\right) ch\,az \sin bz - \left(\frac{b}{a^2+b^2}\right) sh\,az \cos bz\right\}\end{aligned}$$

7. $$\int z\,\mathrm{ch}\,az\cos bz\,dz = z\left\{\left(\frac{a}{a^2+b^2}\right)\mathrm{sh}\,az\cos bz+\left(\frac{b}{a^2+b^2}\right)\mathrm{ch}\,az\sin bz\right\}$$

$$-\left(\frac{a}{a^2+b^2}\right)\left\{\left(\frac{a}{a^2+b^2}\right)\mathrm{ch}\,az\cos bz+\left(\frac{b}{a^2+b^2}\right)\mathrm{sh}\,az\sin bz\right\}$$

$$-\left(\frac{b}{a^2+b^2}\right)\left\{\left(\frac{a}{a^2+b^2}\right)\mathrm{sh}\,az\sin bz-\left(\frac{b}{a^2+b^2}\right)\mathrm{ch}\,az\cos bz\right\}$$

8. $$\int z\,\mathrm{ch}\,az\sin bz\,dz = z\left\{\left(\frac{a}{a^2+b^2}\right)\mathrm{sh}\,az\sin bz-\left(\frac{b}{a^2+b^2}\right)\mathrm{ch}\,az\cos bz\right\}$$

$$-\left(\frac{a}{a^2+b^2}\right)\left\{\left(\frac{a}{a^2+b^2}\right)\mathrm{ch}\,az\sin bz-\left(\frac{b}{a^2+b^2}\right)\mathrm{sh}\,az\cos bz\right\}$$

$$+\left(\frac{b}{a^2+b^2}\right)\left\{\left(\frac{a}{a^2+b^2}\right)\mathrm{sh}\,az\cos bz+\left(\frac{b}{a^2+b^2}\right)\mathrm{ch}\,az\sin bz\right\}$$

9. $$\int z^2\,\mathrm{sh}\,az\sin bz\,dz = z^2\left\{\left(\frac{a}{a^2+b^2}\right)\mathrm{ch}\,az\sin bz-\left(\frac{b}{a^2+b^2}\right)\mathrm{sh}\,az\cos bz\right\}$$

$$+\frac{2z}{(a^2+b^2)^2}\left\{(b^2-a^2)\,\mathrm{sh}\,az\sin bz+2ab\,\mathrm{ch}\,az\cos bz\right\}$$

$$+\frac{2}{(a^2+b^2)^3}\left\{a(a^2-3b^2)\,\mathrm{ch}\,az\sin bz+b(b^2-3a^2)\,\mathrm{sh}\,az\cos bz\right\}$$

10. $$\int z^2 \operatorname{sh} az \cos bz \, dz = \left(\frac{z^2}{a^2+b^2}\right)\left\{a \operatorname{ch} az \cos bz + b \operatorname{sh} az \sin bz\right\}$$

$$- \frac{2z}{(a^2+b^2)^2}\left\{(a^2-b^2) \operatorname{sh} az \cos bz + 2ab \operatorname{ch} az \sin bz\right\}$$

$$- \frac{2}{(a^2+b^2)^3}\left\{b(b^2-3a^2 \operatorname{sh} az \sin bz - a(a^2-3b^2) \operatorname{ch} az \cos bz\right\}$$

11. $$\int z^2 \operatorname{ch} az \cos bz \, dz = \frac{z^2}{(a^2+b^2)}\left\{a \operatorname{sh} az \cos bz + b \operatorname{ch} az \sin bz\right\}$$

$$+ \frac{2z}{(a^2+b^2)^2}\left\{(b^2-a^2) \operatorname{ch} az \cos bz - 2ab \operatorname{sh} az \sin bz\right\}$$

$$- \frac{2}{(a^2+b^2)^3}\left\{b(b^2-3a^2) \operatorname{ch} az \sin bz - a(a^2-3b^2) \operatorname{sh} az \cos bz\right\}$$

12. $$\int z^2 \operatorname{ch} az \sin bz \, dz = \frac{z^2}{(a^2+b^2)}\left\{a \operatorname{sh} az \sin bz - b \operatorname{ch} az \cos bz\right\}$$

$$+ \frac{2z}{(a^2+b^2)^2}\left\{(b^2-a^2) \operatorname{ch} az \sin bz + 2ab \operatorname{sh} az \cos bz\right\}$$

$$+ \frac{2}{(a^2+b^2)^3}\left\{a(a^2-3b^2) \operatorname{sh} az \sin bz + b(b^2-3a^2) \operatorname{ch} az \cos bz\right\}$$

$$\int_0^1 z\cos\frac{m\pi}{2}z\cos\frac{n\pi}{2}\,dz = -\frac{4}{\pi^2(m-n)^2} \quad \text{for } \frac{m+n}{2} \text{ even, } m\neq n$$

$$= -\frac{4}{\pi^2(m+n)^2} \quad \text{for } \frac{m+n}{2} \text{ odd, } m\neq n$$

$$= \frac{1}{4} - \frac{1}{(m\pi)^2} \quad \text{for } m=n.$$

$$\int_0^1 z\sin n\pi z\,\sin m\pi z\,dz = \frac{2mn}{\pi^2(m^2-n^2)^2}\left[(-1)^{m+n}-1\right] \quad \text{for } m\neq n$$

$$= \frac{1}{4} \quad \text{for } m=n.$$

APPENDIX II

SIMPLIFICATION OF SECULAR DETERMINANT FOR SOLUTION OF A PARTICULAR INSTABILITY PROBLEM

CONSIDER THE FOLLOWING IN WHICH $\sinh a = sha$ AND $\cosh a = cha$:

$$\alpha_{mn} = C_n C^m + D_n D^m + E_n E^m$$

$$\alpha_{11} = C_1\left\{\frac{-\pi}{M_1} sha + \frac{4a\pi}{M_1^3}(cha+1)\right\} + D_1\left\{\frac{2a\pi}{M_1^2}(3cha-1) - \frac{\pi}{M_1} sha\left(1+\frac{16a^2}{M_1^2} - \frac{4}{M_1}\right)\right\}$$

$$+E_1\left\{\frac{-\pi}{M_1} cha + \frac{6a\pi}{M_1^2} sha - \frac{4\pi}{M_1^2}(cha+1)\left(\frac{4a^2}{M_1} - 1\right)\right\}$$

$$C_1 = \frac{-\pi}{N_1^2}\left(\frac{1}{sha+a}\right), \quad D_1 = -C_1(cha+1), \quad E_1 = C_1\, sha, \quad N_1 = M_1$$

$$\alpha_{11} = C_1\left\{\frac{-\pi}{M_1} sha + \frac{4a\pi\, cha}{M_1^2} + \frac{4a\pi}{M_1^2} - \frac{6a\pi\, cha}{M_1^2} + \frac{2a\pi}{M_1^2} + \frac{\pi}{M_1} sha\right.$$

$$+ \frac{16\pi a^2 sha}{M_1^3} - \frac{4\pi}{M_1^2} sha - \frac{6a\pi}{M_1^2} ch^2a + \frac{2a\pi\, cha}{M_1^2} + \frac{\pi}{M_1} sha\, cha$$

$$+ \frac{\pi}{M_1^3} 16a^2 sha\, cha - \frac{4\pi}{M_1^2} sha\, cha - \frac{\pi}{M_1} sha\, cha + \frac{6a\pi}{M_1^2} sh^2a$$

$$\left. - \frac{16a^2\pi}{M_1^3} cha\, sha + \frac{4\pi}{M_1^2} sha\, cha - \frac{16a^2\pi}{M_1^3} sha + \frac{4\pi}{M_1^2} sha\right\}$$

$$\alpha_{11} = C_1\{0\} = 0$$

$$\alpha_{22} = C_2\left\{\frac{2\pi}{M_2}\,\mathrm{sh}\,a - \frac{8a\pi}{M_2^2}(\mathrm{ch}\,a-1)\right\} + D_2\left\{-\frac{4a\pi}{M_2^2}(3\,\mathrm{ch}\,a+1)+\frac{2\pi}{M_2}\,\mathrm{sh}\,a\left(1+\frac{16a^2}{M_2^2}-\frac{4}{M_2}\right)\right\}$$

$$+E_2\left\{\frac{2\pi}{M_2}\,\mathrm{ch}\,a - \frac{12a\pi}{M_2^2}\,\mathrm{sh}\,a + \frac{8\pi}{M_2^2}(\mathrm{ch}\,a-1)\left(\frac{4a^2}{M_2}-1\right)\right\}$$

$$C_2 = \frac{2\pi}{N_2^2}\left\{\frac{1}{\mathrm{sh}\,a-a}\right\},\quad D_2 = C_2(\mathrm{ch}\,a-1),\quad E_2 = -C_2\,\mathrm{sh}\,a,\quad M_2 = N_2$$

$$\alpha_{22} = \left\{\frac{2\pi}{M_2}\,\mathrm{sh}\,a - \frac{8a\pi}{M_2^2}\,\mathrm{ch}\,a + \frac{8a\pi}{M_2^2} - \frac{12a\pi}{M_2^2}\,\mathrm{ch}^2 a - \frac{4a\pi}{M_2^2}\,\mathrm{ch}\,a + \frac{2\pi}{M_2}\,\mathrm{sh}\,a\,\mathrm{ch}\,a\right.$$

$$+\frac{32\pi a^2}{M_2^3}\,\mathrm{sh}\,a\,\mathrm{ch}\,a - \frac{8\pi}{M_2^2}\,\mathrm{sh}\,a\,\mathrm{ch}\,a + \frac{12a\pi}{M_2^2}\,\mathrm{ch}\,a + \frac{4a\pi}{M_2^2} - \frac{2\pi\,\mathrm{sh}\,a}{M_2} - \frac{32\pi a^2\,\mathrm{sh}\,a}{M_2^3}$$

$$+\frac{8\pi}{M_2^2}\,\mathrm{sh}\,a - \frac{2\pi}{M_2}\,\mathrm{ch}\,a\,\mathrm{sh}\,a + \frac{12a\pi}{M_2^2}\,\mathrm{sh}^2 a - \frac{32a^2\pi}{M_2^3}\,\mathrm{ch}\,a\,\mathrm{sh}\,a$$

$$\left.+\frac{8\pi}{M_2^2}\,\mathrm{ch}\,a\,\mathrm{sh}\,a + \frac{32\pi a^2}{M_2^3}\,\mathrm{sh}\,a - \frac{8\pi}{M_2^2}\,\mathrm{sh}\,a\right\}C_2$$

$$\alpha_{22} = C_2\{0\} = 0$$

$$\alpha_{12} = C_2\left\{-\frac{\pi}{M_1}\,\mathrm{sh}\,a + \frac{4a\pi}{M_1^2}\,\mathrm{ch}\,a + \frac{4a\pi}{M_1^2}\right\} + D_2\left\{\frac{2a\pi}{M_1^2}(3\,\mathrm{ch}\,a-1)\right.$$

$$\left.-\frac{\pi}{M_1}\,\mathrm{sh}\,a\left(1+\frac{16a^2}{M_1^2}-\frac{4}{M_1}\right)\right\} + E_2\left\{-\frac{\pi}{M_1}\,\mathrm{ch}\,a + \frac{6a\pi}{M_1^2}\,\mathrm{sh}\,a - \frac{4\pi}{M_1^2}(\mathrm{ch}\,a+1)\left(\frac{4a^2}{M_1}-1\right)\right\}$$

$$= C_2\left\{-\frac{\pi}{M_1}\,\mathrm{sh}\,a + \frac{4a\pi}{M_1^2}\,\mathrm{ch}\,a + \frac{4a\pi}{M_1^2} + \frac{6a\pi}{M_1^2} - \frac{2a\pi}{M_1^2}\,\mathrm{ch}\,a - \frac{\pi}{M_1}\,\mathrm{sh}\,a\,\mathrm{ch}\,a - \frac{16a^2\pi}{M_1^3}\,\mathrm{sh}\,a\,\mathrm{ch}\,a\right.$$

$$+\frac{4\pi}{M_1^2}\,\mathrm{sh}\,a\,\mathrm{ch}\,a - \frac{6a\pi}{M_1^2}\,\mathrm{ch}\,a + \frac{2a\pi}{M_1^2} + \frac{\pi}{M_1}\,\mathrm{sh}\,a + \frac{16a^2\pi}{M_1^3}\,\mathrm{sh}\,a - \frac{4\pi}{M_1^2}\,\mathrm{sh}\,a$$

$$\left.+\frac{\pi}{M_1}\,\mathrm{ch}\,a\,\mathrm{sh}\,a - \frac{6a\pi}{M_1^2}\,\mathrm{sh}^2 a + \frac{16a^2\pi}{M_1^3}\,\mathrm{sh}\,a\,\mathrm{ch}\,a + \frac{16a^2\pi}{M_1^3}\,\mathrm{sh}\,a - \frac{4\pi}{M_1^2}\,\mathrm{ch}\,a\,\mathrm{sh}\,a - \frac{4\pi\,\mathrm{sh}\,a}{M_1^2}\right\}$$

$$a_{12} = C_2\left\{ -\frac{4a\pi}{M_1^2}(cha-3) + \frac{8\pi}{M_1^2}\overset{sha}{\wedge}\left(\frac{4a^2}{M_1}-1\right)\right\}$$

$$a_{21} = C_1\left\{\frac{2\pi}{M_2}sha - \frac{8a\pi}{M_2^2}(cha-1)\right\} + D_1\left\{-\frac{4a\pi}{M_2^2}(3cha+1) + \frac{2\pi}{M_2}sha\left(1+\frac{16a^2}{M_2^2}-\frac{4}{M_2}\right)\right\}$$

$$+E_1\left\{\frac{2\pi}{M_2}cha - \frac{12a\pi}{M_2^2}sha + \frac{8\pi}{M_2^2}(cha-1)\left(\frac{4a^2}{M_2}-1\right)\right\}$$

$$= C_1\left\{\frac{2\pi}{M_2}sha - \frac{8a\pi}{M_2^2}cha + \frac{8a\pi}{M_2^2} + \frac{12a\pi}{M_2^2}ch^2a + \frac{4a\pi}{M_2^2}cha - \frac{2\pi}{M_2}sha\,cha\right.$$

$$-\frac{32\pi a^2}{M_2^3}sha\,cha + \frac{8\pi}{M_2}sha\,cha + \frac{12a\pi}{M_2^2}cha + \frac{4a\pi}{M_2^2} - \frac{2\pi}{M_2}sha$$

$$-\frac{32}{M_2^3}\pi a^2 sha + \frac{8\pi}{M_2}sha + \frac{2\pi}{M_2}sha\,cha - \frac{12a\pi}{M_2^2}sh^2a + \frac{32\pi a^2}{M_2^3}sha\,cha$$

$$\left.-\frac{8\pi}{M_2^2}sha\,cha - \frac{32\pi a^2}{M_2^3}sha + \frac{8\pi}{M_2^2}sha\right\}$$

$$a_{21} = C_1\left\{\frac{8a\pi}{M_2^2}(cha+3) + \frac{16\pi}{M_2^2}sha\left(1-\frac{4a^2}{M_2}\right)\right\}$$

Appendix III

Coefficients for Approximative Solution to Problems Concerning the Instability of a Fluid with Homogeneously Distributed Heat Sources

SOME COEFFICIENTS FOR EQUATION (54)

a	2.0	2.5	3.0	3.5	5.0	7.5
C^1	-0.199332868	-0.3168172263	-0.4960498810	-0.7698450147	-2.798851197	-23.44290547
C^2			1.007484995			
D^1	-0.1859343827	-0.2886579660	-0.4428650738	-0.6761169067	-2.391378528	-19.97481865
D^2			0.7702893738			
E^1	-0.1891785569	-0.287920011	-0.439758093	-0.6716829603	-2.386097261	-19.97099443
E^2			0.7977948156			

Appendix III (continued)

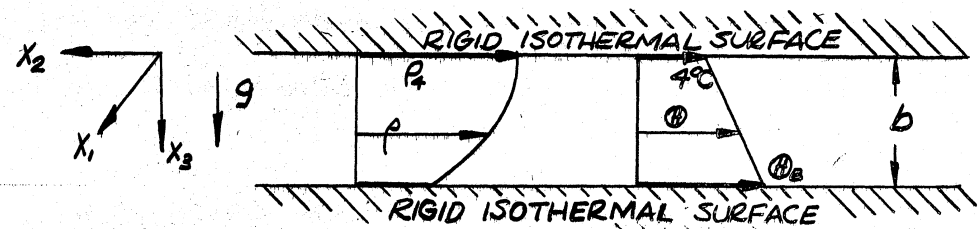
SOME COEFFICIENTS FOR EQUATION (97)

a	2.0	2.5	3.0	3.5	4.0	4.5
B^1	0.31574094	0.45641090	0.65068509	0.92430151	1.31500195	1.87864821
B^3	-0.54891307	-0.82347782	-1.22046037		-2.66066941	
B^5		0.64728587				
C^1	0.40558217	0.52946244	0.71075084	0.97422023	1.35691261	1.91417703
C^3	-0.60347305	-0.85916531	-1.24298574		-2.66765099	
C^5		0.66948629				
D^1	0.20241665	0.29850517	0.43477693	0.63125505	0.91767235	1.33853549
D^3	-0.41235461	-0.623395456	-0.93167713			-2.06710222
E^1	0.24062183	0.32790763	0.45763619	0.64920745	0.93191406	1.34994718
E^3	-0.45731265	-0.65474960	-0.95353673			-2.07764530
E^5		0.59280352				

APPENDIX IV

SOLUTION FOR THE THERMAL INSTABILITY OF WATER AT 4° CELSIUS

CONSIDER THE FOLLOWING SYSTEM:



FOR THIS SYSTEM

$$\bar{\Theta} = 4°C + x_3 \beta ,$$
$$(\bar{\Theta}-4)^2 = -\omega\,(\rho-\rho_4), \qquad (\text{c.f., FIG. 11})$$
$$(\bar{\Theta}-4)^2 = (x_3\,\beta)^2$$

IN WHICH

$$\beta = (\bar{\Theta}_B - 4)/b,$$
$$\omega = 1.25\,(10^5)\,(°C)^2/gm/cm^3,$$
$$\rho_4 = 0.999973\ gm/cm^3$$

AND THE BAR OVER A SYMBOL DENOTES A MEAN QUANTITY. SUBSEQUENT USE OF PRIMED SYMBOLS WILL DESIGNATE A PERTURBATION QUANTITY IN ACCORDANCE WITH THE NOTATION OF CHAPTER II. THUS

$$\Theta = \bar{\Theta} + \Theta', \quad p = \bar{p} + p', \quad \rho = \bar{\rho} + \rho', \quad u_i = u_i'$$

AND

$$\bar{\rho} = \rho_4 \left| 1 - \beta^2 x_3^2/\omega\rho_4 \right|, \qquad \frac{\partial \bar{p}}{\partial x_i} = \left| + \bar{\rho} g \right| \delta_{i3} .$$

THE INTRODUCTION OF THE PERTURBED VARIABLES INTO THE NAVIER-STOKES EQUATIONS, AND THE NEGLECTING OF ALL PRODUCTS OF PERTURBATION TERMS YIELDS

$$\frac{\partial u_i}{\partial t} = -\delta_{i3}\left(\frac{2\beta g}{\omega \rho_4}\right) X_3 \Theta' - \frac{1}{\rho_4}\frac{\partial}{\partial x_i}\left[p' - \left(\lambda + \mu\right)\frac{\partial u_j}{\partial x_j}\right] + \nu \Delta u_i$$

IN WHICH

$$\rho' = -\frac{2\beta X_3 \Theta'}{\omega}, \qquad \rho_4 \approx \bar{\rho}$$

BY SUITABLE CROSS-DIFFERENTIATION TO ELIMINATE p' ONE OBTAINS

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta u_3 = -\left(\frac{2\beta g}{\omega \rho_4}\right) X_3 \Delta_1 \Theta' \qquad (A)$$

IF $\frac{\partial}{\partial X_3}\left(\frac{\partial u_i}{\partial x_i}\right)$ IS SMALL IN COMPARISON WITH Δu_3.

THE CONTINUITY EQUATION IS WRITTEN AS

$$\frac{\partial u_i}{\partial x_i} = \frac{(2\beta X_3/\omega)\left(\frac{\partial \Theta'}{\partial t} + \beta X_3 u_3\right)}{\rho_4\left(1 - X_3^2 \beta^2/\omega \rho_4\right)}$$

AGAIN ONE HAS FOR A STEADY STATE SOLUTION

$$\left|\frac{\partial u_i}{\partial x_i}\right| << |u_3|.$$

THE HEAT CONDUCTION CAN BE WRITTEN AS

$$\frac{1}{K}\frac{D\Theta}{Dt} = \Delta\Theta = \frac{1}{K}\left(\frac{\partial \Theta}{\partial t} + u_i \frac{\partial \Theta}{\partial x_i}\right)$$

OR

$$\left(\frac{\partial}{\partial t} - K\Delta\right)\Theta' = -u_3 \beta \qquad (B)$$

IF PRODUCTS OF TWO PERTURBATION TERMS ARE NEGLECTED, EQUATIONS (A) AND (B) CAN BE NON-DIMENSIONALIZED BY INTRODUCING

$$T = \Theta'/\beta b, \; w = u_3 b/K, \; \tau = tK/b^2, \; (x, y, z) = \left(\frac{X_1}{b}, \frac{X_2}{b}, \frac{X_3}{b}\right).$$

THESE VARIABLES YIELD

$$\left(\frac{K}{\nu}\frac{\partial}{\partial \tau} - \nabla^2\right)\nabla^2 w = -2\mathcal{Y}\, z \nabla_1^2 T$$

AND

$$\left(\frac{\partial}{\partial \tau} - \nabla^2\right) T = -w$$

IN WHICH

$$\mathcal{Y} = \frac{g\beta^2 b^5}{(\nu K \omega \rho)_{4°C}} .$$

BY SETTING

$$w = f\hat{w}e^{\sigma\tau} , \quad T = f\hat{T}e^{\sigma\tau}$$

ONE HAS

$$f_{xx} + f_{yy} + a^2 f = 0 ,$$

$$\left(\sigma - (D^2 - a^2)\right)\hat{T} = -\hat{w}$$

AND

$$\left(\frac{K}{\nu}\sigma - (D^2 - a^2)\right)(D^2 - a^2)\hat{w} = 2\mathcal{Y} a^2 z \hat{T}$$

IF THE "PRINCIPLE OF EXCHANGE OF STABILITIES IS ASSUMED,

$\sigma = 0$, AND

$$(D^2 - a^2)\hat{T} = \hat{w} ,$$

$$(D^2 - a^2)^2\hat{w} = -2\mathcal{Y} a^2 z \hat{T} .$$

THE BOUNDARY CONDITIONS FOR RIGID ISOTHERMAL SURFACES ARE

$$\hat{w} = 0 \text{ AT } z = 0, 1 \; ; \quad \partial\hat{w}/\partial z = 0 \text{ AT } z = 0, 1 ;$$

$$\hat{T} = 0 \text{ AT } z = 0, 1 .$$

THE THERMAL BOUNDARY CONDITIONS ARE SATISFIED BY

$$\hat{T} = \sum_{n=1}^{\infty} A_n \sin n\pi z \quad , \qquad n = 1, 2, 3 \cdots$$

WITH THIS RELATIONSHIP AND THE EQUATION OF MOTION

$$\hat{w} = -2a^2 \mathcal{Y} \sum_{n=1}^{\infty} A_n \Big(B_n \cosh az + C_n \sinh az + D_n z \sinh az + E_n z \cosh az + \frac{1}{N_n^2} z \sin n\pi z + \frac{4n\pi}{N_n^3} \cos n\pi z \Big).$$

THE HEAT CONDUCTION EQUATION REQUIRES THAT

$$-\sum_{n=1}^{\infty} A_n N_n \sin n\pi z = -2a^2 \mathcal{Y} \sum_{n=1}^{\infty} A_n \Big(B_n \cosh az + C_n \sinh az + D_n z \sinh az + E_n z \cosh az + \frac{1}{N_n^2} z \sin n\pi z + \frac{4n\pi}{N_n^3} \cos n\pi z \Big)$$

MULTIPLYING BOTH SIDES OF THIS EQUATION BY $\sin m\pi z$ AND INTEGRATING FROM ZERO TO ONE YIELDS

$$0 = \sum A_n \Big(B_n B^m + C_n C^m + D_n D^m + E_n E^m + \frac{X_{mn}}{N_n^2} + \frac{4n\pi Y_{mn}}{N_n^3} - \frac{\delta_{mn} N_n}{4a^2 \mathcal{Y}} \Big)$$

IN WHICH

$$B_n = -\frac{4n\pi}{N_n^3} ,$$

$$C_n = -\frac{E_n}{a} ,$$

$$D_n = E_n \Big(\frac{1}{a} - \coth a \Big) - \frac{4n\pi}{N_n^3} \Big(\frac{\cos n\pi - \cosh a}{\sinh a} \Big) ,$$

$$E_n = \frac{n\pi a}{N_n^3} \left\{ \frac{4a + 2\sinh 2a - 4\cos n\pi \big[a\cosh a + \sinh a (1 - N_n/4) \big]}{a^2 - \sinh^2 a} \right\},$$

$$B^m = -\left(\frac{m\pi}{M_m}\right)\left((-1)^m \cosh a - 1\right),$$

$$C^m = -\left(\frac{m\pi}{M_m}\right)\left((-1)^m \sinh a\right),$$

$$D^m = C^m - \frac{2a}{M_m} B^m,$$

$$E^m = -\left(\frac{m\pi}{M_m}\right)\left((-1)^m \cosh a\right) - \frac{2a}{M_m} C^m,$$

$$X_{mn} = \frac{2mn\left((-1)^{m+n} - 1\right)}{\pi^2\left(m^2 - n^2\right)^2}, \quad m \neq n$$

$$= 1/4, \quad m = n$$

AND

$$Y_{mn} = 0, \quad m+n = \text{EVEN}$$

$$= \frac{2m}{\pi\left(m^2 - n^2\right)}.$$

USING THESE COEFFICIENTS AND THE METHOD OF CHANDRASEKHAR, ONE OBTAINS A MINIMUM VALUE OF $\mathcal{Y}$ AS

$$\mathcal{Y} = 1701 \quad \text{AT} \quad a = 3.0.$$

A SECOND APPROXIMATION WAS USED TO OBTAIN THIS VALUE.

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